MAT389 Fall 2013, Problem Set 5

Conformal transformations

5.1 Determine the angle of rotation at the point $z = 2+i$ of the transformation $f(z) = z^2$ and illustrate it for some particular curve. Show that the scale factor of the transformation at that point is $2\sqrt{5}$.

The angle of rotation of a holomorphic function $f(z)$ at a point z_0 is $\arg(f'(z_0))$, and the scale factor at z_0 is $|f'(z_0)|$. Since $f'(z) = 2z$, the angle of rotation at $2+i$ is

$$\arg\big(2(2+i)\big) = \arg(4+2i) = \arctan\frac{1}{2}$$

and the scale factor is

$$|2(2+i)| = |4+2i| = \sqrt{16+4} = 2\sqrt{5}.$$

5.2 Show that the angle of rotation at a nonzero point $z = r_0 e^{i\theta_0}$ under the transformation $f(z) = z^n$ $(n \geq 1)$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point.

The angle of rotation and scale factors of $f(z) = z^n$ at $z_0 = r_0 e^{i\theta_0}$ are

$$\arg f'(z_0) = \arg(nz_0^{n-1}) = \arg(nr_0^{n-1}e^{i(n-1)\theta}) = (n-1)\theta, \quad |f'(z_0)| = \left|nr_0^{n-1}e^{i(n-1)\theta}\right| = nr_0^{n-1}.$$

Transformation of harmonic functions

5.3 (i) Show that the transformation $w = e^z$ takes the horizontal strip $0 < \operatorname{Im} z < \pi$ to the (open) upper half plane $\{w \in \mathbb{C} \mid \operatorname{Im} w > 0\}$.

(ii) Use the fact that $h(u,v) = \operatorname{Re} w^2 = u^2 - v^2$ is harmonic on the upper half plane, and the transformation $w = e^z$ to show that $H(x,y) = e^{2x}\cos 2y$ is a harmonic function on the strip.

(i) If $z = x + iy$ is an element of the horizontal strip $0 < \operatorname{Im} z < \pi$, x is arbitrary and $0 < y < \pi$. Since $|e^z| = e^x$ and $\arg e^z = y$, these restrictions yield $|w| > 0$ and $0 < \arg w < \pi$; these inequalities precisely describe the upper half plane.

(ii) Notice that $g(w) = w^2$ is entire, and that $f(z) = e^z$ is holomorphic on the strip. Their composition $g(f(z)) = e^{2z}$ is hence holomorphic on the strip, from it follows that $\operatorname{Re}(e^{2z}) = e^{2x}\cos(2y)$ is indeed harmonic on the strip.

5.4 Suppose that a holomorphic function $w = f(z) = u(x,y) + iv(x,y)$ maps a domain D_z in the z plane onto a domain D_w in the w plane; and let a function $h(u,v)$, with continuous partial derivatives of the first and second order, be defined on D_w. Use the chain rule for partial derivatives to show that if $H(x,y) = h\big(u(x,y), v(x,y)\big)$, then

$$H_{xx}(x,y) + H_{yy}(x,y) = \Big[h_{uu}(u,v) + h_{vv}(u,v)\Big]|f'(z)|^2$$

Hint: in the simplifications you will need to use the Cauchy-Riemann equations for f, and the fact that u and v are harmonic on D_z.

By the chain rule and the product rule,

$$H_x = h_u u_x + h_v v_x,$$

$$H_{xx} = h_{uu}(u_x)^2 + h_{uv}u_x v_x + h_u u_{xx} + h_{vv}(v_x)^2 + h_{vu}u_x v_x + h_v v_{xx}.$$

Similarly,

$$H_y = h_u u_y + h_v v_y,$$

$$H_{yy} = h_{uu}(u_y)^2 + h_{uv}u_y v_y + h_u u_{yy} + h_{vv}(v_y)^2 + h_{vu}u_y v_y + h_v v_{yy}.$$

Adding these together and collecting like terms, we see that

$$\begin{aligned} H_{xx} + H_{yy} = {} & h_{uu}\left[(u_x)^2 + (u_y)^2\right] + h_{vv}\left[(v_x)^2 + (v_y)^2\right] \\ & + (h_{uv} + h_{vu})(u_x v_x + u_y v_y) + h_u(u_{xx} + u_{yy}) + h_v(v_{xx} + v_{yy}). \end{aligned}$$

Since u and v are the real and imaginary parts of a function, f, that is holomorphic on D_z, they satisfy the Cauchy-Riemann equations on said domain. It follows that

$$u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0,$$

which says that the third term in $(*)$ vanishes. The harmonicity of u and v makes the fourth and fifth terms zero too. One more application of the Cauchy-Riemann equations yields

$$\begin{aligned} (u_x)^2 + (u_y)^2 &= (u_x)^2 + (-v_x)^2 = |f'(z)|^2, \\ (v_x)^2 + (v_y)^2 &= (v_x)^2 + (u_x)^2 = |f'(z)|^2, \end{aligned}$$

making $(*)$ into the equation in the statement.

5.5 Let $p(u, v)$ be a function that has continuous partial derivatives of the first and second orders and satisfies *Poisson's equation*

$$p_{uu}(u, v) + p_{vv}(u, v) = \Phi(u, v)$$

in a domain D_w of the w plane, where Φ is a prescribed function. Show how it follows from the previous problem that if a holomorphic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_z onto the domain D_w, then the function

$$P(x, y) = p\big(u(x, y), v(x, y)\big)$$

satisfies the Poisson equation

$$P_{xx}(x, y) + P_{yy}(x, y) = \Phi\big(u(x, y), v(x, y)\big)|f'(z)|^2$$

on D_z.

Setting $h = p$ (and hence $H = P$) in the previous problem, we obtain

$$P_{xx}(x, y) + P_{yy}(x, y) = \Big[p_{uu}(u, v) + p_{vv}(u, v)\Big]|f'(z)|^2$$

Using the Poisson equation satisfied by p yields the desired statement.