MAT389 Fall 2013, Problem Set 8

Integrals of complex-valued functions of a real variable

8.1 In first-year calculus courses, integrals of the form

$$\int_a^b e^{\alpha x} \cos \beta x \, dx, \qquad \int_a^b e^{\alpha x} \sin \beta x \, dx$$

are typically computed by applying integration by parts twice. Notice that they constitute the real and imaginary parts of the integral

$$\int_a^b e^{(\alpha + i\beta)x} \, dx$$

Find expressions for the former by calculating the latter.
Hint: notice that the complex-valued function of the real variable $e^{(\alpha+i\beta)x}$ possesses an antiderivative on the interval $[a, b]$ (in fact, on the whole real line).

Complex integration

8.2 For each of the cases below, compute the integral

$$\int_C f(z) \, dz.$$

(i) C is the semicircle $z = 2e^{i\theta}$, $0 \leq \theta \leq \pi$, and $f(z) = (z + 2)/z$.
(ii) C is the boundary of the square with vertices at the points 0, 1, $1 + i$ and i, taken counterclockwise, and $f(z) = e^{\pi \bar{z}}$.
(iii) C is the unit circle centered at the origin, taken counterclockwise, and $f(z)$ is the principal branch of the multivalued function z^{-1+i}.
(iv) C is the unit circle centered at the origin, taken counterclockwise, and $f(z) = z^n \bar{z}^m$, with $n, m \in \mathbb{Z}$.

8.3 Let C be a simple closed contour, oriented counterclockwise, and R the region enclosed by it. Show that

$$\text{area}(R) = \frac{1}{2i} \oint_C \bar{z} \, dz.$$

Hint: use Green's theorem.

8.4 Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without evaluating the integral, show that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}$$

8.5 Let C_R be the circle of radius $R > 1$ centered at the origin, oriented counterclockwise. Show that

$$\left| \oint_C \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi\, \frac{\pi + \operatorname{Log} R}{R}$$

and hence that the value of this integral approaches zero as R tends to infinity.

Cauchy integral formulas

8.6 For each of the cases below, compute the integral

$$\oint_C f(z)\, dz,$$

where C denotes the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$, oriented counterclockwise.

(i) $f(z) = e^z/(z - \pi i/2)$,
(ii) $f(z) = \cos z/[z(z^2 + 8)]$,
(iii) $f(z) = z/(2z + 1)$,
(iv) $f(z) = \tan(z/2)/(z - x_0)^2$ where $-2 < x_0 < 2$, and
(v) $f(z) = z/(2z + 1)^3$,
(vi) $f(z) = \cosh z/z^4$.

8.7 Show that if f is holomorphic on and inside of a simple closed contour C, and z_0 is not on C, then

$$\oint_C \frac{f'(z)}{z - z_0}\, dz = \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

Warning: note that the statement says that z_0 is not *on* C, not that it is inside of C.

8.8 (i) Use the binomial formula to show that, for any $n \in \mathbb{Z}$, $n \geq 0$, the function

$$P_n(z) = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n} (z^2 - 1)^n$$

is a polynomial of degree n, .

(ii) Let C be any simple closed contour surrounding a fixed point z. Use the Cauchy integral formula for the nth derivative of a holomorphic function to show that

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}}\, ds.$$

(iii) Use the Cauchy integral formula to conclude that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

Note: these polynomials receive the name of *Legendre polynomials*, and they satisfy *Legendre's differential equation*:

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0.$$

The latter appears when solving the (three-dimensional!) Laplace equation in spherical coordinates.

8.9 Let C be a simple closed contour oriented counterclockwise, and f a function that is holomorphic on and inside of C. Provide the details for the derivation of the Cauchy integral formula for the second derivative following these steps:

1. Apply the Cauchy integral formula for f' to show that

$$\frac{f'(z+\Delta z)-f'(z)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\, f(s)\, ds$$

when z is a point inside of C, and $0 < \Delta z < d$, where d is the minimum distance from z to points on C.

2. Use the continuity of f on C to show that the value of the integral

$$\oint_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{2}{(s-z)^3}\right] f(s)\, ds$$

approaches zero as Δz goes to zero.

3. Conclude that

$$f''(z) = \frac{1}{\pi i}\oint_C \frac{f(s)}{(s-z)^2}\, ds.$$

Hint: in the simplications in step 2, retain the difference $s-z$ as a single term. Also, let D be the *maximum* distance from z to points on C.

8.10 Let C be a simple closed contour, and f a continuous function defined on C. Prove that the function defined by the formula

$$g(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}\, dz$$

is holomorphic at all points z interior to C by showing that its derivative is given by

$$g'(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z)^2}\, dz.$$

Hint: simply follow the steps in the proof of the Cauchy integral formula for f'.
Note: suppose that the function f is defined not only on C, but also at every point interior to C. If f is not holomorphic, it may happen that $g(z) \neq f(z)$, even if f is continuous. For example, choose C to be the unit circle, oriented counterclockwise, and

$$f(z) = \begin{cases} 1 & z = 0 \\ \left[1+\exp\left(\frac{1}{1-|z|^2} - \frac{1}{|z|^2}\right)\right]^{-1} & 0 < |z| < 1 \\ 0 & |z| \geq 1 \end{cases}$$

Horrible as it may look, this function is (real-)differentiable on and inside C (and even complex-differentiable on C), but

$$g(0) = \frac{1}{2\pi i}\oint_C \frac{f(s)}{s}\, ds = 0 \neq 1 = f(0).$$