MAT389 Fall 2013, Midterm 2

Nov 12, 2013

Please justify your reasoning. Answers without an explanation will not be given any credit.

Definition: $C_r(z)$ is the circle of radius r centered at z, oriented counterclockwise.

Some useful formulas:

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$
$$\arctan z = \frac{i}{2} \log \frac{i+z}{i-z}$$
$$\operatorname{argtanh} z = \frac{1}{2} \log \frac{1+z}{1-z}$$

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1. [2pt] Calculate all possible values of the following multivalued expressions.

(i) **[1pt]** $(-1)^{1/\pi}$

(ii) **[1pt]** $\arctan(i/2)$

(i) By definition, we have $(-1)^{1/\pi} = e^{(1/\pi)\log(-1)}$. The possible values for the logarithm of -1 are $\log(-1) = i\pi + 2k\pi i$ with $k \in \mathbb{Z}$. Hence,

$$(-1)^{1/\pi} = e^{(1/\pi)\log(-1)} = e^{(1/\pi)i(2k+1)\pi} = e^{(2k+1)i}, \quad k \in \mathbb{Z}.$$

(ii) Using the definition of the arctangent as a multivalued function, we calculate

$$\arctan \frac{i}{2} = \frac{i}{2} \log \frac{i+(i/2)}{i-(i/2)} = \frac{i}{2} \log 3 = \frac{i}{2}(\operatorname{Log} 3 + 2k\pi i) = \frac{i}{2} \operatorname{Log} 3 - k\pi,$$

with $k \in \mathbb{Z}$.

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2. [2pt] Let $f(z)$ denote the principal branch of the multivalued function $z^{1/4}$. Find the image of the right half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ under the transformation

$$w = \operatorname{Log}\left[e^{i\pi/8} f(z)\right].$$

We can see the transformation $w = \operatorname{Log}\left[e^{i\pi/8} f(z)\right]$ as a composition of the transformation $Z = f(z)$, a rotation by $\pi/8$ in the counterclockwise direction —$W = e^{i\pi/8} Z$—, and the principal branch of the logarithm —$w = \operatorname{Log} W$.

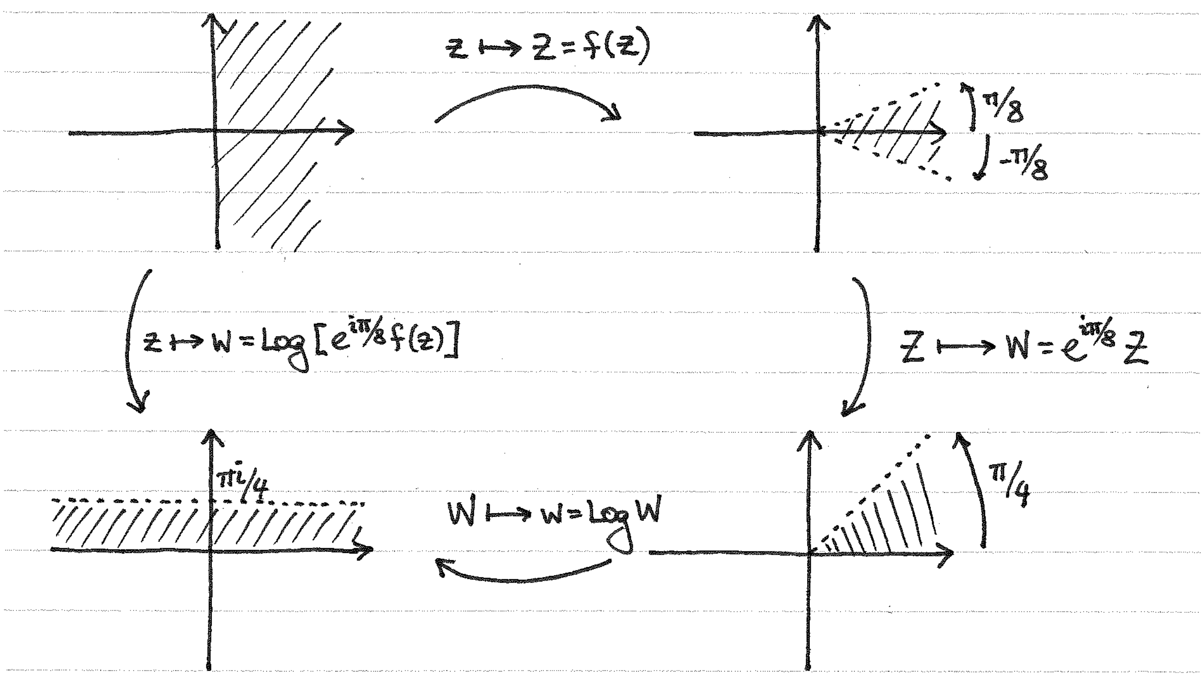

Figure 1: The transformation in Problem 1

The first of these is defined by $f(re^{i\theta}) = \sqrt[4]{r}e^{i\theta/4}$ where $-\pi < \theta < \pi$. In particular the upper-half plane —which can be described as those nonzero z by $-\pi/2 < \arg z < \pi/2$— gets mapped to the wedge $-\pi/8 < \arg Z < \pi/8$. The rotation then takes this region to another wedge: $0 < \arg W < \pi/4$. The principal branch of the logarithm finally sends the latter to the infinite strip $0 < \operatorname{Im} w < \pi/4$.

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3. **[2pt]** Consider the multivalued function $F(z) = \operatorname{argtanh} z$. For each of the conditions below, choose a determination $f(z)$ of $F(z)$ that satisfies said condition. Describe the branch cuts, the discontinuity of your choice of determination as you cross each branch cut, and the maximal domain on which it is holomorphic.

(i) **[1pt]** $f(z)$ is holomorphic at $z = 0$.

(ii) **[1pt]** $f(z)$ is holomorphic for $|z| > 2$.

Writing

$$F(z) = \operatorname{argtanh} z = \frac{1}{2}\log\frac{1+z}{1-z} = \frac{1}{2}\log(1+z) - \frac{1}{2}\log(1-z),$$

it is clear that taking a determination of $F(z)$ is equivalent to specifying two real numbers, α_1 and α_2, and setting

$$f_{(\alpha_1,\alpha_2)}(z) = \frac{1}{2}\log_{(\alpha_1)}(1+z) - \frac{1}{2}\log_{(\alpha_2)}(1-z).$$

The branch cut associated to the first summand is found at the the loci of points $z \in \mathbb{C}$ such that $\arg(1+z) = \alpha_1$ —that is, the half-line emanating from the branch point $z = -1$ with angle α_1.

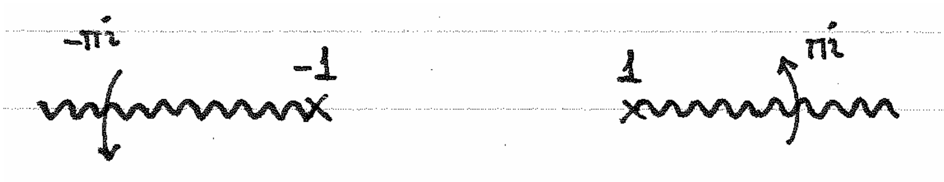

Figure 2: A choice of branch cuts for Problem 2(i)

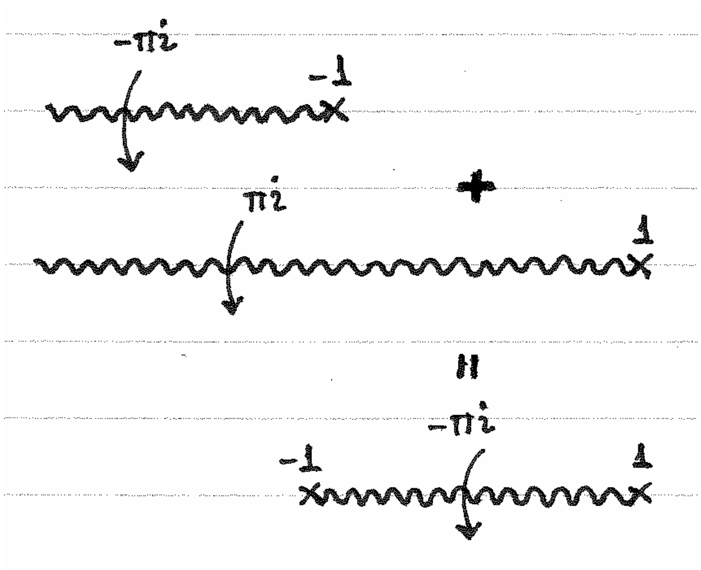

Figure 3: A choice of branch cuts for Problem 2(ii)

Crossing it in the counterclockwise direction picks up an additive factor of $-\pi i$. As for the second summand, the branch cut is given by $\arg(1-z) = \alpha_2$ —equivalently, $\arg(z-1) = \alpha_2 + \pi$—, which is the half-line emanating from $z = 1$ with angle $\alpha_2 + \pi$. Crossing it in the counterclockwise direction results in an additive change of πi for the function (beware the minus sign in front of the logarithm!). The determination $f_{(\alpha_1,\alpha_2)}$ is holomorphic at least everywhere outside these branch cuts.

(i) If we want a determination that is holomorphic at the origin, we just need to ensure that the branch cuts do not pass through that point. One such choice is $\alpha_1 = \alpha_2 = -\pi$. Since no cancellation between the branch cuts occur, $f_{(-\pi,-\pi)}$ is holomorphic exactly outside the branch cuts.

(ii) In this case, we want to "cancel out" the branch cuts —otherwise they run off to ∞ and the determination cannot be holomorphic on the whole of the region $|z| > 2$. This means that we must choose α_1 and α_2 in such a way that $\alpha_1 \equiv \alpha_2 + \pi \pmod{2\pi}$. For example, we might take $\alpha_1 = -\pi$ and $\alpha_2 = 0$. To the left of -1 the discontinuities coming from the two logarithms add up to zero, and hence the resulting function is continuous on the real axis to the left of -1; furthermore, it is holomorphic there, and so $f_{(-\pi,0)}$ is holomorphic on the whole complex plane except for the segment of the real line between the two branch points.

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4. [6pt] For each of the cases below, compute the integral

$$\int_C f(z)\, dz.$$

(i) **[1pt]** C is the line segment from $z = 0$ to $z = 1 + i$, and $f(z) = \operatorname{Re} z$.
(ii) **[1pt]** $C = C_r(z_0)$ ($r > 0$, $z_0 \in \mathbb{C}$), and $f(z) = -\operatorname{Im} z$ (**Hint:** use Green's theorem).
(iii) **[1pt]** $C = C_1(0)$, and $f(z) = (\sin z)/(z - \pi)$.
(iv) **[1pt]** $C = C_3(\sqrt{2})$, and $f(z) = (e^z + z)/(z - 2)$.
(v) **[1pt]** $C = C_{\sqrt{2}}(1)$, and $f(z) = 1/(z^2 - 2i)$.
(vi) **[1pt]** $C = C_{3/2}(2i)$, and $f(z) = z^i/(z - i)^2$, where z^i denotes the principal branch of the corresponding multivalued function.

(i) Parametrizing C by $z(t) = (1 + i)t$ with $0 \le t \le 1$,

$$\int_C \operatorname{Re} z\, dz = \int_0^1 t \cdot (1 + i)\, dt = \frac{1 + i}{2}$$

.

(ii) As hinted, Green's theorem gives

$$\oint_{C_r(z_0)} (-\operatorname{Im} z)\, dz = \oint_{C_r(z_0)} (-y\, dx - iy\, dx) = \iint_{\mathbb{D}_r(z_0)} \left(\frac{\partial(-iy)}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dxdy$$
$$= \iint_{\mathbb{D}_r(z_0)} 1\, dxdy = \pi r^2.$$

(iii) The function $(\sin z)/(z - \pi)$ is holomorphic on the closed unit disk (the only singularity is at $z = \pi$), so the Cauchy-Goursat theorem immediately gives

$$\oint_{C_1(0)} \frac{\sin z}{z - \pi} = 0.$$

(iv) A simple application of the Cauchy integral formula yields

$$\oint_C \frac{e^z + z}{z - 2}\, dz = 2\pi i(e^z + z)\Big|_{z=2} = 2\pi i(e^2 + 2).$$

(v) The denominator in the integrand factors as $z^2 - 2i = (z - 1 - i)(z + 1 + i)$. While the zero of the first factor is inside of $C_{\sqrt{2}}(1)$, that of the second factor is outside. Hence we can write the integrand as

$$\frac{1}{z^2 - 2i} = \frac{1/(z + 1 + i)}{z - (1 + i)},$$

where the denominator is holomorphic on and inside of the contour of integration. We can now use the Cauchy integral formula to obtain

$$\oint_{C_{\sqrt{2}}(1)} \frac{dz}{z^2 - 2i} = 2\pi i\, \frac{1}{z + 1 + i}\Big|_{z=1+i} = \frac{\pi i}{1 + i} = \frac{\pi}{2}\,(1 + i).$$

(vi) Since $C_{3/2}(2i)$ does not intersect the negative real axis —the branch cut for the principal branch of z^i—, the integrand is the quotient of a function that is holomorphic on and insider the contour of integration by a factor $(z-2)^2$. The Cauchy integral formula for the first derivative applies then, and gives

$$\oint_{C_{3/2}(2i)} \frac{z^i}{(z-i)^2} = 2\pi i \left.\frac{dz^i}{dz}\right|_{z=i} = 2\pi i \cdot i z^{i-1}\big|_{z=i} = -2\pi e^{(i-1)\,\mathrm{Log}\, i} = -2\pi e^{(i-1)i\pi/2} = 2\pi i e^{\pi/2}.$$

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5. [2pt] Bound the modulus of the integrals below.

(i) **[1pt]** $\displaystyle\int_{C_1(0)} \frac{dz}{2+\bar{z}^2}$

(ii) **[1pt]** $\displaystyle\int_C \frac{e^z}{|z|^2}\,dz$, where C is the square with vertices $\pm 1 \pm i$.

(i) We bound the denominator from below using the triangle inequality:

$$|2+\bar{z}^2| \geq \big|2 - |\bar{z}^2|\big| = 2-1 = 1.$$

Since the arc-length of the contour of integration is 2π, we have

$$\left|\int_{C_1(0)} \frac{dz}{2+\bar{z}^2}\right| \leq 2\pi.$$

(ii) Observe that the maximum value of $x = \mathrm{Re}\, z$ on the contour of integration is 1 (which is achieved on the line segment between $1+i$ and $1-i$), and that the modulus of every point on C is at least 1. Then the modulus of the numerator is bounded above by $|e^z| = e^x < e$, while the denominator —itself a positive real number— is bounded below by 1. The arc-length of C is equal to 8, and so

$$\left|\int_C \frac{e^z}{|z|^2}\,dz\right| \leq 8 \cdot \frac{e}{1} = 8e.$$

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6. [2pt] Find the value and the location of the maximum of $|\cos z|$ on the square defined by $0 \leq \mathrm{Re}\, z \leq 2\pi$, $0 \leq \mathrm{Im}\, z \leq 2\pi$.

By the Maximum Modulus Principle, the maximum is achieved on the boundary. Using the formula

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

(which follows immediately from $\cos z = \cos x \cosh y - i \sin x \sinh y$), we see that

- $|\cos z|^2 = \cos^2 x$ on the bottom part of the boundary, given by $y=0$, $0 \leq x \leq 2\pi$;
- $|\cos z|^2 = \cos^2 x + \sinh^2 2\pi$ on the top ($y = 2\pi$, $0 \leq x \leq 2\pi$);
- on the vertical components of the boundary, $|\cos z|^2 = 1 + \sinh^2 y$.

It is thus clear that the maximum value of $|\cos z|^2$ is $|\cos z|^2$ is $1 + \sinh^2 2\pi$, which is achieved at the points $2\pi i$, $\pi + 2\pi i$ and $2\pi + 2\pi i$.

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7. **[2pt]** Let f be an entire function. Show that if $|f(z)| > 1$ for all $z \in \mathbb{C}$, then f is constant.

If $|f(z)| > 1$ then f cannot have any zeroes, and so $1/f$ is an entire function. Moreover, it is bounded above by 1. By Liouville's theorem, it is constant. Hence, so is f.

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8. **[2pt]** Let f be an entire function satisfying the inequality $|f(z)| \leq A|z|$ for all $z \in \mathbb{C}$ and some fixed positive constant A. Show that $f(z) = az$, where a is a complex constant, following these steps:

1. **[1pt]** Let $C_R(z_0)$ be the circle of radius $R > 0$ centered at z_0. With the aid of the Cauchy integral formula for the second derivative of f applied on $C_R(z_0)$, show that
$$|f''(z_0)| \leq \frac{2A}{R} + \frac{2A|z_0|}{R^2}$$
.
2. **[0.5pt]** Explain how this bound implies that $|f''(z_0)| = 0$ for all $z_0 \in \mathbb{C}$.
3. **[0.5pt]** Since f'' is identically zero, we have that $f'(z) = a$ —a constant function. In turn, this implies that $f(z) = az + b$. Prove that $b = 0$ under our assumptions.

1. The Cauchy integral formula for the second derivative gives
$$f''(z_0) = \frac{1}{\pi i} \oint_{C_R(z_0)} \frac{f(z)}{(z - z_0)^3}\, dz.$$
The numerator in the integrand is bounded above by
$$|f(z)| < A|z| \leq A|z - z_0| + A|z_0| = AR + A|z_0|.$$
and so
$$|f''(z_0)| = \left| \frac{1}{\pi i} \oint_{C_R(z_0)} \frac{f(z)}{(z - z_0)^3}\, dz \right| \leq \frac{1}{\pi} \frac{AR + A|z_0|}{R^3} \cdot 2\pi R = \frac{2A}{R} + \frac{2A|z_0|}{R^2}$$
.
2. Since the above inequality is valid for arbitrary R, we may take the limit as $R \to +\infty$. The right-hand side goes to zero, and hence so does the left-hand side.
3. At $z = 0$, we have $|b| = |f(0)| < A|0| = 0$, which forces $b = 0$.
