

EXCL Season 4 Assignment 5

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Problem 1. Positive a, b, c satisfy $a + b + c = 1$. Prove that

$$\frac{ab + c}{a + b} + \frac{bc + a}{b + c} + \frac{ca + b}{c + a} \geq 2.$$

Solution. By $a + b + c = 1$ we have $c = 1 - a - b$, thus $ab + c = ab - a - b + 1 = (1 - a)(1 - b)$, and $a + b = 1 - c$. Likewise for $(bc + a)$ and $(ca + b)$, we found that original inequality is equivalent to

$$\frac{(1 - a)(1 - b)}{(1 - c)} + \frac{(1 - b)(1 - c)}{(1 - a)} + \frac{(1 - c)(1 - a)}{(1 - b)} \geq 2.$$

Let $x = 1 - a$, $y = 1 - b$ and $z = 1 - c$, with $x + y + z = 3 - (a + b + c) = 2$, we have

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq 2.$$

By Cauchy-Schwarz,

$$\begin{aligned} \left(\frac{zx}{y} + \frac{xy}{z} + \frac{yz}{x} \right) \left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \right) &\geq (x + y + z)^2 \\ \iff \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} &\geq x + y + z \\ \implies \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} &\geq 2 \end{aligned}$$

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Problem 2. If $a, b, c > 0$, prove that

$$\frac{3(ab + bc + ca)}{2(a + b + c)} \geq \frac{ab}{a + b} + \frac{bc}{b + c} + \frac{ca}{c + a}.$$

Solution. Multiplying by $a + b + c$, we have

$$\iff \frac{3(ab + bc + ca)}{2} \geq (a + b + c) \left(\frac{ab}{a + b} + \frac{bc}{b + c} + \frac{ca}{c + a} \right)$$

$$\begin{aligned}
&\Leftrightarrow \frac{3(ab+bc+ca)}{2} \geq \left(\frac{abc}{a+b} + \frac{abc}{b+c} + \frac{abc}{c+a} \right) + (ab+bc+ca) \\
&\Leftrightarrow \frac{ab+bc+ca}{2} \geq abc \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \\
&\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 2 \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right)
\end{aligned}$$

By AM-HM inequality, we know that $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$, $\frac{1}{b} + \frac{1}{c} \geq \frac{4}{b+c}$ and $\frac{1}{c} + \frac{1}{a} \geq \frac{4}{c+a}$ adding these inequalities we obtain the result. ■

Problem 3. If $a, b, c > 0$, prove that

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

Solution. Since the inequality is homogeneous, we may assume WLOG that $a+b+c=3$. So the inequality we wish to prove is

$$\frac{a}{(3-a)^2} + \frac{b}{(3-b)^2} + \frac{c}{(3-c)^2} \geq \frac{3}{4}.$$

Consider the function $f(x) = \frac{x}{(3-x)^2}$ for $x \neq 3$, the second derivate of this function is $f''(x) = \frac{2x+12}{(3-x)^4}$, which is convex for $x > 0$. Using Jensen's inequality, we have

$$\begin{aligned}
&\frac{f(a) + f(b) + f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right) \\
&\Leftrightarrow f(a) + f(b) + f(c) \geq 3f(1) \\
&\Leftrightarrow f(a) + f(b) + f(c) \geq 3 \left(\frac{1}{(3-1)^2} \right) = \frac{3}{4}
\end{aligned}$$

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Problem 4. If $a, b, c > 0$ are the sides of a triangle, find the minimum value of

$$\frac{\sqrt{a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}} + \frac{\sqrt{b}}{\sqrt{c} + \sqrt{a} - \sqrt{b}} + \frac{\sqrt{c}}{\sqrt{a} + \sqrt{b} - \sqrt{c}}.$$

Solution. Let $x = \sqrt{b} + \sqrt{c} - \sqrt{a}$, $y = \sqrt{c} + \sqrt{a} - \sqrt{b}$ and $z = \sqrt{a} + \sqrt{b} - \sqrt{c}$.

We have that $y+z=2\sqrt{a} \Leftrightarrow \sqrt{a} = \frac{y+z}{2}$, likewise, for \sqrt{b} and \sqrt{c} . Thus, the expression is equal to

$$\frac{x+y}{2z} + \frac{y+z}{2x} + \frac{z+x}{2y} \Leftrightarrow \frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z} \right) \geq \frac{1}{2}(2+2+2) = 3.$$

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