MAT389 Fall 2013, Problem Set 1

1.1 Express the following complex numbers in the form $re^{i\theta}$:

(i) i^3, (ii) $1-i$, (iii) $\sqrt{2}(1+i)$, (iv) $\sqrt{3}-i$, (v) $2-2\sqrt{3}i$.

(i) $i^3 = -i = e^{3\pi i/2}$
(ii) $1-i = \sqrt{2}e^{-\pi i/4}$
(iii) $\sqrt{2}(1+i) = 2e^{\pi i/4}$
(iv) $\sqrt{3}-i = 2e^{-\pi i/6}$
(v) $2-2\sqrt{3}i = 4e^{-\pi i/3}$

1.2 Express the following complex numbers in the form $x+iy$:

(i) $e^{\pi i/4}$, (ii) $5e^{-\pi i}$, (iii) $2e^{3\pi i/2}$, (iv) $e^{4\pi i/3}$, (v) $e^{7\pi i/6}$

(i) $e^{\pi i/4} = \cos(\pi/4) + i\sin(\pi/4) = (\sqrt{2}+i\sqrt{2})/2$
(ii) $5e^{-\pi i} = 5(\cos(-\pi) + i\sin(-\pi)) = 5(-1) = -5$
(iii) $2e^{3\pi i/2} = 2(\cos(3\pi/2) + i\sin(3\pi/2)) = -2i$
(iv) $e^{4\pi i/3} = \cos(4\pi/3) + i\sin(4\pi/3) = (-1-\sqrt{3}i)/2$
(v) $e^{7\pi i/6} = \cos(7\pi/6) + i\sin(7\pi/6) = (-\sqrt{3}-i)/2$

1.3 Calculate:

(i) $\dfrac{1}{i} + \dfrac{1}{1+i}$, (ii) $\dfrac{2}{(1-3i)^2}$, (iii) $(1+\sqrt{3}i)^3$, (iv) $(\sqrt{2}e^{\pi i/2} + \sqrt{2}e^{3\pi i/4})^4$

(i) $\dfrac{1}{i} + \dfrac{1}{1+i} = \dfrac{i+1+i}{i(1+i)} = \dfrac{1+2i}{i-1}$

(ii) $\dfrac{2}{(1-3i)^2} = \dfrac{2(1+3i)^2}{|1-3i|^4} = \dfrac{2(1-9+6i)}{100} = \dfrac{-4+3i}{25}$

(iii) $(1+\sqrt{3}i)^3 = (2e^{\pi i/3})^3 = 8e^{\pi i} = -8$

(iv)

$$\begin{aligned}
(\sqrt{2}e^{\pi i/2} + \sqrt{2}e^{3\pi i/4})^4 &= (\sqrt{2})^4(e^{\pi i/2} + e^{3\pi i/4})^4 \\
&= 4\left((e^{\pi i/2})^4 + 3(e^{\pi i/2})^3 e^{3\pi i/4} + 6(e^{\pi i/2})^2(e^{3\pi i/4})^2 + 3e^{\pi i/2}(e^{3\pi i/4})^3 + (e^{3\pi i/4})^4\right) \\
&= 4(1 + 3e^{\pi i/4} + 6e^{\pi i/2} + 3e^{3\pi i/4} + e^{3\pi i}) \\
&= 4(3i\sqrt{2} + 6i) = (\sqrt{2}+2)12i
\end{aligned}$$

1.4 Show that $\operatorname{Re}(iz) = -\operatorname{Im} z$ for every $z \in \mathbb{C}$.

$$\operatorname{Re}(iz) = \frac{iz + \overline{iz}}{2} = \frac{iz - i\bar{z}}{2} = i\frac{z - \bar{z}}{2} = -\frac{z - \bar{z}}{2i} = -\operatorname{Im}(z).$$

1.5 (a) Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Prove that $\operatorname{Re} z^{-1} > 0$.
(b) Let $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. Prove that $\operatorname{Im} z^{-1} < 0$.

(a) For $z \in \mathbb{C}^{\times}$, $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{\operatorname{Re}(z)}{|z|^2} - \dfrac{\operatorname{Im}(z)i}{|z|^2}$. So, if $\operatorname{Re}(z) > 0$, then $\operatorname{Re}(z^{-1}) = \dfrac{\operatorname{Re}(z)}{|z|^2} > 0$.

(b) With the reasoning above, for $z \in \mathbb{C}^{\times}$ with $\operatorname{Im}(z) > 0$, we have $\operatorname{Im}(z^{-1}) = -\dfrac{\operatorname{Im}(z)}{|z|^2} < 0$.

1.6 Prove *de Moivre's formula*:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \qquad \forall n \in \mathbb{Z}$$

The proof is by induction on n. The statement is clearly true for $n = 0, 1$. So assume that it is true for some $n \geq 1$,; then, it is also true for $n+1$, as the following calculation shows:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n (\cos\theta + i\sin\theta) = (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)\\ &= (\cos n\theta\cos\theta - \sin n\theta\sin\theta) + i(\sin n\theta\cos\theta + \sin\theta\cos n\theta)\\ &= \cos(n+1)\theta + i\sin(n+1)\theta\end{aligned}$$

This proves the statement holds for all nonnegative integers. In order to check its validity for negative integers, suppose $n \geq 1$; then,

$$(\cos\theta + i\sin\theta)^{-n} = [(\cos n\theta + i\sin n\theta)^n]^{-1} = [\cos n\theta + i\sin n\theta]^{-1}$$

But

$$\begin{aligned}&[\cos n\theta + i\sin n\theta][\cos(-n\theta) + i\sin-(n\theta)]\\ &\qquad = [\cos n\theta\cos(-n\theta) - \sin n\theta\sin(-n\theta)] + i[\cos n\theta\sin(-n\theta) + \sin n\theta\cos(-n\theta)]\\ &\qquad = \cos(n\theta - n\theta) + i\sin(n\theta - n\theta) = 1\end{aligned}$$

implies $[\cos n\theta + i\sin n\theta]^{-1} = \cos(-n\theta) + i\sin(-n\theta)$, finishing the proof.

1.7 Calculate the 3rd roots of $-\sqrt{2} - i\sqrt{2}$.

First we write $-\sqrt{2} - i\sqrt{2}$ as $2e^{5\pi i/4}$. If $z = re^{i\theta} \in \mathbb{C}$, with $r > 0$ and $\theta \in \mathbb{R}$, is such that $z^3 = r^3e^{3i\theta} = 2e^{5\pi i/4}$, then

$$r^3 = 2, \quad \text{and} \quad 3\theta = 5\pi/4 \mod 2\pi$$

So $r = \sqrt[3]{2}$ and $\theta \in \{5\pi/12, 5\pi/12 + 2\pi/3, 5\pi/12 + 4\pi/3\} = \{5\pi/12, 13\pi/12, 21\pi/12\}$, and the 3rd roots of $-\sqrt{2} - i\sqrt{2}$ are equal to

$$\sqrt[3]{2}e^{5\pi i/12}, \quad \sqrt[3]{2}e^{13\pi i/12}, \quad \sqrt[3]{2}e^{21\pi i/12} = \sqrt[3]{2}e^{-\pi i/4}$$

1.8 Let $\omega \neq 1$ be an n-th root of unity. Prove that

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

By definition of n-th root of unity, $\omega^n = 1$, so $\omega^n - 1 = 0$. But

$$\omega^n - 1 = (\omega - 1)(1 + \omega + \omega^2 + \cdots + \omega^{n-1})$$

Since $\omega \neq 1$, it has to be that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

1.9 Derive *Lagrange's trigonometric identity*:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad \text{if } \sin\frac{\theta}{2} \neq 0$$

Recall that, for $z \in \mathbb{C}, z \neq 1$, we have

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

In particular, for $z = e^{i\theta}$ with $\theta \neq 0 \mod 2\pi$,

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$$

Taking the real part of the last expression yields

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \operatorname{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right)$$

But

$$\begin{aligned}
\operatorname{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) &= \operatorname{Re}\left(\frac{1 - e^{i(n+1)\theta}}{e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2})}\right) = \operatorname{Re}\left(\frac{e^{-i\theta/2} - e^{i(n+1/2)\theta}}{-2i\sin(\theta/2)}\right) \\
&= \operatorname{Re}\left(\frac{\cos(\theta/2) - i\sin(\theta/2) - \cos[(n+1/2)\theta] - i\sin[(n+1/2)\theta]}{-2i\sin(\theta/2)}\right) \\
&= \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}
\end{aligned}$$

1.10 Find all complex solutions of the following equations:

(i) $1 + z + \cdots + z^7 = 0$, (ii) $(1 - z)^5 = (1 + z)^5$, (iii) $1 - z^2 + z^4 - z^6 = 0$.

(i) Since $(1+z+z^2+\cdots+z^7)(1-z) = 1-z^8$, the solutions of the equation given in the statement are the eighth roots of unity, except for $z = 1$. Namely, $z = e^{2\pi ik/8}$ for $1 \le k \le 7$.

(ii) Note first that the equation $(1 - z)^5 = (1 + z)^5$ is not satisfied for $z = -1$. Hence we can make the substitution $u = \frac{1-z}{1+z}$. Then $u^5 = 1$, whose solutions are $u = e^{2\pi ik/5}$, where $0 \le k \le 4$. Isolating z in terms of u then gives

$$z = \frac{1 - e^{2\pi ik/5}}{1 + e^{2\pi ik/5}} \qquad \text{for } 0 \le k \le 4$$

(iii) Let us make the substitution $u = -z^2$, so the equation $1-z^2+z^4-z^6 = 0$ becomes the equation $1+u+u^2+u^3 = 0$ in the variable u. Since $(1+u+u^2+u^3)(1-u) = 1-u^4$, the solutions of the latter are $u = -1$, $u = i$ and $u = -i$. The solution set of $-z^2 = -1$ is $\{1, -1\}$; that of $-z^2 = i$ is $\{(-\sqrt{2}+i\sqrt{2})/2, (\sqrt{2}-i\sqrt{2})/2\}$; finally, $-z^2 = -1$ yields $\{(\sqrt{2}+i\sqrt{2})/2, (-\sqrt{2}-i\sqrt{2})/2\}$.

1.11 Find a necessary and sufficient condition for the triangle inequality, $|z + w| \le |z| + |w|$, to be an equality. Use this to calculate the maximum of $|z^{10} + a|$ over the unit circle $|z| = 1$, as well as where that maximum is attained.

Recall the proof of the triangle inequality:

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) \\ &\le |z|^2 + |w|^2 + 2|z\bar{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

Equality then occurs if and only if $\mathrm{Re}(z\bar{w}) = |z\bar{w}|$ —that is, if and only if $z\bar{w}$ is real and nonnegative. There are three cases in which this happens: (i) $z = 0$, (ii) $w = 0$, and (iii) $z \ne 0 \ne w$ with

$$\arg(z) = \arg(w) \mod 2\pi$$

In this last case, there exists a positive real number r —namely, $r = |w|/|z|$— such that $w = rz$.

The maximum of $|z^{10} + a|$ over the unit circle is attained when $|z^{10} + a| = |z^{10}| + |a| = 1 + |a|$. If $a = 0$ this happens at every point of the unit circle. Otherwise, $z^{10} = ra$ for $r = |z^{10}|/|a| = 1/|a|$, which yields the solutions

$$z = e^{i\,\mathrm{Arg}(a)/10 + 2\pi ik/10}, \qquad \text{for } 0 \le k \le 9$$

1.12 The usual order relation $>$ on $\mathbb{R}$ satisfies
(a) $x \ne 0$ implies $x > 0$ or $-x > 0$, but not both, and
(b) $x, y > 0$ implies $x + y > 0$ and $xy > 0$.
Show that there does not exist a relation $>$ on $\mathbb{C}$ satisfying (a) and (b).
[Hint: consider i]

Suppose that there exists an order relation $>$ that satisfies (a) and (b). Since $i \neq 0$, we have that either $i > 0$ or $-i > 0$. In both situations, we have the implication $-1 > 0$ because either $-1 = i \cdot i > 0$ or $-1 = (-i) \cdot (-i) > 0$, by assumption (b). But we also $1 = (-1) \cdot (-1) > 0$, which is a contradiction because assumption (a) prohibits that $1 > 0$ and $-1 > 0$ hold at the same time.

1.13 Let $z, w \in \mathbb{C}$. Prove that

$$|z + iw|^2 + |w + iz|^2 = 2\left(|z|^2 + |w|^2\right)$$

Calculate

$$\begin{aligned} |z + iw|^2 &= (z + iw)(\bar{z} - i\bar{w}) = |z|^2 + i(w\bar{z} - z\bar{w}) + |w|^2 \\ |w + iz|^2 &= (w + iz)(\bar{w} - i\bar{z}) = |z|^2 + i(z\bar{w} - w\bar{z}) + |w|^2 \end{aligned}$$

Adding these two lines yields the identity in the statement.

1.14 Give a one-line proof of the fact that $(1+i)^n + (1-i)^n$ is a real number for every $n \in \mathbb{Q}$.

$$\overline{(1+i)^n + (1-i)^n} = \overline{(1+i)^n} + \overline{(1-i)^n} = (\overline{1+i})^n + (\overline{1-i})^n = (1-i)^n + (1+i)^n$$

1.15 Prove, both algebraically and geometrically, that $|z - 1| = |\overline{z} - 1|$.

Algebraically: $|\overline{z} - 1| = |\overline{z - 1}| = |z - 1|$. Geometrically: the points $1, z, \bar{z}$ in the complex plane $\mathbb{C}$ conform the vertices of an isosceles triangle with base perpendicular to the real axis. Thus the distance from z to 1 is the same to the distance from $\bar{z}$ to 1.

1.16 Solve the equation $|e^{i\theta} - 1| = 2$ for θ $(-\pi < \theta \leq \pi)$ and verify the solution geometrically.

Squaring the equation in the statement, we get

$$\begin{aligned} 4 &= |e^{i\theta} - 1|^2 = |\cos\theta - 1 + i\sin\theta|^2 = (\cos\theta - 1)^2 + \sin^2\theta \\ &= \cos^2 - 2\cos\theta + 1 + \sin^2\theta = 2(1 - \cos\theta) \end{aligned}$$

Equivalently, $\cos\theta = -1$ or $\theta = \pi$.

Geometrically, notice that $|e^{i\theta} - 1|$ is the distance between two points on the unit circle: namely, $e^{i\theta}$ and 1. That distance equals 2 only if $e^{i\theta} = -1$, which again gives $\theta = \pi$.

1.17 Give a geometric argument to prove that

$$\left|\frac{z}{|z|} - 1\right| \leq |\operatorname{Arg} z|$$

for any $z \in \mathbb{C}$.

First of all, notice that the complex number $z/|z|$ lies on the unit circle. The non-negative real number $|z/|z| - 1|$ represents the distance from 1 to $z/|z|$. On the other hand, the circular segment going from 1 to $z/|z|$ has arc-length equal to $\operatorname{Arg} z$. It is now enough to notice that the shortest path between two points on the plane is the line segment joining them.

1.18 Let $z_1, z_2, z_3 \in \mathbb{C}$ satisfying

$$z_1 + z_2 + z_3 = 0, \qquad |z_1| = |z_2| = |z_3| = 1.$$

Prove z_1, z_2, z_3 form an equilateral triangle.

After a rotation, we can assume that $z_3 = 1$, so we have that $z_1 + z_2 + 1 = 0$. In particular, $z_2 = -(1 + z_1)$ and $|z_2| = |1 + z_1| = 1$. But,

$$1 = |1 + z_1|^2 = (1 + z_1)(1 + \bar{z}_1) = 1 + z_1 + \bar{z}_1 + |z_1|^2 = 2 + 2\operatorname{Re} z_1$$

so $\operatorname{Re} z_1 = -1/2$.

If $|z_1| = 1$ and $\operatorname{Re}(z_1) = -1/2$, then $\operatorname{Im} z_1$ is either $\sqrt{3}/2$ or $-\sqrt{3}/2$. In the first case, we have

$$z_1 = \frac{-1 + i\sqrt{3}}{2} = e^{2\pi i/3} \qquad \text{and} \qquad z_2 = \frac{-1 - i\sqrt{3}}{2} = e^{-2\pi i/3}$$

In the second,

$$z_1 = \frac{-1 - i\sqrt{3}}{2} = e^{-2\pi i/3} \qquad \text{and} \qquad z_2 = \frac{-1 + i\sqrt{3}}{2} = e^{2\pi i/3}$$

1.19 Consider the Möbius transformation

$$z \mapsto T(z) = \frac{az + b}{cz + d}$$

Verify that the inverse is again a Möbius transformation —namely, the one given by

$$z \mapsto T^{-1}(z) = \frac{dz - b}{-cz + a}$$

We need to check that $(T \circ T^{-1})(z) = z$ and $(T^{-1} \circ T)(z) = z$. For the first identity, write

$$(T \circ T^{-1})(z) = T\left(\frac{dz - b}{-cz + a}\right) = \frac{a\frac{dz-b}{-cz+a} + b}{c\frac{dz-b}{-cz+a} + d} = \frac{(ad - bc)z}{ab - bc} = z$$

For the second,

$$(T^{-1} \circ T)(z) = T^{-1}\left(\frac{az+b}{cz+d}\right) = \frac{d\frac{az+b}{cz+d} - b}{-c\frac{az+b}{cz+d} + a} = \frac{(ad-bc)z}{ab-bc} = z$$

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