MAT389 Fall 2013, Problem Set 4

Wirtinger derivatives

4.1 Use the Cauchy-Riemann equations as expressed using the Wirtinger operator $\partial/\partial\bar{z}$ to find out where each of the functions below is differentiable. Find the corresponding derivatives using $\partial/\partial z$.

(i) $f(z) = (z^3 - 1)\bar{z}$, (ii) $f(z) = (\bar{z}^3 - 1)z$, (iii) $f(z) = e^{\bar{z}}(\bar{z} - iz)$,

(iv) $f(z) = \dfrac{az + b}{c\bar{z} + d}$, where $a, b, c, d \in \mathbb{C}$ and $\bar{a}d - \bar{b}c \neq 0$.

(i) $\partial f/\partial\bar{z} = z^3 - 1$, and so the Cauchy-Riemann equations are only satisfied at the third roots of unity. On the other hand, $\partial f/\partial\bar{z} = 3z^2\bar{z}$. Notice that the real and imaginary parts of this last expression give the partial derivatives $u_x = v_y$ and $u_y = -v_x$. In particular, the latter are continuous everywhere, and hence f is differentiable at 1, $\omega = e^{2\pi i/3}$ and ω^2. The value of the derivatives is now easily calculated:

$$f'(1) = \left.\frac{\partial f}{\partial z}\right|_{z=1} = 3 \cdot 1 \cdot \bar{1} = 3, \qquad f'(\omega) = \left.\frac{\partial f}{\partial z}\right|_{z=\omega} = 3\omega^2\overline{\omega} = 3,$$

$$f'(\omega^2) = \left.\frac{\partial f}{\partial z}\right|_{z=\omega^2} = 3\omega^4\overline{\omega}^2 = 3.$$

(ii) $\partial f/\partial\bar{z} = 3\bar{z}^2 z$ only vanishes at the origin. Since $\partial f/\partial z = \bar{z}^3 - 1$, the partial derivatives $u_x = v_y$ and $u_y = -v_x$ are continuous at the origin, ensuring the differentiability of f there. Moreover,

$$f'(0) = \left.\frac{\partial f}{\partial z}\right|_{z=0} = 0.$$

(iii) The function given does not satisfy the Cauchy-Riemann equations anywhere on the complex plane. Indeed, we have

$$\frac{\partial f}{\partial\bar{z}} = e^{\bar{z}}(\bar{z} - iz + 1) = 0 \qquad \Longleftrightarrow \qquad \bar{z} - iz + 1 = (x + y + 1) - i(x + y) = 0,$$

and it is obvious that the last equation has no solutions.

(iv) The Cauchy-Riemann equations

$$\frac{\partial f}{\partial\bar{z}} = (az + b)\,\frac{-c}{(c\bar{z} + d)^2} = 0.$$

have the point $z = -b/a$ as their unique solution if $a \neq 0$, and no solution otherwise. In the first case, $\partial f/\partial\bar{z} = a/(c\bar{z} + d)$ says that the partial derivatives $u_x = v_y$ and $u_y = -v_x$ are continuous at $z = -b/a$ so long as $\bar{a}d - \bar{b}c \neq 0$. Then,

$$f'\left(-\frac{b}{a}\right) = \frac{a}{c\overline{(-b/a)} + d} = \frac{|a|^2}{\bar{a}d - \bar{b}c}$$

Holomorphic functions

4.2 For each of the functions below, determine the largest domain over which they are holomorphic.

(i) $f(z) = \dfrac{e^{iz}}{z^2 - 2z + 1}$, (ii) $f(z) = \log|z| + i \operatorname{Arg} z$, (iii) $f(z) = (z^3 - 1)\bar{z}$.

(i) Both the numerator and the denominator are entire functions. Their quotient is then holomorphic wherever the denominator does not vanish, i.e., on $\mathbb{C} - \{1\}$.

(ii) This function is the principal branch of the logarithm, $f(z) = \operatorname{Log} z$. Its real and imaginary parts, $u(r, \theta) = \log r$ and $v(r, \theta) = \theta$, satisfy the polar form of the Cauchy-Riemann equations (cf. Problem 3.7)

$$u_r = \frac{1}{r} v_\theta, \qquad \frac{1}{r} u_\theta = -v_r$$

everywhere on $\mathbb{C}^\times$. And although $u(r, \theta)$ has continuous partial derivatives everywhere on $\mathbb{C}^\times$, the same is not the case for $v(r, \theta)$. In fact, the latter is even discontinuous on the negative real axis, $\theta = \pi$. Hence we can only ensure existence of the derivative f' on the set

$$\mathbb{C}^\times - \{z \in \mathbb{C}^\times \mid \operatorname{Im} z = 0, \operatorname{Re} z < 0\}.$$

The latter is an open set, and so $\operatorname{Log} z$ is holomorphic at every point of it.

(iii) In Problem 2.1(i) we saw that this function is differentiable only at the third roots of unity. Hence it is not holomorphic anywhere.

4.3 Prove that the composition of two entire functions is again an entire function.

If f are g are entire functions, their derivatives exist everywhere. Then the chain rule,

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0),$$

ensures the existence of the derivative of the composition $g \circ f$ everywhere.

4.4 Check that the functions below are entire. Can you write them in terms of z in some simple form?

(i) $f(z) = 3x + y + i(3y - x)$, (ii) $f(z) = \sin x \cosh y + i \cos x \sinh y$.

Hint: for (ii), notice that

$$\cosh y = \frac{e^y + e^{-y}}{2} = \frac{e^{-i(iy)} + e^{i(iy)}}{2} = \cos(iy)$$

Can you find a similar identity for $\sinh y$?

(i) Writing $f(z) = (3-i)z$ makes it clear that it is indeed entire, since $\partial f/\partial \bar{z}$ is identically zero and $\partial f/\partial z = 3+i$.

(ii) We have

$$\sinh y = \frac{e^y - e^{-y}}{2} = \frac{e^{-i(iy)} - e^{i(iy)}}{2} = -i\sin(iy),$$

which allows us to rewrite

$$\begin{aligned} f(z) &= \sin x \cosh y + i \cos x \sinh y \\ &= \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin(x+iy) = \sin z, \end{aligned}$$

which we know is entire, since it is a linear combination of two complex exponentials —entire functions themselves. We can also show the above without appealing to the formula for the sine of a sum as follows:

$$\begin{aligned} f(z) &= \sin x \cosh y + i \cos x \sinh y \\ &= \frac{e^{ix} - e^{-ix}}{2i}\frac{e^y + e^{-y}}{2} + i\frac{e^{ix} + e^{-ix}}{2}\frac{e^y - e^{-y}}{2} \\ &= \frac{1}{4i}\Big[(e^{ix} - e^{-ix})(e^y + e^{-y}) - (e^{ix} + e^{-ix})(e^y - e^{-y})\Big] \\ &= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin z. \end{aligned}$$

Harmonic functions

4.5 Check that each of the functions $u(x, y)$ below is harmonic at every $(x, y) \in \mathbb{R}^2$, and find the unique harmonic conjugate, $v(x, y)$, satisfying $v(0, 0) = v_0$. Express the resulting holomorphic (entire, in fact) functions, $f(z) = u(x, y) + iv(x, y)$, in terms of z.

(i) $u(x, y) = ax + by + c$ (where $a, b, c \in \mathbb{R}$) and $v_0 = 0$,
(ii) $u(x, y) = x^2 - y^2 - 2x$ and $v_0 = 1$,
(iii) $u(x, y) = y^3 - 3x^2y$ and $v_0 = 0$,
(iv) $u(x, y) = x^4 - 6x^2y^2 + y^4$ and $v_0 = 0$,
(v) $u(x, y) = e^{2y}\cos 2x$ and $v_0 = 1$.

(i) Since $u(x, y)$ is linear in both x and y, it is clear that $u_{xx} = u_{yy} = 0$, proving the harmonicity of $u(x, y)$ over the whole of $\mathbb{R}^2$. Using the Cauchy-Riemann equations we can find the partial derivatives of a purported harmonic conjugate, $v(x, y)$:

$$v_x = -u_y = -b, \qquad v_y = u_x = a.$$

Integrating the first of these equations gives $v(x, y) = -bx + f(y)$ (since we are integrating with respect to x, the "constant of integration" can depend on y). Differentiating with respect to y, we obtain $v_y = f'(y)$, which, put together with the second of the Cauchy-Riemann equations above, yields $f'(y) = a$, or $f(y) = ay + d$ for some real constant d. The initial condition then implies

$$0 = v(0,0) = -bx + ay + d|_{(x,y)=(0,0)} = d.$$

All in all, we have

$$f(z) = u(x,y) + iv(x,y) = ax + by + c + i(-bx + ay) = (a - ib)z + c.$$

(ii) We have $u_{xx} = 1$ and $u_{yy} = -1$, so $u(x,y)$ satisfies the Laplace equation. From the Cauchy-Riemann equations, we get

$$v_x = -u_y = 2y, \qquad v_y = u_x = 2x - 2.$$

Following the same steps as in (i), we have

$$v = 2xy + f(y) \Rightarrow v_y = 2x + f'(y) \Rightarrow f'(y) = -2 \Rightarrow f(y) = -2y + C \quad (C \in \mathbb{R})$$

From the initial condition $v(0,0) = 1$ we calculate $C = 1$. Finally,

$$f(z) = u(x,y) + iv(x,y) = x^2 - y^2 - 2x + i(2xy - 2y + 1) = z^2 - 2z + i$$

(iii) Once again, the procedure is exactly the same as in (i):

$$u_{xx} + u_{yy} = (-6y) + (6y) = 0.$$

$$v_x = -u_y = 3x^2 - 3y^2, \qquad v_y = u_x = -6xy.$$

$$v = x^3 - 3xy^2 + f(y) \Rightarrow v_y = -6xy + f'(y) \Rightarrow f'(y) = 0 \Rightarrow f(y) = C \quad (C \in \mathbb{R})$$

$$v(0,0) = 0 \Rightarrow C = 0.$$

$$f(z) = u(x,y) + iv(x,y) = y^3 - 3x^2y + i(x^3 - 3xy^2) = i(x + iy)^3 = iz^3.$$

(iv)

$$u_{xx} + u_{yy} = (12x^2) + (-12x^2) = 0.$$

$$v_x = -u_y = 12x^2y - 4y^3, \qquad v_y = u_x = 4x^3 - 12xy^2.$$

$$v = 4x^3y - 4xy^3 + f(y) \Rightarrow v_y = 4x^3 - 12xy^2 + f'(y) \Rightarrow f'(y) = 0 \Rightarrow f(y) = C \quad (C \in \mathbb{R})$$

$$v(0,0) = 0 \Rightarrow C = 0.$$

$$f(z) = u(x,y) + iv(x,y) = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3) = (x + iy)^4 = z^4.$$

(v)

$$u_{xx} + u_{yy} = (-4e^{2y}\cos 2x) + (4e^{2y}\cos 2x) = 0.$$

$$v_x = -u_y = -2e^{2y}\cos 2x, \qquad v_y = u_x = -2e^{2y}\sin 2x.$$

$$v = -e^{2y}\sin 2x + f(y) \Rightarrow v_y = -2e^{2y}\sin 2x + f'(y) \Rightarrow f'(y) = 0 \Rightarrow f(y) = C \quad (C \in \mathbb{R})$$

$$v(0,0) = 1 \Rightarrow C = 1.$$

$$f(z) = u(x,y) + iv(x,y) = e^{2y}\cos 2x + i(1 - e^{2y}\sin 2x) = e^{2y}e^{-2xi} + i = e^{-2iz} + i$$