MAT389 Fall 2013, Problem Set 11

Series

11.1 Given a power series in negative powers of z,

$$S = \sum_{n=0}^{\infty} \frac{b_n}{z^n},$$

show that one of two things happen:

a) the series converges nowhere, or
b) there exists an $R > 0$ such that the series converges:
   (i) converges absolutely on the region $|z| > R$,
   (ii) converges uniformly on the region $|z| \geq r$ for any $r > R$, and
   (iii) diverges on the region $|z| < R$.

Hint: use the change of variables $w = 1/z$ and apply the results on power series in *nonnegative* powers of w that we proved in class.

11.2 Prove that

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & z \neq \pm\pi/2 \\ -\dfrac{1}{\pi} & z = \pm\pi/2 \end{cases}$$

is an entire function.

11.3 Show that if f is holomorphic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function

$$g(z) = \begin{cases} \dfrac{f(z)}{(z - z_0)^{m+1}} & z \neq z_0 \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & z = z_0 \end{cases}$$

is holomorphic at z_0.

11.4 More generally, suppose f and g are holomorphic at z_0, and that

$$f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0, \qquad g(z_0) = g'(z_0) = \cdots = g^{(m)}(z_0) = 0$$

but $g^{(m+1)}(z_0) \neq 0$. Prove *l'Hôpital's rule*:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m+1)}(z_0)}{g^{(m+1)}(z_0)}$$

11.5 Let $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ be power series with radii of convergence R_a and R_b, respectively. Consider the product series $\sum_{n=0}^{\infty} c_n z^n$ defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n \geq 0)$$

Show that its radius of convergence, R_c, satisfies $R_c \geq \min(R_a, R_b)$, and that

$$\sum_{n=0}^{\infty} c_n z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

whenever $|z| < \min(R_a, R_b)$.

Residues

11.6 Calculate the following residues:

(i) $\operatorname*{Res}_{z=0} \dfrac{1}{z+z^2}$, (ii) $\operatorname*{Res}_{z=1} \dfrac{e^{2z}}{(z-1)^2}$,

(iii) $\operatorname*{Res}_{z=0} \dfrac{1-\cosh z}{z^3}$, (iv) $\operatorname*{Res}_{z=0} \dfrac{1-e^{2z}}{z^4}$,

(v) $\operatorname*{Res}_{z=0} z\cos\dfrac{1}{z}$, (vi) $\operatorname*{Res}_{z=0} \dfrac{\sinh z}{z^4(1-z^2)}$,

(vii) $\operatorname*{Res}_{z=1} \dfrac{\sinh z}{z^4(1-z^2)}$, (viii) $\operatorname*{Res}_{z=1} \dfrac{z^2+2}{z-1}$,

(ix) $\operatorname*{Res}_{z=-1/2} \left(\dfrac{z}{2z+1}\right)^3$, (x) $\operatorname*{Res}_{z=i\pi} \dfrac{e^z}{z^2+\pi^2}$.

11.7 Use residues to evaluate the following integrals:

(i) $\displaystyle\oint_{C_1(0)} \frac{e^{-z}}{z^2}\,dz$, (ii) $\displaystyle\oint_{C_1(0)} z^2 e^{1/z}\,dz$,

(iii) $\displaystyle\oint_{C_3(0)} \frac{z+1}{z^2-2z}\,dz$, (iv) $\displaystyle\oint_{C_4(0)} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$,

(v) $\displaystyle\oint_{C_2(2)} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$, (vi) $\displaystyle\oint_{C_2(0)} \frac{dz}{z^3(z+4)}$,

(vii) $\displaystyle\oint_{C_3(-2)} \frac{dz}{z^3(z+4)}$, (viii) $\displaystyle\oint_{C_2(0)} \tan z\,dz$,

(ix) $\displaystyle\oint_{C_2(0)} \frac{dz}{\sinh 2z}$, (x) $\displaystyle\oint_{C_2(0)} \frac{\cosh \pi z}{z(z^2+1)}\,dz$.