EXCL Season 4 Assignment 5

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12 June 2025

Problem 1. Positive a, b, c satisfy $a + b + c = 1$. Prove that

$$\frac{ab+c}{a+b} + \frac{bc+a}{b+c} + \frac{ca+b}{c+a} \geq 2.$$

Solution. By $a+b+c = 1$ we have $c = 1-a-b$, thus $ab+c = ab-a-b+1 = (1-a)(1-b)$, and $a + b = 1 - c$. Likewise for $(bc + a)$ and $(ca + b)$, we found that original inequality is equivalent to

$$\frac{(1-a)(1-b)}{(1-c)} + \frac{(1-b)(1-c)}{(1-a)} + \frac{(1-c)(1-a)}{(1-b)} \geq 2.$$

Let $x = 1 - a$, $y = 1 - b$ and $z = 1 - c$, with $x + y + z = 3 - (a + b + c) = 2$, we have

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq 2.$$

By Cauchy-Schwarz,

$$\begin{aligned}
\left(\frac{zx}{y} + \frac{xy}{z} + \frac{yz}{x}\right)\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) &\geq (x+y+z)^2 \\
\iff \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} &\geq x+y+z \\
\implies \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} &\geq 2. \qquad \blacksquare
\end{aligned}$$

Problem 2. If $a, b, c > 0$, prove that

$$\frac{3(ab+bc+ca)}{2(a+b+c)} \geq \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

Solution. Multiplying by $a + b + c$, we have

$$\iff \frac{3(ab+bc+ca)}{2} \geq (a+b+c)\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right)$$

$$\iff \frac{3(ab+bc+ca)}{2} \geq \left(\frac{abc}{a+b}+\frac{abc}{b+c}+\frac{abc}{c+a}\right)+(ab+bc+ca)$$

$$\iff \frac{ab+bc+ca}{2} \geq abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$\iff \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

By AM-HM inequality, we know that $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$, $\frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c}$ and $\frac{1}{c}+\frac{1}{a} \geq \frac{4}{c+a}$ adding these inequalities we obtain the result. ■

Problem 3. If $a, b, c > 0$, prove that

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

Solution. Since the inequality is homogeneous, we may assume WLOG that $a+b+c=3$. So the inequality we wish to prove is

$$\frac{a}{(3-a)^2}+\frac{b}{(3-b)^2}+\frac{c}{(3-c)^2} \geq \frac{3}{4}.$$

Consider the function $f(x)=\frac{x}{(3-x)^2}$ for $x \neq 3$, the second derivate of this function is $f''(x)=\frac{2x+12}{(3-x)^4}$, which is convex for $x>0$. Using Jensen's inequality, we have

$$\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$$

$$\iff f(a)+f(b)+f(c) \geq 3f(1)$$

$$\iff f(a)+f(b)+f(c) \geq 3\left(\frac{1}{(3-1)^2}\right)=\frac{3}{4}. \quad ■$$

Problem 4. If $a, b, c > 0$ are the sides of a triangle, find the minimum value of

$$\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}.$$

Solution. The answer is 3. Let $x=\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\right)$, $y=(\sqrt{c}+\sqrt{a}-\sqrt{b})$ and $z=(\sqrt{a}+\sqrt{b}-\sqrt{c})$. We have that $y+z=2\sqrt{a} \iff \sqrt{a}=\frac{y+z}{2}$, likewise, for $\sqrt{b}$ and $\sqrt{c}$. Thus, the expression is equal to

$$\frac{x+y}{2z}+\frac{y+z}{2x}+\frac{z+x}{2y} \iff \frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\right) \geq \frac{1}{2}(2+2+2)=3. \quad ■$$