MAT389 Fall 2013, Midterm 2

Nov 12, 2013

Please justify your reasoning. Answers without an explanation will not be given any credit.

Definition: $C_r(z)$ is the circle of radius r centered at z, oriented counterclockwise.

Some useful formulas:

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$
$$\arctan z = \frac{i}{2} \log \frac{i+z}{i-z}$$
$$\operatorname{argtanh} z = \frac{1}{2} \log \frac{1+z}{1-z}$$

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1. **[2pt]** Calculate all possible values of the following multivalued expressions.

(i) **[1pt]** $(-1)^{1/\pi}$

(ii) **[1pt]** $\arctan(i/2)$

2. **[2pt]** Let $f(z)$ denote the principal branch of the multivalued function $z^{1/4}$. Find the image of the right half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ under the transformation

$$w = \operatorname{Log}\left[e^{i\pi/8} f(z)\right].$$

3. **[2pt]** Consider the multivalued function $F(z) = \operatorname{argtanh} z$. For each of the conditions below, choose a determination $f(z)$ of $F(z)$ that satisfies said condition. Describe the branch cuts, the discontinuity of your choice of determination as you cross each branch cut, and the maximal domain on which it is holomorphic.

(i) **[1pt]** $f(z)$ is holomorphic at $z = 0$.

(ii) **[1pt]** $f(z)$ is holomorphic for $|z| > 2$.

4. **[6pt]** For each of the cases below, compute the integral

$$\int_C f(z)\, dz.$$

(i) **[1pt]** C is the line segment from $z = 0$ to $z = 1 + i$, and $f(z) = \operatorname{Re} z$.

(ii) **[1pt]** $C = C_r(z_0)$ ($r > 0$, $z_0 \in \mathbb{C}$), and $f(z) = -\operatorname{Im} z$ (**Hint:** use Green's theorem).

(iii) **[1pt]** $C = C_1(0)$, and $f(z) = (\sin z)/(z - \pi)$.

(iv) **[1pt]** $C = C_3(\sqrt{2})$, and $f(z) = (e^z + z)/(z - 2)$.

(v) **[1pt]** $C = C_{\sqrt{2}}(1)$, and $f(z) = 1/(z^2 - 2i)$.

(vi) **[1pt]** $C = C_{3/2}(2i)$, and $f(z) = z^i/(z - i)^2$, where z^i denotes the principal branch of the corresponding multivalued function.

5. [**2pt**] Bound the modulus of the integrals below.

(i) [**1pt**] $\displaystyle\int_{C_1(0)} \frac{dz}{2+\bar{z}^2}$

(ii) [**1pt**] $\displaystyle\int_C \frac{e^z}{|z|^2}\,dz$, where C is the square with vertices $\pm 1 \pm i$.

6. [**2pt**] Find the value and the location of the maximum of $|\cos z|$ on the square defined by $0 \le \operatorname{Re} z \le 2\pi$, $0 \le \operatorname{Im} z \le 2\pi$.

7. [**2pt**] Let f be an entire function. Show that if $|f(z)| > 1$ for all $z \in \mathbb{C}$, then f is constant.

8. [**2pt**] Let f be an entire function satisfying the inequality $|f(z)| \le A|z|$ for all $z \in \mathbb{C}$ and some fixed positive constant A. Show that $f(z) = az$, where a is a complex constant, following these steps:

1. [**1pt**] Let $C_R(z_0)$ be the circle of radius $R > 0$ centered at z_0. With the aid of the Cauchy integral formula for the second derivative of f applied on $C_R(z_0)$, show that

$$|f''(z_0)| \le \frac{2A}{R} + \frac{2A|z_0|}{R^2}$$

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2. [**0.5pt**] Explain how this bound implies that $|f''(z_0)| = 0$ for all $z_0 \in \mathbb{C}$.
3. [**0.5pt**] Since f'' is identically zero, we have that $f'(z) = a$ —a constant function. In turn, this implies that $f(z) = az + b$. Prove that $b = 0$ under our assumptions.