MAT389 Fall 2013, Problem Set 9

9.1 Let R be a closed bounded region, and let f be a function that is continuous on R and holomorphic everywhere in the interior of R. Assume that $f(z) \neq 0$ for any $z \in R$. Show that $|f(z)|$ achieves a minimum value in R which occurs on the boundary of R and never in the interior.
Hint: apply the maximum modulus principle to the function $F(z) = 1/f(z)$.

Since $f(z) \neq 0$ for any $z \in R$, the function $1/f(z)$ is holomorphic in the interior of R. By the Maximum Modulus Principle, $1/|f(z)|$ achieves a maximum value in R, which must be on the boundary. But a maximum of $1/|f(z)|$ is a minimum of $|f(z)|$.

9.2 Use the function $f(z) = z$ to show that the condition $f(z) \neq 0$ in the previous problem is necessary.

Let R be the closed unit disk. It is clear that the minimum of $|f(z)|$ is achieved at $z = 0$, which is not on the boundary.

9.3 Find the points where the modulus of the function $f(z) = (z+1)^2$ achieves its maximum and minimum values in the closed triangular region with vertices $z = 0$, $z = 2$ and $z = i$.

Since $f(z)$ is holomorphic, the Maximum Modulus Principle implies that $|f(z)|$ achieves its maximum value on the boundary of the triangular region. Moreover, since $f(z) = 0$ only when $z = -1$ —which is outside of the triangular region—, $|f(z)|$ also achieves its minimum on the boundary.

We consider each of the three sides separately.

- We may parametrize the side from $z = 0$ to $z = 2$ by $z = 2t$, $0 \leq t \leq 1$. Then,
$$|f(z)| = |(2t+1)^2| = (2t+1)^2,$$
which achieves its minimum value of 1 at $t = 0$, and its maximum value of 9, at $t = 1$.

- Let us describe the line segment from $z = 2$ to $z = i$ by $z = 2(1-t) + ti$, $0 \leq t \leq 1$. We then have
$$|f(z)| = |(2(1-t) + 1 + ti)^2| = (2t-3)^2 + t^2 = 5t^2 - 12t + 9.$$
Its minimum value of 2 is reached at $t = 1$, and its maximum value of 9, at $t = 0$.

- The line segment from $z = 0$ to $z = i$ is given, for example, by $z = it$, $0 \leq t \leq 1$. The modulus of f there is
$$|f(z)| = |(it+1)^2| = t^2 + 1.$$

The minimum value of 1 attained at $t = 0$, and the maximum value of 2, at $t = 1$.

Hence, $|f(z)|$ achieves its minimum value of 2 at $z = i$, and its maximum value of 9 at $z = 2$.

9.4 Suppose that $f(z)$ is entire and the harmonic function $u(x, y) = \operatorname{Re} f(z)$ has an upper bound. Show that $u(x, y)$ is constant throughout the plane.
Hint: apply Liouville's theorem to the function $g(z) = e^{f(z)}$.

Since $f(z)$ is an entire function, so is $g(z)$. Its modulus is given by $|g(z)| = |e^{f(z)}| = e^{u(x,y)}$. If $u(x, y)$ is bounded on the whole plane, so is $g(x)$. By Liouville's theorem, this means that $g(z)$ is constant, which, in turn, implies that so is $u(x, y)$.
