HYPERCUBE GRAPH ASSOCIAHEDRA

JORDAN ALMETER HYBRID POSTER-SLIDE FOR FPSAC 2020

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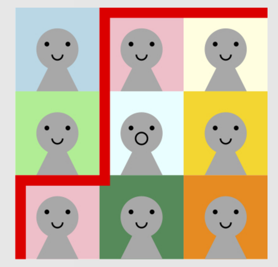
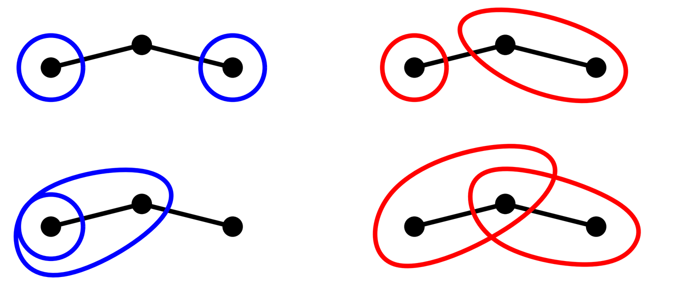
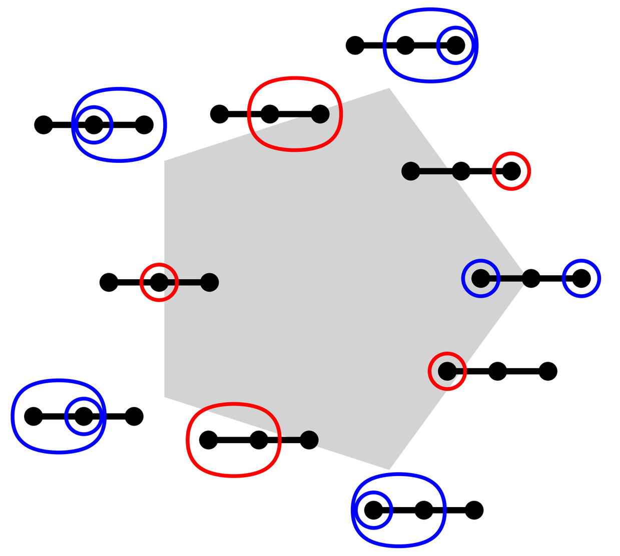


GRAPH ASSOCIAHEDRA [3]

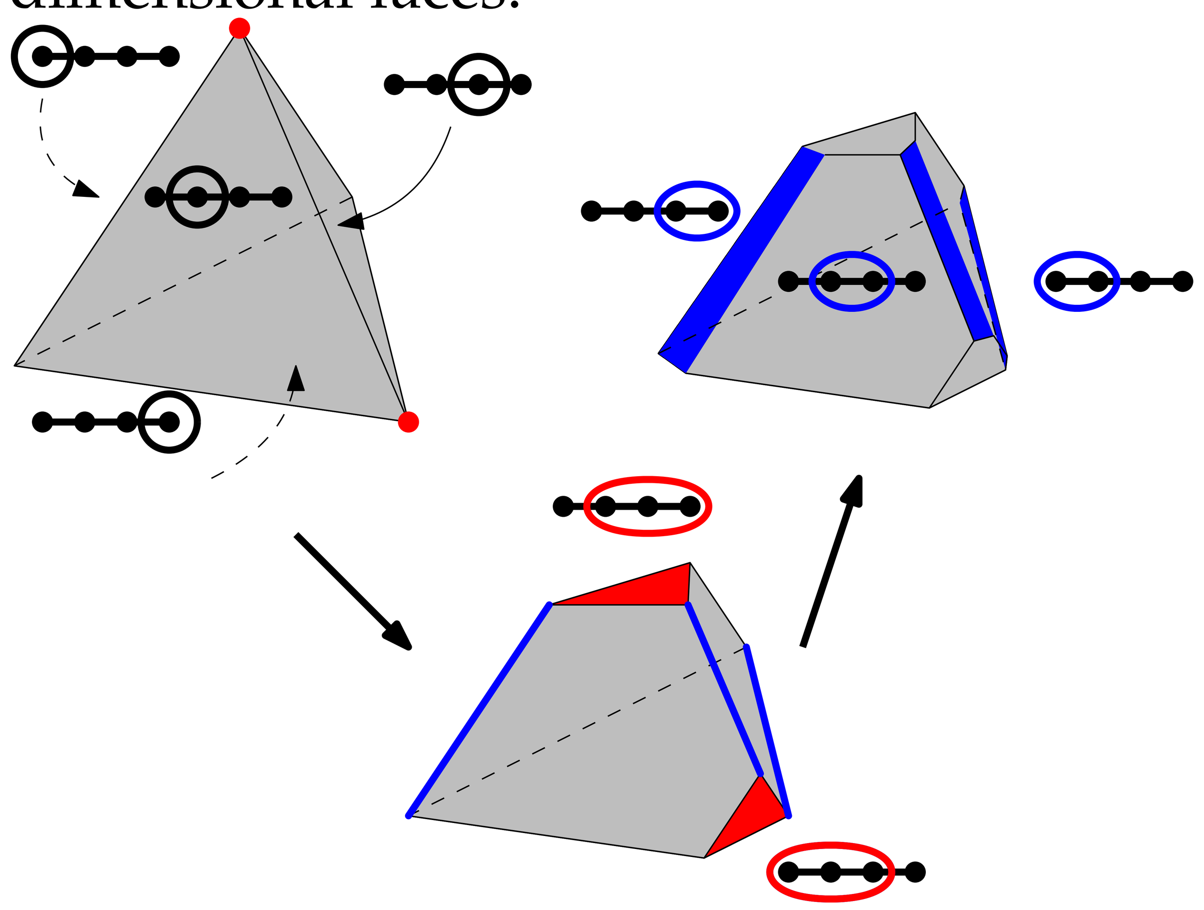
Given a graph G, a ***tube*** is a subset of vertices which induces a connected graph. Two tubes are ***compatible*** if they are either nested, or disjoint and not adjacent.



A ***tubing*** is a collection of pairwise compatible tubes, whose support does not cover the vertex set. The set of tubings forms a simplicial complex, dual to a simple polytope called the ***graph associahedron*** of G, or $\mathcal{K}G$.



The graph associahedron can be found by repeatedly truncating faces of the simplex associated with tubes, starting with lower-dimensional faces.



Graph associahedra are generalized by ***nestohedra***, polytopes dual to ***nested set complexes*** defined by ***building sets***.

$\mathcal{P}$-GRAPH ASSOCIAHEDRA

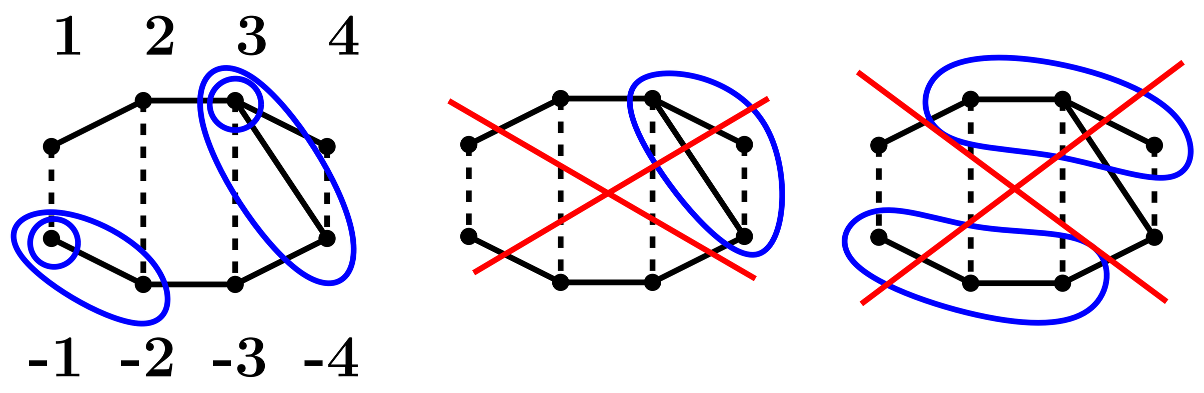
Given a simple polytope $\mathcal{P}$ with facet set S and a graph G on S, define:

- $t \subseteq S$ is a ***tube*** if $G|_t$ is connected and $\bigcap_{s \in t} s \neq \emptyset$.
- A set T of tubes is a ***tubing*** if they are all pairwise compatible, and the intersection of the facets in the support of T is nonempty.

The ***$\mathcal{P}$-graph associahedron*** is a polytope obtained by truncating faces F_t for all tubes t in G, in increasing dimension. Its boundary complex is dual to the tubing complex.

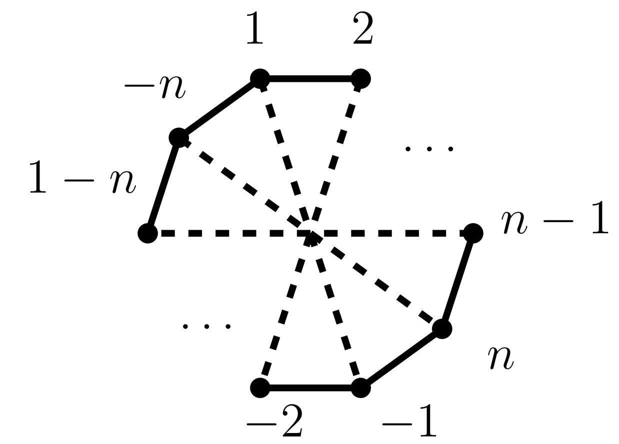
HYPERCUBE GRAPHS

Label the facets of a hypercube $\pm[n]$. For any graph G on $\pm[n]$, draw dashed edges between vertices $\{i, -i\}$. Hypercube graph tubes and tubings must satisfy classical graph associahedron rules, but also must avoid dashed edges within and between tubes.

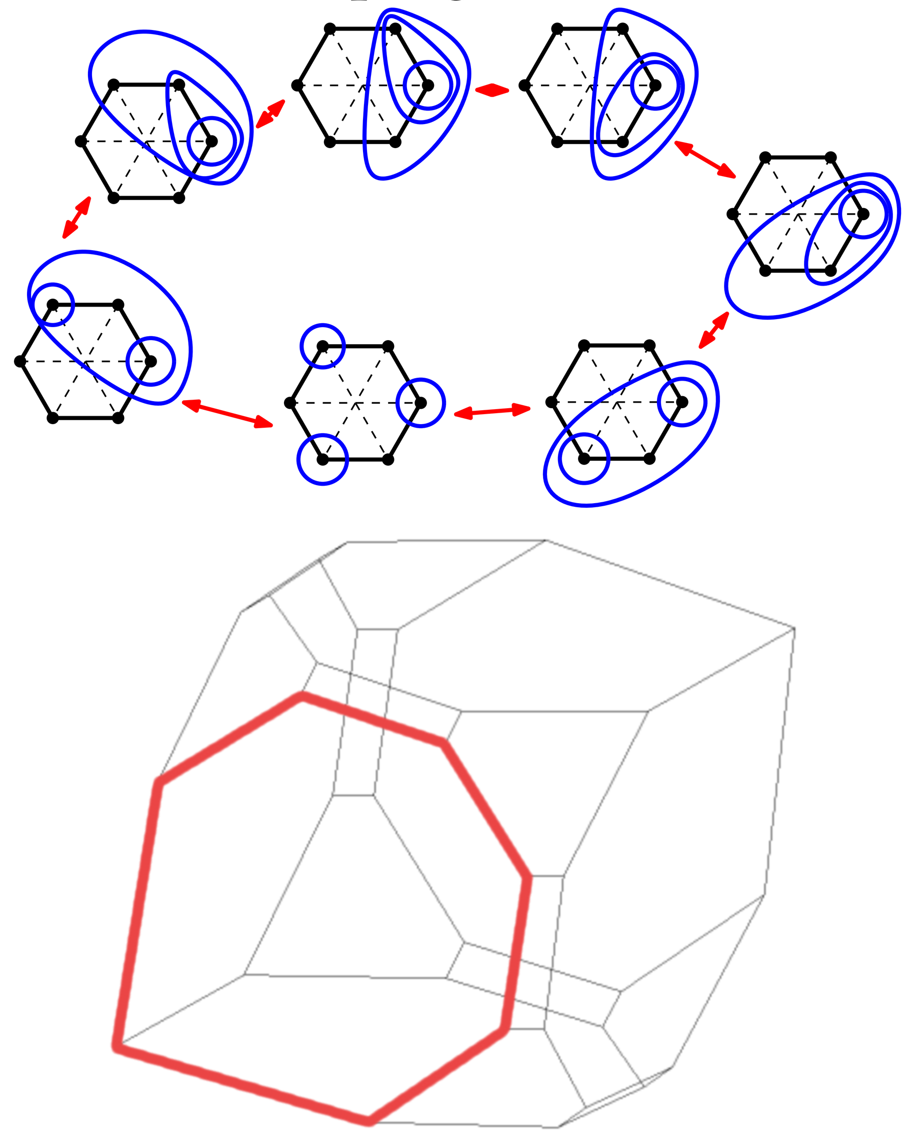


Valid tubing. Invalid tube and tubing.

TWISTED CYCLE GRAPH



The twisted cycle hypercube graph associahedron has 2^{2n-1} vertices. Here we highlight one facet of the three dimensional case, and see a heptagonal recurrence.



EXAMPLES OF HYPERCUBE GRAPH ASSOCIAHEDRA

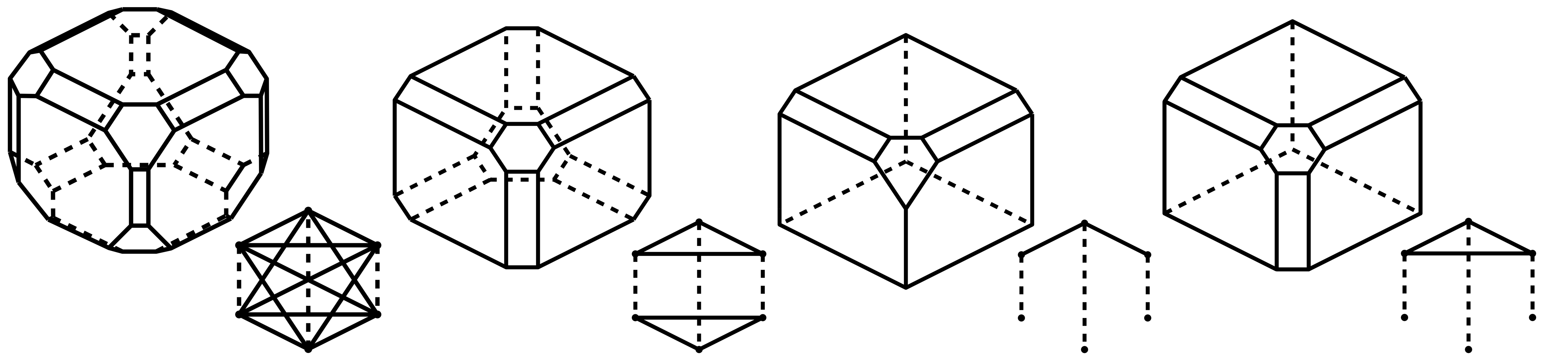
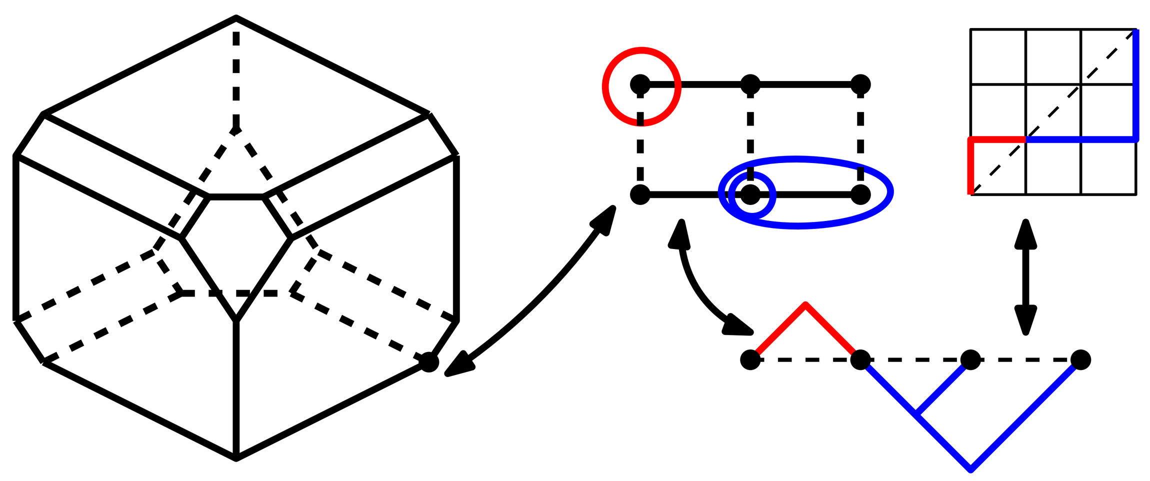


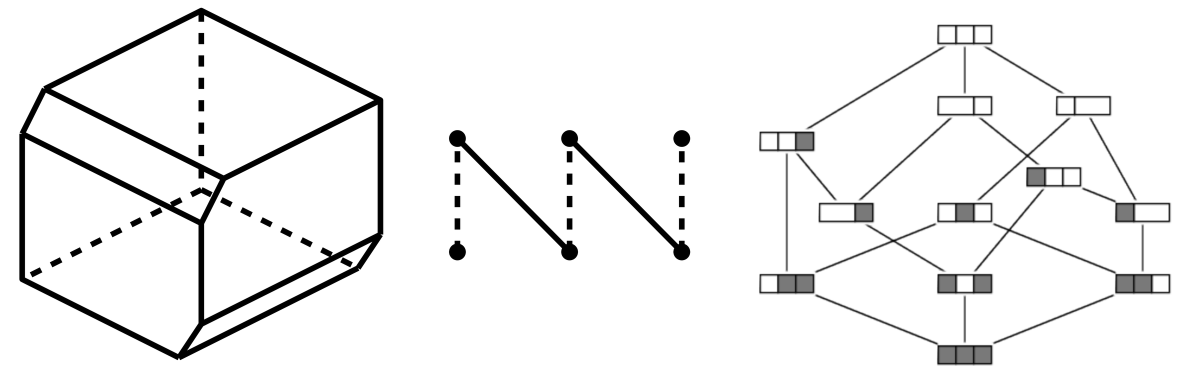
Figure 1: Type B_3 permutahedron, type A_3 permutahedron, type A_3 associahedron, stellohedron.

LINEAR BIASSOCIAHEDRON



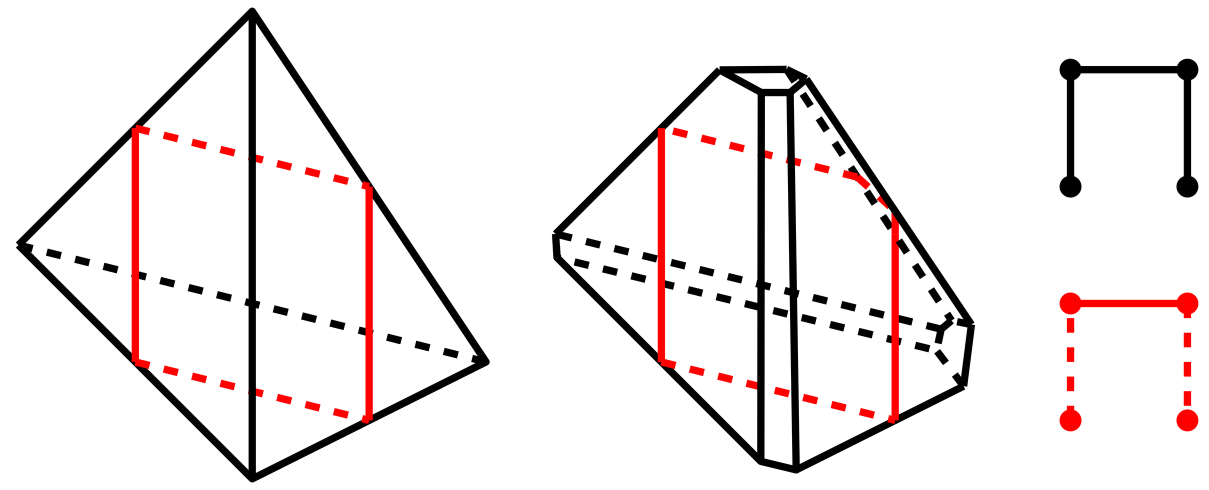
The normal fan of the linear biassociahedron is the refinement of a linear cluster fan with its antipodal fan [2]. Here we show a bijection between its vertices and NE-lattice paths.

PELL PERMUTATIONS



The graph associahedron of a 'zigzag' graph is shown here. By putting an ordering on the vertices, we obtain the Hasse diagram for the lattice of sashes [5] in $n \geq 7$ dimensions, and conjecture that this holds for all n.

SIMPLEX SECTION



Every $\mathcal{P}$-graph associahedron is isomorphic to the intersection of a graph associahedron with the same underlying graph, and a generic n-dimensional subspace; every $\mathcal{P}$-graph tubing complex is a subcomplex of a graph tubing complex.

Hypercube Graph Associahedra, Part 2: $\mathcal{P}$-Nestohedra

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Nestohedra & $\mathcal{P}$-Nestohedra

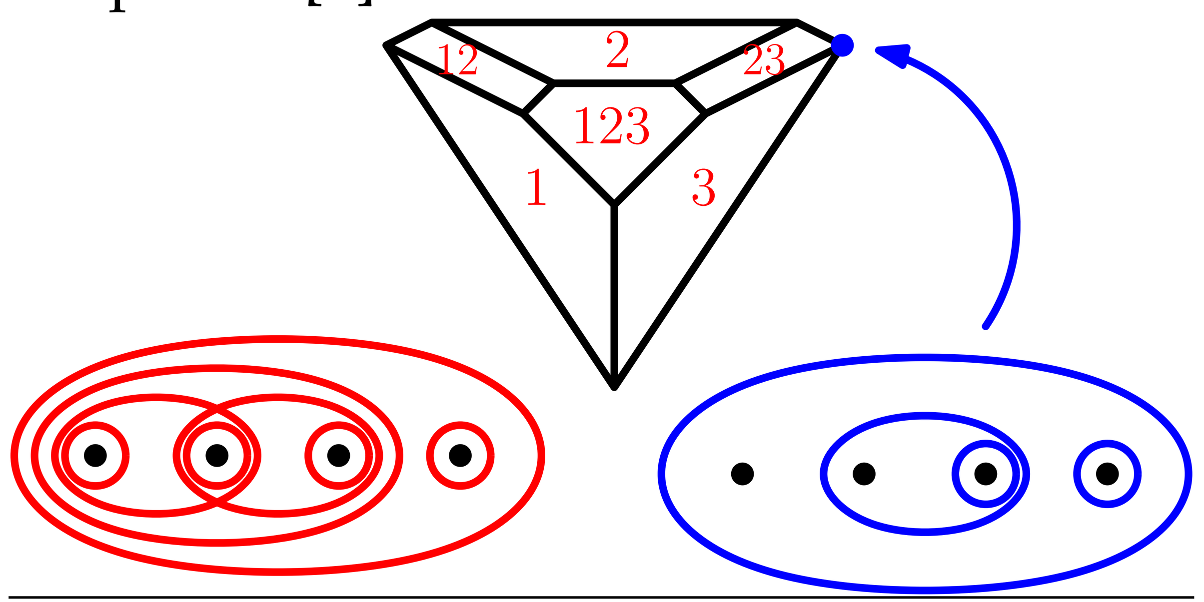
Given a set S, a set $B \subset 2^S$ is a ***classical building set*** if:

1. For any two $I, J \in B$ with $I \cap J \neq \emptyset$, $I \cup J \in \mathcal{B}$
2. $\{i\} \in B$ for all $i \in S$.

A set $N \subseteq B$ is ***nested*** if:

1. For all $I, J \in N$, either $I \subset J, J \subset I$, or $I \cap J = \emptyset$.
2. For any disjoint subset of N of size > 1, their union is not in B.

The ***Nestohedron*** of a building set is a polytope whose poset of nonempty faces is dual to the simplicial complex of nested sets. It can be found either by truncation of faces from B, or Minkowski sums of associated simplices. [7]

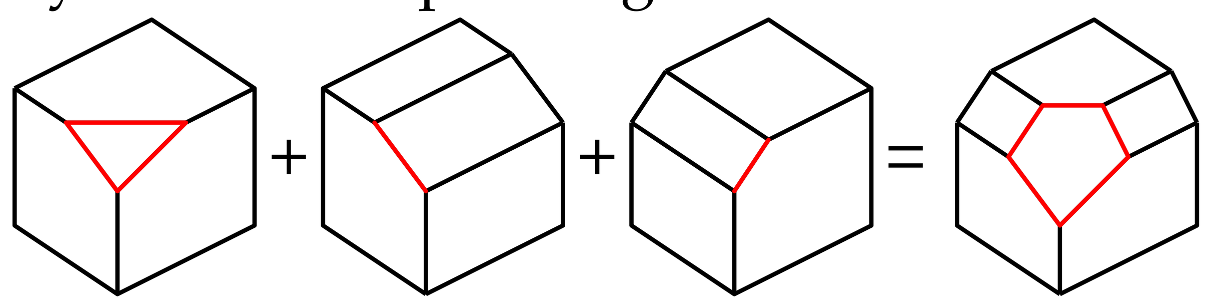


For a simplicial complex Δ on S, $\mathcal{B} \subseteq \Delta$ is a Δ-building set if, for every face in Δ, the restriction of $\mathcal{B}$ to that face is a classical building set. A set $N \subseteq \mathcal{B}$ is nested under $\mathcal{B}$ if it is nested in one of these sets. The nested complex of $\mathcal{B}$ is a subdivision of Δ. [4] [6]

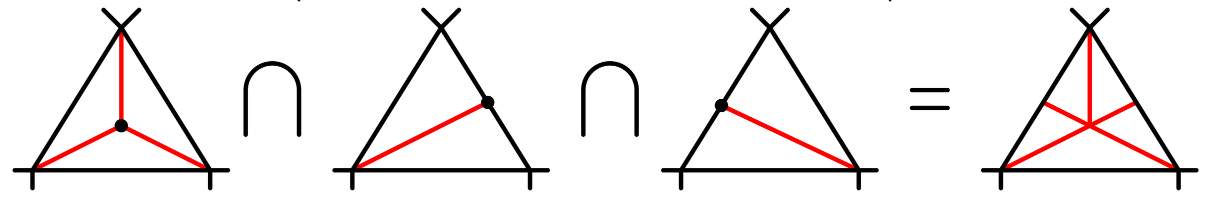
If Δ is dual to the nonempty face poset of a simple polytope $\mathcal{P}$, then the nested complex is dual to a simple polytope, the $\mathcal{P}$-nestohedron of $\mathcal{B}$, or $\mathcal{K}_{\mathcal{P}}\mathcal{B}$.

Decomposing $\mathcal{P}$-Nestohedra

Every $\mathcal{P}$-nestohedron can be written as the Minkowski sum of copies of $\mathcal{P}$, truncated by faces corresponding to sets in $\mathcal{B}$.



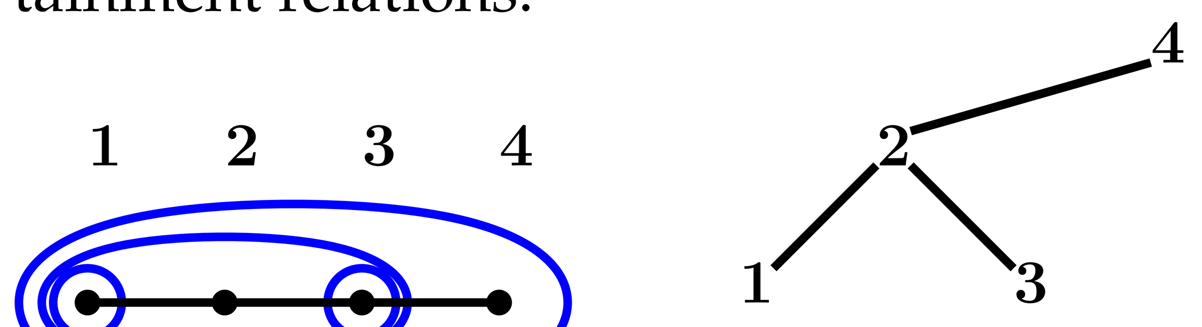
In the normal fan, this corresponds to the coarsest common refinement of stellar subdivisions (one cone here shown).



Every vertex of $\mathcal{P}$ is 'divided' by truncation into a classical nestohedron. More generally, every face of a $\mathcal{P}$-nestohedron is isomorphic to the Cartesian product of an F-nestohedron, for some face F of $\mathcal{P}$, and simplex nestohedra.

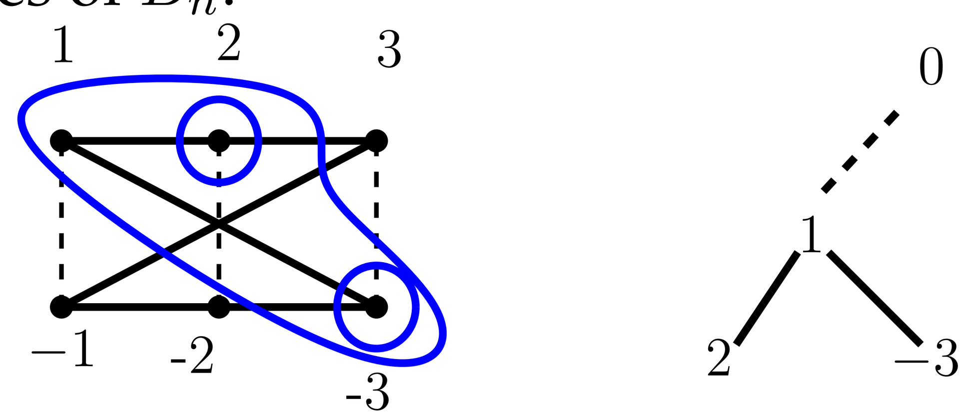
Faces of $\mathcal{P}$-Nestohedra

Maximal cones of normal fans of generalized permutahedra are in bijection with posets on $[n+1]$. For classical nestohedra, we define posets from nested sets by containment relations.



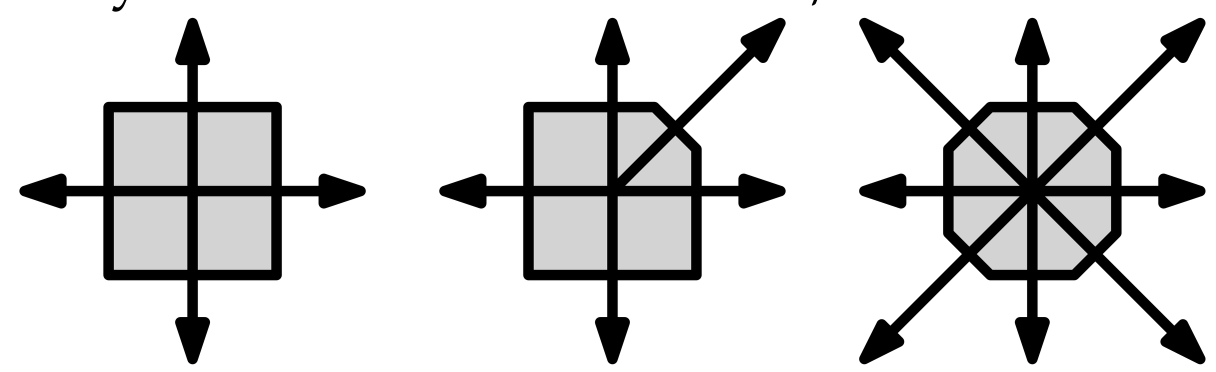
$x_1 < x_2 < x_4, x_3 < x_2$

These posets exist for any $\mathcal{P}$. For a hypercube, these posets can be turned into signed posets by adding 0, which define cones of B_n.

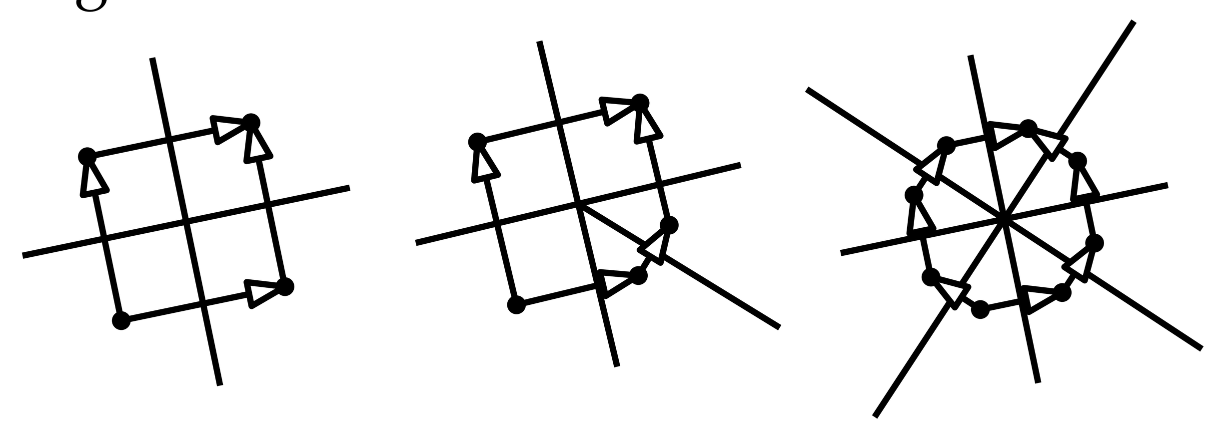


Normal Fan

Truncation is dual to stellar subdivision, so for any simple polytope $\mathcal{P}$ with normal fan $\mathcal{F}_{\mathcal{P}}$, the fan of $\mathcal{K}_{\mathcal{P}}\mathcal{B}$ refines $\mathcal{F}_{\mathcal{P}}$ and coarsens the barycentric subdvision of $\mathcal{F}_{\mathcal{P}}$. The maximal building set gives the barycentric subdivision of $\mathcal{F}_{\mathcal{P}}$.



For the hypercube case, these fans are coarsenings of the type B_n Coxeter fan, and therefore deformations of the B_n permutahedron. Because graph associahedra are deformations of the A_n permutahedron, we wished to generalize them to any arbitrary root system, but this only works when $\mathcal{P}$ is a regular simple polytope, giving us only A_n, B_n, I_n, H_3, H_4 as interesting cases.



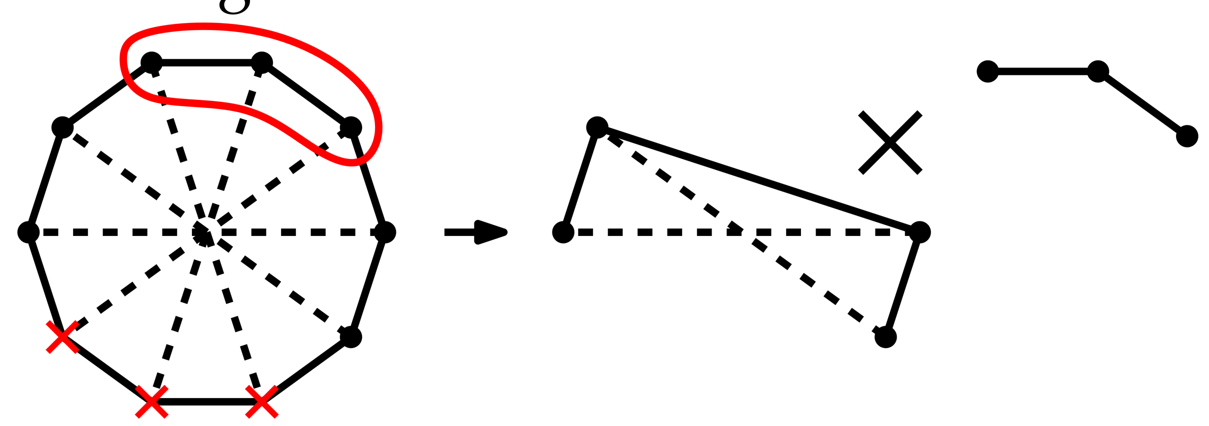
It becomes clear that we can create posets of regions from the normal fans of these polytopes, which are contractions of the type B_n order. Few of these are lattice congruences; it would be exciting to see how [1] generalizes to type B_n.

Thank You!

Thank you to all the organizers of FPSAC 2020 for giving me the opportunity to share my work, and thank you for coming to my poster presentation!

Reconnected Complement

For a building set $\mathcal{B}$ on a simplicial complex, for a face S the set of nested sets containing S can be decomposed into a nested set on $\mathcal{B} \cap 2^S$, and a nested set on the link of S. For $\mathcal{P}$-graph associahedra, this corresponds to taking a tube t, removing all graph vertices that aren't compatible with t, removing t, and connecting any two vertices which were both adjacent to t. This ***reconnected complement*** induces a graphic building set on the link of t.



This helps greatly in enumerating vertices of hypercube graph associahedra, and any family of graphs with self-similarity after taking reconnected complements will give polytopes with some degree of facial self-similarity.

References

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[7] Alexander Postnikov, "Permutohedra, Associahedra, and Beyond." IMRN, 2009