MAT389 Fall 2013, Problem Set 10

Taylor series

10.1 What is the largest disc on which the Taylor series about $z = 0$ for $f(z) = \tanh z$ converges absolutely?

10.2 Recall that the Taylor series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

converge on the whole real line and the interval $[-1, 1]$, respectively. One may think then that the Taylor series for the function

$$f(x) = \arctan\left(\frac{1}{2}\sin x\right)$$

should converge on the whole real line. That this is not so might be puzzling from the perspective of real calculus, but is easily seen using complex analysis. Let

$$F(z) = \arctan\left(\frac{1}{2}\sin z\right) = \frac{i}{2}\log\frac{2i + \sin z}{2i - \sin z}$$

(i) Find a branch of the this multivalued function that restricts to $f(x)$ for real values of z. **Hint:** it is enough to find a branch that is holomorphic at $z = 0$, and whose value at $z = 0$ is $f(0) = 0$.

(ii) Determine the radius of convergence of the Taylor series expansion of $F(z)$ about $z = 0$ by finding the singularity closest to the origin.

(iii) Deduce the radius of convergence of the Taylor series expansion of $f(x)$ about $x = 0$.

10.3 Prove the formulas

$$\sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \cos\left[n(\theta - \phi)\right] = \frac{R^2 - rR\cos(\theta - \phi)}{R^2 - 2rR\cos(\theta - \phi) + r^2}$$

$$\sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \sin\left[n(\theta - \phi)\right] = \frac{rR\sin(\theta - \phi)}{R^2 - 2rR\cos(\theta - \phi) + r^2}$$

where $r < R$ are positive real numbers.
Hint: consider the Taylor series about $z = 0$ of the function $f(z) = (1 - z/s)^{-1}$ with $z = re^{i\theta}$ and $s = Re^{i\phi}$.

Laurent series

10.4 The function $f(z) = (1+z)^{-1}$ is holomorphic everywhere on the complex plane except at the point $z = -1$. Consequently, there are two series expansions about $z = 0$ that converge to it. The first one is its Taylor series expansion about $z = 0$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n,$$

that converges absolutely in the disk $|z| < 1$. The other one is a Laurent series expansion that converges absolutely for $1 < |z| < +\infty$. Compute the latter.

10.5 (i) Let $f(z)$ be a function that is holomorphic in some annular domain about the origin that includes the unit circle, parametrized by $e^{i\phi}$, $-\pi \le \phi \le \pi$. By taking this circle as the contour of integration for the coefficients in the Laurent series expansion of $f(z)$, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\, d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi$$

when z is any point in the annular domain.

(ii) Write $u(\theta) = \operatorname{Re} f(e^{i\theta})$, and show how it follows from the expansion above that

$$\begin{aligned} u(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\, d\phi \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big[a_n\cos(n\theta) + b_n\sin(n\theta)\Big] \end{aligned}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\phi)\cos(n\phi)\, d\phi, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\phi)\sin(n\phi)\, d\phi$$

Note: this a *Fourier series* expansion of the real-valued function $u(\theta)$.

Isolated singularities

10.6 For each of the cases below, write the principal part of the function at its isolated singular point, and determine whether that point is a pole, and essential singular point or a removable singular point.

(i) $f(z) = ze^{1/z}$,
(ii) $f(z) = z^2/(1+z)$,
(iii) $f(z) = (\sin z)/z$,
(iv) $f(z) = (2-z)^{-3}$.

10.7 For an illustration of Picard's Big Theorem, show that the function $f(z) = e^{1/z}$ assumes the value -1 an infinite number of times in any neighborhood of the origin by explicitly computing the solutions to the equation $f(z) = -1$.