MAT389 Fall 2013, Problem Set 2

Curves in $\mathbb{C}$

2.1 Show that every line in $\mathbb{C}$ can be expressed in the form

$$\beta z + \overline{\beta z} + \gamma = 0$$

for some $\beta \in \mathbb{C}^\times$ and $\gamma \in \mathbb{R}$. Why is the condition $\beta \neq 0$ necessary?
Hint: recall that every line in $\mathbb{C}$ can be expressed as the set of solutions of a linear equation of the form $px+qy+r=0$ with p, q not simultaneously zero. What should the relationship between p, q, r on the one hand, and β, γ on the other, be?

2.2 Consider the lines

$$L_1 = \left\{ z \in \mathbb{C} \mid \beta_1 z + \overline{\beta_1 z} + \gamma_1 = 0 \right\}, \qquad L_2 = \left\{ z \in \mathbb{C} \mid \beta_2 z + \overline{\beta_2 z} + \gamma_2 = 0 \right\}$$

where $\beta_1, \beta_2 \in \mathbb{C}^\times$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Prove that L_1 and L_2 are orthogonal if and only if $\operatorname{Re}(\beta_1 \overline{\beta_2}) = 0$.

2.3 Let $\alpha, \beta \in \mathbb{C}$ distinct. Give a geometric argument to show that $|z - \alpha| = |z - \beta|$ is a line.

2.4 Show that every circle in $\mathbb{C}$ can be expressed in the form

$$\alpha z \bar{z} + \beta z + \overline{\beta z} + \gamma = 0$$

for some $\alpha \in \mathbb{R}^\times$, $\beta \in \mathbb{C}^\times$ and $\gamma \in \mathbb{R}$ satisfying $|\beta|^2 > \alpha\gamma$. Why are the conditions $\alpha \neq 0$ and $|\beta|^2 > \alpha\gamma$ necessary?
Hint: recall that every circle in $\mathbb{C}$ can be expressed as the set of solutions of a quadratic equation of the form $m(x^2+y^2)+px+qy+r=0$ with $m \neq 0$. What should the relationship between m, p, q, r on the one hand, and α, β, γ on the other, be?

2.5 Prove that the equation

$$\left| \frac{z-\alpha}{z-\beta} \right| = \lambda, \qquad \alpha, \beta, \in \mathbb{C}, \lambda \in \mathbb{R}_{>0}$$

describes either a circle or a line in $\mathbb{C}$.
Hint: consider the Möbius transformation $z \mapsto (z-\alpha)/(z-\beta)$.

2.6 Let $\alpha, \beta \in \mathbb{C}$ distinct, and let $\lambda \in \mathbb{R}_{>0}$ such that $\lambda > |\alpha - \beta|$. What geometric figure is described by the equation $|z-\alpha| + |z-\beta| = \lambda$? What goes wrong if $\lambda \leq |\alpha - \beta|$?

2.7 Let $\alpha, \beta \in \mathbb{C}$ distinct, and let $\lambda \in \mathbb{R}^\times$. What geometric figure is described by the equation $|z-\alpha| - |z-\beta| = \lambda$? What happens when $\lambda = 0$?

Circles in Σ

Recall that the Riemann sphere is defined as the set

$$\Sigma = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$$

Let P be the plane defined by

$$P = \{(a, b, c) \in \mathbb{R}^3 \mid Aa + Bb + Cc = D\}$$

2.8 Show that P passes through the North pole, $N = (0, 0, 1)$, if and only if $C = D$.

2.9 Prove that $P \cap \Sigma \neq \varnothing$ if and only if $A^2 + B^2 + C^2 \geq D^2$ as follows:

1. Convince yourself that $P \cap \Sigma \neq \varnothing$ if and only if the point of P closest to the origin —call it p— satisfies $d(0, p) \leq 1$, where d is the usual, Euclidean distance in $\mathbb{R}^3$ and $0 \in \mathbb{R}^3$ means the origin.
   Hint: a picture should suffice.

2. Show that
$$d(0, p) = \sqrt{\frac{D^2}{A^2 + B^2 + C^2}}$$
   Hint: recall from multivariable calculus that the vector $\langle A, B, C\rangle$ is normal to P, and that, in fact, $p = (\lambda A, \lambda B, \lambda C)$ for the appropriate value of λ.

Notice that $P \cap \Sigma$ consists of a single point (p, in fact) if and only if $A^2 + B^2 + C^2 = D^2$. Hence, if $A^2 + B^2 + C^2 > D^2$, then $P \cap \Sigma$ is an actual circle in Σ (that is, its radius is strictly positive).

The stereographic projection

Remember that stereographic projection establishes a bijection between the Riemann sphere Σ and the extended complex plane $\hat{\mathbb{C}}$ as follows:

$$\begin{aligned} \Sigma &\xrightarrow{\quad\pi\quad} \hat{\mathbb{C}} \\ (a, b, c) &\longmapsto \frac{a + ib}{1 - c} \\ N = (0, 0, 1) &\longmapsto \infty \end{aligned} \qquad \begin{aligned} \hat{\mathbb{C}} &\xrightarrow{\quad\varphi\quad} \Sigma \\ z = x + iy &\longmapsto \left(\tfrac{2x}{x^2+y^2+1}, \tfrac{2y}{x^2+y^2+1}, \tfrac{x^2+y^2-1}{x^2+y^2+1}\right) \\ \infty &\longmapsto N = (0, 0, 1) \end{aligned}$$

2.10 Using stereographic projection, we can transport maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to maps $\Sigma \to \Sigma$. Describe geometrically the self-map of the Riemann sphere obtained from inversion $z \mapsto 1/z$ on the extended complex plane.
Hint: remember that $1/z = \overline{z}/|z|^2$. Calculate the inverse of $(a+ib)/(1-c)$ and judiciously use the equation defining the Riemann sphere to simplify the result. The geometric interpretation arises easily from that.

Möbius transformations

2.11 Show that any Möbius transformation of the form

$$T(z) = e^{i\theta}\frac{z - z_0}{z - \overline{z_0}}, \qquad \theta \in \mathbb{R}, \quad \operatorname{Im} z_0 > 0$$

sends the real line to the circle of radius 1. In other words, $|T(z)| = 1$ whenever $\operatorname{Im} z = 0$.

2.12 Find a Möbius transformation $\mathbb{D} \to \mathbb{D}$ that takes $1/2$ to $1/3$.
Hint: recall that the Möbius transformations preserving the unit disc $\mathbb{D}$ take the form

$$T(z) = e^{i\theta}\frac{z + \alpha}{\overline{\alpha}z + 1}$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ satisfying $|\alpha| < 1$.

2.13 Find a Möbius transformation $\mathbb{D} \to \mathbb{H}$ that takes the origin to the point $3 + 2i$.
Hint: first, use the Möbius transformation

$$z \mapsto \frac{z + i}{iz + 1}$$

to take $\mathbb{D}$ into $\mathbb{H}$ —and the origin to i. Then find the appropriate Möbius transformation $\mathbb{H} \to \mathbb{H}$ that takes $i \mapsto 3 + 2i$. Recall that the Möbius transformation preserving the upper half-plane $\mathbb{H}$ are always of the form

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R}, \quad ad - bc > 0$$

2.14 We saw in class that

$$T : z \mapsto \frac{z - z_1}{z_2 - z_1}\frac{z_2 - z_3}{z - z_3}$$

is the unique Möbius transformation that takes $z_1 \mapsto 0, z_2 \mapsto 1, z_3 \mapsto \infty$. Similarly,

$$U : w \mapsto \frac{w - w_1}{w_2 - w_1}\frac{w_2 - w_3}{w - w_3}$$

is the unique Möbius transformation that takes $w_1 \mapsto 0, w_2 \mapsto 1, w_3 \mapsto \infty$. If follows that $U^{-1} \circ T$ is the unique Möbius transformation that takes $z_1 \mapsto w_1, z_2 \mapsto w_2, z_3 \mapsto w_3$. It is easy to compute the latter explicitly by isolating w in the equation

$$U(w) = T(z) \quad \Leftrightarrow \quad \frac{w - w_1}{w_2 - w_1}\frac{w_2 - w_3}{w - w_3} = \frac{z - z_1}{z_2 - z_1}\frac{z_2 - z_3}{z - z_3}$$

Use this line of reasoning to find the unique Möbius transformation taking $-i \mapsto -1$, $1 \mapsto 0$, $i \mapsto 1$.

Topology in the complex plane

2.15 For each of the choices of S below, do the following:

1. classify all the points of $\mathbb{C}$ according to whether they are interior, exterior or boundary points of S;
2. decide whether S is open, closed, both open and closed, or neither open nor closed;
3. decide whether S is connected;
4. decide whether S is simply-connected;
5. decide whether S is bounded.

(i) $S = \{z \in \mathbb{C}^\times \mid 0 < \operatorname{Arg} z < \pi/2\}$;
(ii) $S = \{z \in \mathbb{C} \mid |z| \geq |z-4|\}$;
(iii) $S = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \delta\}$, where $z_0 \in \mathbb{C}$ and $\delta \in \mathbb{R}_{>0}$;
(iv) $S = \left\{z \in \mathbb{C} \mid \operatorname{Re}(z^2) > 0\right\} \cup \{0\}$;
(v) $S = \mathbb{C}$.

2.16 Find the accumulation points of each of the following subsets of $\mathbb{C}$:

(i) $S = \{i^n \mid n \in \mathbb{N}\}$;
(ii) $S = \{i^n/n \mid n \in \mathbb{N}\}$;
(iii) $S = \{z \in \mathbb{C}^\times \mid 0 \leq \operatorname{Arg} z < \pi/2\}$;
(iv) $S = \{(-1)^n(1+i)(n-1)/n \mid n \in \mathbb{N}\}$.

2.17 Prove that the *interior* of a subset $S \subset \mathbb{C}$,

$$\mathring{S} = \{z \in S \mid z \text{ is an interior point of } S\}\,,$$

is open, and that, in fact, it is the biggest open subset of $\mathbb{C}$ contained in S.

2.18 Prove that the *closure* of a subset $S \subset \mathbb{C}$,

$$\overline{S} = \{z \in S \mid z \text{ is an interior or boundary point of } S\} = \mathring{S} \cup \partial S = S \cup \partial S$$

is closed, and that, in fact, it is the smallest closed subset of $\mathbb{C}$ containing S.
Note: here ∂S denotes the *boundary* of S.

2.19 Show that the intersection of finitely many open subsets of $\mathbb{C}$ is open.
Hint: start by proving that the intersection of two open subsets of $\mathbb{C}$ is open, and then apply induction.

2.20 Show that the complement of an open subset of $\mathbb{C}$ is closed, and viceversa. Use this and the previous problem to conclude that the union of finitely many closed subsets of $\mathbb{C}$ is closed.

2.21 Show that $\partial S = \overline{S} \cap \overline{S^c}$, and hence ∂S is closed.

2.22 Consider the following family of open subsets of $\mathbb{C}$:

$$S_n = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 + \frac{1}{n} \right\}, \qquad n \in \mathbb{N}$$

Is the intersection $S = \bigcap_{n \in \mathbb{N}} S_n$ open?