Incident Wave @ Obstacles Not Centered @ The origin

I. $\underline{u}_{inc}$ in Global coordinates

Recall: We define the incident wave $\underline{u}_{inc}$ (which is a vector displacement) in terms of two potentials ϕ_{inc} and ψ_{inc}, the compressional and shear scalar potentials, respectively, as follows:

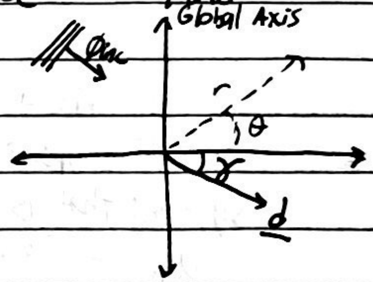
$$\boxed{\underline{u}_{inc} = \nabla_{r,\theta}\,\phi_{inc} + \nabla_{r,\theta,z} \times \psi_{inc}\,\hat{z}}$$

For our specific problem, we define these potentials as

① $\phi_{inc}(r,\theta) = e^{ik_p \underline{x}\cdot\underline{d}} = e^{ik_p (r\cos\theta,\, r\sin\theta)\cdot(\cos\gamma,\, \sin\gamma)}$

$$\Rightarrow \quad \boxed{\phi_{inc}(r,\theta) = e^{ik_p r\cos(\theta-\gamma)}}$$

where γ is the "angle of incidence" of ϕ_{inc} in the global-polar-coordinate system; and (r,θ) are global polar coordinates.



② $\psi_{inc}(r,\theta) \equiv 0$

Thus, in global polar coordinates, we can write $\underline{u}_{inc}$ as

$$\boxed{\underline{u}_{inc}(r,\theta) = \begin{bmatrix} (u_{inc})_r(r,\theta) \\ (u_{inc})_\theta(r,\theta) \end{bmatrix} = \nabla_{r,\theta}\,\phi_{inc} = \begin{bmatrix} \frac{\partial \phi_{inc}}{\partial r}(r,\theta) \\ \frac{1}{r}\frac{\partial \phi_{inc}}{\partial \theta}(r,\theta) \end{bmatrix} = \begin{bmatrix} ik_p\cos(\theta-\gamma)\,\phi_{inc}(r,\theta) \\ -ik_p\sin(\theta-\gamma)\,\phi_{inc}(r,\theta) \end{bmatrix}}$$

(Here, the notation $(u_{inc})_r$ means "the (compressional) displacement in the radial direction", and $(u_{inc})_\theta$ means "the (shear) displacement in the angular direction".)

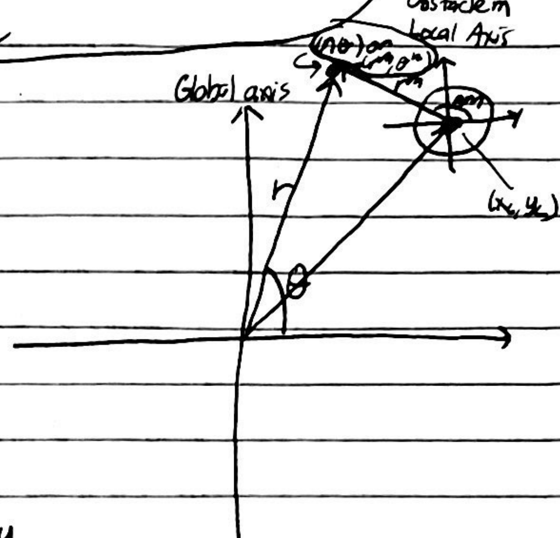
II. What about local coordinates?

In our multiple-scattering problem, not all obstacles are centered at the origin. So we introduce "local" coordinate systems centered at the middle of each obstacle. We write these as (r^m, θ^m) for the local polar coordinates for obstacle m.

Note: We could write (r^m, θ^m) as functions of (r, θ) or vice versa; for example, we could write

$$(r, \theta) := (r(r^m, \theta^m), \theta(r^m, \theta^m)).$$

This will be useful in a minute



In order to talk about "hard" boundary conditions, we want to decompose $\underline{u}_{inc}$ into a component in the r^m-radial direction & a component in the θ^m angular direction (rather than in the r & θ -directions).

So we need to find a new way to decompose $\underline{u}_{inc}$ in this new coordinate system; that is, we

$$u_{inc} = \begin{bmatrix} (u_{inc})_r \\ (u_{inc})_\theta \end{bmatrix}_{(r,\theta) \text{ coordinates}} \Rightarrow \begin{bmatrix} (u_{inc})_{r^m} \\ (u_{inc})_{\theta^m} \end{bmatrix}_{(r^m,\theta^m) \text{ coordinates}}$$

we need to find a way to get this "mapping", essentially

KEY IDEA: We will decompose u_{inc} in the (r^m, θ^m)-local coordinate system, and find a mapping from $(r^m, \theta^m) \mapsto (r, \theta)$ that will allow a suitable change-of-coordinates transformation.

Steps

① Find a way to express (r, θ) as a function of (r^m, θ^m)

- Let $(x, y) = (r\cos\theta, r\sin\theta)$ be any point in global coordinates
- Let (x_c, y_c) be the global Cartesian coordinates of the center of obstacle m
- Let $(x^m, y^m) = (r^m\cos\theta^m, r^m\sin\theta^m)$ be the same point as (x, y), but expressed as m-local coordinates.

Notice: $(x,y) = (x^m, y^m) + (x_c, y_c)$

$$= (x^m + x_c, y^m + y_c)$$

$$= (r^m\cos\theta^m + x_c, r^m\sin\theta^m + y_c)$$

- Converting this back to global polar coordinates:

☆ $r(r^m,\theta^m) = \sqrt{x^2+y^2}$

$$= \sqrt{(r^m\cos\theta^m + x_c)^2 + (r^m\sin\theta^m + y_c)^2}$$

$$= \sqrt{(r^m)^2\cos^2\theta^m + 2r^m x_c\cos\theta^m + x_c^2 + (r^m)^2\sin^2\theta^m + 2r^m y_c\sin\theta^m + y_c^2}$$

$$= \sqrt{(r^m)^2(\cos^2\theta^m + \sin^2\theta^m) + 2r^m(x_c\cos\theta^m + y_c\sin\theta^m) + (x_c^2+y_c^2)}$$

$$\Rightarrow \boxed{r(r^m,\theta^m) = \sqrt{(r^m)^2 + 2r^m(x_c\cos\theta^m + y_c\sin\theta^m) + (x_c^2+y_c^2)}}$$

☆ $\theta(r^m,\theta^m) = \operatorname{atan2}(y,x)$

$$\Rightarrow \boxed{\theta(r^m,\theta^m) = \operatorname{atan2}(r^m\sin\theta^m + y_c, r^m\cos\theta^m + x_c)}$$

where
$$\operatorname{atan2}(y,x) := \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x>0 \\ \arctan\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \text{ and } y\geq 0 \\ \arctan\left(\frac{y}{x}\right)-\pi & \text{if } x<0 \text{ and } y<0 \\ +\frac{\pi}{2} & \text{if } x=0 \text{ and } y>0 \\ -\frac{\pi}{2} & \text{if } x=0 \text{ and } y<0 \\ \text{undefined} & \text{if } x=0 \text{ and } y=0 \end{cases}$$

is the function which gives a precise angle $(\in[-\pi,\pi])$ for any point $(x,y)\neq(0,0)$.

Now, we have expressed (r,θ) as a function of (r^m,θ^m).

② **Decompose u_{inc} in the m-local coordinate system:**

$$u_{inc}(r,\theta) = \begin{bmatrix} (u_{inc})_{r^m}(r,\theta) \\ (u_{inc})_{\theta^m}(r,\theta) \end{bmatrix} = \nabla_{r^m\theta^m}\phi_{inc} = \begin{bmatrix} \frac{\partial}{\partial r^m}\phi_{inc}(r,\theta) \\ \frac{1}{r^m}\frac{\partial}{\partial\theta^m}\phi_{inc}(r,\theta) \end{bmatrix}$$

③ Expand out total derivatives:

$$\begin{bmatrix} (u_{inc})_{r^m}(r,\theta) \\ (u_{inc})_{\theta^m}(r,\theta) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r^m}\phi_{inc}(r(r^m,\theta^m),\ \theta(r^m,\theta^m)) \\ \frac{1}{r^m}\frac{\partial}{\partial\theta^m}\phi_{inc}(r(r^m,\theta^m),\ \theta(r^m,\theta^m)) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\partial\phi_{inc}}{\partial r}\right)\left(\frac{\partial r}{\partial r^m}\right) + \left(\frac{\partial\phi_{inc}}{\partial\theta}\right)\left(\frac{\partial\theta}{\partial r^m}\right) \\ \frac{1}{r^m}\left[\left(\frac{\partial\phi_{inc}}{\partial r}\right)\left(\frac{\partial r}{\partial\theta^m}\right) + \left(\frac{\partial\phi_{inc}}{\partial\theta}\right)\left(\frac{\partial\theta}{\partial\theta^m}\right)\right] \end{bmatrix}$$

④ Compute all required derivatives

$$\phi_{inc}(r,\theta) = e^{ik_p r\cos(\theta-\gamma)}$$

$$\to \frac{\partial\phi_{inc}}{\partial r} = ik_p\cos(\theta-\gamma)e^{ik_p r\cos(\theta-\gamma)} = ik_p\cos(\theta-\gamma)\,\phi_{inc}(r,\theta)$$

$$\to \frac{\partial\phi_{inc}}{\partial\theta} = -ik_p r\sin(\theta-\gamma)e^{ik_p r\cos(\theta-\gamma)} = -ik_p r\sin(\theta-\gamma)\,\phi_{inc}(r,\theta)$$

$$r(r^m,\theta^m) = \sqrt{(r^m)^2 + 2r^m(x_c\cos\theta^m + y_c\sin\theta^m) + (x_c^2+y_c^2)}$$

$$\to \frac{\partial r}{\partial r^m} = \frac{2r^m + 2(x_c\cos\theta^m + y_c\sin\theta^m)}{2\sqrt{(r^m)^2 + 2r^m(x_c\cos\theta^m + y_c\sin\theta^m) + (x_c^2+y_c^2)}}$$

$$= \frac{r^m + x_c\cos\theta^m + y_c\sin\theta^m}{r(r^m,\theta^m)}$$

$$\to \frac{\partial r}{\partial\theta^m} = \frac{2r^m(-x_c\sin\theta^m + y_c\cos\theta^m)}{2\sqrt{(r^m)^2 + 2r^m(x_c\cos\theta^m + y_c\sin\theta^m) + (x_c^2+y_c^2)}}$$

$$= \frac{r^m(-x_c\sin\theta^m + y_c\cos\theta^m)}{r(r^m,\theta^m)}$$

$$\theta(r^m,\theta^m) = \mathrm{atan2}(r^m\sin\theta^m + y_c,\ r^m\cos\theta^m + x_c)$$

$$\to \frac{\partial\theta}{\partial r^m} = \left(\frac{\partial\,\mathrm{atan2}(y,x)}{\partial x}\right)\left(\frac{\partial x}{\partial r^m}\right) + \left(\frac{\partial\,\mathrm{atan2}(y,x)}{\partial y}\right)\left(\frac{\partial y}{\partial r^m}\right)$$

$$= \left(\frac{-y}{x^2+y^2}\right)(\cos\theta^m) + \left(\frac{x}{x^2+y^2}\right)(\sin\theta^m)$$

Note: Here,

$$x^2+y^2 = (r^m\cos\theta^m + x_c)^2 + (r^m\sin\theta^m + y_c)^2$$

$$= (r^m)^2\cos^2\theta^m + 2r^m x_c\cos\theta^m + x_c^2 + (r^m)^2\sin^2\theta^m + 2r^m y_c\sin\theta^m + y_c^2$$

$$= (r^m)^2 + 2r^m(x_c\cos\theta^m + y_c\sin\theta^m) + (x_c^2+y_c^2)$$

$$= \left[r(r^m,\theta^m)\right]^2$$

$$\Rightarrow \frac{\partial \theta}{\partial r^m} = \left(\frac{-r^m \sin\theta^m - y_c}{[r(r^m,\theta^m)]^2}\right)(\cos\theta^m) + \left(\frac{r^m\cos\theta^m + x_c}{[r(r^m,\theta^m)]^2}\right)(\sin\theta^m)$$

$$= \frac{\cancel{-r^m \sin\theta^m \cos\theta^m} - y_c\cos\theta^m + \cancel{r^m \sin\theta^m \cos\theta^m} + x_c \sin\theta^m}{[r(r^m,\theta^m)]^2}$$

$$= \frac{1}{r^2}(x_c \sin\theta^m - y_c\cos\theta^m)$$

$$\frac{\partial \theta}{\partial \theta^m} = \left(\frac{\partial \operatorname{atan2}(y,x)}{\partial x}\right)\left(\frac{\partial x}{\partial \theta^m}\right) + \left(\frac{\partial \operatorname{atan2}(y,x)}{\partial y}\right)\left(\frac{\partial y}{\partial \theta^m}\right)$$

$$= \left(\frac{-y}{x^2+y^2}\right)(-r^m\sin\theta^m) + \left(\frac{x}{x^2+y^2}\right)(r^m\cos\theta^m)$$

$$= \frac{1}{r^2}\left[(-r^m\sin\theta^m - y_c)(-r^m\sin\theta^m) + (r^m\cos\theta^m + x_c)(r^m\cos\theta^m)\right]$$

$$= \frac{1}{r^2}\left[(r^m)^2\sin^2\theta^m + r^m y_c \sin\theta^m + (r^m)^2\cos^2\theta^m + r^m x_c\cos\theta^m\right]$$

$$= \frac{r^m}{r^2}\left[r^m + (x_c\cos\theta^m + y_c\sin\theta^m)\right]$$

⑤ Substitute & Simplify

$$\begin{bmatrix} (u_{inc})_{r^m}(r,\theta) \\ (u_{inc})_{\theta^m}(r,\theta) \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial \phi_{inc}}{\partial r}\right)\left(\frac{\partial r}{\partial r^m}\right) + \left(\frac{\partial \phi_{inc}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial r^m}\right) \\ \frac{1}{r^m}\left[\left(\frac{\partial \phi_{inc}}{\partial r}\right)\left(\frac{\partial r}{\partial \theta^m}\right) + \left(\frac{\partial \phi_{inc}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \theta^m}\right)\right] \end{bmatrix}$$

$$\Rightarrow (u_{inc})_{r^m}(r,\theta) = \Big\{ (iK_p\cos(\theta-\gamma)\phi_{inc})\left[\frac{1}{r}(r^m + x_c\cos\theta^m + y_c\sin\theta^m)\right]$$

$$+ (-iK_p \cancel{r}\sin(\theta-\gamma)\phi_{inc})\left[\frac{1}{\cancel{r^2}_{(r)}}(x_c\sin\theta^m - y_c\cos\theta^m)\right]\Big\}$$

$$= \frac{1}{r} iK_p \phi_{inc}(r,\theta)\Big[\cos(\theta-\gamma)(r^m + x_c\cos\theta^m + y_c\sin\theta^m) - \sin(\theta-\gamma)(x_c\sin\theta^m - y_c\cos\theta^m)\Big]$$

Recall:
- $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$
- $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b)$

$$= \frac{1}{r} iK_p \phi_{inc}(r,\theta)\Big[r^m\cos(\theta-\gamma) + \underline{x_c\cos(\theta-\gamma)\cos\theta^m} + \underline{y_c\cos(\theta-\gamma)\sin\theta^m} \underline{- x_c\sin(\theta-\gamma)\sin\theta^m} + \underline{y_c\sin(\theta-\gamma)\cos\theta^m}\Big]$$

$$\Rightarrow \boxed{(u_{inc})_{r^m} = \frac{1}{r} iK_p \phi_{inc}(r,\theta)\left[r^m\cos(\theta-\gamma) + x_c\cos(\theta^m+\theta-\gamma) + y_c\sin(\theta^m+\theta-\gamma)\right]}$$

$$\bullet\ (u_{inc})_{\theta^m}(r,\theta) = \frac{1}{\cancel{r^m}}\Big\{(iK_p\cos(\theta-\gamma)\phi_{inc})\left[\frac{1}{r}(\cancel{r^m}(-x_c\sin\theta^m + y_c\cos\theta^m))\right]$$

$$+ (-iK_p \cancel{r}\sin(\theta-\gamma)\phi_{inc})\left[\frac{\cancel{r^m}}{\cancel{r^2}_{(r)}}(r^m + (x_c\cos\theta^m + y_c\sin\theta^m))\right]\Big\}$$

$$= \frac{1}{r} iK_p \phi_{inc}(r,\theta)\Big[\cos(\theta-\gamma)(-x_c\sin\theta^m + y_c\cos\theta^m) - \sin(\theta-\gamma)(r^m + x_c\cos\theta^m + y_c\sin\theta^m)\Big]$$

$$= \frac{1}{r} iK_p \phi_{inc}(r,\theta)\Big[\underline{-x_c\cos(\theta-\gamma)\sin\theta^m} + \underline{y_c\cos(\theta-\gamma)\cos\theta^m} - r^m\sin(\theta-\gamma) \underline{- x_c\sin(\theta-\gamma)\cos\theta^m} \underline{- y_c\sin(\theta-\gamma)\sin(\theta^m)}\Big]$$

$$\Rightarrow \boxed{(u_{inc})_{\theta^m}(r,\theta) = \frac{1}{r} iK_p \phi_{inc}(r,\theta)\left[-x_c\sin(\theta^m+\theta-\gamma) + y_c\cos(\theta^m+\theta-\gamma) - r^m\sin(\theta-\gamma)\right]}$$

CONCLUSION

We can represent the displacement $\underline{u}_{inc}$ (a vector) in (r^m, θ^m)-local coordinates as

$$\boxed{\underline{u}_{inc} = \begin{bmatrix} (u_{inc})_{r^m} \\ (u_{inc})_{\theta^m} \end{bmatrix}(r,\theta) = \frac{1}{r} i K_p \phi_{inc}(r,\theta) \begin{bmatrix} r^m \cos(\theta-\gamma) + x_c \cos(\theta^m+\theta-\gamma) + y_c \sin(\theta^m+\theta-\gamma) \\ -r^m \sin(\theta-\gamma) - x_c \sin(\theta^m+\theta-\gamma) + y_c \cos(\theta^m+\theta-\gamma) \end{bmatrix}}$$

where
- (r,θ) is the global polar coordinate of the point of evaluation,
- (r^m,θ^m) is the local polar coordinate of the point of evaluation,
- γ is the (global) angle of incidence of ϕ_{inc},
- $\phi_{inc}(r,\theta)$ is evaluated in global coordinates, and
- and (x_c, y_c) is the global cartesian coordinate of the center of obstacle m.

<u>Notice</u>: If the obstacle <u>is</u> at the origin, then $(x_c, y_c) = (0,0)$ and $r^m = r$. So u_{inc} simplifies to

$$\underline{u}_{inc} = \frac{1}{r} i K_p \phi_{inc} \begin{bmatrix} r\cos(\theta-\gamma) + (0) + (0) \\ -r\sin(\theta-\gamma) - (0) + (0) \end{bmatrix}$$

$$= \begin{bmatrix} i K_p \phi_{inc} \cos(\theta-\gamma) \\ -i K_p \phi_{inc} \sin(\theta-\gamma) \end{bmatrix}$$

which exactly matches the original formula given in global polar coordinates.