Consequences of transmit buffers on *Priority Queueing*

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1 The model

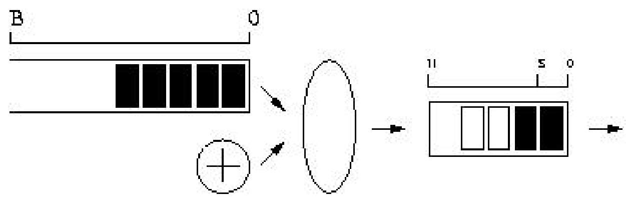

Fig. 1 – The model.

2 Analysis of the Markov chain

3 Joint p.g.f. of the tx-queue and tx-ring sizes

In this section, we derive the steady-state joint pgf of both the tx-queue and the tx-ring. The time is slotted, and the beginning of each slot corresponds to the end of the transmission of a packet. We denote q_k (resp. r_k) the size of the tx-queue (resp. tx-ring) during slot k. The joint pgf of q_k and r_k is denoted $U_k(z_1, z_2) \triangleq \mathbb{E}\left[z_1^{q_k} \cdot z_2^{r_k}\right]$. System evolution is described by the following equations :

$$q_{k+1} = \begin{cases} (q_k - N + S - 1)^+ + a_k & \text{si } r_k = S \quad (1a) \\ q_k + a_k & \text{sinon} \quad (1b) \end{cases}$$

$$r_{k+1} = \begin{cases} N & \text{si } r_k = S \quad (2a) \\ r_k - 1 & \text{sinon} \quad (2b) \end{cases}$$

$$U_{k+1}(z_1, z_2) = \mathbb{E}\left[z_1^{q_{k+1}} \cdot z_2^{r_{k+1}}\right]$$

$$\begin{aligned} U_{k+1}(z_1, z_2) = \; & \mathbb{E}_{q_k \le (N-S+1), r_k = S}\left[z_1^{q_{k+1}} \cdot z_2^{rk+1}\right] \cdot \mathbb{P}\left[q_k \le (N-S+1), r_k = S\right] \\ + \; & \mathbb{E}_{q_k > (N-S+1), r_k = S}\left[z_1^{q_{k+1}} \cdot z_2^{rk+1}\right] \cdot \mathbb{P}\left[q_k > (N-S+1), r_k = S\right] \\ + \; & \mathbb{E}_{r_k \neq S}\left[z_1^{q_{k+1}} \cdot z_2^{rk+1}\right] \cdot \mathbb{P}\left[r_k \neq S\right] \end{aligned}$$

$$\begin{aligned} U_{k+1}(z_1, z_2) = \; & \mathbb{E}\left[z_1^{a_k}\right] \cdot z_2^N \cdot \mathbb{P}\left[q_k \le (N-S+1), r_k = S\right] \\ + \; & \mathbb{E}\left[z_1^{a_k}\right] \cdot \mathbb{E}\left[z_1^{q_k}\right] \cdot \left(\frac{z_2}{z_1}\right)^N \cdot z_1^{S-1} \cdot \mathbb{P}\left[q_k > (N-S+1), r_k = S\right] \\ + \; & \mathbb{E}\left[z_1^{a_k}\right] \cdot \mathbb{E}\left[z_1^{q_k} \cdot z_2^{r_k}\right] \cdot \frac{1}{z_2} \cdot \mathbb{P}\left[r_k \neq S\right] \end{aligned}$$

$$
\begin{aligned}
U_{k+1}(z_1,z_2) \quad = & \quad A_k(z_1)\cdot z_2^N \cdot \mathbb{P}\left[q_k \le (N-S+1), r_k = S\right] \\
+ & \quad A_k(z_1)\cdot \left(\frac{z_2}{z_1}\right)^N \cdot \mathbb{E}\left[z_1^{q_k}\right]\cdot {z_1}^{S-1}\cdot \mathbb{P}\left[q_k > (N-S+1), r_k = S\right] \\
+ & \quad A_k(z_1)\cdot U_k(z_1,z_2)\cdot \frac{1}{z_2}\cdot (1-\mathbb{P}\left[r_k = S\right])
\end{aligned}
$$

Since every packet has the same size :

$$
\mathbb{P}\left[r_k = S\right] = \frac{1}{N-S+1}
$$

Denoting $\mathcal{P} = \mathbb{P}\left[q_k \le (N-S+1), r_k = S\right]$, we have :

$$
U_{k+1}(z_1,z_2) = A_k(z_1)\cdot \left[z_2^N\cdot \mathcal{P} + \left(\frac{z_2}{z_1}\right)^N \cdot z_1^{S-1}\cdot \mathbb{E}\left[z_1^{q_k}\right]\cdot \left(\frac{1}{N-S+1} - \mathcal{P}\right) + \frac{U_k(z_1,z_2)}{z_2}\cdot \left(\frac{N-S}{N-S+1}\right)\right] \tag{3}
$$

From (3) we have :

$$
U_{k+1}(z_1,z_1) = A_k(z_1)\cdot \left[z_1^N\cdot \mathcal{P} + \mathbb{E}\left[z_1^{q_k}\right]\cdot z_1^{S-1}\cdot \left(\frac{1}{N-S+1} - \mathcal{P}\right) + \frac{U_k(z_1,z_1)}{z_1}\cdot \left(\frac{N-S}{N-S+1})\right)\right]
$$

Thus :

$$
\mathbb{E}\left[z_1^{q_k}\right] = \frac{\frac{U_{k+1}(z_1,z_1)}{A_k(z_1)} - z_1^N\cdot \mathcal{P} - \frac{U_k(z_1,z_1)}{z_1}\cdot \left(\frac{N-S}{N-S+1}\right)}{\left(\frac{1}{N-S+1} - \mathcal{P}\right)\cdot z_1^{S-1}} \tag{4}
$$

With the result of equation (4), equation (3) becomes :

$$
\begin{aligned}
U_{k+1}(z_1,z_2) = A_k(z_1)\cdot \Bigg[z_2^N\cdot \mathcal{P} + \left(\frac{z_2}{z_1}\right)^N \cdot \left(\frac{U_{k+1}(z_1,z_1)}{A_k(z_1)} - z_1^N\cdot \mathcal{P} - \frac{U_k(z_1,z_1)}{z_1}\cdot \left(\frac{N-S}{N-S+1}\right)\right) \\
+ \frac{U_k(z_1,z_2)}{z_2}\cdot \frac{N-S}{N-S+1}\Bigg]
\end{aligned}
$$

or

$$
U_{k+1}(z_1,z_2) = U_{k+1}(z_1,z_1)\cdot \left(\frac{z_2}{z_1}\right)^N + \frac{N-S}{N-S+1}\cdot A_k(z_1)\cdot \left[\frac{U_k(z_1,z_2)}{z_2} - \frac{U_k(z_1,z_1)}{z_1}\cdot \left(\frac{z_2}{z_1}\right)^N\right] \tag{5}
$$

Since we are interested in the steady-state p.g.f. of the system contents, we define $U(z_1,z_2)$ as $\lim_{k\to\infty} U_k(z_1,z_2)$. $A(z)$ is defined in the same way. Applying this limit to equation (5), we find :

$$
U(z_1,z_2) = U(z_1,z_1)\cdot \left(\frac{z_2}{z_1}\right)^N + \frac{N-S}{N-S+1}\cdot A(z_1)\cdot \left[\frac{U(z_1,z_2)}{z_2} - \frac{U(z_1,z_1)}{z_1}\cdot \left(\frac{z_2}{z_1}\right)^N\right]
$$

$$
U(z_1,z_2) = U(z_1,z_1)\cdot \left(\frac{z_2}{z_1}\right)^N \cdot \frac{1 - \frac{N-S}{N-S+1}\cdot \frac{A(z_1)}{z_1}}{1 - \frac{N-S}{N-S+1}\cdot \frac{A(z_1)}{z_2}} \tag{6}
$$

$$
R(z) = U(1,z) = z^N\cdot \frac{1 - \frac{N-S}{N-S+1}}{1 - \frac{N-S}{N-S+1}\cdot \frac{1}{z}}
$$

$$
R'(z) = \frac{1}{N-S+1}\cdot \frac{N\cdot z^{N-1}\cdot \left(1 - \frac{1}{z}\cdot \frac{N-S}{N-S+1}\right) - z^{N-2}\cdot \frac{N-S}{N-S+1}}{\left(1 - z\cdot \frac{N-S}{N-S+1}\right)^2}
$$

We obtain $R'(1) = S$, though it should be :

$$R'(1) = \mathbb{E}\left[z^{r_\infty}\right] = \frac{\sum_{i=S}^{N} i}{N-S+1} = \frac{\sum_{i=0}^{N-S} i + S}{N-S+1} = \frac{\frac{(N-S)(N-S+1)}{2} + S(N-S+1)}{N-S+1} = S + \frac{N+S}{2} \tag{7}$$

Considering the p.g.f. of the tx-queue size :

$$Q(z) = U(z,1) = U(z,z) \cdot \left(\frac{1}{z}\right)^N \cdot \frac{1 - \frac{N-S}{N-S+1} \cdot \frac{A(z)}{z}}{1 - \frac{N-S}{N-S+1} \cdot A(z)}$$