MAT389 Fall 2013, Problem Set 1

1.1 Express the following complex numbers in the form $re^{i\theta}$:

(i) i^3, (ii) $1-i$, (iii) $\sqrt{2}(1+i)$, (iv) $\sqrt{3}-i$, (v) $2-2\sqrt{3}i$.

1.2 Express the following complex numbers in the form $x+iy$:

(i) $e^{\pi i/4}$, (ii) $5e^{-\pi i}$, (iii) $2e^{3\pi i/2}$, (iv) $e^{4\pi i/3}$, (v) $e^{7\pi i/6}$

1.3 Calculate:

(i) $\dfrac{1}{i}+\dfrac{1}{1+i}$, (ii) $\dfrac{2}{(1-3i)^2}$, (iii) $(1+\sqrt{3}i)^3$, (iv) $(\sqrt{2}e^{\pi i/2}+\sqrt{2}e^{3\pi i/4})^4$

1.4 Show that $\operatorname{Re}(iz) = -\operatorname{Im} z$ for every $z \in \mathbb{C}$.

1.5 (a) Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Prove that $\operatorname{Re} z^{-1} > 0$.
(b) Let $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. Prove that $\operatorname{Im} z^{-1} < 0$.

1.6 Prove *de Moivre's formula*:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \qquad \forall n \in \mathbb{Z}$$

1.7 Calculate the 3rd roots of $-\sqrt{2}-i\sqrt{2}$.

1.8 Let $\omega \neq 1$ be an n-th root of unity. Prove that

$$1+\omega+\omega^2+\cdots+\omega^{n-1}=0$$

1.9 Derive *Lagrange's trigonometric identity*:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad \text{if } \sin\frac{\theta}{2}\neq 0$$

1.10 · Find all complex solutions of the following equations:

(i) $1+z+\cdots+z^7=0$, (ii) $(1-z)^5=(1+z)^5$, (iii) $1-z^2+z^4-z^6=0$.

1.11 Find a necessary and sufficient condition for the triangle inequality, $|z+w| \leq |z|+|w|$, to be an equality. Use this to calculate the maximum of $|z^{10}+a|$ over the unit circle $|z|=1$, as well as where that maximum is attained.

1.12 The usual order relation $>$ on $\mathbb{R}$ satisfies

(a) $x \neq 0$ implies $x > 0$ or $-x > 0$, but not both, and

(b) $x, y > 0$ implies $x + y > 0$ and $xy > 0$.

Show that there does not exist a relation $>$ on $\mathbb{C}$ satisfying (a) and (b).
[Hint: consider i]

1.13 Let $z, w \in \mathbb{C}$. Prove that

$$|z + iw|^2 + |w + iz|^2 = 2\left(|z|^2 + |w|^2\right)$$

1.14 Give a one-line proof of the fact that $(1+i)^n + (1-i)^n$ is a real number for every $n \in \mathbb{Q}$.

1.15 Prove, both analytically and geometrically, that $|z - 1| = |\overline{z} - 1|$.

1.16 Solve the equation $|e^{i\theta} - 1| = 2$ for θ $(-\pi < \theta \leq \pi)$ and verify the solution geometrically.

1.17 Give a geometric argument to prove that

$$\left|\frac{z}{|z|} - 1\right| \leq |\operatorname{Arg} z|$$

for any $z \in \mathbb{C}$.

1.18 Let $z_1, z_2, z_3 \in \mathbb{C}$ satisfying

$$z_1 + z_2 + z_3 = 0, \qquad |z_1| = |z_2| = |z_3| = 1.$$

Prove z_1, z_2, z_3 form an equilateral triangle.

1.19 Consider the Möbius transformation

$$z \mapsto T(z) = \frac{az + b}{cz + d}$$

Verify that the inverse is again a Möbius transformation —namely, the one given by

$$z \mapsto T^{-1}(z) = \frac{dz - b}{-cz + a}$$