MAT389 Fall 2013, Midterm 1

Oct 8, 2013

Please justify your reasoning. Answers without an explanation will not be given any credit.

1. **[2pt]** Find all solutions to each of the following equations, and plot them:

(i) **[0.5pt]** $(z+1)^4 + i = 0$,

(ii) **[0.5pt]** $\operatorname{Re}(z+5) = \operatorname{Im}(z-i)$,

(iii) **[0.5pt]** $\operatorname{Re}\left(\frac{z}{1+i}\right) = 0$,

(iv) **[0.5pt]** $\operatorname{Re}(z^2+5) = 0$.

2. **[2pt]** Describe the following subsets of the complex plane, and plot them:

(i) **[0.5pt]** $\{z \in \mathbb{C} \mid |\operatorname{Re} z| + |\operatorname{Im} z| < 1\}$,

(ii) **[0.5pt]** $\{z \in \mathbb{C}^\times \mid 0 \leq \operatorname{Arg} z \leq \pi, |z| \leq 1\}$,

(iii) **[0.5pt]** $\{w \in \mathbb{C} \mid w = e^z \text{ for some } z \text{ with } \operatorname{Re} z \leq 0 \text{ and } 0 \leq \operatorname{Im} z \leq \pi\}$

(iv) **[0.5pt]** $\{z \in \mathbb{C} \mid \operatorname{Re} z^2 > 0\}$

3. **[2pt]** Prove that

$$\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = 1$$

whenever $|z| = 1$.

Hint: if $|z| = 1$, then $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

4. **[1pt]** Express $f(z) = z^3 + z + 1$ as $f(z) = u(x,y) + iv(x,y)$.

5. **[2pt]** Find the unique Möbius transformation, T, that takes $i \mapsto -i$, $0 \mapsto 0$, $-1 \mapsto \infty$.

Hint: remember that the unique Möbius transformation that takes $z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$ is given by the rule

$$z \mapsto \frac{z-z_1}{z_2-z_1}\frac{z_2-z_3}{z-z_3}$$

6. **[1pt]** Show that $f(z) = 1/z$ takes the line $z - \bar{z} = 2i$ to the circumference of radius $1/2$ centered at the point $-i/2$.

7. **[2pt]** Consider the transformation $f(z) = z + 1/z$.

(i) **[1pt]** What is the image under f of the unit circle, $|z| = 1$?

(ii) **[1pt]** What about the image of the open punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$?

8. **[2pt]** Let $S = \{z \in \mathbb{C}^\times \mid 0 \leq \operatorname{Arg} z \leq \pi/2\} \cup \{1/2 - i\} \subset \mathbb{C}$.

(i) **[0.5pt]** Classify all points in $\mathbb{C}$ as interior, exterior or boundary with respect to S.

(ii) **[0.25pt]** What are the accumulation points of S?

(iii) **[0.25pt]** Is S bounded?

(iv) **[0.25pt]** Is S open? Is S closed?

(v) **[0.25pt]** Is S connected?

(vi) **[0.25pt]** Is S compact?

(vii) **[0.25pt]** Is S a domain? Is S a region?

Note: for (i) and (ii), you do not need to justify your answer.

9. **[1pt]** Let Log denote the principal branch of the logarithm, defined by

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z$$

for $z \in \mathbb{C}^\times$. Give a counterexample to the identity

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

10. **[1pt]** Enunciate the Cauchy-Riemann equations (in rectangular coordinates). We know that if a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 \in \mathring{\Omega}$, then its real and imaginary parts satisfy these equations. The latter is only a necessary condition for differentiability of f at z_0, though. According to the theorem we proved in class, what are sufficient conditions for f to be differentiable at z_0?

11. **[2pt]** Find out where each of the functions below is holomorphic:

(i) **[0.4pt]** $f(z) = z + \dfrac{1}{z}$, (ii) **[0.4pt]** e^{e^z}, (iii) **[0.4pt]** $\dfrac{1}{e^z - 1}$,

(iv) **[0.4pt]** $|z|^2$, (v) **[0.4pt]** $\dfrac{z}{z^n - 2}$, $(n \in \mathbb{N})$

12. **[2pt]** Check that $u(x, y) = x^3 - 3xy^2$ is harmonic at every $(x, y) \in \mathbb{R}^2$. Calculate the unique harmonic conjugate, $v(x, y)$, that satisfies $v(0, 0) = 0$.