MAT389 Fall 2013, Problem Set 7

Complex exponentials

7.1 Find all the possible values of x^i for $x \in \mathbb{R}^\times$.
Hint: consider the cases $x < 0$ and $x > 0$ separately.

We have
$$x^i = e^{i \log x} = e^{i(\operatorname{Log}|x| + i \arg x)} = e^{-\arg x} e^{i \operatorname{Log}|x|}.$$
If $x > 0$, $\arg(x) = 2\pi k$ for $k \in \mathbb{Z}$, so
$$x^i = e^{-2\pi k} e^{i \operatorname{Log} x}.$$
If $x < 0$, $\arg(x) = (2k+1)\pi$ for $k \in \mathbb{Z}$ and
$$x^i = e^{-(2k+1)\pi} e^{i \operatorname{Log}(-x)}.$$

7.2 Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What restrictions must be placed on the constant c so that the values of $|i^c|$ are all the same?

The values of i^c can be calculated as
$$i^c = i^{a+bi} = e^{(a+bi)\log i} = e^{(a+bi) i \arg i} = e^{-b \arg i} e^{ai \arg i}.$$
Hence $|i^c| = e^{-b \arg i}$. Since $\arg i$ is multiple-valued, $|i^c|$ is constant if and only if $b = 0$, i.e. c is real.

Inverse trigonometric functions

7.3 Find expressions for the derivatives of the multivalued functions.
$$\operatorname{argsinh} z = \log\left[z + (z^2+1)^{1/2}\right], \qquad \operatorname{argcosh} z = \log\left[z + (z^2-1)^{1/2}\right],$$
$$\operatorname{argtanh} z = \frac{1}{2} \log \frac{1+z}{1-z}.$$
Hint: notice that, as in the case of inverse trigonometric functions, the derivatives might be different for different branches of these functions.

The multivaluedness of these three functions comes from that of of the logarithm, and that of square roots —which ultimately comes from that of the logarithm. In the first case, we have nothing to worry about, since the derivative of any branch of the logarithm is the (univalued) function $1/z$. When dealing with square roots, we need to be careful though. Let $f(z)$ be a (univalued) function. Choosing a branch of $[f(z)]^{1/2}$ is tantamount to choosing $\alpha \in \mathbb{R}$; the corresponding branch is then given by the rule $e^{(1/2)\log_{(\alpha)} f(z)}$. We can now apply the chain rule to find that the derivative of this function to be

$$\frac{d}{dz}e^{(1/2)\log_{(\alpha)} f(z)} = \frac{1}{2}e^{(1/2)\log_{(\alpha)} f(z)}\frac{d}{dz}\log_{(\alpha)} f(z) = \frac{1}{2}e^{(1/2)\log_{(\alpha)} f(z)}\frac{f'(z)}{f(z)} = \frac{f'(z)}{2e^{(1/2)\log_{(\alpha)} f(z)}}$$

That is, we can write

$$\frac{d}{dz}[f(z)]^{1/2} = \frac{f'(z)}{2[f(z)]^{1/2}}$$

as long as we interpret the square roots on both sides to be the same branches of the multivalued function $[f(z)]^{1/2}$.

With this in mind, we can write

$$\begin{aligned}
\frac{d}{dz}\operatorname{argsinh} z &= \frac{d}{dz}\log\left[z+(z^2+1)^{1/2}\right] = \frac{1}{z+(z^2+1)^{1/2}}\left[1+\frac{2z}{2(z^2+1)^{1/2}}\right] \\
&= \frac{1}{z+(z^2+1)^{1/2}}\cdot\frac{(z^2+1)^{1/2}+z}{(z^2+1)^{1/2}} = \frac{1}{(z^2+1)^{1/2}} \\
\\
\frac{d}{dz}\operatorname{argcosh} z &= \frac{d}{dz}\log\left[z+(z^2-1)^{1/2}\right] = \frac{1}{z+(z^2-1)^{1/2}}\left[1+\frac{2z}{2(z^2-1)^{1/2}}\right] \\
&= \frac{1}{z+(z^2-1)^{1/2}}\cdot\frac{(z^2-1)^{1/2}+z}{(z^2-1)^{1/2}} = \frac{1}{(z^2-1)^{1/2}} \\
\frac{d}{dz}\operatorname{argtanh} z &= \frac{1}{2}\frac{d}{dz}\Big[\log(1+z)-\log(1-z)\Big] = \frac{1}{2}\left[\frac{1}{1+z}-\frac{1}{1-z}\right] = \frac{1}{1-z^2}
\end{aligned}$$

Combining branch cuts

7.4 Every polynomial of degree d with complex coeffcients can be written as a product

$$P(z) = a_d\prod_{j=1}^{d}(z-z_j)$$

where the $z_i \in \mathbb{C}$ are its d roots, and $a_d \in \mathbb{C}$ is the coefficient of its highest degree term —this statement is equivalent to the Fundamental Theorem of Algebra.
Fix $R > \max_{j=1,\dots,d}|z_j|$, and let $C_R = \{z \in \mathbb{C} \mid |z| = R\}$. In each of the following cases, either choose a determination of the function $f(z) = P(z)^{1/n}$ so that it is holomorphic at every point of C_R or explain why it is impossible.

(i) $n = 2$, $d = 1$: $z_1 = 0$.
(ii) $n = 2$, $d = 2$: $z_1 = i$, $z_2 = -i$.
(iii) $n = 2$, $d = 6$: $z_j = \omega^j$ with $\omega = e^{\pi i/3}$.
(iv) $n = 3$, $d = 3$: $z_1 = -1$, $z_2 = 0$, $z_3 = 1$.
(v) $n = 3$, $d = 3$: $z_1 = -1$, $z_2 = i$, $z_3 = 1$.
(vi) $n = 2$, $d = 2$: $z_1 = z_2 = 1$.

(i) The function $f(z) = z^{1/2}$ has a single branch point at $z = 0$. Any branch cut emanating from it must intersect the C_R and extend beyond it. Hence no branch can be made holomorphic everywhere outside C_R.

(ii) Writing $f(z) = (z-i)^{1/2}(z+i)^{1/2}$, we see that this function has two branch points. For each of the two factors, choose the branch cut going south from the corresponding branch point (see Figure 1). Since crossing any of these cuts picks up a multiplicative (-1) factor, they cancel out south of the point $z = -i$. Hence this determination of $f(z)$ is holomorphic everywhere outside the line segment between the two branch points. In particular, it is holomorphic outside the circle C_R as long as $R > 1$.

Notice that the same argument can be used to construct a branch of any function of the form $f(z) = (z - z_1)^{1/2}(z - z_2)^{1/2}$ that is holomorphic everywhere outside the line segment joining z_1 and z_2 (see Figure 2).

(iii) The multivalued function $f(z) = \prod_{j=1}^{6}(z - \omega^j)^{1/2}$ has six branch points. We can pair them up and use the last observation in (ii). For example, let

$$f_1(z) = (z-\omega)^{1/2}(z-\omega^2)^{1/2}, \quad f_2(z) = (z-\omega^3)^{1/2}(z-\omega^4)^{1/2}, \quad f_3(z) = (z-\omega^5)^{1/2}(z-\omega^6)^{1/2}$$

Each one of these can be made into a function that is holomorphic outside of the line segment joining the two roots. The product $f(z) = f_1(z)f_2(z)f_3(z)$ is hence holomorphic outside of C_R when $R > 1$ (see Figure 3).

(iv) In $f(z) = z^{1/3}(z-1)^{1/3}(z+1)^{1/3}$, choose the principal determination of each of the factors. The branch cuts all point in the negative real direction. Crossing any of them in the counterclockwise direction picks up a multiplicative $e^{2\pi i/3}$. To the left of the point $z = -1$ we cross all three of them, and so they cancel out there (see Figure 4). We conclude that this branch of $f(z)$ is holomorphic everywhere outside the line segment joining the points $z = 1$ and $z = -1$.

(v) In the case of $f(z) = (z-1)^{1/3}(z+1)^{1/3}(z-i)^{1/3}$, the fact that the three branch points do not lie on a line precludes us from canceling out the branch cuts. The strategy of (iv) does not work. One could try canceling two branch cuts (e.g., that from $z = -1$ in the positive real direction, and that from $z = 1$ in the negative real direction cancel outside of the line segment joining the two points), but the third one (emanating from $z = i$ in the example) would then always extend to infinity.

(vi) The function $f(z) = [(z-1)^2]^{1/2}$ can actually be extended as a holomorphic function to the whole complex plane. Depending on which branch we take, we get the entire function $z - 1$ or $-(z-1) = 1 - z$.

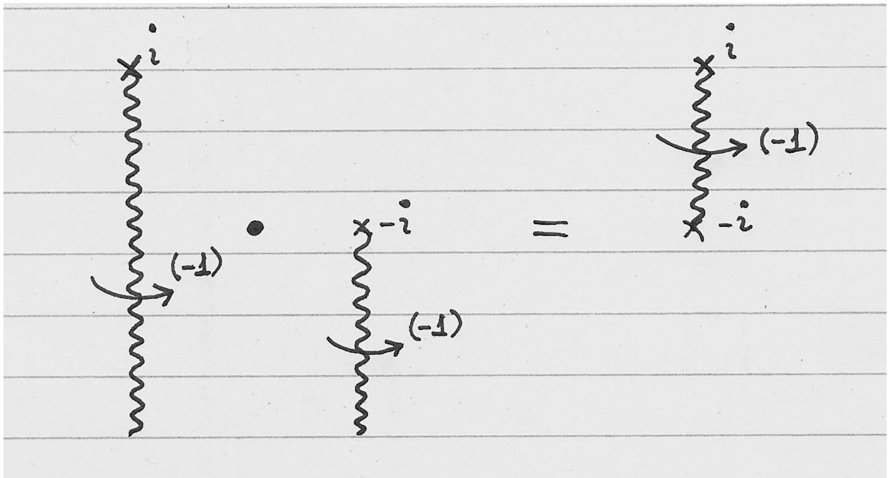

Figure 1: Branch cuts for 7.4 (ii)

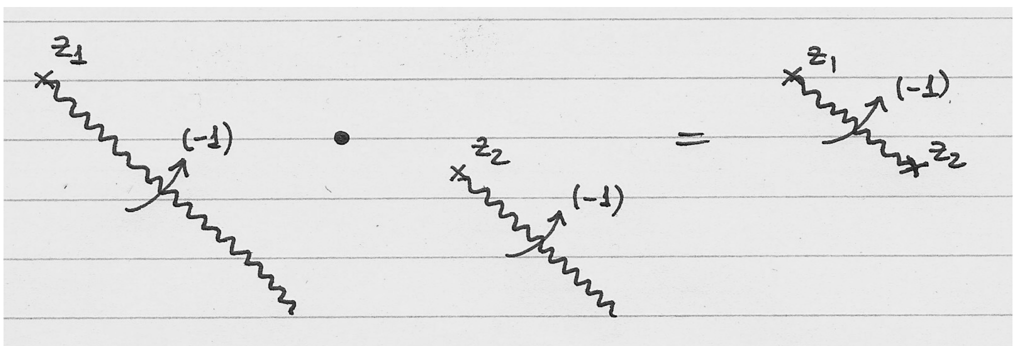

Figure 2: Branch cuts for the observation at the end of 7.4 (ii)

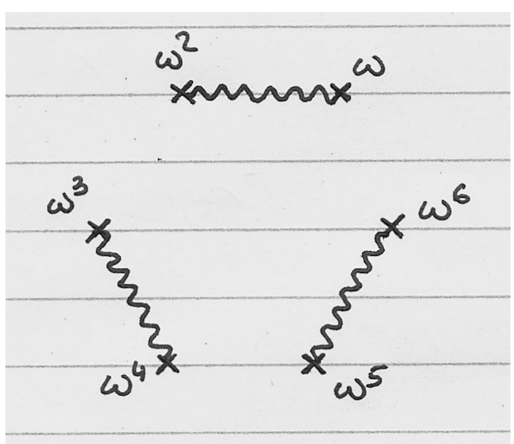

Figure 3: Branch cuts for 7.4 (iii)

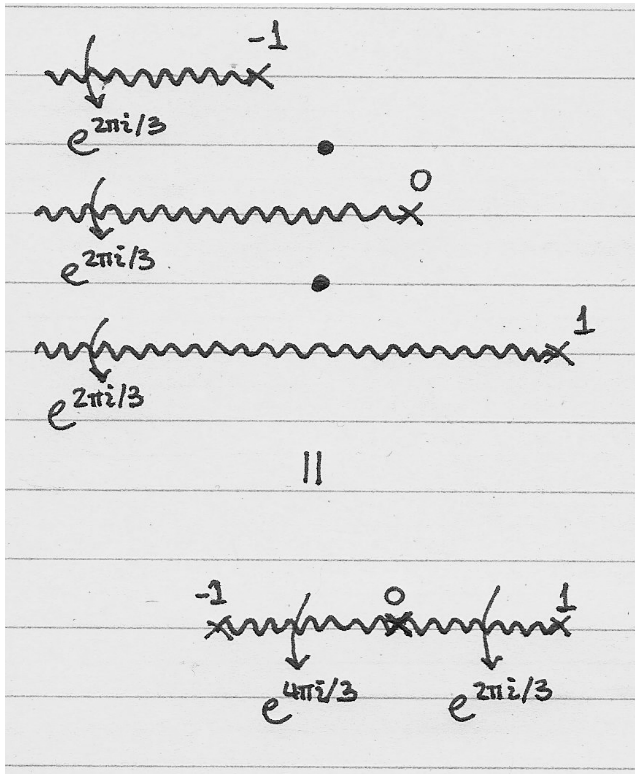

Figure 4: Branch cuts for 7.4 (iv)

Riemann surfaces

7.5 For your choice of branch cuts in (ii) in the previous problem, explain how to glue the two sheets into a Riemann surface for the corresponding function.

The Riemann surface for $f(z) = (z-i)^{1/2}(z+i)^{1/2}$ consists of two sheets. Each one of them looks like the complex plane with the segment joining the points $z = i$ and $z = -i$ removed. The left 'lip' of each one of them should be glued to the right 'lip' of the other (see Figure 5).

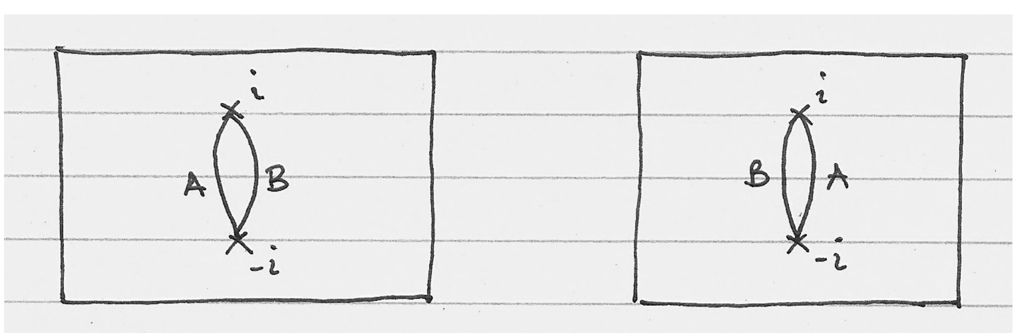

Figure 5: Riemann surface for 7.5

Arcs and contours

7.6 Find a parametrization of the ellipse given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

that traverses it in the counter-clockwise direction.

Let $z(t) = x(t) + iy(t) = a\cos t + ib\sin t$. Then,

$$\frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1,$$

so the image of $z(t)$ is contained in the ellipse. As t goes from 0 to 2π, this sweeps the whole ellipse.

7.7 Why is $z(t) = (1+i)t^3$ for $-1 \le t \le 1$ not a smooth arc? Give another parametrization of the same line segment in the complex plane that is indeed a smooth arc.

Since $z'(t) = 3(1+i)t^2$, it follows that $z'(0) = 0$, so $z(t)$ is not a smooth arc. However, the line segment from $-1-i$ to $1+i$ can be parametrized by the arc $u(t) = (1+i)t$, $-1 \le t \le 1$, which is smooth because $u'(t) = 1 + i \neq 0$.

7.8 Why is $z(t) = t^2 + t^3 i$ for $-1 \le t \le 1$ not a smooth arc? Can you give a parametrization of the same curve in the complex plane that is a smooth arc?

Since $z'(t) = 2t + 3t^2 i$, it follows that $z'(0) = 0$, so $z(t)$ is not a smooth arc. Its image is contained in the curve $x^3 = y^2$. This has a “kink” at the point $(0,0)$ —which corresponds to $t = 0$—, and so no reparametrization can be smooth.
