MAT389 Fall 2013, Problem Set 3

Functions

3.1 For each of the functions defined below, describe the domain of definition that is understood:

(i) $f(z) = \dfrac{1}{z^2+1}$, (ii) $f(z) = \operatorname{Arg}\dfrac{1}{z}$, (iii) $f(z) = \dfrac{z}{z+\bar{z}}$, (iv) $f(z) = \dfrac{1}{1-|z|^2}$.

Limits

3.2 Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, and let z_0 be an interior point of Ω. Prove that the limit $\lim_{z\to z_0} f(z)$ is unique (if it exists).
Hint: write out what it means for $\lim_{z\to z_0} f(z) = \alpha$ and $\lim_{z\to z_0} f(z) = \beta$. Using this information, what can you say about $|\alpha - \beta|$?

3.3 The following statement is false:

$$\lim_{z\to z_0} f(z) = \infty \Longrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) \text{ exist}$$

Can you provide a counterexample?

Derivatives

3.4 Apply the definition of derivative to prove that $f(z) = \operatorname{Re} z$ is nowhere differentiable.

3.5 Where is $f(z) = |z|^2$ differentiable?

3.6 Let f denote the function whose values are $f(0) = 0$ and

$$f(z) = \frac{(\bar{z})^2}{z}, \qquad \text{for } z \neq 0$$

Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but that the derivative of f fails to exist there.
Hint: to prove that the derivative does not exist, calculate the limit in the definition when approaching $z = 0$ horizontally ($\Delta y = 0$), and along the line $y = x$ ($\Delta x = \Delta y$).

3.7 Suppose $u(x,y)$ and $v(x,y)$ have first-order partial derivatives with respect to x and y at some point $z_0 = (x_0, y_0) \neq (0,0)$.

(i) Use the change of coordinates $x = r\cos\theta, y = r\sin\theta$ and the chain rule to show that

$$\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta, & u_\theta &= -u_x r\sin\theta + u_y r\cos\theta, \\ v_r &= v_x \cos\theta + v_y \sin\theta, & v_\theta &= -v_x r\sin\theta + v_y r\cos\theta. \end{aligned} \tag{$*$}$$

at the point $z = z_0$.

(ii) Using these identities and the Cauchy-Riemann equations in rectangular coordinates, deduce the *polar form of the Cauchy-Riemann equations*:

$$u_r = \frac{1}{r}\,v_\theta, \qquad \frac{1}{r}\,u_\theta = -v_r$$

(iii) Solve the equations $(*)$ for u_x, u_y, v_x and v_y to show that

$$\begin{aligned} u_x &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, & u_y &= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}, \\ v_x &= v_r \cos\theta - v_\theta \frac{\sin\theta}{r}, & v_y &= v_r \sin\theta + v_\theta \frac{\cos\theta}{r}. \end{aligned} \tag{$**$}$$

(iv) Use these identities to deduce the Cauchy-Riemann equations in rectangular coordinates from their polar form.

(v) Prove that the derivative of $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ at $z = z_0$ can be expressed in any of the following two forms:

$$f'(z_0) = (\cos\theta_0 - i\sin\theta_0)\Big[u_r(r_0,\theta_0) + iv_r(r_0,\theta_0)\Big] = \frac{-i}{z_0}\Big[u_\theta(r_0,\theta_0) + iv_\theta(r_0,\theta_0)\Big]$$