MAT389 Fall 2013, Problem Set 3

Functions

3.1 For each of the functions defined below, describe the domain of definition that is understood:

$$\text{(i) } f(z) = \frac{1}{z^2+1}, \qquad \text{(ii) } f(z) = \operatorname{Arg}\frac{1}{z}, \qquad \text{(iii) } f(z) = \frac{z}{z+\bar{z}}, \qquad \text{(iv) } f(z) = \frac{1}{1-|z|^2}.$$

(i) The function is well-defined at all points in which the denominator does not vanish. Thus the domain is the set $\mathbb{C} - \{\pm i\}$.

(ii) The domain of both $1/z$ and the principal argument $\operatorname{Arg} z$ is $\mathbb{C}^\times$. On the other hand, $1/z$ does not take the value 0 at any point of the complex plane, so the largest set on which $f(z)$ is defined is $\mathbb{C}^\times$ itself.

(iii) $z + \bar{z} = 0$ if and only if z lies on the imaginary axis. Hence, the domain of $f(z)$ can be taken to be $\mathbb{C} - \{z \in \mathbb{C} \mid \operatorname{Re}(z) = 0\}$.

(iv) Once again, the problem is the vanishing of the denominator. That happens over the circle $|z| = 1$, and so $\operatorname{dom} f = \mathbb{C} - \{z \in \mathbb{C} \mid |z| = 1\}$.

Limits

3.2 Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, and let z_0 be an interior point of Ω. Prove that the limit $\lim_{z\to z_0} f(z)$ is unique (if it exists).
Hint: write out what it means for $\lim_{z\to z_0} f(z) = \alpha$ and $\lim_{z\to z_0} f(z) = \beta$. Using this information, what can you say about $|\alpha - \beta|$?

By definition, we have

$$\forall \epsilon > 0, \exists \delta_1 > 0 \text{ such that } 0 < |z - z_0| < \delta_1 \Longrightarrow |f(z) - \alpha| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta_2 > 0 \text{ such that } 0 < |z - z_0| < \delta_2 \Longrightarrow |f(z) - \beta| < \epsilon$$

Fixed $\epsilon > 0$, we can find $z^* \in \mathbb{C}$ such that $|f(z^*) - \alpha| < \epsilon$ and $|f(z^*) - \beta| < \epsilon$ simultaneously. Indeed, it is enough for z^* to satisfy $0 < |z^* - z_0| < \min\{\delta_1, \delta_2\}$. By the triangle inequality,

$$|\alpha - \beta| = |(f(z^*) - \beta) - (f(z^*) - \alpha)| \leq |f(z) - \alpha| + |f(z) - \beta| < 2\epsilon$$

Since ϵ is arbitrary, the only possibilty is $|\alpha - \beta| = 0$; equivalently, $\alpha = \beta$.

3.3 The following statement is false:

$$\lim_{z\to z_0} f(z) = \infty \Longrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) \text{ exist}$$

Can you provide a counterexample?

Let $z_0 = 0$, and choose

$$f(z) = \frac{1}{|x|} + i\sin\frac{1}{y}$$

We have $\lim_{z\to 0} 1/|x| = \infty$, but $\lim_{z\to 0}\sin(1/y)$ does not exist. On the other hand, $\lim_{z\to 0} f(z) = \infty$ because $\lim_{z\to 0} 1/f(z) = 0$. Indeed,

$$0 \le \lim_{z\to 0}\left|\frac{1}{f(z)}\right|^2 = \lim_{z\to 0}\frac{1}{1/x^2 + \sin^2(1/y)} \le \lim_{z\to 0}\frac{1}{1/x^2} = \lim_{z\to 0} x^2 = 0.$$

So $\lim_{z\to 0}|1/f(z)|^2 = 0$, which is equivalent to saying that $\lim_{z\to 0} 1/f(z) = 0$.

Derivatives

3.4 Apply the definition of derivative to prove that $f(z) = \operatorname{Re} z$ is nowhere differentiable.

Remember that a function $f(z)$ is differentiable at $z = z_0$ if and only if the limit

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists. For $f(z) = \operatorname{Re} z$, we have

$$\lim_{\Delta z\to 0}\frac{\operatorname{Re}(z_0+\Delta z) - \operatorname{Re} z_0}{\Delta z} = \lim_{\Delta z\to 0}\frac{\operatorname{Re} z_0 + \operatorname{Re}\Delta z - \operatorname{Re} z_0}{\Delta z} = \lim_{\Delta z\to 0}\frac{\operatorname{Re}\Delta z}{\Delta z}$$

Since the last expression is independent of z_0, the existence of the limit determines the differentiability of $f(z)$ at every point of the complex plane. But $\operatorname{Re}(\Delta z)/\Delta z$ approaches different values along different trajectories $\Delta z \to 0$. For example, when $\Delta z = \Delta x$, we have

$$\lim_{\Delta z\to 0}\frac{\operatorname{Re}(\Delta z)}{\Delta z} = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} = 1,$$

while for $\Delta z = i\Delta y$ it is

$$\lim_{\Delta z\to 0}\frac{\operatorname{Re}(\Delta z)}{\Delta z} = \lim_{\Delta y\to 0}\frac{0}{i\Delta y} = 0.$$

3.5 Where is $f(z) = |z|^2$ differentiable?

We have $f(z) = u(x,y) + iv(x,y)$ with $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. The Cauchy-Riemann equations for $f(z)$ are then

$$u_x = 2x = 0 = v_y \qquad \text{and} \qquad u_y = 2y = 0 = -v_x,$$

whose only solution is $(x,y) = (0,0)$. On the other hand, the partial derivatives u_x, u_y, v_x, v_y are certainly continuous at that point, from which we conclude $f(z)$ is differentiable only at $z = 0$.

3.6 Let f denote the function whose values are $f(0) = 0$ and

$$f(z) = \frac{(\bar{z})^2}{z}, \qquad \text{for } z \neq 0$$

Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but that the derivative of f fails to exist there.
Hint: to prove that the derivative does not exist, calculate the limit in the definition when approaching $z = 0$ horizontally ($\Delta y = 0$), and along the line $y = x$ ($\Delta x = \Delta y$).

For $z \neq 0$, we calculate

$$f(z) = \frac{(\bar{z})^2}{z} = \frac{(\bar{z})^3}{|z|^2} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2 + y^2} = \frac{x^3 - 3xy^2 - i(3x^2y - y^3)}{x^2 + y^2}$$

Hence,

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \qquad v(x, y) = \begin{cases} \frac{y^3 - 3yx^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

The partial derivatives are

$$u_x(0, 0) = 1, \quad u_y(0, 0) = 0, \quad v_x(0, 0) = 0, \quad v_y(0, 0) = 1,$$

and so the Cauchy-Riemann equations are satisfied at $z = 0$.

On the other hand, the limit

$$\lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

does not exist. Indeed, if $\Delta z = i\Delta y$, we have

$$\lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2} = \lim_{\Delta y \to 0} \frac{(-i\Delta y)^2}{(i\Delta y)^2} = 1,$$

while, for $\Delta z = \Delta x + i\Delta x = (1 + i)\Delta x$,

$$\lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2} = \lim_{\Delta x \to 0} \frac{[(1 - i)\Delta x]^2}{[(1 + i)\Delta x]^2} = -1.$$

3.7 Suppose $u(x, y)$ and $v(x, y)$ have first-order partial derivatives with respect to x and y at some point $z_0 = (x_0, y_0) \neq (0, 0)$.

(i) Use the change of coordinates $x = r\cos\theta, y = r\sin\theta$ and the chain rule to show that

$$\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta, & u_\theta &= -u_x r\sin\theta + u_y r\cos\theta, \\ v_r &= v_x \cos\theta + v_y \sin\theta, & v_\theta &= -v_x r\sin\theta + v_y r\cos\theta. \end{aligned} \tag{$*$}$$

at the point $z = z_0$.

(ii) Using these identities and the Cauchy-Riemann equations in rectangular coordinates, deduce the *polar form of the Cauchy-Riemann equations*:

$$u_r = \frac{1}{r}\, v_\theta, \qquad \frac{1}{r}\, u_\theta = -v_r$$

(iii) Solve the equations $(*)$ for u_x, u_y, v_x and v_y to show that

$$\begin{aligned} u_x &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, & u_y &= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}, \\ v_x &= v_r \cos\theta - v_\theta \frac{\sin\theta}{r}, & v_y &= v_r \sin\theta + v_\theta \frac{\cos\theta}{r}. \end{aligned} \tag{$**$}$$

(iv) Use these identities to deduce the Cauchy-Riemann equations in rectangular coordinates from their polar form.

(v) Prove that the derivative of $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ at $z = z_0$ can be expressed in any of the following two forms:

$$f'(z_0) = (\cos\theta_0 - i\sin\theta_0)\Big[u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)\Big] = \frac{-i}{z_0}\Big[u_\theta(r_0, \theta_0) + iv_\theta(r_0, \theta_0)\Big]$$

(i) The chain rule implies that

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta, & u_\theta &= u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta, \\ v_r &= v_x x_r + v_y y_r = v_x \cos\theta + v_y \sin\theta, & v_\theta &= v_x x_\theta + v_y y_\theta = -v_x r\sin\theta + v_y r\cos\theta. \end{aligned}$$

(ii) The Cauchy Riemann equations $u_x = v_y, u_y = -v_x$ imply

$$u_r = u_x \cos\theta + u_y \sin\theta = v_y \cos\theta - v_x \sin\theta = \frac{1}{r} v_\theta$$

$$v_r = v_x \cos\theta + v_y \sin\theta = -u_y \cos\theta + u_x \sin\theta = -\frac{1}{r} u_\theta$$

(iii) Write $(*)$ in matrix notation as

$$\begin{pmatrix} u_r & v_r \\ u_\theta & v_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$$

and isolate the matrix containing the partial derivatives of u and v with respect to x and y:

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}^{-1} \begin{pmatrix} u_r & v_r \\ u_\theta & v_\theta \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r & v_r \\ u_\theta & v_\theta \end{pmatrix}$$

(iv) Assuming the Cauchy-Riemann equations in polar coordinates

$$u_r = \frac{1}{r}\, v_\theta, \qquad \frac{1}{r}\, u_\theta = -v_r,$$

we have

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} = \frac{1}{r}\, v_\theta \cos\theta + v_r \sin\theta = v_y$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r} = \frac{1}{r}\, v_\theta \sin\theta - v_r \cos\theta = -v_x$$

(v) We know that

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iv_x(x_0, y_0).$$

Thus

$$\begin{aligned} f'(z_0) &= u_x(x_0, y_0) + iv_x(x_0, y_0) \\ &= u_r(r_0, \theta_0)\cos\theta_0 + v_r(r_0, \theta_0)\sin\theta_0 + i\Big[v_r(r_0, \theta_0)\cos\theta_0 - u_r(r_0, \theta_0)\sin\theta_0\Big] \\ &= (\cos\theta_0 - i\sin\theta_0)\Big[u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)\Big] \end{aligned}$$

$$\begin{aligned} f'(z_0) &= v_y(x_0, y_0) - iv_x(x_0, y_0) \\ &= \frac{-u_\theta(r_0, \theta_0)\sin\theta_0 + v_\theta(r_0, \theta_0)\cos\theta_0}{r_0} - i\,\frac{u_\theta(r_0, \theta_0)\cos\theta_0 + v_\theta(r_0, \theta_0)\sin\theta_0}{r_0} \\ &= \frac{-\sin\theta_0 - i\cos\theta_0}{r_0}\Big[u_\theta(r_0, \theta_0) + iv_\theta(r_0, \theta_0)\Big] = \frac{-i}{r_0 e^{i\theta_0}}\Big[u_\theta(r_0, \theta_0) + iv_\theta(r_0, \theta_0)\Big] \\ &= \frac{-i}{z_0}\Big[u_\theta(r_0, \theta_0) + iv_\theta(r_0, \theta_0)\Big] \end{aligned}$$