MAT389 Fall 2013, Problem Set 7

Complex exponentials

7.1 Find all the possible values of x^i for $x \in \mathbb{R}^\times$.
Hint: consider the cases $x < 0$ and $x > 0$ separately.

7.2 Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What restrictions must be placed on the constant c so that the values of $|i^c|$ are all the same?

Inverse trigonometric functions

7.3 Find expressions for the derivatives of the multivalued functions.

$$\operatorname{argsinh} z = \log\left[z + (z^2+1)^{1/2}\right], \qquad \operatorname{argcosh} z = \log\left[z + (z^2-1)^{1/2}\right],$$

$$\operatorname{argtanh} z = \frac{1}{2}\log\frac{1+z}{1-z}.$$

Hint: notice that, as in the case of inverse trigonometric functions, the derivatives might be different for different branches of these functions.

Combining branch cuts

7.4 Every polynomial of degree d with complex coeffcients can be written as a product

$$P(z) = a_d \prod_{j=1}^{d} (z - z_j)$$

where the $z_i \in \mathbb{C}$ are its d roots, and $a_d \in \mathbb{C}$ is the coefficient of its highest degree term —this statement is equivalent to the Fundamental Theorem of Algebra.
Fix $R > \max_{j=1,\ldots,d} |z_j|$, and let $C_R = \{z \in \mathbb{C} \mid |z| = R\}$. In each of the following cases, either choose a determination of the function $f(z) = P(z)^{1/n}$ so that it is holomorphic at every point of C_R or explain why it is impossible.

(i) $n = 2$, $d = 1$: $z_1 = 0$.
(ii) $n = 2$, $d = 2$: $z_1 = i$, $z_2 = -i$.
(iii) $n = 2$, $d = 6$: $z_j = \omega^j$ with $\omega = e^{\pi i/3}$.
(iv) $n = 3$, $d = 3$: $z_1 = -1$, $z_2 = 0$, $z_3 = 1$.
(v) $n = 3$, $d = 3$: $z_1 = -1$, $z_2 = i$, $z_3 = 1$.
(vi) $n = 2$, $d = 2$: $z_1 = z_2 = 1$.

Riemann surfaces

7.5 For your choice of branch cuts in (ii) in the previous problem, explain how to glue the two sheets into a Riemann surface for the corresponding function.

Arcs and contours

7.6 Find a parametrization of the ellipse given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

that traverses it in the counter-clockwise direction.

7.7 Why is $z(t) = (1+i)t^3$ for $-1 \le t \le 1$ not a smooth arc? Give another parametrization of the same line segment in the complex plane that is indeed a smooth arc.

7.8 Why is $z(t) = t^2 + t^3 i$ for $-1 \le t \le 1$ not a smooth arc? Can you give a parametrization of the same curve in the complex plane that is a smooth arc?