# MAT389 Fall 2013, Problem Set 5

## Conformal transformations

**5.1** Determine the angle of rotation at the point $z = 2+i$ of the transformation $f(z) = z^2$ and illustrate it for some particular curve. Show that the scale factor of the transformation at that point is $2\sqrt{5}$.

**5.2** Show that the angle of rotation at a nonzero point $z = r_0 e^{i\theta_0}$ under the transformation $f(z) = z^n$ $(n \geq 1)$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point.

## Transformation of harmonic functions

**5.3**
(i) Show that the transformation $w = e^z$ takes the horizontal strip $0 < \operatorname{Im} z < \pi$ to the (open) upper half plane $\{w \in \mathbb{C} \mid \operatorname{Im} w > 0\}$.
(ii) Use the fact that $h(u,v) = \operatorname{Re} w^2 = u^2 - v^2$ is harmonic on the upper half plane, and the transformation $w = e^z$ to show that $H(x,y) = e^{2x}\cos 2y$ is a harmonic function on the strip.

**5.4** Suppose that a holomorphic function $w = f(z) = u(x,y) + iv(x,y)$ maps a domain $D_z$ in the $z$ plane onto a domain $D_w$ in the $w$ plane; and let a function $h(u,v)$, with continuous partial derivatives of the first and second order, be defined on $D_w$. Use the chain rule for partial derivatives to show that if $H(x,y) = h\big(u(x,y), v(x,y)\big)$, then

$$H_{xx}(x,y) + H_{yy}(x,y) = \Big[h_{uu}(u,v) + h_{vv}(u,v)\Big]|f'(z)|^2$$

**Hint:** in the simplifications you will need to use the Cauchy-Riemann equations for $f$, the fact that $u$ and $v$ are harmonic on $D_z$, and that the continuity of the partial derivatives of $h$ ensure $h_{uv} = h_{vu}$.

**5.5** Let $p(u,v)$ be a function that has continuous partial derivatives of the first and second orders and satisfies *Poisson's equation*

$$p_{uu}(u,v) + p_{vv}(u,v) = \Phi(u,v)$$

in a domain $D_w$ of the $w$ plane, where $\Phi$ is a prescribed function. Show how it follows from the previous problem that if a holomorphic function $w = f(z) = u(x,y) + iv(x,y)$ maps a domain $D_z$ onto the domain $D_w$, then the function

$$P(x,y) = p\big(u(x,y), v(x,y)\big)$$

satisfies the Poisson equation

$$P_{xx}(x,y) + P_{yy}(x,y) = \Phi\big(u(x,y), v(x,y)\big)|f'(z)|^2$$

on $D_z$.