MAT389 Fall 2013, Problem Set 8

Integrals of complex-valued functions of a real variable

8.1 In first-year calculus courses, integrals of the form

$$\int_a^b e^{\alpha x}\cos\beta x\,dx, \qquad \int_a^b e^{\alpha x}\sin\beta x\,dx$$

are typically computed by applying integration by parts twice. Notice that they constitute the real and imaginary parts of the integral

$$\int_a^b e^{(\alpha+i\beta)x}\,dx$$

Find expressions for the former by calculating the latter.
Hint: notice that the complex-valued function of the real variable $e^{(\alpha+i\beta)x}$ possesses an antiderivative on the interval $[a,b]$ (in fact, on the whole real line).

An antiderivative of $e^{(\alpha+i\beta)x}$ on the interval $[a,b]$ is $(\alpha+i\beta)^{-1}e^{(\alpha+i\beta)x}$. Thus

$$\int_a^b e^{(\alpha+i\beta)x}\,dx = \left.\frac{e^{(\alpha+i\beta)x}}{\alpha+i\beta}\right|_a^b = \frac{e^{(\alpha+i\beta)b}-e^{(\alpha+i\beta)a}}{\alpha+i\beta}$$

Taking the real and imaginary part in both sides of the equality, we get

$$\int_a^b e^{\alpha x}\cos\beta x\,dx = \frac{\alpha(e^{\alpha b}\cos\beta b - e^{\alpha a}\cos\beta a)+\beta(e^{\alpha b}\sin\beta b - e^{\alpha a}\sin\beta a)}{\alpha^2+\beta^2}$$

$$\int_a^b e^{\alpha x}\sin\beta x\,dx = \frac{-\beta(e^{\alpha b}\cos\beta b - e^{\alpha a}\cos\beta a)+\alpha(e^{\alpha b}\sin\beta b - e^{\alpha a}\sin\beta a)}{\alpha^2+\beta^2}$$

Complex integration

8.2 For each of the cases below, compute the integral

$$\int_C f(z)\,dz.$$

(i) C is the semicircle $z=2e^{i\theta}$, $0\le\theta\le\pi$, and $f(z)=(z+2)/z$.
(ii) C is the boundary of the square with vertices at the points 0, 1, $1+i$ and i, taken counterclockwise, and $f(z)=e^{\pi\bar{z}}$.
(iii) C is the unit circle centered at the origin, taken counterclockwise, and $f(z)$ is the principal branch of the multivalued function z^{-1+i}.
(iv) C is the unit circle centered at the origin, taken counterclockwise, and $f(z)=z^n\bar{z}^m$, with $n,m\in\mathbb{Z}$.

(i)

$$\int_C \frac{z+2}{z}\,dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\,2ie^{i\theta}\,d\theta = 2i\int_0^{\pi}(e^{i\theta}+1)\,d\theta = -4+2i\pi.$$

(ii) We can parametrize the four sides of the square given by the arcs

$$\begin{aligned}
\gamma_1(t) &= t, && 0 \le t \le 1\\
\gamma_2(t) &= 1+it, && 0 \le t \le 1\\
\gamma_3(t) &= (1-t)+i, && 0 \le t \le 1\\
\gamma_4(t) &= (1-t)i, && 0 \le t \le 1
\end{aligned}$$

With this choice, we have

$$\begin{aligned}
\oint_C e^{\pi\bar z}\,dz &= \int_{\gamma_1} e^{\pi\bar z}\,dz + \int_{\gamma_2} e^{\pi\bar z}\,dz + \int_{\gamma_3} e^{\pi\bar z}\,dz + \int_{\gamma_4} e^{\pi\bar z}\,dz\\
&= \int_0^1 e^{\pi t}\,dt + i\int_0^1 e^{\pi(1-it)}\,dt - \int_0^1 e^{\pi(1-t-i)}\,dt - i\int_0^1 e^{-\pi(1-t)i}\,dt\\
&= \frac{1}{\pi}\left(e^{\pi t} - e^{\pi(1-it)} + e^{\pi(1-t-i)} - e^{-\pi(1-t)i}\right)\Big|_0^1 = \frac{4(e^{\pi}-1)}{\pi}
\end{aligned}$$

(iii) Remember that the principal branch of z^{-1+i} is defined by $e^{(-1+i)\operatorname{Log} z} = e^{(-1+i)(\operatorname{Log}|z|+i\operatorname{Arg} z)}$. Then,

$$\oint_C z^{-1+i}\,dz = \int_{-\pi}^{\pi} e^{(-1+i)i\theta} ie^{i\theta}\,d\theta = \int_{-\pi}^{\pi} ie^{-\theta}\,d\theta = -ie^{-\theta}\Big|_{-\pi}^{\pi} = -i(e^{-\pi}-e^{\pi}) = 2i\sinh\pi$$

(iv) Using the usual parametrization of the unit circle, we obtain

$$\oint_C z^n\bar z^m\,dz = \int_0^{2\pi} e^{in\theta}e^{-im\theta}ie^{i\theta}\,d\theta = \int_0^{2\pi} ie^{(n-m+1)i\theta}\,d\theta = \begin{cases}0 & \text{if } n-m+1\neq 0\\ 2\pi i & \text{if } n-m+1=0\end{cases}$$

There is a cute little trick to reduce this result to a family integrals we have computed in class. Notice that, on the unit circle, the complex conjugate of a number coincides with its inverse. Hence,

$$\oint_C z^n\bar z^m\,dz = \oint_C z^{n-m}\,dz = \begin{cases}0 & \text{if } n-m\neq -1\\ 2\pi i & \text{if } n-m=-1\end{cases}$$

8.3 Let C be a simple closed contour, oriented counterclockwise, and R the region enclosed by it. Show that

$$\operatorname{area}(R) = \frac{1}{2i}\oint_C \bar z\,dz.$$

Hint: use Green's theorem.

The complex integral in the statement can be written as

$$\frac{1}{2i}\oint_C \bar z\,dz = \frac{1}{2i}\oint_C (x\,dx + y\,dy) + \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

By Green's theorem, the right hand side of the above equation equals

$$\frac{1}{2i}\iint_R \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) dxdy + \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) dxdy = 0 + \iint_R dxdy = \text{area}(R),$$

as claimed.

8.4 Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2 - 1}\right| \le \frac{\pi}{3}$$

The triangle inequality and the fact that $|z| = 2$ on C give

$$|z^2 - 1| \ge ||z|^2 - 1| = |4 - 1| = 3.$$

Hence the integrand is bound by $1/3$ on C. On the other hand, the arc-length of C is π. Thus,

$$\left|\int_C \frac{dz}{z^2 - 1}\right| \le \frac{\pi}{3}$$

8.5 Let C_R be the circle of radius $R > 1$ centered at the origin, oriented counterclockwise. Show that

$$\left|\oint_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| < 2\pi\,\frac{\pi + \operatorname{Log} R}{R}$$

and hence that the value of this integral approaches zero as R tends to infinity.

For $z \in C_R$,

$$\left|\frac{\operatorname{Log} z}{z^2}\right| = \left|\frac{\operatorname{Log}|z| + i\operatorname{Arg} z}{z^2}\right| \le \frac{\operatorname{Log}|z| + |\operatorname{Arg} z|}{|z|^2} \le \frac{\operatorname{Log} R + \pi}{R^2}$$

Since the arc-length of C_R is $2\pi R$, we have

$$\left|\oint_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| \le 2\pi\,\frac{\pi + \operatorname{Log} R}{R}$$

As $R \to \infty$, the latter limits to zero, i.e.,

$$\lim_{R\to\infty} \oint_{C_R} \frac{\operatorname{Log} z}{z^2} = 0.$$

Cauchy integral formulas

8.6 For each of the cases below, compute the integral

$$\oint_C f(z)\,dz,$$

where C denotes the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$, oriented counterclockwise.

(i) $f(z) = e^z/(z - \pi i/2)$,
(ii) $f(z) = \cos z/[z(z^2 + 8)]$,
(iii) $f(z) = z/(2z + 1)$,
(iv) $f(z) = \tan(z/2)/(z - x_0)^2$ where $-2 < x_0 < 2$, and
(v) $f(z) = z/(2z + 1)^3$,
(vi) $f(z) = \cosh z/z^4$.

(i) The point $z = \pi i/2$ is in the interior of the square C. The Cauchy integral formula then yields

$$\oint_C \frac{e^z\,dz}{(z - \pi i/2)} = 2\pi i e^{\pi i/2} = -2\pi.$$

(ii) The singularities of $f(z) = \cos z/[z(z^2 + 8)]$ are located at $z = 0$ and $z = \pm\sqrt{8}$. Only the first one is inside of C, and so we have

$$\oint_C \frac{\cos z\,dz}{z(z^2 + 8)} = \oint_C \frac{(\cos z)/(z^2 + 8)}{z}\,dz = 2\pi i\,\frac{\cos 0}{0^2 + 8} = \frac{\pi i}{4}$$

(iii)

$$\oint_C \frac{z\,dz}{2z + 1} = \oint_C \frac{z/2}{z + 1/2}\,dz = 2\pi i \frac{-1/2}{2} = -\frac{\pi i}{2}$$

(iv) The function $\tan(z/2)$ is holomorphic except when $z = (2k+1)\pi/2$, $k \in \mathbb{Z}$. Notice that none of these point lie on or inside C. Applying the Cauchy integral for the first derivative, we obtain

$$\oint_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz = 2\pi i\,\frac{d}{dz}\tan\frac{z}{2}\bigg|_{z=x_0} = \pi i \sec^2\frac{x_0}{2}$$

(v) Here we use the Cuchy integral formula for the second derivative:

$$\oint_C \frac{z\,dz}{(2z + 1)^3} = \oint_C \frac{z/8}{(z + 1/2)^3}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\frac{z}{8}\bigg|_{z=-1/2} = 0.$$

(vi) The Cauchy integral formula for the third derivative applies in this case:

$$\oint_C \frac{\cosh z}{z^4}\,dz = \frac{2\pi i}{3!}\,\frac{d^3}{dz^3}(\cosh z)\bigg|_{z=0} = \frac{\pi i}{3}\sinh 0 = 0.$$

8.7 Show that if f is holomorphic on and inside of a simple closed contour C, and z_0 is not on C, then

$$\oint_C \frac{f'(z)}{z - z_0}\, dz = \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

Warning: note that the statement says that z_0 is not *on* C, not that it is inside of C.

If the function f is holomorphic on and inside of C, so is its derivative f'. Now, if the point z_0 is outside of the contour C, both of the integrands above are are holomorphic on and inside of C. The Cauchy–Goursat theorem then ensures that both integrals are zero.

On the other hand, when z_0 is an interior point of C, we can apply the appropriate Cauchy integral formulas to each sides. The left hand side gives

$$\oint_C \frac{f'(z)}{z - z_0}\, dz = 2\pi i f'(z_0),$$

while the right hand side is

$$\oint_C \frac{f(z)}{(z - z_0)^2}\, dz = 2\pi i f'(z_0).$$

8.8 (i) Use the binomial formula to show that, for any $n \in \mathbb{Z}$, $n \geq 0$, the function

$$P_n(z) = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n} (z^2 - 1)^n$$

is a polynomial of degree n, .

(ii) Let C be any simple closed contour surrounding a fixed point z. Use the Cauchy integral formula for the nth derivative of a holomorphic function to show that

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}}\, ds.$$

(iii) Use the Cauchy integral formula to conclude that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

Note: these polynomials receive the name of *Legendre polynomials*, and they satisfy *Legendre's differential equation*:

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx} P_n(x)\right] + n(n+1)P_n(x) = 0.$$

The latter appears when solving the (three-dimensional!) Laplace equation in spherical coordinates.

(i) An explicit calculation proves the desired statement:

$$\begin{aligned} P_n(z) &= \frac{1}{n!\, 2^n} \frac{d^n}{dz^n} (z^2 - 1)^n = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k z^{2(n-k)} \\ &= \frac{1}{n!\, 2^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^n}{dz^n} z^{2(n-k)} = \frac{1}{n!\, 2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (-1)^k \frac{(2n-2k)!}{(n-2k)!} z^{n-2k} \end{aligned}$$

(ii) Applying the Cauchy integral formula for the nth derivative to the function $f(z) = (z^2 - 1)^n$ yields

$$\oint_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds = \frac{2\pi i}{n!}\frac{d^n}{dz^n}(z^2-1)^n = 2^{n+1}\pi i P_n(z).$$

(iii) For $z = 1$, a factor of $(s-1)^n$ cancels out between the numerator and the denominator:

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(s^2-1)^n}{(s-1)^{n+1}}\,ds = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(s+1)^n}{s-1}\,ds = \frac{2\pi i}{2^{n+1}\pi i}(1+1)^n = 1.$$

In the case $z = -1$, the common factor of numerator and denominator is $(s+1)^n$:

$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(s^2-1)^n}{(s+1)^{n+1}}\,ds = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(s-1)^n}{s+1}\,ds = \frac{2\pi i}{2^{n+1}\pi i}(-1-1)^n = (-1)^n.$$

8.9 Let C be a simple closed contour oriented counterclockwise, and f a function that is holomorphic on and inside of C. Provide the details for the derivation of the Cauchy integral formula for the second derivative following these steps:

1. Apply the Cauchy integral formula for f' to show that

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\,f(s)\,ds$$

when z is a point inside of C, and $0 < \Delta z < d$, where d is the minimum distance from z to points on C.

2. Use the continuity of f on C to show that the value of the integral

$$\oint_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{2}{(s-z)^3}\right] f(s)\,ds$$

approaches zero as Δz goes to zero.

3. Conclude that

$$f''(z) = \frac{1}{\pi i}\oint_C \frac{f(s)}{(s-z)^3}\,ds.$$

Hint: in the simplications in step 2, retain the difference $s - z$ as a single term. Also, let D be the *maximum* distance from z to points on C.

1. With the given hypothesis, both z and $z+\Delta z$ are interior to the contour C. We can thus apply the Cauchy integral formula for f' twice and combine the resulting denominators.

$$\begin{aligned}\frac{f'(z+\Delta z) - f'(z)}{\Delta z} &= \frac{1}{2\pi i\,\Delta z}\oint_C \left[\frac{f(s)}{(s-z-\Delta z)^2} - \frac{f(s)}{(s-z)^2}\right] ds \\ &= \frac{1}{2\pi i\,\Delta z}\oint_C \left[\frac{(s-z)^2 - (s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right] f(s)\,ds \\ &= \frac{1}{2\pi i}\oint_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\,f(s)\,ds\end{aligned}$$

2. Writing the two fractions in the statement under a common denominator we have

$$\oint_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{1}{(s-z)^3}\right] f(s)\,ds = \oint_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\, f(s)\,ds$$

To obtain an upper bound for the modulus of the denominator in this last integral, let D be the maximum distance from z to points on C (as suggested in the hint). Then,

$$|3(s-z)\Delta z - 2(\Delta z)^2| \leq 3|s-z||\Delta z| + 2|\Delta z|^2 \leq 3D|\Delta z| + 2|\Delta z|^2.$$

Since $|s-z| \geq d > |\Delta z|$, we have

$$|s-z-\Delta z| \geq \Big||s-z| - |\Delta z|\Big| = |s-z| - |\Delta z| \geq d - |\Delta z|,$$

Let M be an upper bound for $|f(s)|$ for $s \in C$. Denoting by L the arc-length of C, and putting it all together, we have

$$\left|\oint_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\, f(s)\,ds\right| \leq \frac{3D|\Delta z| + 2|\Delta z|^2}{(d-|\Delta z|)^2 d^3} ML$$

Notice that this last expression can be written as Δz times a factor that has a finite limit as Δz goes to zero. Hence,

$$\lim_{\Delta z \to 0} \oint_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{2}{(s-z)^3}\right] f(s)\,ds = 0.$$

3. Write

$$\begin{aligned}
&\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\oint_C \frac{f(s)}{(s-z)^3}\,ds \\
&\qquad = \frac{1}{2\pi i}\oint_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\, f(s)\,ds - \frac{1}{\pi i}\oint_C \frac{f(s)}{(s-z)^3}\,ds \\
&\qquad = \frac{1}{2\pi i}\oint_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{2}{(s-z)^3}\right] f(s)\,ds
\end{aligned}$$

Taking the limit as $\Delta z \to 0$ finishes the proof:

$$\begin{aligned}
f''(z) - \frac{1}{\pi i}\oint_C \frac{f(s)}{(s-z)^3}\,ds &= \lim_{\Delta z \to 0}\left[\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\oint_C \frac{f(s)}{(s-z)^3}\,ds\right] \\
&= \lim_{\Delta z \to 0}\oint_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{2}{(s-z)^3}\right] f(s)\,ds = 0.
\end{aligned}$$

8.10 Let C be a simple closed contour, and f a continuous function defined on C. Prove that the function defined by the formula

$$g(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z}\, ds$$

is holomorphic at all points z interior to C by showing that its derivative is given by

$$g'(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)^2}\, ds.$$

Hint: simply follow the steps in the proof of the Cauchy integral formula for the first derivative of a holomorphic function.

Note: suppose that the function f is defined not only on C, but also at every point interior to C. If f is not holomorphic, it may happen that $g(z) \neq f(z)$. For example, choose C to be the unit circle, oriented counterclockwise, and

$$f(z) = \begin{cases} 1 & z = 0 \\ \left[1 + \exp\left(\frac{1}{1-|z|^2} - \frac{1}{|z|^2}\right)\right]^{-1} & 0 < |z| < 1 \\ 0 & |z| \geq 1 \end{cases}$$

Horrible as it may look, this function is continuous (and even complex-differentiable) on C. But

$$g(0) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s}\, ds = 0 \neq 1 = f(0).$$

Let z be a point interior to C, and denote by d the minimum distance from z to points on C. For any Δz such that $0 < |\Delta z| < d$, write

$$\begin{aligned}
\frac{g(z+\Delta z) - g(z)}{\Delta z} - \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)^2}\, ds &= \frac{1}{2\pi i\, \Delta z} \oint_C \left[\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right] f(s) - \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)^2}\, ds \\
&= \frac{1}{2\pi i} \oint_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right] f(s)\, ds \\
&= \frac{1}{2\pi i} \oint_C \frac{\Delta z}{(s-z-\Delta z)(s-z)^2} f(s)\, ds.
\end{aligned}$$

We claim that the last integral limits to zero as $\Delta z \to 0$. Indeed, we have

$$|s - z - \Delta z| \geq \Big||s-z| - |\Delta z|\Big| = |s-z| - |\Delta z| \geq d - |\Delta z|,$$

for $|s - z| \geq d > |\Delta z|$. Because f is continuous on C, it achieves a maximum —that is, there exists an $M \geq 0$ such that $|f(s)| \leq M$ for all $s \in C$. Denoting by L the arc-length of C, we have

$$\left|\oint_C \frac{\Delta z}{(s-z-\Delta z)(s-z)^2} f(s)\, ds\right| \leq \frac{|\Delta z|}{(d-|\Delta z|)d^2} ML$$

As $\Delta z \to 0$, this last expression also tends to zero. Hence,

$$g'(z) - \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z)^2}\,ds = \lim_{\Delta z \to 0}\left[\frac{g(z+\Delta z) - g(z)}{\Delta z} - \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z)^2}\,ds\right]$$
$$= \lim_{\Delta z \to 0}\oint_C \frac{\Delta z}{(s-z-\Delta z)(s-z)^2}\,f(s)\,ds = 0$$

Thus the derivative of g exists at every point z interior to C, i.e., g is holomorphic inside of C.