MAT389 Fall 2013, Problem Set 9

9.1 Let R be a closed bounded region, and let f be a function that is continuous on R and holomorphic everywhere in the interior of R. Assume that $f(z) \neq 0$ for any $z \in R$. Show that $|f(z)|$ achieves a minimum value in R which occurs on the boundary of R and never in the interior.
Hint: apply the maximum modulus principle to the function $F(z) = 1/f(z)$.

9.2 Use the function $f(z) = z$ to show that the condition $f(z) \neq 0$ in the previous problem is necessary.

9.3 Find the points where the modulus of the function $f(z) = (z+1)^2$ achieves its maximum and minimum values in the closed triangular region with vertices $z = 0$, $z = 2$ and $z = i$.

9.4 Suppose that $f(z)$ is entire and the harmonic function $u(x, y) = \operatorname{Re} f(z)$ has an upper bound. Show that $u(x, y)$ is constant throughout the plane.
Hint: apply Liouville's theorem to the function $g(z) = e^{f(z)}$.