MAT389 Fall 2013, Problem Set 2

Curves in $\mathbb{C}$

2.1 Show that every line in $\mathbb{C}$ can be expressed in the form

$$\beta z + \overline{\beta z} + \gamma = 0$$

for some $\beta \in \mathbb{C}^\times$ and $\gamma \in \mathbb{R}$. Why is the condition $\beta \neq 0$ necessary?
Hint: recall that every line in $\mathbb{C}$ can be expressed as the set of solutions of a linear equation of the form $px+qy+r=0$ with p, q not simultaneously zero. What should the relationship between p, q, r on the one hand, and β, γ on the other, be?

Write $z = x + yi$, and $\beta = a + bi$. Then

$$\begin{aligned} \beta z + \overline{\beta z} + \gamma = 0 &\iff (a+bi)(x+yi) + (a-bi)(x-yi) + \gamma = 0 \\ &\iff (ax-by) + (ax-by) + (bx+ay)i + (-bx-ay)i + \gamma = 0 \\ &\iff (2a)x + (-2b)y + \gamma = 0. \end{aligned}$$

Hence, if we set $\gamma = r$ and $\beta = (p - iq)/2$, then the line $px + qy + r = 0$ is the set of solutions to $\beta z + \overline{\beta z} + \gamma = 0$.

2.2 Consider the lines

$$L_1 = \left\{ z \in \mathbb{C} \mid \beta_1 z + \overline{\beta_1 z} + \gamma_1 = 0 \right\}, \qquad L_2 = \left\{ z \in \mathbb{C} \mid \beta_2 z + \overline{\beta_2 z} + \gamma_2 = 0 \right\}$$

where $\beta_1, \beta_2 \in \mathbb{C}^\times$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Prove that L_1 and L_2 are orthogonal if and only if $\operatorname{Re}(\beta_1 \overline{\beta_2}) = 0$.

The direction vectors of the lines L_1 and L_2 are given by $\langle \operatorname{Im}\beta_1, \operatorname{Re}\beta_1 \rangle$ and $\langle \operatorname{Im}\beta_2, \operatorname{Re}\beta_2 \rangle$, respectively. The lines are orthogonal if and only if the dot product of their direction vectors vanishes:

$$0 = \langle \operatorname{Im}\beta_1, \operatorname{Re}\beta_1 \rangle \cdot \langle \operatorname{Im}\beta_2, \operatorname{Re}\beta_2 \rangle = \operatorname{Im}\beta_1 \operatorname{Im}\beta_2 + \operatorname{Re}\beta_1 \operatorname{Re}\beta_2 = \operatorname{Re}(\beta_1 \overline{\beta_2})$$

2.3 Let $\alpha, \beta \in \mathbb{C}$ distinct. Give a geometric argument to show that $|z - \alpha| = |z - \beta|$ is a line.

The equation $|z - \alpha| = |z - \beta|$ describes the set of points that are equidistant to α and β —namely, the perpendicular bisector to the line segment joining α and β.

2.4 Show that every circle in $\mathbb{C}$ can be expressed in the form

$$\alpha z\overline{z} + \beta z + \overline{\beta z} + \gamma = 0$$

for some $\alpha \in \mathbb{R}^\times$, $\beta \in \mathbb{C}^\times$ and $\gamma \in \mathbb{R}$ satisfying $|\beta|^2 > \alpha\gamma$. Why are the conditions $\alpha \neq 0$ and $|\beta|^2 > \alpha\gamma$ necessary?

Hint: recall that every circle in $\mathbb{C}$ can be expressed as the set of solutions of a quadratic equation of the form $m(x^2+y^2)+px+qy+r = 0$ with $m \neq 0$. What should the relationship between m, p, q, r on the one hand, and α, β, γ on the other, be?

Write $z = x + yi$, and $\beta = a + bi$. Then

$$\alpha z\overline{z} + \beta z + \overline{\beta z} + \gamma = 0 \iff \alpha(x^2 + y^2) + (2ax - 2by) + \gamma = 0.$$

Setting $\alpha = m$, $p = 2a$, $q = -2b$, $\gamma = r$, it follows that the circle $m(x^2 + y^2) + px + qy + r = 0$ is the set of solutions to $\alpha z\overline{z} + \beta z + \overline{\beta z} + \gamma = 0$. Completing squares, we see that $\alpha z\overline{z} + \beta z + \overline{\beta z} + \gamma = 0$ is equivalent to

$$\left(x + \frac{a}{\alpha}\right)^2 + \left(y - \frac{b}{\alpha}\right)^2 = \frac{|\beta|^2}{\alpha^2} - \frac{\gamma}{\alpha},$$

so the quantity on the right hand side of this equation must be non-negative for solutions to exist: if $|\beta|^2 = \gamma\alpha$, there is exactly one solution, while for $|\beta|^2 > \gamma\alpha$ we obtain an actual circle. On other hand, if $\alpha = 0$, the equation in the statement describes a line (see Problem 2.1).

2.5 Prove that the equation

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda, \qquad \alpha, \beta, \in \mathbb{C}, \lambda \in \mathbb{R}_{>0}$$

describes either a circle or a line in $\mathbb{C}$.

Hint: consider the Möbius transformation $z \mapsto (z-\alpha)/(z-\beta)$.

Let $S = \{z \in \mathbb{C} \mid |z| = \lambda\}$. Clearly S is the circle of radius λ. Let T be the Möbius transformation $z \mapsto (z-\alpha)/(z-\beta)$. Consider the set $S' = T^{-1}(S)$, which is described by the equation in the statement. Since T is a Möbius transformation, so is T^{-1}, so S' is a circle or a line.

2.6 Let $\alpha, \beta \in \mathbb{C}$ distinct, and let $\lambda \in \mathbb{R}_{>0}$ such that $\lambda > |\alpha - \beta|$. What geometric figure is described by the equation $|z - \alpha| + |z - \beta| = \lambda$? What goes wrong if $\lambda \leq |\alpha - \beta|$?

Geometrically, $|z - \alpha| + |z - \beta| = \gamma$ means that the sum of the distance from z to α and that from z to β is the constant γ. The locus of such points is an ellipse with foci α and β, and major axis of length λ. Since $|\alpha - \beta| \leq |z - \alpha| + |z - \beta|$, if $\lambda < |\alpha - \beta|$, there are no solutions, while, if $\lambda = |\alpha - \beta|$, the only solutions are the points on the line segment from α to β.

2.7 Let $\alpha, \beta \in \mathbb{C}$ distinct, and let $\lambda \in \mathbb{R}^\times$. What geometric figure is described by the equation $|z-\alpha| - |z-\beta| = \lambda$? What happens when $\lambda = 0$?

The geometric figure described by the equation in the statement is one of the two branches of the hyperbola with foci α and β. If $\lambda = 0$, it degenerates into a line (see Problem 2.3).

Circles in Σ

Recall that the Riemann sphere is defined as the set

$$\Sigma = \left\{(a,b,c) \in \mathbb{R}^3 \mid a^2+b^2+c^2 = 1\right\}$$

Let P be the plane defined by

$$P = \left\{(a,b,c) \in \mathbb{R}^3 \mid Aa+Bb+Cc = D\right\}$$

2.8 Show that P passes through the North pole, $N = (0,0,1)$, if and only if $C = D$.

P passes through the North pole if and only if the point $(0,0,1)$ satisfies the defining equation $Aa+Bb+Cc = D$, i.e., if and only if $C = D$.

2.9 Prove that $P \cap \Sigma \neq \varnothing$ if and only if $A^2+B^2+C^2 \geq D^2$ as follows:

1. Convince yourself that $P \cap \Sigma \neq \varnothing$ if and only if the point of P closest to the origin —call it p— satisfies $d(0,p) \leq 1$, where d is the usual, Euclidean distance in $\mathbb{R}^3$ and $0 \in \mathbb{R}^3$ means the origin.
   Hint: a picture should suffice.

2. Show that
$$d(0,p) = \sqrt{\frac{D^2}{A^2+B^2+C^2}}$$
   Hint: recall from multivariable calculus that the vector $\langle A,B,C\rangle$ is normal to P, and that, in fact, $p = (\lambda A, \lambda B, \lambda C)$ for the appropriate value of λ.

Notice that $P \cap \Sigma$ consists of a single point (p, in fact) if and only if $A^2+B^2+C^2 = D^2$. Hence, if $A^2+B^2+C^2 > D^2$, then $P \cap \Sigma$ is an actual circle in Σ (that is, its radius is strictly positive).

1. If $d(0,p) = 1$, then P and Σ intersect only at p. If $d(0,p) > 1$, P and Σ do not intersect, by the definition of p as the closest point to Σ. If $d(0,p) < 1$, it is visually clear that P and Σ intersect, and that they do so in more than one point.

2. Let p' be the point on P such that $\overrightarrow{0p'}$ is orthogonal to P. We claim that $p = p'$. If not, the line segment $0p$ is the hypotenuse of the right-angled triangle $\triangle 0pp'$, so $d(0,p') < d(0,p)$, contradicting the definition of p. Hence, $p = p'$.

   Since $\overrightarrow{0p}$ is orthogonal to P, it is parallel to the normal vector to P. Hence, $\overrightarrow{0p} = \lambda\langle A, B, C\rangle$ for some $\lambda \in \mathbb{R}$. Since p lies on P, λ must satisfy $A(\lambda A) + B(\lambda B) + C(\lambda C) = D$, so

$$\lambda = \frac{D}{A^2+B^2+C^2} \quad \Longrightarrow \quad d(0,p)^2 = \|\overrightarrow{0p}\|^2 = |\lambda|^2(A^2+B^2+C^2) = \frac{D^2}{A^2+B^2+C^2}.$$

The stereographic projection

Remember that stereographic projection establishes a bijection between the Riemann sphere Σ and the extended complex plane $\hat{\mathbb{C}}$ as follows:

$$\begin{aligned} \Sigma &\xrightarrow{\quad\pi\quad} \hat{\mathbb{C}} \\ (a,b,c) &\longmapsto \frac{a+ib}{1-c} \\ N=(0,0,1) &\longmapsto \infty \end{aligned} \qquad\qquad \begin{aligned} \hat{\mathbb{C}} &\xrightarrow{\qquad\quad\varphi\quad\qquad} \Sigma \\ z = x+iy &\longmapsto \left(\tfrac{2x}{x^2+y^2+1}, \tfrac{2y}{x^2+y^2+1}, \tfrac{x^2+y^2-1}{x^2+y^2+1}\right) \\ \infty &\longmapsto N=(0,0,1) \end{aligned}$$

2.10 Using stereographic projection, we can transport maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to maps $\Sigma \to \Sigma$. Describe geometrically the self-map of the Riemann sphere obtained from inversion $z \mapsto 1/z$ on the extended complex plane.
Hint: remember that $1/z = \bar{z}/|z|^2$. Calculate the inverse of $(a+ib)/(1-c)$ and judiciously use the equation defining the Riemann sphere to simplify the result. The geometric interpretation arises easily from that.

If $z = (a+ib)/(1-c)$, then

$$|z|^2 = z\bar{z} = \frac{a+ib}{1-c}\frac{a-ib}{1-c} = \frac{a^2+b^2}{(1-c)^2} = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c}$$

and

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a-ib}{1-c}\frac{1-c}{1+c} = \frac{a-ib}{1+c}$$

Thus the transformation $z \mapsto 1/z$ takes

$$\Sigma \ni (a,b,c) \mapsto (a,-b,-c) \in \Sigma$$

That is, inversion on the complex plane corresponds to the composition of two reflections on the Riemann sphere: one with respect to the ac-plane (switching the sign of b) and another one with respect to the ab-plane (which corresponds to switching the sign of c).

Möbius transformations

2.11 Show that any Möbius transformation of the form

$$T(z) = e^{i\theta}\frac{z - z_0}{z - \overline{z_0}}, \qquad \theta \in \mathbb{R}, \quad \operatorname{Im} z_0 \neq 0$$

sends the real line to the circle of radius 1. In other words, $|T(z)| = 1$ whenever $\operatorname{Im} z = 0$.

If $\operatorname{Im} z = 0$, then $z = \bar{z}$ and $\overline{z - z_0} = \bar{z} - \overline{z_0} = z - \overline{z_0}$. Since the modulus of a complex number is the same as that of its complex-conjugate, we have

$$|z - z_0| = |\overline{z - z_0}| = |z - \overline{z_0}| \quad \Longrightarrow \quad \left|\frac{z - z_0}{z - \overline{z_0}}\right| = 1$$

Since $|e^{i\theta}| = 1$, it follows that $|T(z)| = 1$ whenever $\operatorname{Im} z = 0$.

The condition $\operatorname{Im} z_0 \neq 0$ ensures that T is not constant. Indeed, if $\operatorname{Im} z_0 = 0$, then $z - z_0 = z - \overline{z_0}$ (for any value of z!), and so $T(z) = e^{i\theta}$.

2.12 Find a Möbius transformation $\mathbb{D} \to \mathbb{D}$ that takes $1/2$ to $1/3$.
Hint: recall that the Möbius transformations preserving the unit disc $\mathbb{D}$ take the form

$$T(z) = e^{i\theta}\frac{z + \alpha}{\overline{\alpha} z + 1}$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ satisfying $|\alpha| < 1$.

Note that

$$T_\alpha : z \mapsto \frac{z + \alpha}{\overline{\alpha} z + 1}$$

sends $0 \mapsto \alpha$ and $-\alpha \mapsto 0$. Hence, $T_{-1/2}$ sends $1/2 \mapsto 0$, and $T_{1/3}$ sends $0 \mapsto 1/3$. The composition $T_{1/3} \circ T_{-1/2}$ preserves the unit disk, $\mathbb{D}$, and sends $1/2$ to $1/3$. The explicit computation yields

$$\left(T_{1/3} \circ T_{-1/2}\right)(z) = \frac{1 - 5z}{z - 5}$$

2.13 Find a Möbius transformation $\mathbb{D} \to \mathbb{H}$ that takes the origin to the point $3 + 2i$.
Hint: first, use the Möbius transformation

$$z \mapsto \frac{z + i}{iz + 1}$$

to take $\mathbb{D}$ into $\mathbb{H}$ —and the origin to i. Then find the appropriate Möbius transformation $\mathbb{H} \to \mathbb{H}$ that takes $i \mapsto 3 + 2i$. Recall that the Möbius transformation preserving the upper half-plane $\mathbb{H}$ are always of the form

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R}, \quad ad - bc > 0$$

Let T_1 be the Möbius transformation $z \mapsto \frac{z+i}{iz+1}$, sending $\mathbb{D}$ onto $\mathbb{H}$ and 0 to i. Let T_2 be the Möbius transformation $z \mapsto 2z+i$. Then T_2 preserves $\mathbb{H}$, and $T_2(i) = 3+2i$. Hence, the Möbius transformation

$$T(z) = (T_2 \circ T_1)(z) = \frac{(2+3i)z + (3+i)}{iz+1}$$

sends $\mathbb{D}$ onto $\mathbb{H}$ and maps 0 to $3+2i$.

2.14 We saw in class that

$$T : z \mapsto \frac{z - z_1}{z_2 - z_1} \frac{z_2 - z_3}{z - z_3}$$

is the unique Möbius transformation that takes $z_1 \mapsto 0, z_2 \mapsto 1, z_3 \mapsto \infty$. Similarly,

$$U : w \mapsto \frac{w - w_1}{w_2 - w_1} \frac{w_2 - w_3}{w - w_3}$$

is the unique Möbius transformation that takes $w_1 \mapsto 0, w_2 \mapsto 1, w_3 \mapsto \infty$. If follows that $U^{-1} \circ T$ is the unique Möbius transformation that takes $z_1 \mapsto w_1, z_2 \mapsto w_2, z_3 \mapsto w_3$. It is easy to compute the latter explicitly by isolating w in the equation

$$U(w) = T(z) \qquad \Longleftrightarrow \qquad \frac{w - w_1}{w_2 - w_1} \frac{w_2 - w_3}{w - w_3} = \frac{z - z_1}{z_2 - z_1} \frac{z_2 - z_3}{z - z_3}$$

Use this line of reasoning to find the unique Möbius transformation taking $-i \mapsto -1$, $1 \mapsto 0$, $i \mapsto 1$.

Let $z_1 = -i$, $z_2 = 1$, $z_3 = i$, and $w_1 = -1$, $w_2 = 0$, $w_3 = 1$, and define T and U as above. Explicitly,

$$T(z) = \frac{z+i}{1+i} \frac{1-i}{z-i} = \frac{1-iz}{z-i}, \qquad \text{and} \qquad U(w) = \frac{w+1}{0+1} \frac{0-1}{w-1} = \frac{w+1}{1-w}.$$

Then

$$U(w) = T(z) \qquad \Longleftrightarrow \qquad \frac{w+1}{1-w} = \frac{1-iz}{z-i} \qquad \Longleftrightarrow \qquad w = i\,\frac{1-z}{1+z}.$$

Topology in the complex plane

2.15 For each of the choices of S below, do the following:

1. classify all the points of $\mathbb{C}$ according to whether they are interior, exterior or boundary points of S;
2. decide whether S is open, closed, both open and closed, or neither open nor closed;
3. decide whether S is connected;
4. decide whether S is simply-connected;
5. decide whether S is bounded.

(i) $S = \{z \in \mathbb{C}^{\times} \mid 0 < \operatorname{Arg} z < \pi/2\}$;
(ii) $S = \{z \in \mathbb{C} \mid |z| \geq |z-4|\}$;
(iii) $S = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \delta\}$, where $z_0 \in \mathbb{C}$ and $\delta \in \mathbb{R}_{>0}$;
(iv) $S = \left\{z \in \mathbb{C} \mid \operatorname{Re}(z^2) > 0\right\} \cup \{0\}$;
(v) $S = \mathbb{C}$.

1. (i) The interior of S is S itself, the open first quadrant. The boundary consists of the rays from 0 along the positive real axis and the positive imaginary axis, including the origin. The rest of the points of $\mathbb{C}$ are exterior.

   (ii) S is the set of points which are at least as far from 0 as they are from 4 —that is,
   $$S = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 2\}.$$
   Hence, the interior of S is $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq 2\}$, the exterior is $\{z \in \mathbb{C} \mid \operatorname{Re} z < 2\}$, and the boundary is $\{z \in \mathbb{C} \mid \operatorname{Re} z = 2\}$.

   (iii) S is the open disk of radius δ and centre z_0, minus the point z_0. All points of S are interior. The boundary is the union of the one-point set $\{z_0\}$ with the circle $\{z \in \mathbb{C} \mid |z - z_0| = \delta\}$. The exterior is the complex plane minus the closed disk $\{z \in \mathbb{C} \mid |z - z_0| \leq \delta\}$.

   (iv) S is the set of points in the open first quadrant that lie below the line $y = x$, plus the origin. In other words, it is the open wedge in the first quadrant formed by the lines $y = x$ and $y = 0$, plus the origin. The interior is the open wedge itself; the boundary consists of the two rays emanating from 0 in the direction of 1, and $1 + i$, respectively, including the origin. The rest of the complex plane makes up the exterior of S.

   (v) The interior of S is the whole of S. Consequently, the boundary and exterior are empty.

2. (i), (iii) and (v) are open. (ii) and (v) are closed. (iv) is neither open nor closed. Note that (v) is both open and closed.

3. (i)-(v) are all connected.

4. (i), (ii), (iv) and (v) are simply connected.

5. Only (iii) is bounded.

2.16 Find the accumulation points of each of the following subsets of $\mathbb{C}$:

(i) $S = \{i^n \mid n \in \mathbb{N}\}$;
(ii) $S = \{i^n/n \mid n \in \mathbb{N}\}$;
(iii) $S = \{z \in \mathbb{C}^\times \mid 0 \leq \operatorname{Arg} z < \pi/2\}$;
(iv) $S = \{(-1)^n(1+i)(n-1)/n \mid n \in \mathbb{N}\}$.

The accumulation points are:

(i) $\emptyset$, (ii) $\{0\}$, (iii) $\{z \in \mathbb{C}^\times \mid 0 \leq \operatorname{Arg} z \leq \pi/2\} \cup \{0\}$, (iv) $\{\pm(1+i)\}$.

2.17 Prove that the *interior* of a subset $S \subset \mathbb{C}$,

$$\mathring{S} = \{z \in S \mid z \text{ is an interior point of } S\}\,,$$

is open, and that, in fact, it is the biggest open subset of $\mathbb{C}$ contained in S.

Let x be an interior point of S. By definition, there is a neighborhood $D_\epsilon(x)$ of x completely contained in S. We need to show that there is likewise a neighborhoof of x which is a subset of $\mathring{S}$. We claim that $D_{\epsilon/2}(x)$ is such a neighborhood: if $y \in D_{\epsilon/2}(x)$, then $D_{\epsilon/2}(y) \subset D_\epsilon(x) \subset S$, so in fact $y \in \mathring{S}$. We have shown that $D_{\epsilon/2}(x) \subset \mathring{S}$, so $\mathring{S}$ is open.

Now suppose that $U \subset S$ is open. Let $x \in U$. Then there is an open neighborhoof $D_\epsilon(x) \subset U \subset S$, so $x \in \mathring{S}$. This shows that $U \subset \mathring{S}$, and so $\mathring{S}$ is the largest open subset contained in S.

2.18 Prove that the *closure* of a subset $S \subset \mathbb{C}$,

$$\overline{S} = \{z \in S \mid z \text{ is an interior or boundary point of } S\} = \mathring{S} \cup \partial S = S \cup \partial S$$

is closed, and that, in fact, it is the smallest closed subset of $\mathbb{C}$ containing S.
Note: here ∂S denotes the *boundary* of S.

To show that $\overline{S}$ is closed, it is enough to show that if $x \notin \overline{S}$, then x is an exterior point of $\overline{S}$. Suppose $x \notin \overline{S}$. Then x is an exterior point of S, and there is a neighborhood $D_\epsilon(x)$ which is fully contained in S^c. We claim that $D_\epsilon(x)$ and ∂S must be disjoint. Suppose $y \in D_\epsilon(x)$. Since $D_\epsilon(x)$ is open, there is a smaller neighborhood $D_{\epsilon'}(y)$ that is a subset of $D_\epsilon(x)$. Since $D_\epsilon(x) \subset S^c$, it follows that $D_{\epsilon'}(y) \subset S^c$, so y cannot be in ∂S. This shows that $D_\epsilon(x)$ and ∂S are disjoint, so that x is not an accumulation point of $\overline{S}$ and hence not an interior nor boundary point of $\overline{S}$.

Finally, if V is closed and $S \subset V$, then $\partial S \subset V$ as well: by definition every $x \in \partial S$ is an accumulation point of S, and therefore of V, so every $x \in \partial S$ is an element of V. Hence, $S \subset V$ and $\partial S \subset V$, so $\overline{S} \subset V$.

2.19 Show that the intersection of finitely many open subsets of $\mathbb{C}$ is open.
Hint: start by proving that the intersection of two open subsets of $\mathbb{C}$ is open, and then apply induction.

Let us first prove that the intersection of two open sets is open. Let U, V be open sets. Let $x \in U \cap V$. Since $x \in U$, there are neighborhoods $D_{\epsilon_1}(x) \subset U$ and $D_{\epsilon_2}(x) \subset V$. Since

$$D_{\min(\epsilon_1,\epsilon_2)}(x) \subset U \cap V,$$

it follows that $U \cap V$ is open.

Now suppose that the intersection of k open sets is open. Let $U_1, \ldots, U_{k+1}$ be open sets. By hypothesis, $U_1 \cap U_2 \cap \cdots \cap U_k$ is open, so it follows that

$$U_1 \cap U_2 \cap \cdots \cap U_{k+1} = (U_1 \cap \cdots \cap U_k) \cap U_{k+1}$$

is open. Hence the intersection of any finite number open sets is open.

2.20 Show that the complement of an open subset of $\mathbb{C}$ is closed, and viceversa. Use this and the previous problem to conclude that the union of finitely many closed subsets of $\mathbb{C}$ is closed.

The first part is either trivial or easy, depending on the definition of a *closed* set. To make things interesting, use the definition that $S \subset \mathbb{C}$ is closed if it contains all of its accumulation points. Let U be open, and x be an accumulation point of U^c. Then every ϵ-neighborhood of x contains a point $y \in U^c$ different from x. Hence, no ϵ-neighborhood of x can be contained in U, so $x \notin U$. This proves that U^c is closed. Reversing the roles of U and U^c proves the converse.

Finally, if $U_1, \ldots, U_n$ are closed sets, $U_1^c, \ldots, U_n^c$ are hence open, so by Problem 2.19, $U_1^c \cap \ldots \cap U_n^c$ is also open. Therefore, $U_1 \cup \ldots \cup U_n = (U_1^c \cap \ldots U_n^c)^c$ is closed.

2.21 Show that $\partial S = \overline{S} \cap \overline{S^c}$, and hence ∂S is closed.

First, note that $x \in \partial S$ means every open disk in x contains a point in S and S^c. This is the same definition as for the boundary of S^c, so it follows that $\partial S = \partial S^c$.

By Problem 2.18, $\overline{S} = \partial S \cup \mathring{S}$ and $\overline{S^c} = \partial S^c \cup \mathring{S^c}$. Hence,

$$\begin{aligned}\overline{S} \cap \overline{S^c} &= (\partial S \cup \mathring{S}) \cap (\partial S^c \cup \mathring{S^c}) = (\partial S \cup \mathring{S}) \cap (\partial S \cup \mathring{S^c}) \\ &= \partial S \cup (\mathring{S} \cap \mathring{S^c}) = \partial S \cup \emptyset = \partial S\end{aligned}$$

In the last line, we have used that $\mathring{S}$ and $\mathring{S^c}$ are disjoint, since the first is the set of interior points of S and the second is the set of exterior points of S.

2.22 Consider the following family of open subsets of $\mathbb{C}$:

$$S_n = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 + \frac{1}{n} \right\}, \qquad n \in \mathbb{N}$$

Is the intersection $S = \bigcap_{n\in\mathbb{N}} S_n$ open?

No, the intersection is the set $\{z \in \mathbb{C} \mid |z| \leq 1\}$, which is the closed unit disk.