MAT389 Fall 2013, Problem Set 4

Wirtinger derivatives

4.1 Use the Cauchy-Riemann equations as expressed using the Wirtinger operator $\partial/\partial\bar{z}$ to find out where each of the functions below is differentiable. Find the corresponding derivatives using $\partial/\partial z$.

(i) $f(z) = (z^3 - 1)\bar{z}$, (ii) $f(z) = (\bar{z}^3 - 1)z$, (iii) $f(z) = e^{\bar{z}}(\bar{z} - iz)$,

(iv) $f(z) = \dfrac{az + b}{c\bar{z} + d}$, where $a, b, c, d \in \mathbb{C}$ and $\bar{a}d - \bar{b}c \neq 0$.

Holomorphic functions

4.2 For each of the functions below, determine the largest domain over which they are holomorphic.

(i) $f(z) = \dfrac{e^{iz}}{z^2 - 2z + 1}$, (ii) $f(z) = \log|z| + i\operatorname{Arg} z$, (iii) $f(z) = (z^3 - 1)\bar{z}$.

4.3 Prove that the composition of two entire functions is again an entire function.

4.4 Check that the functions below are entire. Can you write them in terms of z in some simple form?

(i) $f(z) = 3x + y + i(3y - x)$, (ii) $f(z) = \sin x \cosh y + i \cos x \sinh y$.

Hint: for (ii), notice that

$$\cosh y = \frac{e^y + e^{-y}}{2} = \frac{e^{-i(iy)} + e^{i(iy)}}{2} = \cos(iy)$$

Can you find a similar identity for $\sinh y$?

Harmonic functions

4.5 Check that each of the functions $u(x, y)$ below is harmonic at every $(x, y) \in \mathbb{R}^2$, and find the unique harmonic conjugate, $v(x, y)$, satisfying $v(0, 0) = v_0$. Express the resulting holomorphic (entire, in fact) functions, $f(z) = u(x, y) + iv(x, y)$, in terms of z.

(i) $u(x, y) = ax + by + c$ (where $a, b, c \in \mathbb{R}$) and $v_0 = 0$,
(ii) $u(x, y) = x^2 - y^2 - 2x$ and $v_0 = 1$,
(iii) $u(x, y) = y^3 - 3x^2y$ and $v_0 = 0$,
(iv) $u(x, y) = x^4 - 6x^2y^2 + y^4$ and $v_0 = 0$,
(v) $u(x, y) = e^{2y}\cos 2x$ and $v_0 = 1$.