MAT389 Fall 2013, Problem Set 6

The exponential

6.1 Suppose that a function $f(z) = u(x,y) + iv(x,y)$ satisfies the following two conditions:
(1) $f(x + i0) = e^x$, and
(2) f is entire, with derivative $f'(z) = f(z)$.

Follow the steps below to show that $f(z)$ must be the function

$$f(z) = e^x(\cos\theta + i\sin\theta)$$

(1) Obtain the equations $u_x = u$ and $v_x = v$ and then use them to show that there exist real-valued functions ϕ and ψ of the real variable y such that

$$u(x,y) = e^x\phi(y), \quad \text{and} \quad v(x,y) = e^x\psi(y).$$

(2) Use the fact that u is harmonic to obtain the differential equation $\phi''(y) + \phi(y) = 0$ and thus show that $\phi(y) = A\cos y + B\sin y$, where A and B are complex numbers.
(3) After pointing out why $\psi(y) = A\sin y - B\cos y$ and noting that

$$u(x,0) + iv(x,0) = e^x,$$

find A and B. Conclude that

$$u(x,y) = e^x\cos y, \quad \text{and} \quad v(x,y) = e^x\sin y.$$

6.2 If e^z is purely imaginary, what restriction is placed on z?

6.3 Describe the behavior of e^z as
(1) $x \to -\infty$, with y fixed; and
(2) $y \to +\infty$, with x fixed.

Trigonometric and hyperbolic functions

6.4 Show that $e^{iz} = \cos z + i\sin z$ for every complex number z.

6.5 Use the identities

$$\sin(z_1 \pm z_2) = \sin z_1\cos z_2 \pm \cos z_1\sin z_2, \qquad \cos(z_1 \pm z_2) = \cos z_1\cos z_2 \mp \sin z_1\sin z_2$$

and the relationship between trigonometric and hyperbolic functions,

$$-i\sinh(iz) = \sin z, \quad -i\sin(iz) = \sinh z, \quad \cosh(iz) = i\cos z, \quad \cos(iz) = \cosh(z)$$

to deduce expressions for $\sinh(z_1 \pm z_2)$ and $\cosh(z_1 \pm z_2)$.

6.6 Find all the zeros and singularities of the function $f(z) = \tanh z = \sinh z/\cosh z$,

6.7 Find all roots of the equations

(i) $\cosh z = 1/2$, (ii) $\sinh z = i$, (iii) $\cosh z = -2$.

6.8 Show that the image of the line segment given by

$$-\pi \leq x \leq \pi, \quad \text{and} \quad y = c$$

for some fixed $c > 0$ under the transformation $w = \sin z$ is given by the ellipse with equation

$$\left(\frac{u}{\cosh c}\right)^2 + \left(\frac{v}{\sinh c}\right)^2 = 1.$$

6.9 Find a conformal transformation $w = f(z)$ that takes the semi-infinite strip $0 < x < \pi/2$, $y > 0$ onto the upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.
Hint: start by considering the image of the domain given under $Z = \sin z$. Do you know of a conformal transformation $w = g(Z)$ that takes the resulting domain to the entire upper half-plane?

The logarithm

6.10 Find the image under Log of the following complex numbers:

(i) i, (ii) $-ei$, (iii) $1 - i$, (iv) $-1 + i\sqrt{3}$.

6.11 Find the image under $\log_{(\pi/2)}$ of the wedge $\{z \in \mathbb{C}^{\times} \mid 0 < \operatorname{Arg} z < \pi/4\}$.

6.12 Find the image of the (open) upper half-plane $\mathbb{H}$ under the transformation

$$w = \operatorname{Log} \frac{z-1}{z+1}$$

Hint: break up the transformation above as follows:

$$Z = \frac{z-1}{z+1}, \quad w = \operatorname{Log} Z,$$

and notice that the first of these is a Möbius transformation