

# The Dunford-Pettis theorem

Jordan Bell

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## 1 Weak topology and weak-\* topology

If  $(E, \tau)$  is a topological vector space, we denote by  $E^*$  the set of continuous linear maps  $E \rightarrow \mathbb{C}$ , the **dual space of  $E$** . The **weak topology on  $E$** , denoted  $\sigma(E, E^*)$ , is the coarsest topology on  $E$  with which each function  $x \mapsto \lambda x$ ,  $\lambda \in E^*$ , is continuous  $E \rightarrow \mathbb{C}$ . Thus,  $\sigma(E, E^*) \subset \tau$ . If  $(E, \tau)$  is a locally convex space, it follows by the Hahn-Banach separation theorem that  $E^*$  separates  $X$ , and hence  $|\lambda|, \lambda \in E^*$ , is a separating family of seminorms on  $E$  that induce the topology  $\sigma(E, E^*)$ . Therefore, if  $(E, \tau)$  is a locally convex space, then  $(E, \sigma(E, E^*))$  is a locally convex space.

If  $(E, \tau)$  is a topological vector space, the **weak-\* topology on  $E^*$** , denoted  $\sigma(E^*, E)$ , is the coarsest topology on  $E^*$  with which each function  $\lambda \mapsto \lambda x$ ,  $x \in E$ , is continuous  $E^* \rightarrow \mathbb{C}$ . It is a fact that  $E^*$  with the topology  $\sigma(E^*, E)$  is a locally convex space.

If  $E$  is a normed space, then  $\|\lambda\|_{op} = \sup_{\|x\| \leq 1} |\lambda x|$  is a norm on the dual space  $E^*$ , and that  $E^*$  with this norm is a Banach space. The **Banach-Alaoglu theorem** states that  $\{\lambda \in E^* : \|\lambda\|_{op} \leq 1\}$  is a compact subset of  $(E^*, \sigma(E^*, E))$ .

If  $(X, \Sigma, \mu)$  is a  $\sigma$ -finite measure space, for  $g \in L^\infty(\mu)$  define  $\phi_g \in (L^1(\mu))^*$  by  $\phi_g(f) = \int_X f g d\mu$ . The map  $g \mapsto \phi_g$  is an isometric isomorphism  $L^\infty(\mu) \rightarrow (L^1(\mu))^*$ .<sup>1</sup>

Let  $(X, \Sigma, \mu)$  be a probability space. If  $\Psi \in (L^\infty(\mu))^*$  and  $A \mapsto \Psi(\chi_A)$  is countably additive on  $\Sigma$ , then there is some  $f \in L^1(\mu)$  such that

$$\Psi(g) = \int_X g f d\mu, \quad g \in L^\infty(\mu),$$

and  $\|\Psi\|_{op} = \|f\|_1$ .<sup>2</sup> Also, an additive function  $F$  on an algebra of sets  $\mathcal{A}$  is countably additive if and only if whenever  $A_n$  is a decreasing sequence of elements of  $\mathcal{A}$  with  $\bigcap_{n=1}^\infty A_n = \emptyset$ , we have  $\lim_{n \rightarrow \infty} F(A_n) = 0$ .<sup>3</sup> Using that  $\mu$  is countably additive we get the following.

<sup>1</sup>Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 190, Theorem 6.15.

<sup>2</sup>V. I. Bogachev, *Measure Theory*, volume I, p. 263, Proposition 4.2.2.

<sup>3</sup>V. I. Bogachev, *Measure Theory*, volume I, p. 9, Proposition 1.3.3.

**Theorem 1.** Suppose that  $(X, \Sigma, \mu)$  be a probability space and that  $\Psi \in (L^\infty(\mu))^*$ , and suppose that for each  $\epsilon > 0$  there is some  $\delta > 0$  such that  $E \in \Sigma$  and  $\mu(E) \leq \delta$  imply that  $|\Psi(\chi_A)| \leq \epsilon$ . Then there is some  $f \in L^1(\mu)$  such that

$$\Psi(g) = \int_X g f d\mu, \quad g \in L^\infty(\mu).$$

## 2 Normed spaces

If  $E$  is a normed space, its dual space  $E^*$  with the operator norm is a Banach space, and  $E^{**} = (E^*)^*$  with the operator norm is a Banach space. Define  $i : E \rightarrow E^{**}$  by

$$i(x)(\lambda) = \lambda(x), \quad x \in E, \quad \lambda \in E^*.$$

It follows from the Hahn-Banach extension theorem that  $i : E \rightarrow E^{**}$  is an isometric linear map.

If  $E$  and  $F$  are normed spaces and  $T : E \rightarrow F$  is a bounded linear map, we define the **transpose**  $T^* : F^* \rightarrow E^*$  by  $T^*\lambda = \lambda \circ T$  for  $\lambda \in F^*$ . If  $T$  is an isometric isomorphism, then  $T^* : F^* \rightarrow E^*$  is an isometric isomorphism, where  $E^*$  and  $F^*$  are each Banach spaces with the operator norm. In particular, we have said that when  $(X, \Sigma, \mu)$  is a  $\sigma$ -finite measure space, then the map  $\phi : L^\infty(\mu) \rightarrow (L^1(\mu))^*$  defined for  $g \in L^\infty(\mu)$  by

$$\phi_g(f) = \int_X f g d\mu, \quad f \in L^1(\mu),$$

is an isometric isomorphism, and hence  $\phi^* : (L^1(\mu))^{**} \rightarrow (L^\infty(\mu))^*$  is an isometric isomorphism. Therefore, for  $E = L^1(\mu)$  we have that

$$\phi^* \circ i : L^1(\mu) \rightarrow (L^\infty(\mu))^* \tag{1}$$

is an isometric linear map. For  $f \in L^1(\mu)$  and  $g \in L^\infty(\mu)$ ,

$$\begin{aligned} (\phi^* \circ i)(f)(g) &= (\phi^*(i(f)))(g) \\ &= (i(f) \circ \phi)(g) \\ &= i(f)(\phi_g) \\ &= \phi_g(f). \end{aligned}$$

The **Eberlein-Smulian theorem** states that if  $E$  is a normed space and  $A$  is a subset of  $E$ , then  $A$  is weakly compact if and only if  $A$  is weakly sequentially compact.<sup>4</sup>

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<sup>4</sup>Robert E. Megginson, *An Introduction to Banach Space Theory*, p. 248, Theorem 2.8.6.

### 3 Equi-integrability

Let  $(X, \Sigma, \mu)$  be a probability space and let  $\mathcal{F}$  be a subset of  $L^1(\mu)$ . We say that  $\mathcal{F}$  is **equi-integrable** if for every  $\epsilon > 0$  there is some  $\delta > 0$  such that for any  $A \in \Sigma$  with  $\mu(A) \leq \delta$  and for all  $f \in \mathcal{F}$ ,

$$\int_A |f| d\mu \leq \epsilon.$$

If  $\mathcal{F}$  is a bounded subset of  $L^1(\mu)$ , it is a fact that  $\mathcal{F}$  being equi-integrable is equivalent to

$$\lim_{C \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > C\}} |f| d\mu = 0. \quad (2)$$

The following theorem gives a condition under which a sequence of integrable functions is bounded and equi-integrable.<sup>5</sup>

**Theorem 2.** *Let  $(X, \Sigma, \mu)$  be a probability space and let  $f_n$  be a sequence in  $L^1(\mu)$ . If for each  $A \in \Sigma$  the sequence  $\int_A f_n d\mu$  has a finite limit, then  $\{f_n\}$  is bounded in  $L^1(\mu)$  and is equi-integrable.*

### 4 The Dunford-Pettis theorem

A subset  $A$  of a topological space  $X$  is said to be **relatively compact** if  $A$  is contained in some compact subset of  $X$ . When  $X$  is a Hausdorff space, this is equivalent to the closure of  $A$  being a compact subset of  $X$ .

The following is the **Dunford-Pettis theorem**.<sup>6</sup>

**Theorem 3** (Dunford-Pettis theorem). *Suppose that  $(X, \Sigma, \mu)$  is a probability space and that  $\mathcal{F}$  is a bounded subset of  $L^1(\mu)$ .  $\mathcal{F}$  is equi-integrable if and only if  $\mathcal{F}$  is a relatively compact subset of  $L^1(\mu)$  with the weak topology.*

*Proof.* Suppose that  $\mathcal{F}$  is equi-integrable, and let  $T = \phi^* \circ i : L^1(\mu) \rightarrow (L^\infty(\mu))^*$  be the isometric linear map in (1), for which

$$T(f)(g) = \int_X f g d\mu, \quad f \in L^1(\mu), \quad g \in L^\infty(\mu).$$

Then  $T(\mathcal{F})$  is a bounded subset of  $(L^\infty(\mu))^*$ , so is contained in some closed ball  $B$  in  $(L^\infty(\mu))^*$ . By the Banach-Alaoglu theorem,  $B$  is weak-\* compact, and therefore the weak-\* closure  $\mathcal{H}$  of  $T(\mathcal{F})$  is weak-\* compact. Let  $F \in \mathcal{H}$ .

<sup>5</sup>V. I. Bogachev, *Measure Theory*, volume I, p. 269, Theorem 4.5.6.

<sup>6</sup>V. I. Bogachev, *Measure Theory*, volume I, p. 285, Theorem 4.7.18; Fernando Albiac and Nigel J. Kalton, *Topics in Banach Space Theory*, p. 109, Theorem 5.2.9; R. E. Edwards, *Functional Analysis: Theory and Applications*, p. 274, Theorem 4.21.2; P. Wojtaszczyk, *Banach Spaces for Analysts*, p. 137, Theorem 12; Joseph Diestel, *Sequences and Series in Banach Spaces*, p. 93; François Trèves, *Topological Vector Spaces, Distributions and Kernels*, p. 471, Theorem 46.1.

There is a net  $F_\alpha = T(f_\alpha)$  in  $T(\mathcal{F})$ ,  $\alpha \in I$ , such that for each  $g \in L^\infty(\mu)$ ,  $F_\alpha(g) \rightarrow F(g)$ , i.e.,

$$\int_X f_\alpha g d\mu \rightarrow F(g), \quad g \in L^\infty(\mu). \quad (3)$$

Let  $\epsilon > 0$ . Because  $\mathcal{F}$  is equi-integrable, there is some  $\delta > 0$  such that when  $A \in \Sigma$  and  $\mu(A) \leq \delta$ ,

$$\sup_{\alpha \in I} \int_A |f_\alpha| d\mu \leq \epsilon,$$

which gives

$$|F(\chi_A)| = \lim_\alpha \left| \int_X f_\alpha \chi_A d\mu \right| = \lim_\alpha \left| \int_A f_\alpha d\mu \right| \leq \sup_{\alpha \in I} \int_A |f_\alpha| d\mu \leq \epsilon.$$

By Theorem 1, this tells us that there is some  $f \in L^1(\mu)$  for which

$$F(g) = \int_X g f d\mu, \quad g \in L^\infty(\mu),$$

and hence  $F = T(f)$ . This shows that  $\mathcal{H} \subset T(L^1(\mu))$ , and

$$\int_X f_\alpha g d\mu \rightarrow \int_X f g d\mu, \quad g \in L^\infty(\mu)$$

tells us that  $f_\alpha \rightarrow f$  in  $\sigma(L^1(\mu), (L^1(\mu))^*)$ , in other words  $T^{-1}(F_\alpha)$  converges weakly to  $T(f)$ . Thus  $T^{-1} : \mathcal{H} \rightarrow L^1(\mu)$  is continuous, where  $\mathcal{H}$  has the subspace topology  $\tau_{\mathcal{H}}$  inherited from  $(L^\infty(\mu))^*$  with the weak-\* topology and  $L^1(\mu)$  has the weak topology.  $(\mathcal{H}, \tau_{\mathcal{H}})$  is a compact topological space, so  $T^{-1}(\mathcal{H})$  is a weakly compact subset of  $L^1(\mu)$ . But  $\mathcal{F} \subset T^{-1}(\mathcal{H})$ , which establishes that  $\mathcal{F}$  is a relatively weakly compact subset of  $L^1(\mu)$ .

Suppose that  $\mathcal{F}$  is a relatively compact subset of  $L^1(\mu)$  with the weak topology and suppose by contradiction that  $\mathcal{F}$  is not equi-integrable. Then by (2), there is some  $\eta > 0$  such that for all  $C_0$  there is some  $C \geq C_0$  such that

$$\sup_{f \in \mathcal{F}} \int_{\{|f| > C\}} |f| d\mu > \eta,$$

whence for each  $n$  there is some  $f_n \in \mathcal{F}$  with

$$\int_{\{|f_n| > n\}} |f_n| d\mu \geq \eta, \quad (4)$$

On the other hand, because  $\mathcal{F}$  is relatively weakly compact, the Eberlein-Smulian theorem tells us that  $\mathcal{F}$  is relatively weakly sequentially compact, and so there is a subsequence  $f_{a(n)}$  of  $f_n$  and some  $f \in L^1(\mu)$  such that  $f_{a(n)}$  converges weakly to  $f$ . For  $A \in \Sigma$ , as  $\chi_A \in L^\infty(\mu)$  we have

$$\lim_{n \rightarrow \infty} \int_A f_{a(n)} d\mu = \int_A f d\mu,$$

and thus Theorem 2 tells us that the collection  $\{f_{a(n)}\}$  is equi-integrable, contradicting (4). Therefore,  $\mathcal{F}$  is equi-integrable.  $\square$

**Corollary 4.** *Suppose that  $(X, \Sigma, \mu)$  is a probability space. If  $\{f_n\} \subset L^1(\mu)$  is bounded and equi-integrable, then there is a subsequence  $f_{a(n)}$  of  $f_n$  and some  $f \in L^1(\mu)$  such that*

$$\int_X f_{a(n)} g d\mu \rightarrow \int_X f g d\mu, \quad g \in L^\infty(\mu).$$

*Proof.* The Dunford-Pettis theorem tells us that  $\{f_n\}$  is relatively weakly compact, so by the Eberlein-Smulian theorem,  $\{f_n\}$  is relatively weakly sequentially compact, which yields the claim.  $\square$

## 5 Separable topological spaces

It is a fact that if  $E$  is a separable topological vector space and  $K$  is a compact subset of  $(E^*, \sigma(E^*, E))$ , then  $K$  with the subspace topology inherited from  $(E^*, \sigma(E^*, E))$  is metrizable. Using this and the Banach-Alaoglu theorem, if  $E$  is a separable normed space it follows that  $\{\lambda \in E^* : \|\lambda\|_{op} \leq 1\}$  with the subspace topology inherited from  $(E, \sigma(E^*, E))$  is compact and metrizable, and hence is sequentially compact.<sup>7</sup> In particular, when  $E$  is a separable normed space, a bounded sequence in  $E^*$  has a weak-\* convergent subsequence.

If  $X$  is a separable metrizable space and  $\mu$  is a  $\sigma$ -finite Borel measure on  $X$ , then the Banach space  $L^p(\mu)$  is separable for each  $1 \leq p < \infty$ .<sup>8</sup>

**Theorem 5.** *Suppose that  $X$  is a separable metrizable space and  $\mu$  is a  $\sigma$ -finite Borel measure on  $X$ . If  $\{g_n\}$  is a bounded subset of  $L^\infty(\mu)$ , then there is a subsequence  $g_{a(n)}$  of  $g_n$  and some  $g \in L^\infty(\mu)$  such that*

$$\int_X f g_{a(n)} d\mu \rightarrow \int_X f g d\mu, \quad f \in L^1(\mu).$$

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<sup>7</sup>A second-countable  $T_1$  space is compact if and only if it is sequentially compact: Stephen Willard, *General Topology*, p. 125, 17G.

<sup>8</sup>René L. Schilling, *Measures, Integrals and Martingales*, p. 270, Corollary 23.20.