MAT389 Fall 2013, Midterm 1

Oct 8, 2013

Please justify your reasoning. Answers without an explanation will <u>not</u> be given any credit.

1. **[2pt]** Find all solutions to each of the following equations, and plot them:

(i) **[0.5pt]** $(z+1)^4 + i = 0$,

(ii) **[0.5pt]** $\operatorname{Re}(z+5) = \operatorname{Im}(z-i)$,

(iii) **[0.5pt]** $\operatorname{Re}\left(\frac{z}{1+i}\right) = 0$,

(iv) **[0.5pt]** $\operatorname{Re}(z^2+5) = 0$.

(i) Rewriting the equation given as $(z+1)^4 = -i$, we have that $z = -1 + (-i)^{1/4}$, where $(-i)^{1/4}$ denotes any fourth root of $-i$. We find the latter by expressing $-i$ in polar form:

$$-i = e^{3\pi i/2} \quad \Rightarrow \quad (-i)^{1/4} = e^{i(3\pi/8+\pi k/2)}, \quad k = 0, 1, 2, 3.$$

Hence, the solutions of the original equation are the four points

$$z = -1 + e^{i(3\pi/8+\pi k/2)} = -1 + \cos\left(\frac{3\pi}{8} + \frac{\pi k}{2}\right) + i \sin\left(\frac{3\pi}{8} + \frac{\pi k}{2}\right),$$

where k takes the value 0, 1, 2 and 3.

(ii) Expressing $z = x + iy$, we get

$$\operatorname{Re}(z+5) = \operatorname{Re}(x+iy+5) = x+5, \qquad \operatorname{Im}(z-i) = \operatorname{Im}(x+iy-i) = y-1;$$

that is, the solution set of the equation in the statement is the line $y = x + 6$.

(iii) As in (ii), we calculate

$$\operatorname{Re}\frac{z}{1+i} = \operatorname{Re}\frac{x+iy}{1+i} = \operatorname{Re}\left(\frac{x+iy}{1+i}\cdot\frac{1-i}{1-i}\right) = \operatorname{Re}\frac{(x+iy)(1-i)}{2}$$
$$= \operatorname{Re}\frac{x+y+i(y-x)}{2} = \frac{x+y}{2}$$

Thus the given equation is satisfied on the line $x + y = 0$.

(iv) Once more we write z in terms of its real and imaginary parts:

$$\operatorname{Re}(z^2+5) = \operatorname{Re}\left[(x+iy)^2+5\right] = \operatorname{Re}\left(x^2+y^2+2xyi+5\right) = x^2+y^2+5$$

The equation $x^2+y^2+5 = 0$ describes a branch of a hyperbola.

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2. **[2pt]** Describe the following subsets of the complex plane, and plot them:

(i) **[0.5pt]** $\{z \in \mathbb{C} \mid |\operatorname{Re} z| + |\operatorname{Im} z| < 1\}$,

(ii) **[0.5pt]** $\{z \in \mathbb{C}^\times \mid 0 \le \operatorname{Arg} z \le \pi, |z| \le 1\}$,

(iii) **[0.5pt]** $\{w \in \mathbb{C} \mid w = e^z \text{ for some } z \text{ with } \operatorname{Re} z \le 0 \text{ and } 0 \le \operatorname{Im} z \le \pi\}$

(iv) **[0.5pt]** $\{z \in \mathbb{C} \mid \operatorname{Re} z^2 > 0\}$

(i) The equality $|x|+|y|=1$ describes a square with vertices at the points $\pm 1, \pm i$. The inequality in the statement defines the interior of said square.

(ii) We have $0 \leq \operatorname{Arg} z \leq \pi$ if and only if z belongs to the (closed) upper half-plane minus the origin, while the region $|z|<1$ is the interior of the unit circle centered at the origin. The desired subset is the intersection of these two.

(iii) Since w belongs to the image of the exponential function, it is nonzero. $\operatorname{Re} z \leq 0$ implies $|w| = e^{\operatorname{Re} z} \leq e^0 = 1$, while $0 \leq \operatorname{Im} z \leq \pi$ gives $0 \leq \arg w \leq \pi$. The region is then the same as that of (ii).

(iv) Since $\operatorname{Re} z^2 = x^2 - y^2 = (x+y)(x-y)$, the inequality $\operatorname{Re} z^2$ is satisfied if either both $x+y>0$ and $x-y>0$, or both $x+y<0$ and $x-y<0$. In the first case we have $-x<y<x$ —which is the wedge on the right of the picture—, while in the second we have $x<y<-x$ —the left wedge.

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3. [2pt] Prove that

$$\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = 1$$

whenever $|z|=1$.
Hint: if $|z|=1$, then $z=e^{i\theta}$ for some $\theta \in \mathbb{R}$.

If $z=e^{i\theta}$, we have $z^{-1}=e^{-i\theta}=\bar{z}$. Hence,

$$\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = \left|\frac{1}{z}\frac{az+b}{\bar{b}+z^{-1}\bar{a}}\right| = \left|\frac{1}{z}\frac{az+b}{\overline{az+b}}\right| = \frac{1}{|z|}\frac{|az+b|}{|\overline{az+b}|} = 1,$$

where in the last step we have used the fact that the modulus of a complex number coincides with that of its complex-conjugate.

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4. [1pt] Express $f(z)=z^3+z+1$ as $f(z)=u(x,y)+iv(x,y)$.

Write $z=x+iy$, expand the powers using the binomial theorem and recall that $i^2=-1$ and $i^3=i$:

$$\begin{aligned} f(z) &= z^3+z+1 = (x+iy)^3+(x+iy)+1 \\ &= x^3+3ix^2y-3xy^2-iy^3+x+iy+1 \\ &= (x^3-3xy^2+x+1)+i(3x^2y-y^3+y) \end{aligned}$$

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5. [2pt] Find the unique Möbius transformation, T, that takes $i \mapsto -i$, $0 \mapsto 0$, $-1 \mapsto \infty$.
Hint: remember that the unique Möbius transformation that takes $z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$ is given by the rule

$$z \mapsto \frac{z-z_1}{z_2-z_1}\frac{z_2-z_3}{z-z_3}$$

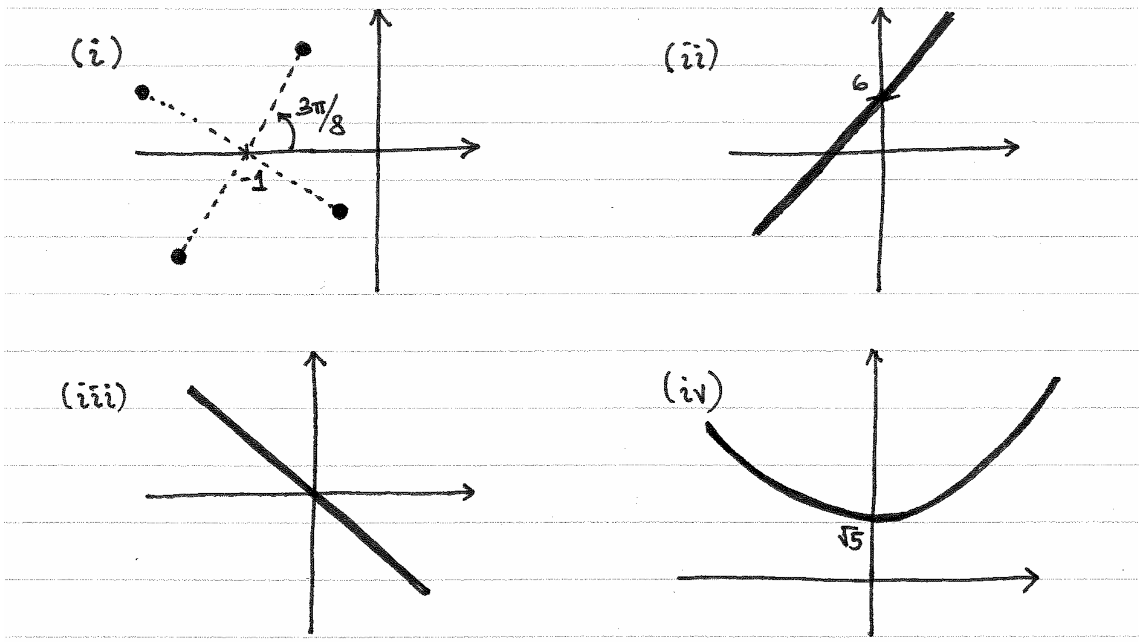

Figure 1: Solution sets to equations in Problem 1

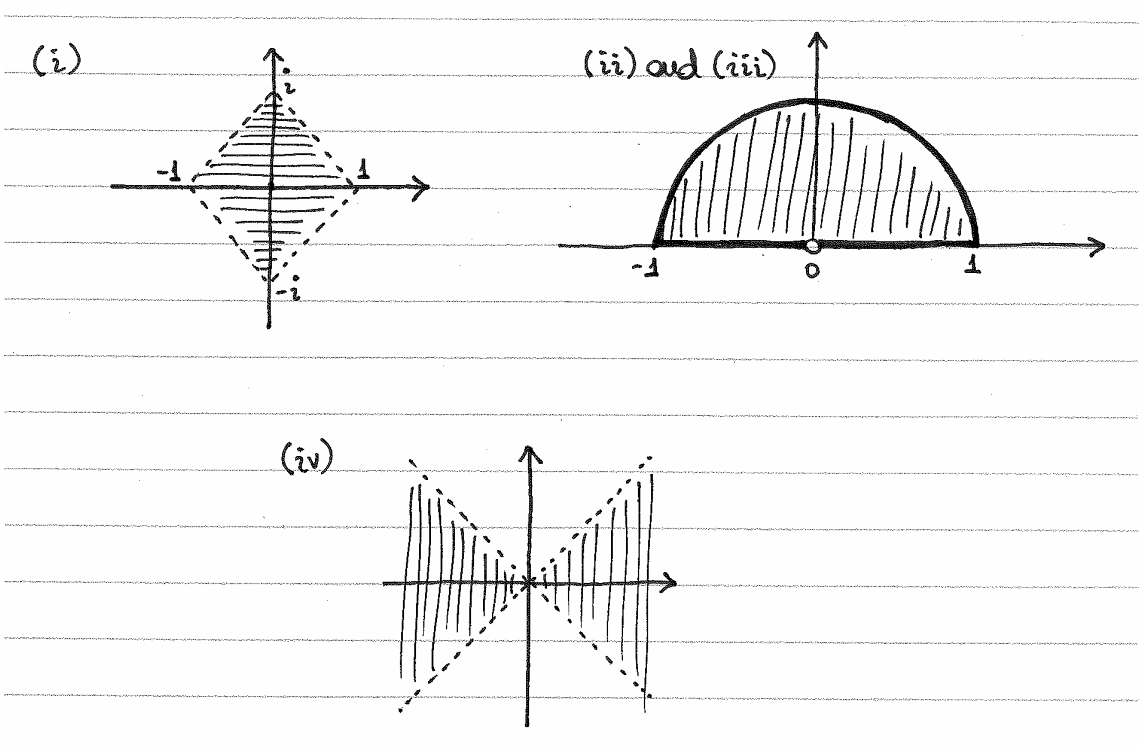

Figure 2: Regions in Problem 2

According to the formula in the hint, the unique Möbius transformation, T_1, taking $i \mapsto 0$, $0 \mapsto 1$, $-1 \mapsto \infty$ is given by

$$T_1(z) = \frac{z-i}{0-i}\,\frac{0-(-1)}{z-(-1)} = i\,\frac{z-i}{z+1}$$

Similarly, the unique Möbius transformation, T_2, that maps $-i \mapsto 0$, $0 \mapsto 1$, $\infty \mapsto \infty$ is

$$T_2(w) = \frac{w-(-i)}{0-(-i)} = -iw + 1$$

The inverse of a Möbius transformation is again a Möbius transformation, as is the composition of any two of them. Thus $w = (T_2^{-1} \circ T_1)(z)$ is the function we are looking for:

$$T_2(w) = T_1(z) \iff -iw + 1 = i\,\frac{z-i}{z+1} \iff w = -(1+i)\,\frac{z}{z+1}$$

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6. **[1pt]** Show that $f(z) = 1/z$ takes the line $z - \bar{z} = 2i$ to the circumference of radius $1/2$ centered at the point $-i/2$.

Let $w = f(z) = 1/z$. The transformation f takes the equation $z - \bar{z} = 2i$ to $1/w - 1/\bar{w} = 2i$. We can massage this last equation to get

$$\frac{1}{w} - \frac{1}{\bar{w}} = 2i \iff \bar{w} - w = 2iw\bar{w} \iff -\operatorname{Im} w = |w|^2.$$

If $w = u + iv$, this results in

$$-v = u^2 + v^2.$$

Completing the square yields

$$u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4},$$

which is precisely the equation of the circumference of radius $1/2$ centered at the point $-i/2$.

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7. **[2pt]** Consider the transformation $f(z) = z + 1/z$.

(i) **[1pt]** What is the image under f of the unit circle, $|z| = 1$?

(ii) **[1pt]** What about the image of the open punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$?

(i) If z belongs to the unit circle, it can be written as $e^{i\theta}$ for $0 \le \theta < 2\pi$. Then,

$$f(e^{i\theta}) = e^{i\theta} + \frac{1}{e^{i\theta}} = e^{i\theta} + e^{-i\theta} = 2\cos\theta.$$

As θ goes from 0 to π (that is, the upper-half of the circle), $f(z)$ moves from 2 to -2 along the real axis; as θ goes from 0 to π, $f(z)$ returns along the same real axis to the point 2.

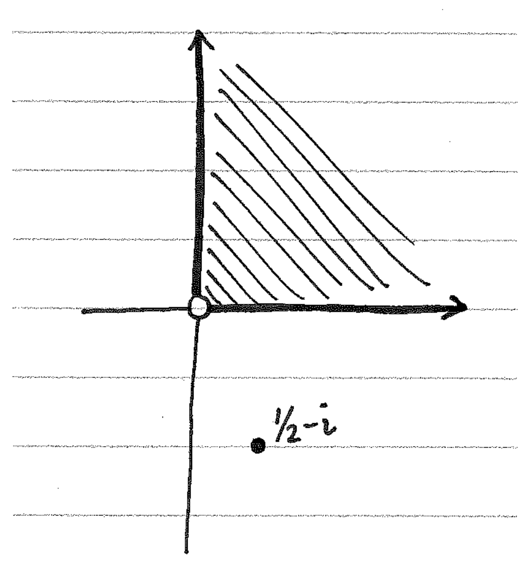

Figure 3: The set S in Problem 8

(ii) For $z = re^{i\theta}$ with $0 < r < 1$, we have

$$f(re^{i\theta}) = re^{i\theta} + \frac{1}{re^{i\theta}} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

We see that for each fixed value $0 < r < 1$, the circle of radius r maps to the ellipse with equation

$$\left(\frac{x}{1/r + r}\right)^2 + \left(\frac{y}{1/r - r}\right)^2 = 1$$

As r moves between 0 and 1, these cover the whole of the complex plane except for the line segment on the real axis between the points 2 and -2.

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8. **[2pt]** Let $S = \{z \in \mathbb{C}^\times \mid 0 \le \operatorname{Arg} z \le \pi/2\} \cup \{1/2 - i\} \subset \mathbb{C}$.

(i) **[0.5pt]** Classify all points in $\mathbb{C}$ as interior, exterior or boundary with respect to S.
(ii) **[0.25pt]** What are the accumulation points of S?
(iii) **[0.25pt]** Is S bounded?
(iv) **[0.25pt]** Is S open? Is S closed?
(v) **[0.25pt]** Is S connected?
(vi) **[0.25pt]** Is S compact?
(vii) **[0.25pt]** Is S a domain? Is S a region?

Note: for (i) and (ii), you do not need to justify your answer.

(i) The interior of S is $\{z \in \mathbb{C}^\times \mid 0 < \operatorname{Arg} z < \pi/2\}$. The boundary is formed by the origin, the positive real axis, the positive imaginary axis and the point $1/2 - i$. The rest of the points in the complex plane are exterior to S.

(ii) The set of accumulation points of S is the closed first quadrant of the complex plane.

(iii) No: all points of the form $x(1+i)$ with $x > 0$ belong to S, and their moduli are unbounded.

(iv) It is not open, since it contains points in its boundary —e.g., any point in the positive real axis. It is also not closed, since it does not contain all of its boundary points —e.g., the origin.

(v) No: the point $1/2 - i$ is disconnected from the rest of S.

(vi) No: a compact set is both bounded and closed, neither of which S is.

(vii) No: both domains and regions are connected, which S is not.

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9. **[1pt]** Let Log denote the principal branch of the logarithm, defined by

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z$$

for $z \in \mathbb{C}^{\times}$. Give a counterexample to the identity

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

Take $z_1 = z_2 = e^{3\pi i/2}$. Then

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2},$$

while

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} e^{3\pi i/2} = -\frac{\pi}{2}.$$

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10. **[1pt]** Enunciate the Cauchy-Riemann equations (in rectangular coordinates). We know that if a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 \in \mathring{\Omega}$, then its real and imaginary parts satisfy these equations. The latter is only a necessary condition for differentiability of f at z_0, though. According to the theorem we proved in class, what are sufficient conditions for f to be differentiable at z_0?

The Cauchy-Riemann equations are $u_x = v_y$, $u_y = -v_x$. A function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = z_0$ if, at that point, its real and imaginary parts have continuous partial derivatives and satisfy the Cauchy-Riemann equations.

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11. **[2pt]** Find out where each of the functions below is holomorphic:

(i) **[0.4pt]** $f(z) = z + \dfrac{1}{z}$, (ii) **[0.4pt]** e^{e^z}, (iii) **[0.4pt]** $\dfrac{1}{e^z - 1}$,

(iv) **[0.4pt]** $|z|^2$, (v) **[0.4pt]** $\dfrac{z}{z^n - 2}$, $(n \in \mathbb{N})$

(i) The function z is entire, and so $1/z$ is holomorphic everywhere except at $z = 0$. Since the sum of holomorphic functions is again holomorphic, $f(z) = z + 1/z$ is holomorphic on $\mathbb{C}^\times$.

(ii) The composition of entire functions is entire —hence so is $f(z) = e^{e^z}$, which is the composition of the exponential function with itself.

(iii) The denominator is an entire function that vanishes at $z = 2\pi ik$ for $k \in \mathbb{Z}$. Hence $f(z) = 1/(e^z - 1)$ is holomorphic outside of those points.

(iv) The function $f(z) = |z|^2 = z\bar{z}$ has $\partial f/\partial \bar{z} = z$, so it is differentiable only at the origin. Consequently, it is nowhere holomorphic.

(v) Both the numerator and the denominator are entire, so the quotient only fails to be holomorphic at the n-th roots of 2 —that is, at $z = \sqrt[n]{2}e^{2\pi ik/n}$ for $k = 0, \ldots, n-1$.

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12. **[2pt]** Check that $u(x, y) = x^3 - 3xy^2$ is harmonic at every $(x, y) \in \mathbb{R}^2$. Calculate the unique harmonic conjugate, $v(x, y)$, that satisfies $v(0, 0) = 0$.

We calculate

$$u_x = 3x^2 - 3y^2, \quad u_{xx} = 6x, \quad u_y = -6xy, \quad u_{yy} = -6x,$$

and so $u_{xx} + u_{yy} = 6x - 6x = 0$. A harmonic conjugate must satisfy the Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_x.$$

From the second of these, we have $v_x = 6xy$. Integrating, we obtain $v = 3x^2y + g(y)$, where g is a function of y only. Differentiating with respect to y, we get $v_y = 3x^2 + g'(y)$. Using the first of the Cauchy-Riemann equations, we find that $g'(y) = -3y^2$, or $g(y) = -y^3 + C$, for C a constant. The initial condition $v(0, 0) = 0$ fixes $C = 0$, and so

$$v(x, y) = 3x^2y - y^3.$$

Notice that this pair (u, v) is nothing but that formed by the real and imaginary parts of the function $f(z) = z^3$.
