MAT389 Fall 2013, Problem Set 6

The exponential

6.1 Suppose that a function $f(z) = u(x, y) + iv(x, y)$ satisfies the following two conditions:

(1) $f(x + i0) = e^x$, and

(2) f is entire, with derivative $f'(z) = f(z)$.

Follow the steps below to show that $f(z)$ must be the function

$$f(z) = e^x(\cos\theta + i\sin\theta)$$

(1) Obtain the equations $u_x = u$ and $v_x = v$ and then use them to show that there exist real-valued functions ϕ and ψ of the real variable y such that

$$u(x, y) = e^x\phi(y), \quad \text{and} \quad v(x, y) = e^x\psi(y).$$

(2) Use the fact that u is harmonic to obtain the differential equation $\phi''(y) + \phi(y) = 0$ and thus show that $\phi(y) = A\cos y + B\sin y$, where A and B are real numbers.

(3) After pointing out why $\psi(y) = A\sin y - B\cos y$ and noting that

$$u(x, 0) + iv(x, 0) = e^x,$$

find A and B. Conclude that

$$u(x, y) = e^x\cos y, \quad \text{and} \quad v(x, y) = e^x\sin y.$$

(1) If the function f is entire, then $f'(z) = u_x + iv_x$ at every point z. But $f'(z) = f(z) = u + iv$, and thus $u_x = u$ and $v_x = v$. Note that

$$(ue^{-x})_x = u_x e^{-x} - ue^{-x} = ue^{-x} - ue^{-x} = 0,$$

which implies that $u(x, y)e^{-x}$ is independent of x. Hence $u(x, y) = e^x\phi(y)$ for some real-valued function $\phi(y)$. Likewise, $v(x, y) = e^x\psi(y)$ for some real-valued function ψ.

(2) Since the function u is harmonic, we have

$$u_{xx} + u_{yy} = e^x\phi(y) + e^x\phi''(y) = 0 \quad \Longrightarrow \quad \phi''(y) + \phi(y) = 0.$$

The general solution of this equation is $\phi(y) = A\cos y + B\sin y$, with A and B real numbers. The same argument applied to v yields $\psi(y) = C\cos y + D\sin y$ with C and D real.

(3) The Cauchy-Riemann equations imply that

$$e^x(A\cos y + B\sin y) = u_x = v_y = e^x(-C\sin y + D\cos y),$$

$$\Longrightarrow \quad A\cos y + B\sin y = D\cos y - C\sin y.$$

As the functions $\cos y$ and $\sin y$ are linearly independent, we have that $A = D$ and $B = -C$, and $\psi(y) = A\sin y - B\cos y$.

On the other hand $u(x, 0) + iv(x, 0) = e^x$, so

$$u(x, 0) = e^x \phi(0) = e^x (A \cos 0 + B \sin 0) = Ae^x = e^x$$

and

$$v(x, 0) = e^x \psi(0) = e^x (A \sin 0 - B \cos 0) = -Be^x = 0,$$

which imply that $A = 1$ and $B = 0$; therefore $u(x, y) = e^x \cos x$ and $v(x, y) = e^x \sin x$.

6.2 If e^z is purely imaginary, what restriction is placed on z?

If $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$ is purely imaginary, then $e^x \cos y = 0$. Since $e^x \neq 0$, it has to be that $\cos y = 0$. Hence the complex numbers for which e^z is purely imaginary are those of the form

$$z = x + \frac{(2k+1)\pi i}{2}, \quad x \in \mathbb{R}, k \in \mathbb{Z}.$$

6.3 Describe the behavior of e^z as
(1) $x \to -\infty$, with y fixed; and
(2) $y \to +\infty$, with x fixed.

(1) When y is fixed, the limit $\lim_{x \to -\infty} e^z$ exists and is equal to zero:

$$\lim_{x \to -\infty} e^z = \lim_{x \to -\infty} e^x e^{iy} = e^{iy} \lim_{x \to -\infty} e^x = 0.$$

(2) But when x is fixed, the limit $\lim_{y \to \infty} e^z$ does not exist. For example, for the sequence of numbers $y_n = 2n\pi$ with $n \in \mathbb{Z}$, we have

$$\lim_{n \to \infty} e^{x+y_n} = e^x \lim_{n \to \infty} e^{2n\pi i} = e^x$$

but for the sequence $y_n' = (2n+1)\pi$ with $n \in \mathbb{N}$,

$$\lim_{n \to \infty} e^{x+y_n'} = e^x \lim_{n \to \infty} e^{(2n+1)\pi i} = -e^x.$$

In fact, the image of the line $x = x_0$ is the circle of radius e^{x_0}, and $f(x_0 + iy)$ winds around it once every $2\pi i$.

Trigonometric and hyperbolic functions

6.4 Show that $e^{iz} = \cos z + i\sin z$ for every complex number z.

Recall that for any complex number z we define

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z := \frac{e^{iz} + e^{-iz}}{2}.$$

Thus,

$$\cos z + i\sin z = \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2} = e^{iz}.$$

6.5 Use the identities

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \qquad \cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

and the relationship between trigonometric and hyperbolic functions,

$$-i\sinh(iz) = \sin z, \quad -i\sin(iz) = \sinh z, \quad \cosh(iz) = i\cos z, \quad \cos(iz) = \cosh(z)$$

to deduce expressions for $\sinh(z_1 \pm z_2)$ and $\cosh(z_1 \pm z_2)$.

For the hyperbolic sine, we have

$$\begin{aligned}
\sinh(z_1 \pm z_2) &= -i\sin\big[i(z_1 \pm z_2)\big] = -i\big[\sin(iz_1)\cos(iz_2) \pm \cos(iz_1)\sin(iz_2)\big] \\
&= -i\big[i\sinh z_1 \cosh z_2 \pm i\cosh z_1 \sinh z_2\big] \\
&= \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2.
\end{aligned}$$

For the hyperbolic cosine,

$$\begin{aligned}
\cosh(z_1 \pm z_2) &= \cos\big[i(z_1 \pm z_2)\big] = \cos(iz_1)\cos(iz_2) \mp \sin(iz_1)\sin(iz_2) \\
&= \cosh z_1 \cosh z_2 \mp (i\sinh z_1)(i\sinh z_2) \\
&= \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2.
\end{aligned}$$

6.6 Find all the zeros and singularities of the function $f(z) = \tanh z = \sinh z/\cosh z$,

The zeros of f are those of the numerator. We saw in class that the solutions to the equation $\sinh z = 0$ are those complex numbers of the form $z = \pi k i$ for $k \in \mathbb{Z}$.

Since f is a quotient of entire functions, the only singularities occur at those points where the denominator vanishes. Again, we saw in class that the zeroes of $\cosh z$ occur at $z = (2k+1)\pi i/2$ for $k \in \mathbb{Z}$.

6.7 Find all roots of the equations

(i) $\cosh z = 1/2$, (ii) $\sinh z = i$, (iii) $\cosh z = -2$.

(i) After the change of variables $u = e^z$, the equation

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{1}{2} \qquad \Longleftrightarrow \qquad e^z + e^{-z} = 1$$

becomes $u^2 - u + 1 = 0$, whose solutions are $u = (1 \pm i\sqrt{3})/2$. For the plus sign, we obtain

$$z = \log \frac{1 + i\sqrt{3}}{2} = \operatorname{Log}\left|\frac{1 + i\sqrt{3}}{2}\right| + i \arg \frac{1 + i\sqrt{3}}{2} = i\left(\frac{\pi}{3} + 2\pi k\right), \quad k \in \mathbb{Z}.$$

For the minus sign,

$$z = \log \frac{1 - i\sqrt{3}}{2} = \operatorname{Log}\left|\frac{1 - i\sqrt{3}}{2}\right| + i \arg \frac{1 - i\sqrt{3}}{2} = i\left(-\frac{\pi}{3} + 2\pi k\right), \quad k \in \mathbb{Z}.$$

(ii) Performing the substitution $u = e^z$ as in (i), we obtain the equation $u^2 - 2u - 1 = 0$, with solutions $u = 1 \pm \sqrt{2}$. For the plus and minus signs, respectively, we get

$$z = \log(1 + \sqrt{2}) = \operatorname{Log}\left|1 + \sqrt{2}\right| + i \arg(1 + \sqrt{2}) = \operatorname{Log}(1 + \sqrt{2}) + 2\pi k i, \quad k \in \mathbb{Z},$$

$$z = \log(1 - \sqrt{2}) = \operatorname{Log}\left|1 - \sqrt{2}\right| + i \arg(1 - \sqrt{2}) = \operatorname{Log}(\sqrt{2} - 1) + (2k + 1)\pi i, \quad k \in \mathbb{Z}.$$

(iii) We take $u = e^z$ yet again to obtain $u^2 + 4u + 1 = 0$. The solutions to the latter are $u = -2 \pm \sqrt{3}$ and hence

$$z = \log(-2 + \sqrt{3}) = \operatorname{Log}\left|-2 + \sqrt{3}\right| + i \arg(-2 + \sqrt{3}) = \operatorname{Log}(2 - \sqrt{3}) + (2k + 1)\pi i, \quad k \in \mathbb{Z},$$

$$z = \log(-2 - \sqrt{3}) = \operatorname{Log}\left|-2 - \sqrt{3}\right| + i \arg(-2 - \sqrt{3}) = \operatorname{Log}(2 + \sqrt{3}) + (2k + 1)\pi i, \quad k \in \mathbb{Z}.$$

6.8 Show that the image of the line segment given by

$$-\pi \le x \le \pi, \quad \text{and} \quad y = c$$

for some fixed $c > 0$ under the transformation $w = \sin z$ is given by the ellipse with equation

$$\left(\frac{u}{\cosh c}\right)^2 + \left(\frac{v}{\sinh c}\right)^2 = 1.$$

Let $z(t) = t + ic$, $t \in [-\pi, \pi]$, be a parametrization of the line segment given. Then

$$\begin{aligned} w(t) &= \sin z(t) = \sin(t + ic) = \sin t \cos ic + \cos t \sin ic \\ &= \sin t \cosh c + i \cos t \sinh c \end{aligned}$$

is a parametrization of its image under the transformation $w = \sin z$. Notice that

$$\frac{u(t)}{\cosh c} = \frac{\operatorname{Re} w(t)}{\cosh c} = \sin t, \qquad \frac{v(t)}{\sinh c} = \frac{\operatorname{Im} z(t)}{\sinh c} = \cos t.$$

The fundamental trigonometric identity $\sin^2 t + \cos^2 t = 1$ then implies the equation

$$\left(\frac{u(t)}{\cosh c}\right)^2 + \left(\frac{v(t)}{\sinh c}\right)^2 = 1.$$

Thus the image of the segment is contained in the ellipse described by the equation above.

To see that the image is the whole ellipse, observe that both $\sin t$ and $\cos t$ are 2π-periodic, and that $-\pi \leq t \leq \pi$ covers exactly one full period.

6.9 Find a conformal transformation $w = f(z)$ that takes the semi-infinite strip $0 < x < \pi/2$, $y > 0$ onto the upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.
Hint: start by considering the image of the domain given under $Z = \sin z$. Do you know of a conformal transformation $w = g(Z)$ that takes the resulting domain to the entire upper half-plane?

We saw in class that the image of the given semi-infinite strip in the z-plane under $Z = \sin z$ is the (open) first quadrant in the Z-plane, and that the transformation is conformal at every point in it —since it is holomorphic and its derivative vanished nowhere on it. Now compose with the transformation $w = Z^2$, which takes the first quadrant to the whole upper-half plane. Since the latter is conformal everywhere except at $Z = 0$, the composed function $w = \sin^2 z$ is conformal on the strip.

The logarithm

6.10 Find the image under Log of the following complex numbers:

(i) i, (ii) $-ei$, (iii) $1 - i$, (iv) $-1 + i\sqrt{3}$.

(i) $\operatorname{Log} i = \operatorname{Log} 1 + i\pi/2 = i\pi/2$.
(ii) $\operatorname{Log}(-ei) = \operatorname{Log} e - i\pi/2 = 1 - i\pi/2$.
(iii) $\operatorname{Log}(1 - i) = \operatorname{Log} \sqrt{2} - i\pi/4$.
(iv) $\operatorname{Log}\left(-1 + i\sqrt{3}\right) = \operatorname{Log} 2 + 2\pi i/3$.

6.11 Find the image under $\log_{(\pi/2)}$ of the wedge $\{z \in \mathbb{C}^\times \mid 0 < \operatorname{Arg} z < \pi/4\}$.

For $z \in \mathbb{C}^\times$,

$$\log_{(\pi/2)} z := \operatorname{Log}|z| + i \arg_{(\pi/2)} z, \quad \text{where } \pi/2 < \arg_{(\pi/2)} z < \pi/2 + 2\pi = 5\pi/2.$$

But $0 < \operatorname{Arg} z < \pi/4$ if and only if $0 + 2\pi = 2\pi < \arg_{(\pi/2)} z < \pi/4 + 2\pi = 9\pi/4$. It follows that the image of the wedge under $\log_{(\pi/2)}$ is the horizontal strip $\{w \in \mathbb{C} \mid 2\pi < \operatorname{Im} w < 9\pi/4\}$.

6.12 Find the image of the (open) upper half-plane $\mathbb{H}$ under the transformation

$$w = \operatorname{Log} \frac{z-1}{z+1}$$

Hint: break up the transformation above as follows:

$$Z = \frac{z-1}{z+1}, \quad w = \operatorname{Log} Z,$$

and notice that the first of these is a Möbius transformation

Recall that Möbius transformations of the form

$$Z = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc > 0$$

map the upper half-plane in the z-plane onto the upper half-plane in the Z-plane. On the other hand, $w = \operatorname{Log} Z$ maps the upper half-plane in the Z-plane onto the strip $\{w \in \mathbb{C} \mid 0 < \operatorname{Im} w < \pi\}$. Hence the latter is also the image of the upper half-plane in the z-plane under the composed transformation.
