MAT389 Fall 2013, Problem Set 12

Rouché's theorem

12.1 Determine the number of zeroes of the following polynomials inside the unit circle:

(i) $z^6 - 5z^4 + z^3 - 2z$, (ii) $2z^4 - 2z^3 + 2z^2 - 2z + 9$.

(i) For all z in the unit circle we have

$$|z^6 + z^3 - 2z| \leq |z|^6 + |z|^3 + 2|z| = 4 < 5 = |5z^4|.$$

The polynomial $5z^4$ has four zeroes inside the unit circle, counting multiplicities. By Rouché's theorem, so does $z^6 - 5z^4 + z^3 - 2z$.

(ii) On the unit circle, we have

$$|2z^4 - 2z^3 + 2z^2 - 2z| \leq 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8 < 9.$$

Hence the polynomial in the statement has the same number of zeroes inside the unit circle as the constant function 9 —that is, none.

12.2 Determine the number of roots of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the region $1 \leq |z| < 2$.

Observe that on the unit circle

$$|2z^5 + z + 1| \leq 2|z|^5 + |z| + 1 = 4 < 6 = |6z^2|$$

Thus $2z^5 - 6z^2 + z + 1$ and $6z^2$ have an equal number of zeroes inside the circle $|z| = 1$ —two.

On the other hand, we find that, on the circle $|z| = 2$,

$$|-6z^2 + z + 1| \leq 6|z|^2 + |z| + 1 = 27 < 64 = |2z^5|.$$

By Rouche's theorem, the polynomial $2z^5 - 6z^2 + z + 1$ has five zeroes within the circle $|z| = 2$. Since two of them occur strictly inside the unit circle, there are three in the region $1 \leq |z| < 2$.

12.3 Show that if c is a complex number such that $|c| > e$, the equation $cz^n = e^z$ has n roots inside the unit circle.

For $|z| = 1$, the inequalities $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$ yield $|-e^z| = e^{\operatorname{Re} z} \leq e < |c| = |cz^n|$. Rouché's theorem then ensures that the equations $cz^n = 0$ and $cz^n - e^z = 0$ have the same number of solutions inside the unit circle —that is, n.

Trigonometric integrals

12.4 Use residues to establish the following integration formulas:

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$ (ii) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\,\pi$

(iii) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}$ (iv) $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(a^2-1)^{3/2}}$ $(a>1)$

(v) $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\,\pi$ $(n \in \mathbb{Z}_{>0})$

Note: beware the limits of integration!

The basic strategy for calculating integrals of this kind is to use the formula

$$\int_0^{2\pi} F(\cos\theta,\sin\theta)d\theta = \oint_{C_1(0)} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

that we proved in class, followed by an application of the Residue theorem.

(i) We follow our nose:

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \oint_{C_1(0)} \frac{1}{5-2i(z-z^{-1})}\frac{dz}{iz} = \oint_{C_1(0)} \frac{dz}{2z^2+5iz-2}$$

The denominator of this last integrand factors as $2z^2+5iz-2 = 2(z+2i)(z+i/2)$; of its two zeroes, only the second lies inside the contour of integration, and so

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \operatorname*{Res}_{z=-i/2} \frac{1}{2(z+2i)(z+i/2)} = 2\pi i \left.\frac{1}{2(z+2i)}\right|_{z=-i/2} = \frac{2\pi}{3}$$

(ii) Although the limits of integration are slightly different, the result is the same due to the periodicity of $\sin\theta$:

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} = \oint_{C_1(0)} \frac{1}{1-(z-z^{-1})^2/4}\frac{dz}{iz} = \oint_{C_1(0)} \frac{4iz\,dz}{z^4-6z^2+1}$$

Note that

$$z^4-6z^2+1 = \left[z^2-(3+2\sqrt{2})\right]\left[z^2-(3-2\sqrt{2})\right]$$

The zeroes of the first factor must have modulus greater than one and lie outside the unit circle, since $3+2\sqrt{2}>1$. Those of the second factor are a bit tricky to find; we make the ansatz that they are of the form $a+b\sqrt{2}$, with $a,b\in\mathbb{Q}$. Then,

$$(a+b\sqrt{2})^2 = (a^2+2b^2)+2ab\sqrt{2} = 3-2\sqrt{2} \iff \begin{cases} a^2+2b^2=3 \\ 2ab=-2 \end{cases} \iff a=-b=\pm 1.$$

That is, they are $1-\sqrt{2}$ and $-1+\sqrt{2}$. We can now compute residues:

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} &= 2\pi i \left[\operatorname*{Res}_{z=1-\sqrt{2}} \frac{4iz}{z^4-6z^2+1} + \operatorname*{Res}_{z=-1+\sqrt{2}} \frac{4iz}{z^4-6z^2+1} \right] \\
&= 2\pi i \left[\left. \frac{4iz}{(z^2-3-2\sqrt{2})(z+1-\sqrt{2})} \right|_{z=1-\sqrt{2}} + \left. \frac{4iz}{(z^2-3-2\sqrt{2})(z-1+\sqrt{2})} \right|_{z=-1+\sqrt{2}} \right] \\
&= 2\pi i \left(-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}} \right) = \sqrt{2}\pi
\end{aligned}$$

(iii) Our original formula does not apply quite so directly here, but the modification we need to make is easy: substitute $\cos m\theta$ by $(z^m + z^{-m})/2$. Then,

$$\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5-4\cos 2\theta} = \oint_{C_1(0)} \frac{(z^3+z^{-3})^2/4}{5-2(z^2+z^{-2})} \frac{dz}{iz} = \oint_{C_1(0)} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)}\, dz$$

The singularities of this last integrand lying inside the unit circle are located at 0 and $\pm 1/\sqrt{2}$. The residues at the last two points are computed in a straightforward manner:

$$\operatorname*{Res}_{z=1/\sqrt{2}} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \left. \frac{i(z^6+1)^2}{8z^5(z^2-2)(z+1/\sqrt{2})} \right|_{z=1/\sqrt{2}} = -\frac{27}{64}\, i$$

$$\operatorname*{Res}_{z=-1/\sqrt{2}} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \left. \frac{i(z^6+1)^2}{8z^5(z^2-2)(z-1/\sqrt{2})} \right|_{z=-1/\sqrt{2}} = -\frac{27}{64}\, i$$

At $z=0$, the easiest approach consists of expanding each factor —other than the z^5, of course— in a Taylor series about the origin:

$$\frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \frac{i}{8z^5}(1+2z^6+z^{12}) \left(1 + \frac{z^2}{2} + \frac{z^4}{4} + O(z^6) \right) \left(1 + 2z^2 + 4z^4 + O(z^6) \right)$$

Since we need to identify the coefficient of the z^{-1} term, we look for the terms in the product of these three series that have degree four. There are three of them:

$$\frac{i}{8z^5} \cdot 1 \cdot \frac{z^4}{4} \cdot 1, \qquad \frac{i}{8z^5} \cdot 1 \cdot \frac{z^2}{2} \cdot 2z^2, \qquad \frac{i}{8z^5} \cdot 1 \cdot 1 \cdot 4z^4.$$

Hence,

$$\operatorname*{Res}_{z=0} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \frac{21}{32}\, i$$

and

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5-4\cos 2\theta} &= 2\pi i \left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=1/\sqrt{2}} + \operatorname*{Res}_{z=-1/\sqrt{2}} \right) \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} \\
&= 2\pi i \left(\frac{21}{32}\, i - \frac{27}{64}\, i - \frac{27}{64}\, i \right) = \frac{3\pi}{8}
\end{aligned}$$

(iv) Following our basic strategy, we obtain

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \oint_{C_1(0)} \frac{1}{[a+(z+z^{-1})/2]^2} \frac{dz}{iz} = \oint_{C_1(0)} \frac{-4iz}{(z^2+2az+1)^2}\, dz$$

We have

$$z^2 + 2az + 1 = 0 \iff z = -a \pm \sqrt{a^2 - 1}$$

(the condition $a > 1$ ensures the square root above is a real number). For the negative sign, we obtain a point outside the unit circle, while the positive sign lands us inside of it —and gives rise to a double pole. Hence,

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} &= 2\pi i \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{-4iz}{(z^2 + 2az + 1)^2} \\
&= 2\pi i \, \frac{d}{dz} \left. \frac{-4iz}{(z + a + \sqrt{a^2-1})^2} \right|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{2a\pi}{(a^2 - 1)^{3/2}}
\end{aligned}$$

and

$$\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{a\pi}{(a^2 - 1)^{3/2}}$$

(v) Once again, we just apply our basic strategy:

$$\int_0^{2\pi} \sin^{2n}\theta \, d\theta = \oint_{C_1(0)} \frac{(z - z^{-1})^{2n}}{(2i)^{2n}} \frac{dz}{iz} = \frac{(-1)^n}{2^{2n} i} \oint_{C_1(0)} \frac{(z^2 - 1)^{2n}}{z^{2n+1}} \, dz$$

By the binomial theorem,

$$\frac{(z^2 - 1)^{2n}}{z^{2n+1}} = \frac{1}{z^{2n+1}} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} z^{2k}$$

The z^{-1} term occurs when $k = n$ in the above sum, and it has coefficient

$$\operatorname*{Res}_{z=0} \frac{(z^2 - 1)^{2n}}{z^{2n+1}} = (-1)^n \binom{2n}{n} = (-1)^n \frac{(2n)!}{(n!)^2}$$

Consequently,

$$\int_0^{2\pi} \sin^{2n}\theta \, d\theta = \frac{(-1)^n}{2^{2n} i} \cdot 2\pi i \operatorname*{Res}_{z=0} \frac{(z^2 - 1)^{2n}}{z^{2n+1}} = \frac{(2n)!}{2^{2n-1}(n!)^2} \, \pi$$

and

$$\int_0^{\pi} \sin^{2n}\theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \sin^{2n}\theta \, d\theta = \frac{(2n)!}{2^{2n}(n!)^2} \, \pi$$

Improper integrals

12.5 In each of the following cases, establish the convergence of the given integral and calculate its value.

(i) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2 \, dx}{(x^2 + 1)^2}$

(ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x \, dx}{(x^2 + 1)(x^2 + 2x + 2)}$

(iii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^3 \sin x}{x^4 + 16} \, dx$

(iv) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x \, dx}{(x + \alpha)^2 + \beta^2}$ $(\beta > 0)$

Remember that if the integral

$$I = \int_{-\infty}^{+\infty} f(x)\, dx$$

converges, its value is equal to its Cauchy principal value

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\, dx = \lim_{R \to +\infty} \int_{-R}^{R} f(x)\, dx.$$

We can compute the latter by

1. replacing the real-valued function $f(x)$ by a complex-valued function $f(z)$ that restricts to $f(x)$ on the positive real axis;
2. integrating $f(z)$ on the contour of Figure 1a, making sure that

   $$\int_{C_R} f(z)\, dz \longrightarrow 0$$

   as $R \to +\infty$; and
3. applying the Residue theorem to conclude that

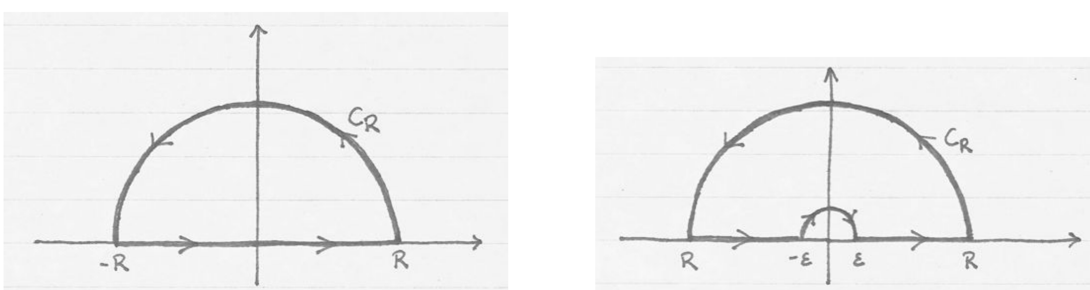
   $$I = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} f(z),$$

   where the sum ranges over the singularities of $f(z)$ lying inside the contour of integration.

We might need a slight variation of the above procedure:

(a) If $f(x)$ involved sines or cosines, we replace them by complex exponentials instead. Then, $f(z)$ restricts to $f(x)$ on the real axis only after taking real/imaginary part.

(b) If $f(z)$ has a singularity at $z = 0$, we modify the contour of integration to that of Figure 1b. We then need to either prove that the integral over C_ϵ also vanishes as $\epsilon \to 0$, or compute its limiting value.



(a) The basic contour

(b) Avoiding the origin

Figure 1: Contours for $\int_{-\infty}^{+\infty} f(x)\, dx$

(i) We apply the basic strategy above —that is choose $f(z) = z^2/(z^2+1)^2$ and integrate over the contour on Figure 1a. On C_R, we have $|z^2+1| = |R^2-1| = R^2-1$ (as long as R is greather than 1), so

$$\left| \int_{C_R} \frac{z^2}{(z^2+1)^2}\, dz \right| \leq \frac{R^2}{(R^2-1)^2} \pi R \xrightarrow[R \to +\infty]{} 0$$

Of the two poles of $f(z)$, only that at $z = i$ is inside the contour of integration. The Residue theorem now gives

$$I = 2\pi i \operatorname*{Res}_{z=i} \frac{z^2}{(z^2+1)^2} = 2\pi i \, \frac{-i}{4} = \frac{\pi}{2}$$

(ii) Let $f(z) = z/[(z^2+1)(z^2+2z+2)]$, and use the basic contour again. On C_R, and for R sufficiently large, we have the bounds $|z^2 - 1| \geq R^2 - 1$ and

$$|z^2 + 2z + 2| = |z - (-1+i)|\,|z - (-1-i)| \geq |R - \sqrt{2}|^2$$

and hence

$$\left| \int_{C_R} \frac{z\,dz}{(z^2+1)(z^2+2z+2)} \right| \leq \frac{R}{(R^2-1)(R-\sqrt{2})^2}\, \pi R \xrightarrow[R\to+\infty]{} 0$$

We can now calculate residues to obtain

$$\begin{aligned} I &= 2\pi i \left(\operatorname*{Res}_{z=i} \frac{z}{(z^2+1)(z^2+2z+2)} + \operatorname*{Res}_{z=-1+i} \frac{z}{(z^2+1)(z^2+2z+2)} \right) \\ &= 2\pi i \left(\frac{1-2i}{10} + \frac{-1+3i}{10} \right) = -\frac{\pi}{5} \end{aligned}$$

(iii) Choose $f(z) = z^3 e^{iz}/(z^4+16)$. We first prove that the integral over C_R vanishes as $R \to +\infty$.

$$\left| \int_{C_R} \frac{z^3 e^{iz}}{z^4+16}\, dz \right| = \left| \int_0^{\pi} \frac{R^3 e^{3i\theta} e^{iRe^{i\theta}}}{R^4 e^{4i\theta} + 16}\, iRe^{i\theta}\, d\theta \right| \leq \int_0^{\pi} \frac{R^4 \left| e^{iRe^{i\theta}} \right|}{|R^4 e^{4i\theta} + 16|}\, d\theta$$

With $|e^{iRe^{i\theta}}| = e^{-R\sin\theta}$ and $\left|R^4 e^{4i\theta} + 16\right| \geq R^4 - 16$, and using Jordan's inequality, we have

$$\left| \int_{C_R} \frac{z^3 e^{iz}}{z^4+16}\, dz \right| \leq \frac{R^4}{R^4-16} \int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{R^4}{R^4-16}\, \frac{\pi}{R} \xrightarrow[R\to+\infty]{} 0$$

On the real line, we have

$$\int_{-\infty}^{+\infty} \frac{x^3 e^{ix}}{x^4+16}\, dx = \int_{-\infty}^{+\infty} \frac{x^3 \cos x}{x^4+16}\, dx + iI$$

That is, we can find the integral we are interested in by taking the imaginary part:

$$I = \operatorname{Im} \int_{-\infty}^{+\infty} \frac{x^3 e^{ix}}{x^4+16}\, dx$$

On the other hand, of the four zeroes of the denominator only two of them lie on the upper half plane —those at $z = \sqrt{2}(\pm 1 + i)$. Then,

$$\begin{aligned} I &= \operatorname{Im} \left[2\pi i \left(\operatorname*{Res}_{z=\sqrt{2}(1+i)} \frac{z^3 e^{iz}}{z^4+16} + \operatorname*{Res}_{z=\sqrt{2}(-1+i)} \frac{z^3 e^{iz}}{z^4+16} \right) \right] \\ &= \operatorname{Im} \left[\left. \frac{2\pi i z^3 e^{iz}}{(z^4+16)/(z-\sqrt{2}-\sqrt{2}i)} \right|_{z=\sqrt{2}(1+i)} + \left. \frac{2\pi i z^3 e^{iz}}{(z^4+16)/(z+\sqrt{2}-\sqrt{2}i)} \right|_{z=\sqrt{2}(-1+i)} \right] \\ &= \operatorname{Im} \left[2\pi i\, \frac{e^{\sqrt{2}(1+i)}}{4} + 2\pi i\, \frac{e^{\sqrt{2}(-1+i)}}{4} \right] = \pi \cos\sqrt{2} \cosh\sqrt{2} \end{aligned}$$

(iv) Once again, we substitute the cosine for a complex exponential, taking $f(z) = e^{iz}/[(z+\alpha)^2+\beta^2]$.

$$\left|\int_{C_R} \frac{e^{iz}\,dz}{(z+\alpha)^2+\beta^2}\right| = \left|\int_0^\pi \frac{e^{iRe^{i\theta}}\,iRe^{i\theta}\,d\theta}{(Re^{i\theta}+\alpha)^2+\beta^2}\right| \le \int_0^\pi \frac{R\left|e^{iRe^{i\theta}}\right|}{|(Re^{i\theta}+\alpha)^2+\beta^2|}\,d\theta$$

Since $(z+\alpha)^2+\beta^2 = [z-(-\alpha+i\beta)][z-(-\alpha-i\beta)]$, we can bound

$$\left|(Re^{i\theta}+\alpha)^2+\beta^2\right| = \left|Re^{i\theta}+\alpha-i\beta\right|\left|Re^{i\theta}+\alpha+i\beta\right| \ge |R-\alpha^2-\beta^2|^2$$

and

$$\begin{aligned}\left|\int_{C_R} \frac{e^{iz}\,dz}{(z+\alpha)^2+\beta^2}\right| &\le \frac{R}{(R-\alpha^2-\beta^2)^2}\int_0^\pi e^{-R\sin\theta}\,d\theta \\ &< \frac{R}{(R-\alpha^2-\beta^2)^2}\,\frac{\pi}{R} \xrightarrow[R\to+\infty]{} 0\end{aligned}$$

On the real line, we have

$$\int_{-\infty}^{+\infty} \frac{e^{ix}\,dx}{(x+\alpha)^2+\beta^2} = I + i\int_{-\infty}^{+\infty} \frac{\sin x\,dx}{(x+\alpha)^2+\beta^2}$$

Hence,

$$\begin{aligned} I &= \mathrm{Re}\left[2\pi i \operatorname*{Res}_{z=-\alpha+i\beta} \frac{e^{iz}}{(z+\alpha)^2+\beta^2}\right] = \mathrm{Re}\left[2\pi i \left.\frac{e^{iz}}{z+\alpha+i\beta}\right|_{z=-\alpha+i\beta}\right] \\ &= Re\left[\frac{\pi}{\beta}\,e^{-\beta-i\alpha}\right] = \frac{\pi}{\beta}\,e^{-\beta}\cos\alpha\end{aligned}$$

12.6 Compute the following integrals:

(i) $\displaystyle\int_0^{+\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

(ii) $\displaystyle\int_0^{+\infty} \frac{\cos x}{x^2+\alpha^2}\,dx \ (\alpha > 0)$

(iii) $\displaystyle\int_0^{+\infty} \frac{x\,dx}{x^5+1}$

(iv) $\displaystyle\int_0^{+\infty} \frac{x^5\,dx}{x^{10}+1}$

(v) $\displaystyle\int_0^{+\infty} \frac{\log x}{(x^2+1)^2}\,dx$

(vi) $\displaystyle\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx$

(vii) $\displaystyle\int_0^{+\infty} \frac{x^{1/4}}{x^3+1}\,dx$

(viii) $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}\,dx}{x^2+2x+5}$

Hint: for (vi), notice that $2\sin^2 x = 1-\cos 2x = \mathrm{Re}(1-e^{2ix})$, and integrate the function $f(z) = (1-e^{2iz})/z^2$ over the appropriate contour.

Our basic strategy for computing integrals of the type

$$I = \int_0^{+\infty} f(x)\, dx$$

consists of:

1. replacing the real-valued function $f(x)$ by a complex-valued function $f(z)$ that restricts to $f(x)$ on the positive real axis;
2. choosing a contour as in Figure 2a in such a way that

$$\int_{L_1} f(z)\, dz \longrightarrow I, \qquad \int_{L_2} f(z)\, dz \longrightarrow kI \quad (k \neq 1), \qquad \int_{C_R} f(z)\, dz \longrightarrow 0$$

   as $R \to +\infty$; and
3. applying the Residue theorem to conclude that

$$(1+k)I = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} f(z),$$

   where the sum ranges over the singularities of $f(z)$ lying inside the contour of integration.

Several variants are important:

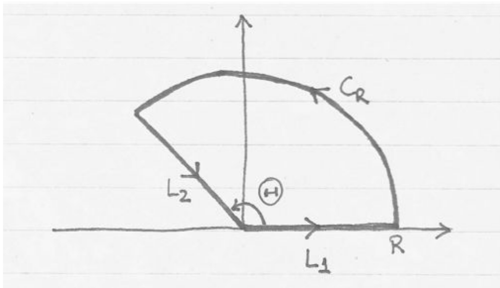
(a) We might need to choose a function $f(z)$ that restricts to $f(x)$ on the positive real axis only after taking real/imaginary part.
(b) The integral over L_2 might only be proportional to I after taking real/imaginary part.
(c) The function $f(z)$ might have a singularity at $z = 0$. We deal with this issue by modifying our contour of integration, carving a small circular arc about the origin as in Figure 2b. We then need to either prove that the integral over C_ϵ also vanishes as $\epsilon \to 0$, or compute its limiting value.
(d) If $\Theta = 2\pi$ and $f(z)$ has a branch cut along the positive real axis, we use the "keyhole" contour of Figure 2c and let $\delta \to 0$.

(i) We choose

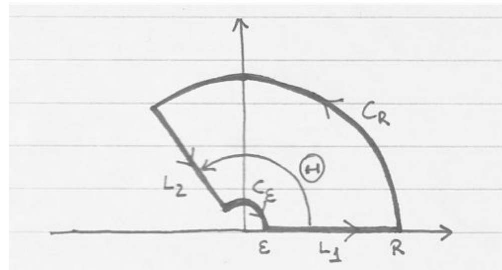
$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

Since this function is even, we take $\Theta = \pi$, in which case we obtain $k = 1$. Over C_R, we have $|z^2| = R^2$, $|z^2+1| \geq |R^2-1| = R^2-1$ and $|z^2+4| \geq |R^2-4| = R^2-4$. Hence,

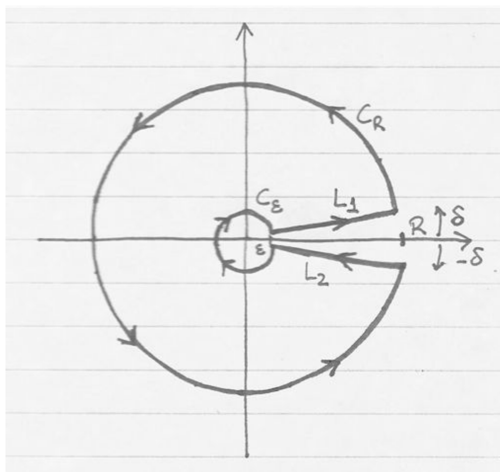
$$\left| \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\, dz \right| \leq \frac{R^2}{(R^2-1)(R^2-4)}\, \pi R \xrightarrow[R\to+\infty]{} 0$$



(a) The basic contour



(b) Avoiding the origin



(c) The case $\Theta = 2\pi$ with a branch cut

Figure 2: Contours for $\int_0^{+\infty} f(x)\,dx$

and

$$\begin{aligned}
2I &= 2\pi i \left(\operatorname*{Res}_{z=i} \frac{z^2}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{z^2}{(z^2+1)(z^2+4)} \right) \\
&= 2\pi i \left(\left. \frac{z^2}{(z+i)(z^2+4)} \right|_{z=i} + \left. \frac{z^2}{(z^2+1)(z+2i)} \right|_{z=2i} \right) \\
&= 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right) = \frac{\pi}{3} \\
\implies & \int_0^{+\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}
\end{aligned}$$

(ii) We use variant (a) of the basic strategy, with $f(z) = e^{iz}/(z^2+\alpha^2)$ and $\Theta = \pi$. Parametrizing C_R by $z = Re^{i\theta}$ for $0 \le \theta \le \pi$, we have

$$\left| \int_{C_R} \frac{e^{iz}}{z^2+\alpha^2}\,dz \right| = \left| \int_0^\pi \frac{e^{iRe^{i\theta}}}{R^2 e^{2i\theta}+\alpha^2}\, iRe^{i\theta} \right| \le \int_0^\pi \frac{R\left|e^{iRe^{i\theta}}\right|}{|R^2e^{2i\theta}+\alpha^2|}\,d\theta$$

In the denominator, we take the bound

$$\left|R^2 e^{2i\theta} + \alpha^2\right| = \left|Re^{i\theta} + \alpha i\right| \left|Re^{i\theta} - \alpha i\right| \ge (R-\alpha)^2,$$

while in the numerator, it is $|e^{iRe^{i\theta}}| = e^{-R\sin\theta}$. Jordan's inequality gives

$$\left|\int_{C_R} \frac{e^{iz}}{z^2+\alpha^2}\,dz\right| \le \frac{R}{(R-\alpha)^2}\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{R}{(R-\alpha)^2}\,\frac{\pi}{R} \xrightarrow[R\to+\infty]{} 0$$

The integration over the real line yields

$$\int_{-\infty}^{0} \frac{e^{ix}}{x^2+\alpha^2}\,dx + \int_0^{+\infty} \frac{e^{ix}}{x^2+\alpha^2}\,dx = \int_0^{\infty} \frac{e^{-ix}+e^{ix}}{x^2+\alpha^2}\,dx = 2I$$

We can now apply to Residue theorem to get

$$2I = 2\pi i \operatorname*{Res}_{z=\alpha i} \frac{e^{ix}}{x^2+\alpha^2} = 2\pi i \left.\frac{e^{ix}}{x+\alpha i}\right|_{z=\alpha i} = 2\pi i \frac{e^{-\alpha}}{2\alpha i} = \frac{\pi}{\alpha}e^{-\alpha}$$

$$\implies \int_0^{+\infty} \frac{\cos x}{x^2+\alpha^2}\,dx = \frac{\pi}{2\alpha}e^{-\alpha}$$

(iii) Let $f(z) = z/(z^5+1)$. Under the transformation $z \mapsto e^{2\pi i/5}z$, the function $f(z)$ gets multiplied by the constant factor $e^{2\pi i/5}$. Thus we can apply the basic strategy with $\Theta = 2\pi/5$, which results in $k = -e^{4\pi i/5}$; indeed,

$$\int_{L_2} \frac{z}{z^5+1}\,dz = \int_R^0 \frac{e^{2\pi i/5}x}{x^5+1}\,e^{2\pi i/5}\,dx = -e^{4\pi i/5}\int_0^R \frac{x}{x^5+1}\,dx \xrightarrow[R\to+\infty]{} -e^{4\pi i/5}I$$

Over C_R, we have $|z| = R$, $|z^5+1| \ge |R^5-1| = R^5-1$ and

$$\left|\int_{C_R} \frac{z}{z^5+1}\,dz\right| \le \frac{R}{R^5-1}\,\frac{2\pi}{5}\,R \xrightarrow[R\to+\infty]{} 0$$

The denominator in $f(z)$ has only one singularity inside the contour of integration: a simple pole at $z = e^{\pi i/5}$. Hence,

$$(1-e^{4\pi i/5})I = 2\pi i \operatorname*{Res}_{z=e^{\pi i/5}} \frac{z}{z^5+1} = 2\pi i \left.\frac{z}{(z^5+1)/(z-e^{\pi i/5})}\right|_{z=e^{\pi i/5}} = 2\pi i\,\frac{1}{5}e^{-3\pi i/5}$$

$$\implies \int_0^{+\infty} \frac{x\,dx}{x^5+1} = \frac{\pi}{5\sin(2\pi/5)}$$

(For the calculation of the denominator in the residue, see the appendix to this problem).

(iv) Let $f(z) = z^5/(z^{10}+1)$. Since the denominator is even, we might try $\Theta = \pi$; however, that produces $k = -1$ (after all, the original $f(x)$ is odd). Following on the example of the last integral, we take $\Theta = 2\pi/10 = \pi/5$ instead and find that, with this choice, $k = e^{\pi i/5}$. It is easy to see that the integral on C_R vanishes in the limit:

$$\left|\int_{C_R} \frac{z^5}{z^{10}+1}\,dz\right| \le \frac{R^5}{R^{10}-1}\,\frac{\pi}{5}\,R \xrightarrow[R\to+\infty]{} 0$$

Only one of the poles of $f(z)$ lie inside of the contour of integration, namely $z = e^{\pi i/10}$. Hence,

$$(1+e^{\pi i/5})I = 2\pi i \operatorname*{Res}_{z=e^{\pi i/10}} \frac{z^5}{z^{10}+1} = 2\pi i \left.\frac{z^5}{(z^{10}+1)/(z-e^{\pi i/10})}\right|_{z=e^{\pi i/10}} = 2\pi i\,\frac{1}{10}\,e^{-2\pi i/5}$$

$$\implies \int_0^{+\infty} \frac{x^5\,dx}{x^{10}+1} = \frac{\pi}{10\cos(\pi/10)}$$

(Once again, refer to the appendix for the computation of the denominator in the residue).

(v) Take $f(z) = \log_{(-\pi/2)} /(z^2+1)^2$. Since the denominator stays the same after $z \mapsto -z$, we can take $\Theta = \pi$. Two difficulties arise. The first one is that $f(z)$ has a branch point at the origin, and so we will need to invoke variant (c) of our basic strategy. Moreover, the integral over L_2 is proportional to that over L_1 only after taking real parts —that is, we need to blend in variant (b) as well.
Let us start by proving that the integrals over C_R and C_ϵ are zero in the limits $R \to +\infty$ and $\epsilon \to 0$, respectively. For the first, and with R sufficiently large, we have

$$|\log_{(-\pi/2)} z| = |\operatorname{Log} R + i\theta| \leq \operatorname{Log} R + \pi, \quad |z^2+1| \geq |R^2-1| = R^2-1$$

$$\left| \int_{C_R} \frac{\log_{(-\pi/2)} z}{(z^2+1)^2}\, dz \right| \leq \frac{\operatorname{Log} R + \pi}{(R^2-1)^2}\, \pi R \xrightarrow[R\to+\infty]{} 0$$

On the small circular arc of radius $\epsilon < 1$, it is

$$|\log_{(-\pi/2)} z| = |\operatorname{Log} \epsilon + i\theta| \leq -\operatorname{Log} \epsilon + \pi, \quad |z^2+1| \geq |\epsilon^2-1| = 1-\epsilon^2$$

$$\left| \int_{C_\epsilon} \frac{\log_{(-\pi/2)} z}{(z^2+1)^2}\, dz \right| \leq \frac{-\operatorname{Log} \epsilon + \pi}{(1-\epsilon^2)^2}\, \pi \epsilon \xrightarrow[\epsilon\to 0]{} 0$$

Over the straight legs of the contour, we get

$$\int_{L_1} \frac{\log_{(-\pi/2)} z}{(z^2+1)^2}\, dz = \int_\epsilon^R \frac{\operatorname{Log} x}{(x^2+1)^2}\, dx \longrightarrow I$$

$$\int_{L_2} \frac{\log_{(-\pi/2)} z}{(z^2+1)^2}\, dz = \int_{-R}^{-\epsilon} \frac{\operatorname{Log}(-x) + i\pi}{(x^2+1)^2}\, dx \longrightarrow I + i\pi \int_0^{+\infty} \frac{dx}{(x^2+1)^2}$$

as both $R \to +\infty$ and $\epsilon \to 0$. The point $z = i$ is a double pole of $f(z)$, so

$$\begin{aligned} 2I + i\pi \int_0^{+\infty} \frac{dx}{(x^2+1)^2} &= 2\pi i \operatorname*{Res}_{z=i} \frac{\log_{(-\pi/2)} z}{(z^2+1)^2} = 2\pi i \frac{d}{dz} \frac{\log_{(-\pi/2)} z}{(z+i)^2}\bigg|_{z=i} \\ &= 2\pi i \left. \frac{z + i - 2z \log_{(-\pi/2)} z}{z(z+i)^3} \right|_{z=i} = -\frac{\pi}{2} + \frac{\pi^2}{4} i \end{aligned}$$

Taking the real part of this last expression yields the integral we seek:

$$\int_0^{+\infty} \frac{\log x}{(x^2+1)^2}\, dx = -\frac{\pi}{4}$$

As a bonus, we obtain

$$\int_0^{+\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

from the imaginary part

(vi) As per the hint, take $f(z) = (1 - e^{2iz})/z^2$. Notice that this function has a simple pole at the origin, and so we will need to use variant (c) of the basic strategy. Since $f(x)$ is even, we try $\Theta = \pi$. This choice yields, in the limit,

$$\int_{L_1} \frac{1 - e^{2iz}}{z^2} = \int_\epsilon^R \frac{1 - e^{2ix}}{x^2}\, dx \longrightarrow 2I - i \int_0^{+\infty} \frac{\sin 2x}{x^2}\, dx$$

$$\int_{L_2} \frac{1-e^{2iz}}{z^2} = \int_{-R}^{-\epsilon} \frac{1-e^{2ix}}{x^2}\,dx = \int_{\epsilon}^{R} \frac{1-e^{-2ix}}{x^2}\,dx \longrightarrow 2I + i\int_0^{+\infty} \frac{\sin 2x}{x^2}\,dx$$

The integral over C_R vanishes:

$$\left|\int_{C_R} \frac{1-e^{2iz}}{z^2}\,dz\right| \leq \frac{2}{R^2}\,\pi R \xrightarrow[R\to+\infty]{} 0$$

That over C_ϵ, however, does not:

$$\int_{C_\epsilon} \frac{1-e^{2iz}}{z^2}\,dz = -\pi i \operatorname*{Res}_{z=0} \frac{1-e^{2iz}}{z^2} = -2\pi$$

Since $f(z)$ has no poles inside the contour of integration, we have $4I - 2\pi = 0$, or

$$\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

(vii) Let $f(z) = z^{1/4}/(z^3+1)$, where the numerator is the determination $z^{1/4} = e^{(1/4)\log_{(-\pi/2)} z}$ that has the branch cut along the negative imaginary axis —so we will need an indented contour like that of Figure 2b— and restricts to $x^{1/4}$ on the positive real axis. Since the denominator is invariant under the transformation $z \mapsto e^{2\pi i/3}z$, we set $\Theta = 2\pi/3$, which gives

$$\int_{L_1} \frac{z^{1/4}}{z^3+1}\,dz = \int_{\epsilon}^{R} \frac{x^{1/4}}{x^3+1}\,dx \longrightarrow I$$

$$\int_{L_2} \frac{z^{1/4}}{z^3+1}\,dz = \int_{R}^{\epsilon} \frac{e^{\pi i/6}x^{1/4}}{x^3+1}\,e^{2\pi i/3}dx = -e^{5\pi i/6}\int_{\epsilon}^{R} \frac{x^{1/4}}{x^3+1}\,dx \longrightarrow -e^{5\pi i/6}I$$

On the other hand, the integrals over C_R and C_ϵ go to zero as $R\to+\infty$ and $\epsilon\to 0$:

$$\left|\int_{C_R} \frac{z^{1/4}}{z^3+1}\,dz\right| \leq \frac{R^{1/4}}{R^3-1}\,\frac{2\pi}{3}\,R \xrightarrow[R\to+\infty]{} 0$$

$$\left|\int_{C_\epsilon} \frac{z^{1/4}}{z^3+1}\,dz\right| \leq \frac{\epsilon^{1/4}}{1-\epsilon^3}\,\frac{2\pi}{3}\,\epsilon \xrightarrow[\epsilon\to 0]{} 0$$

Only the pole at $z = e^{\pi i/3}$ is inside the contour of integration, so

$$(1-e^{5\pi i/6})I = 2\pi i \operatorname*{Res}_{z=e^{\pi i/3}} \frac{z^{1/4}}{z^3+1} = 2\pi i\left.\frac{z^{1/4}}{(z^3+1)(z-e^{\pi i/3})}\right|_{z=e^{\pi i/3}} = 2\pi i\,\frac{1}{3}e^{-7\pi i/12}$$

$$\Longrightarrow \int_0^{+\infty} \frac{x^{1/4}}{x^3+1}\,dx = \frac{\pi}{3\sin(5\pi/12)}$$

(The denominator can be calculated by hand, but the argument of the appendix still applies).

(viii) Here we need to resort to variant (d) of the basic strategy, since the polynomial z^2+2z+5 is not invariant under any transformation of the form $z \mapsto e^{i\Theta}z$ with $\Theta \notin 2\pi\mathbb{Z}$. Thus, we let $f(z) = z^{1/2}/(z^2+2z+5)$, where by $z^{1/2}$ we mean the determination $z^{1/2} = e^{(1/2)\log_{(0)} z}$ that has its branch cut along the positive real axis. On the two legs of the contour, we have

$$\int_{L_1} \frac{z^{1/2}\,dz}{z^2+2z+5} = \int_{\epsilon}^{R} \frac{e^{i\delta/2}x^{1/2}e^{i\delta}\,dx}{e^{2i\delta}x^2+2e^{i\delta}x+5} \longrightarrow I$$

$$\int_{L_2} \frac{z^{1/2}\,dz}{z^2+2z+5} = \int_R^{\epsilon} \frac{e^{i(2\pi-\delta)/2}x^{1/2}e^{i(2\pi-\delta)}\,dx}{e^{2i(2\pi-\delta)}x^2+2e^{i(2\pi-\delta)}x+5} \longrightarrow I$$

where the limit is as $R \to +\infty$, $\epsilon \to 0$ and $\delta \to 0$. Over the two circular arcs,

$$\left| \int_{C_R} \frac{z^{1/2}\,dz}{z^2+2z+5} \right| \le \frac{R^{1/2}}{(R-\sqrt{5})^2}\,(2\pi-2\delta)R \xrightarrow[R\to+\infty]{} 0$$

$$\left| \int_{C_\epsilon} \frac{z^{1/2}\,dz}{z^2+2z+5} \right| \le \frac{\epsilon^{1/2}}{(\sqrt{5}-\epsilon)^2}\,(2\pi-2\delta)\epsilon \xrightarrow[\epsilon\to 0]{} 0$$

where we have used the bounds

$$|z^2+2z+5| = |z+1-2i||z+1+2i| \ge |R-\sqrt{5}|^2 \quad \text{for } R \ge \sqrt{5}$$

$$|z^2+2z+5| = |z+1-2i||z+1+2i| \ge |\sqrt{5}-\epsilon|^2 \quad \text{for } \epsilon \le \sqrt{5}$$

Hence,

$$\begin{aligned}
2I &= 2\pi i \left(\operatorname*{Res}_{z=-1+2i} \frac{z^{1/2}}{z^2+2z+5} + \operatorname*{Res}_{z=-1-2i} \frac{z^{1/2}}{z^2+2z+5} \right) \\
&= 2\pi i \left(\left. \frac{z^{1/2}}{z+1+2i} \right|_{z=-1+2i} + \left. \frac{z^{1/2}}{z+1-2i} \right|_{z=-1-2i} \right) \\
&= 2\pi i \left(\frac{(-1+2i)^{1/2}}{4i} - \frac{(-1-2i)^{1/2}}{4i} \right) = \frac{\pi}{2} \left[(-1+2i)^{1/2} - (-1+2i)^{1/2} \right] \\
\implies \int_0^{+\infty} \frac{\sqrt{x}\,dx}{x^2+2x+5} &= \frac{\pi}{4} \left[(-1+2i)^{1/2} - (-1+2i)^{1/2} \right] = \frac{\pi}{4}\sqrt{2(\sqrt{5}+1)}
\end{aligned}$$

Appendix: in (iii), (iv) and (vii), the following quantity appeared when calculating the appropriate residue:

$$\left. \frac{z^k+1}{z-e^{\pi i/k}} \right|_{z=e^{\pi i/k}} = \prod_{j=1}^{k-1} \left(z - e^{(2j+1)\pi i/k} \right)$$

Although a brute force computation is possible for low values of k, it quickly becomes unmanageable. A slick way of computing it —for all values of k at once!— is to apply l'Hôpital's rule, which we proved in Problem 11.4 —the result in Problem 11.3 is, in fact, enough—:

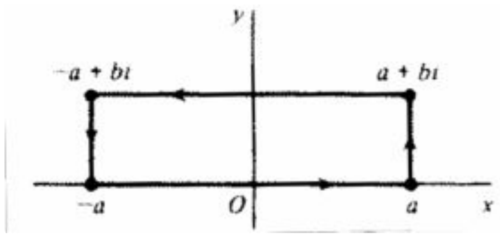
$$\left. \frac{z^k+1}{z-e^{\pi i/k}} \right|_{z=e^{\pi i/k}} = \left. \frac{kz^{k-1}}{1} \right|_{z=e^{\pi i/k}} = ke^{(k-1)\pi i/k}$$

12.7 Derive the integration formula

$$\int_0^{+\infty} e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

by integrating the function $f(z) = e^{-z^2}$ around the rectangular contour C in the figure, and then letting $a \to +\infty$. Use the well-known integration formula

$$\int_0^{+\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.$$



First of all, note that, since $f(z)$ is entire, $\int_C f(z) dz = 0$. Hence the integrals over each of the segments in the contour (with the orientation given) sum up to zero. Label them as follows:

- C_1, along the real axis;
- C_2, the vertical segment on the right;
- C_2, the top of the contour; and
- C_4, the vertical segment on the left.

We claim that the integrals over C_2 and C_4 vanish in the limit $a \to +\infty$. Indeed, on both of them we have $|e^{-z^2}| = e^{-a^2+y^2} \le e^{-a^2+b^2}$, so

$$\left| \int_{C_2} e^{-z^2} \, dz \right| \le e^{-a^2+b^2} b \qquad \text{and} \qquad \left| \int_{C_4} e^{-z^2} \, dz \right| \le e^{-a^2+b^2} b.$$

On the real axis we get the well-know Gaussian integral:

$$\int_{C_1} e^{-z^2} \, dz = \int_{-a}^{a} e^{-x^2} \, dx \xrightarrow[a \to +\infty]{} \int_{-\infty}^{+\infty} e^{-x^2} \, dx = 2 \int_0^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

The integral on C_4 yields

$$\int_{C_4} e^{-z^2} \, dz = \int_a^{-a} e^{-x^2+b^2-2ibx} \, dx = -e^{b^2} \int_{-a}^{a} e^{-x^2} (\cos 2bx - i \sin 2bx) \, dx$$

$$\xrightarrow[a \to +\infty]{} -e^{b^2} \int_{-\infty}^{+\infty} e^{-x^2} \cos 2bx \, dx + i e^{b^2} \, \text{P.V.} \int_{-\infty}^{+\infty} e^{-x^2} \sin 2bx \, dx$$

From the real part of $\int_C f(z) dz = 0$ we get

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos 2bx \, dx = \sqrt{\pi} e^{b^2} \implies \int_0^{+\infty} e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2},$$

since the integrand is an even function.

12.8 Show that

$$\int_{-\infty}^{+\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \frac{\pi}{2\cosh(\pi/2)}$$

Hint: integrate $f(z) = e^{iz}/(e^z + e^{-z})$ over the rectangle with vertices at $\pm R$ and $\pm R + i\pi$.

Let C denote the integration contour defined in the hint. We refer to the line segments that make it up as follows:

- C_1, from $-R$ to R along the real axis;
- C_2, the vertical segment from R to $R + i\pi$;
- C_2, the top of the contour —from $R + \pi$ to $-R + i\pi$; and
- C_4, the vertical segment from $-R + i\pi$ to $-R$.

The points of C_2 are of the form $z = R + iy$ with $0 \le y \le \pi$. Hence the following bounds hold there:

$$|e^{iz}| = |e^{i(R+iy)}| = e^{-y} \le 1,$$

$$|e^z + e^{-z}| \ge \big||e^z| - |e^{-z}|\big| = |e^R - e^{-R}| = \sinh R.$$

We can thus show that

$$\left|\int_{C_2} \frac{e^{iz}\, dz}{e^z + e^{-z}}\right| \le \frac{\pi}{\sinh R} \xrightarrow[R\to+\infty]{} 0$$

We can treat the integral along C_4 similarly:

$$|e^{iz}| = |e^{i(-R+iy)}| = e^{-y} \le 1,$$

$$|e^z + e^{-z}| \ge \big||e^z| - |e^{-z}|\big| = |e^{-R} - e^R| = \sinh R;$$

$$\left|\int_{C_4} \frac{e^{iz}\, dz}{e^z + e^{-z}}\right| \le \frac{\pi}{\sinh R} \xrightarrow[R\to+\infty]{} 0$$

Along C_1 we have

$$\int_{C_1} \frac{e^{iz}\, dz}{e^z + e^{-z}} = \int_{-R}^{R} \frac{e^{ix}\, dx}{e^x + e^{-x}} \xrightarrow[R\to+\infty]{} \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{e^x + e^{-x}}$$

The integral we are interested in is the real part of the latter. The integral on C_3 is closely related to it too:

$$\int_{C_3} \frac{e^{iz}}{e^z + e^{-z}}\, dz = \int_{R}^{-R} \frac{e^{i(x+i\pi)}}{e^{x+i\pi} + e^{-x-i\pi}}\, dx = e^{-\pi} \int_{-R}^{R} \frac{e^{ix}}{e^x + e^{-x}} \xrightarrow[R\to+\infty]{} e^{-\pi} \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{e^x + e^{-x}}$$

For the residue calculation, notice that $e^z + e^{-z} = 2\cosh z = 0$ for $z = (2k+1)\pi i/2$ with $k \in \mathbb{Z}$. Of these, only one is inside the contour of integration —when $k = 0$. Using our knowledge of Taylor series, we have

$$\begin{aligned}
\operatorname*{Res}_{z=\pi i/2} \frac{e^{iz}}{e^z + e^{-z}} &= \operatorname*{Res}_{z=\pi i/2} \frac{e^{iz}}{2\cosh z} = \operatorname*{Res}_{z=\pi i/2} \frac{e^{iz}}{2i\sum_{n=0}^{\infty}(z - \pi i/2)^{2n+1}/(2n+1)!} \\
&= \operatorname*{Res}_{z=\pi i/2} \frac{1}{z - \pi i/2} \frac{e^{iz}}{2i\sum_{n=0}^{\infty}(z - \pi i/2)^{2n}/(2n+1)!} \\
&= \left.\frac{e^{iz}}{2i\sum_{n=0}^{\infty}(z - \pi i/2)^{2n}/(2n+1)!}\right|_{z=\pi i/2} = \frac{e^{-\pi/2}}{2i}
\end{aligned}$$

Putting it all together,

$$(1+e^{-\pi})\int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{e^x+e^{-x}} = 2\pi i \operatorname*{Res}_{z=\pi i/2}\frac{e^{iz}}{e^z+e^{-z}} = \pi e^{-\pi/2}$$
$$\implies \int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{e^x+e^{-x}} = \frac{\pi e^{-\pi/2}}{(1+e^{-\pi})} = \frac{\pi}{\cosh(\pi/2)}$$

The real part of this identity is the result we were looking for —and the imaginary part of the integral on the left is zero.

Laplace transforms

12.9 Find the inverse Laplace transforms of the following functions:

(i) $F(s)=\dfrac{1}{3-5s}$ (ii) $F(s)=\dfrac{2s-2}{(s+1)(s^2+2s+5)}$

(iii) $F(s)=\dfrac{2s^3}{s^4-4}$ (iv) $F(s)=\dfrac{s^2-a^2}{(s^2+a^2)^2}$

Let $F(s)$ be a function that is holomorphic everywhere on the complex plane except for a finite collection of isolated singularities, z_k $(k=1,\dots,n)$. Recall that the inverse Laplace transform of $F(s)$ is defined as

$$\mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\ \text{P.V.}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds,$$

where γ is a any real number satisfying $\gamma > \operatorname{Re} z_k$ for all k. We calculate this by applying the Residue theorem to the contour in Figure 3. If we can prove that the integral over C_R vanishes as $R\to+\infty$, we can conclude that

$$\mathcal{L}^{-1}\{F(s)\}(t) = \sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}\left[e^{st}F(s)\right].$$

(i) The function given has just one (simple) pole at $s=3/5$, and so we fix $\gamma>3/5$. Parametrizing C_R as $s=\gamma+Re^{i\theta}$ for $\pi/2<\theta<3\pi/2$, we have

$$\left|\int_{C_R}\frac{e^{st}}{3-5s}\,ds\right| = \left|\int_{\pi/2}^{3\pi/2}\frac{e^{t(\gamma+Re^{i\theta})}}{3-5s}\,iRe^{i\theta}\,d\theta\right| \le \int_{\pi/2}^{3\pi/2}\frac{\left|e^{t(\gamma+Re^{i\theta})}\right|}{|3-5\gamma-5Re^{i\theta}|}\,R\,d\theta$$

As long as R is sufficiently big, we can bound the denominator from below by

$$\left|3-5\gamma-5Re^{i\theta}\right| \ge \left||3-5\gamma|-5R\right| = 5R-5\gamma+3$$

On the other hand, the numerator can be written as

$$\left|e^{t(\gamma+Re^{i\theta})}\right| = e^{\gamma t}e^{Rt\cos\theta}$$

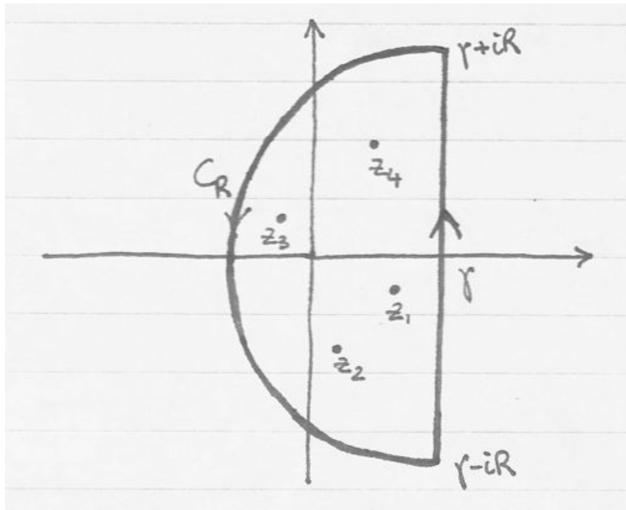


Figure 3: The contour for the inverse Laplace transform

With the change of variables $\theta = \psi + \pi/2$, Jordan's inequality yields

$$\left|\int_{C_R} \frac{e^{st}}{3-5s}\,ds\right| \le \int_{\pi/2}^{3\pi/2} \frac{Re^{\gamma t}e^{Rt\cos\theta}}{5R-5\gamma+3}\,d\theta = \frac{Re^{\gamma t}}{5R-5\gamma+3}\int_0^{\pi} e^{-Rt\sin\psi}\,d\psi < \frac{Re^{\gamma t}}{5R-5\gamma+3}\,\frac{\pi}{Rt}$$

The latter goes to zero as $R \to +\infty$, and so

$$\mathcal{L}^{-1}\{F(s)\}(t) = \operatorname*{Res}_{s=3/5} \frac{e^{st}}{3-5s} = -\frac{1}{5}\,e^{3t/5}$$

(ii) Since $s^2+2s+5 = \big[s-(-1+2i)\big]\big[s-(-1-2i)\big]$, the function $F(s)$ has three poles: $s=-1$, $s=-1+2i$ and $s=-1-2i$. We thus need to take $\gamma > -1$. For simplicity, me may choose $\gamma = 0$. As above, we parametrize C_R by $s = \gamma + Re^{i\theta} = Re^{i\theta}$ for $\pi/2 < \theta < 3\pi/2$, so that

$$\begin{aligned}
&\left|\int_{C_R} \frac{(2s-2)e^{st}}{(s+1)(s^2+2s+5)}\,ds\right| \\
&\quad= \left|\int_{\pi/2}^{3\pi/2} \frac{2(Re^{i\theta}-1)e^{tRe^{i\theta}}}{(Re^{i\theta}+1)\big[Re^{i\theta}-(-1+2i)\big]\big[Re^{i\theta}-(-1-2i)\big]}\,iRe^{i\theta}\,d\theta\right| \\
&\quad\le \int_{\pi/2}^{3\pi/2} \frac{2R\left|Re^{i\theta}-1\right|\left|e^{tRe^{i\theta}}\right|}{|Re^{i\theta}+1|\,|Re^{i\theta}+1-2i|\,|Re^{i\theta}+1+2i|}\,d\theta
\end{aligned}$$

In the numerator of this last integrand, we have

$$\left|Re^{i\theta}-1\right| \le R+1$$

$$\left|e^{tRe^{i\theta}}\right| = e^{Rt\cos\theta}$$

In the denominator, we bound

$$\left|Re^{i\theta}+1\right| \ge \left|R-1\right| = R-1$$

$$\left|Re^{i\theta}+1-2i\right| \geq \big|R-|1-2i|\big| = R-\sqrt{5}$$

$$\left|Re^{i\theta}+1+2i\right| \geq \big|R-|1+2i|\big| = R-\sqrt{5}$$

as long as R is big enough. Putting this all together, performing the change of variables $\theta = \psi + \pi/2$ and using Jordan's inequality, we have

$$\begin{aligned}
\left|\int_{C_R} \frac{(2s-2)e^{st}}{(s+1)(s^2+2s+5)}\,ds\right| &\leq \int_{\pi/2}^{3\pi/2} \frac{2R(R+1)e^{Rt\cos\theta}}{(R-1)(R-\sqrt{5})^2}\,d\theta \\
&= \frac{2R(R+1)}{(R-1)(R-\sqrt{5})^2}\int_0^{\pi} e^{-Rt\sin\psi}\,d\psi \\
&< \frac{2R(R+1)}{(R-1)(R-\sqrt{5})^2}\,\frac{\pi}{Rt}
\end{aligned}$$

Since this goes to zero in the limit, we conclude that

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \left(\operatorname*{Res}_{s=-1} + \operatorname*{Res}_{s=-1+2i} + \operatorname*{Res}_{s=-1-2i}\right) \frac{(2s-2)e^{st}}{(s+1)(s^2+2s+5)} \\
&= -e^{-t} + \frac{1-i}{2}\,e^{-t}e^{2it} + \frac{1+i}{2}\,e^{-t}e^{-2it} \\
&= e^{-t}\big[\cos 2t + \sin 2t - 1\big]
\end{aligned}$$

(iii) We have four poles, located at the points $s = \pm\sqrt{2}$ and $s = \pm\sqrt{2}i$, so we fix $\gamma > \sqrt{2}$. Calculating as above, we have

$$\begin{aligned}
\left|\int_{C_R} \frac{2s^3e^{st}}{s^4-4}\,ds\right| &= \left|\int_{\pi/2}^{3\pi/2} \frac{2(\gamma+Re^{i\theta})^3e^{t(\gamma+Re^{i\theta})}}{(\gamma+Re^{i\theta})^4-4}\,iRe^{i\theta}\,d\theta\right| \\
&\leq \int_{\pi/2}^{3\pi/2} \frac{2R\left|\gamma+Re^{i\theta}\right|^3\left|e^{t(\gamma+Re^{i\theta})}\right|}{|(\gamma+Re^{i\theta})^4-4|}\,d\theta
\end{aligned}$$

In the numerator, we write

$$\left|\gamma+Re^{i\theta}\right| \leq R+\gamma$$

$$\left|e^{t(\gamma+Re^{i\theta})}\right| = e^{\gamma t}e^{Rt\cos\theta}$$

For R sufficiently big, we have $\left|\gamma+Re^{i\theta}\right| \geq R-\gamma$ and

$$\left|(\gamma+Re^{i\theta})^4-4\right| \geq \left|\left|\gamma+Re^{i\theta}\right|^4-4\right| = \left|\gamma+Re^{i\theta}\right|^4-4 \geq (R-\gamma)^4-4$$

Once again we use this information, the change of variables $\theta = \psi+\pi/2$ and Jordan's inequality to obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{2s^3e^{st}}{s^4-4}\,ds\right| &\leq \int_{\pi/2}^{3\pi/2} \frac{2R(R+\gamma)^3e^{\gamma t}e^{Rt\cos\theta}}{(R-\gamma)^4-4}\,d\theta \\
&= \frac{2R(R+\gamma)^3e^{\gamma t}}{(R-\gamma)^4-4}\int_0^{\pi} e^{-Rt\sin\psi}\,d\psi \\
&< \frac{2R(R+\gamma)^3e^{\gamma t}}{(R-\gamma)^4-4}\,\frac{\pi}{Rt} \xrightarrow[R\to+\infty]{} 0
\end{aligned}$$

The residue computation is tedious but straightforward. It yields

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \left(\operatorname*{Res}_{s=\sqrt{2}} + \operatorname*{Res}_{s=-\sqrt{2}} + \operatorname*{Res}_{s=\sqrt{2}i} + \operatorname*{Res}_{s=-\sqrt{2}i}\right) \frac{2s^3 e^{st}}{s^4-4} \\
&= \frac{1}{2}\, e^{\sqrt{2}t} + \frac{1}{2}\, e^{-\sqrt{2}t} + \frac{1}{2}\, e^{i\sqrt{2}t} + \frac{1}{2}\, e^{-i\sqrt{2}t} = \cosh\sqrt{2}t + \cos\sqrt{2}t
\end{aligned}$$

(iv) $F(s)$ has double poles at $s = \pm ai$, so we need $\gamma > 0$. Then,

$$\begin{aligned}
\left|\int_{C_R} \frac{(s^2-a^2)e^{st}}{(s^2+a^2)^2}\right| &= \left|\int_{\pi/2}^{3\pi/2} \frac{(\gamma+Re^{i\theta}-a)(\gamma+Re^{i\theta}+a)e^{t(\gamma+Re^{i\theta})}}{(\gamma+Re^{i\theta}-ai)^2(\gamma+Re^{i\theta}+ai)^2}\, iRe^{i\theta}\, d\theta\right| \\
&\le \int_{\pi/2}^{3\pi/2} \frac{R\left|\gamma+Re^{i\theta}-a\right|\left|\gamma+Re^{i\theta}+a\right|\left|e^{t(\gamma+Re^{i\theta})}\right|}{\left|\gamma+Re^{i\theta}-ai\right|^2\left|\gamma+Re^{i\theta}+ai\right|^2}\, d\theta
\end{aligned}$$

For the bounds, we take

$$\left|\gamma+Re^{i\theta}-a\right| \le R+|\gamma-a|$$

$$\left|\gamma+Re^{i\theta}+a\right| \le R+|\gamma+a|$$

$$\left|e^{t(\gamma+Re^{i\theta})}\right| = e^{\gamma t}e^{Rt\cos\theta}$$

$$\left|\gamma+Re^{i\theta}-ai\right| \ge \big|R-|\gamma-ai|\big| = R-|\gamma-ai|$$

$$\left|\gamma+Re^{i\theta}+ai\right| \ge \big|R-|\gamma+ai|\big| = R-|\gamma+ai|$$

as long as R is big enough. With these,

$$\begin{aligned}
\left|\int_{C_R} \frac{(s^2-a^2)e^{st}}{(s^2+a^2)^2}\right| &\le \int_{\pi/2}^{3\pi/2} \frac{R(R+|\gamma-a|)(R+|\gamma+a|)e^{\gamma t}e^{Rt\cos\theta}}{(R-|\gamma-ai|)^2(R-|\gamma+ai|)^2}\, d\theta \\
&= \frac{R(R+|\gamma-a|)(R+|\gamma+a|)e^{\gamma t}}{(R-|\gamma-ai|)^2(R-|\gamma+ai|)^2}\int_0^{\pi} e^{-Rt\sin\psi}\, d\psi \\
&< \frac{R(R+|\gamma-a|)(R+|\gamma+a|)e^{\gamma t}}{(R-|\gamma-ai|)^2(R-|\gamma+ai|)^2}\,\frac{\pi}{Rt} \xrightarrow[R\to+\infty]{} 0
\end{aligned}$$

We can thus evaluate the inverse Laplace transform by calculating residues:

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \operatorname*{Res}_{s=ai} \frac{(s^2-a^2)e^{st}}{(s^2+a^2)^2} + \operatorname*{Res}_{s=-ai} \frac{(s^2-a^2)e^{st}}{(s^2+a^2)^2} \\
&= \frac{d}{dz}\, \frac{(s^2-a^2)e^{st}}{(s+ai)^2}\bigg|_{s=ai} + \frac{d}{dz}\, \frac{(s^2-a^2)e^{st}}{(s-ai)^2}\bigg|_{s=-ai} \\
&= \frac{1}{2}\, te^{iat} + \frac{1}{2}\, te^{-iat} = t\cos at
\end{aligned}$$

12.10 Using Laplace transforms, solve the following initial value problems:

(i) $y'' + y = \sin 4t,\ \ y(0) = 0,\ y'(0) = 1,$

(ii) $y'' + y' + 2y = e^{-t}\cos 2t,\ \ y(0) = 1,\ y'(0) = -1,$

Recall the Laplace transform of sines and cosines,

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}, \quad \mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2},$$

and the following basic properties,

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\}(s) = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{e^{ct}f(t)\}(s) = F(s-c)$$

where $F(s) = \mathcal{L}f(t)(s)$.

(i) Thanks to the formulas above and the linearity of the Laplace transform, we have

$$\begin{aligned}
\mathcal{L}\{y''(t)\}(s) + \mathcal{L}\{y(t)\}(s) &= \mathcal{L}\{\sin 4t\}(s) \\
s^2Y(s) - 1 + Y(s) &= \frac{4}{s^2+16} \\
Y(s) &= \frac{s^2+20}{(s^2+1)(s^2+16)}
\end{aligned}$$

Notice that $Y(s)$ has poles at $s = \pm i$ and $s = \pm 4i$. Hence, for $\gamma > 0$, we write

$$\begin{aligned}
\left|\int_{C_R} \frac{(s^2+20)e^{st}}{(s^2+1)(s^2+16)}\,ds\right| &= \left|\int_{\pi/2}^{3\pi/2} \frac{\left[(\gamma+Re^{i\theta})^2+20\right]e^{t(\gamma+Re^{i\theta})}}{\left[(\gamma+Re^{i\theta})^2+1\right]\left[(\gamma+Re^{i\theta})^2+16\right]}\, iRe^{i\theta}\,d\theta\right| \\
&\le \int_{\pi/2}^{3\pi/2} \frac{R\left|(\gamma+Re^{i\theta})^2+20\right|\left|e^{t(\gamma+Re^{i\theta})}\right|}{|(\gamma+Re^{i\theta})^2+1|\,|(\gamma+Re^{i\theta})^2+16|}\,d\theta
\end{aligned}$$

For large R, we calculate the following bounds:

$$\begin{aligned}
\left|(\gamma+Re^{i\theta})^2+20\right| &= \left|\gamma+Re^{i\theta}+i\sqrt{20}\right|\left|\gamma+Re^{i\theta}-i\sqrt{20}\right| \\
&\le \left(R+|\gamma+i\sqrt{20}|\right)\left(R+|\gamma-i\sqrt{20}|\right) \le \left(R+\gamma+\sqrt{20}\right)^2
\end{aligned}$$

$$\left|e^{t(\gamma+Re^{i\theta})}\right| = e^{\gamma t}e^{Rt\cos\theta}$$

$$\begin{aligned}
\left|(\gamma+Re^{i\theta})^2+1\right| &= \left|\gamma+Re^{i\theta}+i\right|\left|\gamma+Re^{i\theta}-i\right| \\
&\ge (R-|\gamma+i|)\,(R-|\gamma-i|) \ge (R-|\gamma-1|)^2
\end{aligned}$$

$$\begin{aligned}
\left|(\gamma+Re^{i\theta})^2+16\right| &= \left|\gamma+Re^{i\theta}+4i\right|\left|\gamma+Re^{i\theta}-4i\right| \\
&\ge (R-|\gamma+4i|)\,(R-|\gamma-4i|) \ge (R-|\gamma-4|)^2
\end{aligned}$$

Then,

$$\left|\int_{C_R} \frac{(s^2+20)e^{st}}{(s^2+1)(s^2+16)}\, ds\right| \le \int_{\pi/2}^{3\pi/2} \frac{R(R+\gamma+\sqrt{20})^2 e^{\gamma t} e^{Rt\cos\theta}}{(R-|\gamma-1|)^2(R-|\gamma-4|)^2}\, d\theta$$
$$= \frac{R(R+\gamma+\sqrt{20})^2 e^{\gamma t}}{(R-|\gamma-1|)^2(R-|\gamma-4|)^2} \int_0^{\pi} e^{-Rt\sin\psi}\, d\psi$$
$$< \frac{R(R+\gamma+\sqrt{20})^2 e^{\gamma t}}{(R-|\gamma-1|)^2(R-|\gamma-4|)^2}\, \frac{\pi}{Rt} \xrightarrow[R\to+\infty]{} 0$$

We can thus calculate the inverse Laplace transform by evaluating residues:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) &= \left(\operatorname*{Res}_{z=i} + \operatorname*{Res}_{z=-i} + \operatorname*{Res}_{z=4i} + \operatorname*{Res}_{z=-4i}\right) \frac{(s^2+20)e^{st}}{(s^2+1)(s^2+16)} \\ &= \frac{19}{30i}\, e^{it} - \frac{19}{30i}\, e^{-it} - \frac{1}{30i}\, e^{4it} + \frac{1}{30i}\, e^{-4it} \\ &= \frac{1}{15}\big[19\sin t - \sin 4t\big] \end{aligned}$$

(ii) [This problem is computationally heavy. Don't worry too much about the explicit calculation of the residues: I used a computer to do it!] The Laplace transform of the given initial value problem yields

$$\begin{aligned} \mathcal{L}\{y''(t)\}(s) + \mathcal{L}\{y'(t)\}(s) + 2\mathcal{L}\{y(t)\}(s) &= \mathcal{L}\{e^{-t}\cos 2t\}(s) \\ s^2Y(s) - s + 1 + sY(s) - 1 + 2Y(s) &= \frac{s+1}{(s+1)^2+4} \\ Y(s) &= \frac{s^3+2s^2+6s+1}{(s^2+2s+5)(s^2+s+2)} \end{aligned}$$

The poles of this function are located at

$$s^2+2s+5=0 \iff s=-1\pm 2i, \quad \text{and} \quad s^2+s+2=0 \iff s=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$$

We thus need $\gamma > -1/2$. It will be convenient for our calculations below to set $\gamma = 0$. Then,

$$\begin{aligned} &\left|\int_{C_R} \frac{(s^3+2s^2+6s+1)e^{st}}{(s^2+2s+5)(s^2+s+2)}\, ds\right| \\ &\quad = \left|\int_{\pi/2}^{3\pi/2} \frac{\left(R^3e^{3i\theta}+2R^2e^{2i\theta}+6Re^{i\theta}+1\right) e^{tRe^{i\theta}}}{\left(R^2e^{2i\theta}+2Re^{i\theta}+5\right)\left(R^2e^{2i\theta}+Re^{i\theta}+2\right)}\, iRe^{i\theta}\, d\theta\right| \\ &\quad \le \int_{\pi/2}^{3\pi/2} \frac{R\left|R^3e^{3i\theta}+2R^2e^{2i\theta}+6Re^{i\theta}+1\right| \left|e^{tRe^{i\theta}}\right|}{\left|R^2e^{2i\theta}+2Re^{i\theta}+5\right| \left|R^2e^{2i\theta}+Re^{i\theta}+2\right|}\, d\theta \end{aligned}$$

Although this integral might seem a bit daunting, we can work out each piece simply:

$$\left|R^3e^{3i\theta}+2R^2e^{2i\theta}+6Re^{i\theta}+1\right| \le R^3+2R^2+6R+1$$

$$\left|e^{tRe^{i\theta}}\right| = e^{Rt\cos\theta}$$

$$\left|R^2e^{2i\theta}+2Re^{i\theta}+5\right| = |R+1+2i|\,|R+1-2i| \geq \left(R-\sqrt{5}\right)^2$$

$$\left|R^2e^{2i\theta}+Re^{i\theta}+2\right| = \left|R+\frac{1}{2}+\frac{\sqrt{7}}{2}i\right|\left|R+\frac{1}{2}-\frac{\sqrt{7}}{2}i\right| \geq \left(R-\sqrt{2}\right)^2$$

Once again, we have assumed above that R is sufficiently big.

$$\begin{aligned}
\left|\int_{C_R}\frac{(s^3+2s^2+6s+1)e^{st}}{(s^2+2s+5)(s^2+s+2)}\,ds\right| &\leq \int_{\pi/2}^{3\pi/2}\frac{R(R^3+2R^2+6R+1)e^{Rt\cos\theta}}{\left(R-\sqrt{5}\right)^2\left(R-\sqrt{2}\right)^2}\,d\theta \\
&= \frac{R(R^3+2R^2+6R+1)}{\left(R-\sqrt{5}\right)^2\left(R-\sqrt{2}\right)^2}\int_0^{\pi}e^{-Rt\sin\psi}\,d\psi \\
&< \frac{R(R^3+2R^2+6R+1)}{\left(R-\sqrt{5}\right)^2\left(R-\sqrt{2}\right)^2}\,\frac{\pi}{Rt} \xrightarrow[R\to+\infty]{} 0
\end{aligned}$$

The residue calculation yields

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\}(t) \\
&= \left(\operatorname*{Res}_{z=-1+2i}+\operatorname*{Res}_{z=-1-2i}+\operatorname*{Res}_{z=(-1+\sqrt{7}i)/2}+\operatorname*{Res}_{z=(-1-\sqrt{7}i)/2}\right)\frac{(s^3+2s^2+6s+1)e^{st}}{(s^2+2s+5)(s^2+s+2)} \\
&= \frac{-1+i}{8}\,e^{(-1+2i)t}-\frac{1+i}{8}\,e^{(-1-2i)t}+\frac{3\sqrt{7}+5i}{5\sqrt{7}+7i}\,e^{(-1+\sqrt{7}i)t/2}+\frac{-3\sqrt{7}+5i}{-5\sqrt{7}-7i}\,e^{(-1-\sqrt{7}i)t/2} \\
&= -\frac{1}{4}\,e^{-t}\big[\cos 2t+\sin 2t\big]+\frac{1}{4}e^{-t/2}\left[5\cos\frac{\sqrt{7}}{2}t-\frac{\sqrt{7}}{7}\sin\frac{\sqrt{7}}{2}t\right]
\end{aligned}$$