MAT389 Fall 2013, Problem Set 12

Rouché's theorem

12.1 Determine the number of zeroes of the following polynomials inside the unit circle:

(i) $z^6 - 5z^4 + z^3 - 2z$, (ii) $2z^4 - 2z^3 + 2z^2 - 2z + 9$.

12.2 Determine the number of roots of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the region $1 \leq |z| < 2$.

12.3 Show that if c is a complex number such that $|c| > e$, the equation $cz^n = e^z$ has n roots inside the unit circle.

Trigonometric integrals

12.4 Use residues to establish the following integration formulas:

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}$ (ii) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta} = \sqrt{2}\,\pi$

(iii) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5 - 4\cos 2\theta} = \frac{3\pi}{8}$ (iv) $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{a\pi}{(a^2 - 1)^{3/2}}$ $(a > 1)$

(v) $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\,\pi$ $(n \in \mathbb{Z}_{>0})$

Note: beware the limits of integration!

Improper integrals

12.5 In each of the following cases, establish the convergence of the given integral and calculate its value.

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2 + 1)^2}$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x\, dx}{(x^2 + 1)(x^2 + 2x + 2)}$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4 + 16}\, dx$ (iv) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x + \alpha)^2 + \beta^2}$ $(\beta > 0)$

12.6 Compute the following integrals:

(i) $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$ (ii) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+\alpha^2}\,dx \ (\alpha>0)$

(iii) $\displaystyle\int_0^\infty \frac{x\,dx}{x^5+1}$ (iv) $\displaystyle\int_0^\infty \frac{x^5\,dx}{x^{10}+1}$

(v) $\displaystyle\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx$ (vi) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

(vii) $\displaystyle\int_0^\infty \frac{x^{1/4}}{x^3+1}\,dx$ (viii) $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{x^2+2x+5}$

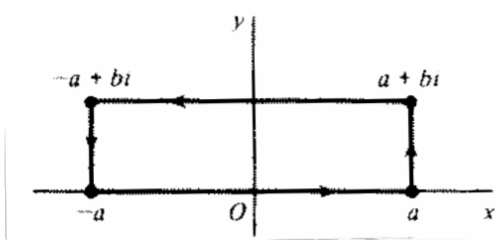
Hint: for (vi), notice that $2\sin x^2 = 1-\cos 2x = \mathrm{Re}(1-e^{2ix})$, and integrate the function $f(z) = (1-e^{2iz})/z^2$ over the appropriate contour.

12.7 Derive the integration formula

$$\int_0^{+\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

by integrating the function $f(z) = e^{-z^2}$ around the rectangular contour C in the figure, and then letting $a \to +\infty$. Use the well-known integration formula

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$



12.8 Show that

$$\int_{-\infty}^{+\infty} \frac{\cos x}{e^x+e^{-x}}\,dx = \frac{\pi}{2\cosh(\pi/2)}$$

Hint: integrate $f(z) = e^{iz}/(e^z+e^{-z})$ over the rectangle with vertices at $\pm R$ and $\pm R + i\pi$.

Laplace transforms

12.9 Find the inverse Laplace transforms of the following functions:

(i) $F(s) = \dfrac{1}{3-5s}$ (ii) $F(s) = \dfrac{2s-2}{(s+1)(s^2+2s+5)}$

(iii) $F(s) = \dfrac{2s^3}{s^4-4}$ (iv) $F(s) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$

12.10 Using Laplace transforms, solve the following initial value problems:

(i) $y'' + y = \sin 4t,\;\; y(0) = 0,\; y'(0) = 1,$

(ii) $y'' + y' + 2y = e^{-t}\cos 2t,\;\; y(0) = 1,\; y'(0) = -1,$