Chebyshev polynomials

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1 Chebyshev polynomials of first kind

On the one hand,

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \sum_{0\le\nu\le n} i^\nu \binom{n}{\nu} \cos^{n-\nu}\theta \sin^\nu\theta \\ &= \sum_{0\le 2k\le n} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta)\sin^{2k}(\theta) \\ &+ i \sum_{0\le 2k+1\le n} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(\theta)\sin^{2k+1}(\theta).\end{aligned}$$

On the other hand,

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

Therefore

$$\begin{aligned}\cos n\theta &= \sum_{0\le k\le n/2} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta)\sin^{2k}(\theta) \\ &= \sum_{0\le k\le n/2} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta)(1-\cos^2\theta)^k \\ &= \sum_{0\le k\le n/2} \binom{n}{2k} \cos^{n-2k}(\theta)(\cos^2\theta - 1)^k \\ &= \sum_{0\le k\le n/2} \binom{n}{2k} \cos^{n-2k}(\theta) \sum_{0\le j\le k} \binom{k}{j} \cos^{2k-2j}(\theta)(-1)^j \\ &= \sum_{0\le j\le n/2} (-1)^j \cos^{n-2j}(\theta) \sum_{j\le k\le n/2} \binom{n}{2k}\binom{k}{j}.\end{aligned}$$

Now,

$$\sum_{j\le k\le n/2} \binom{n}{2k}\binom{k}{j} = 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j}.$$

Hence

$$\cos n\theta = \sum_{0\le j\le n/2} (-1)^j \cos^{n-2j}(\theta) 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j}.$$

For $z \in \mathbb{C}$ let

$$T_n(z) = \sum_{0\le j\le n/2} (-1)^j 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j} z^{n-2j}. \tag{1}$$

Note

$$T_n(z)[z^n] = 2^{n-1}z^n.$$

Theorem 1.

$$T_n(\cos\theta) = \cos(n\theta)$$

and

$$T_m \circ T_n = T_{mn}.$$

Proof. For $\theta \in \mathbb{R}$,

$$T_n(\cos\theta) = \cos(n\theta).$$

Then

$$T_m(T_n(\cos\theta)) = T_m(\cos(n\theta)) = \cos(mn\theta) = T_{mn}(\theta).$$

That is, for $z \in [-1,1]$ we have $T_m(T_n(z)) = T_{mn}(z)$. Then by analytic continuation it follows that this is true for all z. □

Theorem 2.

$$T_n(z) + T_{n-2}(z) = 2zT_{n-1}(z).$$

Proof. Using $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

$$\cos(n\theta) = \cos(\theta + (n-1)\theta) = \cos\theta\cos((n-1)\theta) - \sin\theta\sin((n-1)\theta)$$

and

$$\cos((n-2)\theta) = \cos(-\theta + (n-1)\theta) = \cos\theta\cos((n-1)\theta) + \sin\theta\sin((n-1)\theta).$$

Then

$$\cos(n\theta) + \cos((n-2)\theta) = 2\cos\theta\cos((n-1)\theta).$$

Therefore

$$\begin{aligned} T_n(\cos\theta) + T_{n-2}(\cos\theta) &= \cos(n\theta) + \cos((n-2)\theta) \\ &= 2\cos\theta\cos((n-1)\theta) \\ &= 2\cos\theta \cdot T_{n-1}(\cos\theta). \end{aligned}$$

That is, for $z \in [-1,1]$,

$$T_n(z) + T_{n-2}(z) = 2zT_{n-1}(z),$$

and by analytic continuation this is true for all $z \in \mathbb{C}$. □

2 Chebyshev polynomials of second kind

Define

$$nU_{n-1}(z) = T_n'(z). \tag{2}$$

Theorem 3.

$$U_{n-1}(\cos\theta) = \frac{\sin(n\theta)}{\sin\theta}$$

and

$$(1-z^2)T_n''(z) = nU_n(z) - n(n+1)T_n(z).$$

Proof. On the one hand,

$$(T_n(\cos\theta))' = -\sin\theta \cdot T_n'(\cos\theta).$$

On the other hand,

$$(T_n(\cos\theta))' = (\cos(n\theta))' = -n\sin(n\theta).$$

Hence

$$T_n'(\cos\theta) = n\frac{\sin(n\theta)}{\sin\theta}, \qquad U_{n-1}(\cos\theta) = \frac{\sin(n\theta)}{\sin\theta}.$$

Now,

$$(T_n'(\cos\theta))' = -\sin\theta \cdot T_n''(\cos\theta)$$

and

$$\begin{aligned}
(T_n'(\cos\theta))' &= n\frac{n\cos(n\theta)\sin\theta - \sin(n\theta)\cos\theta}{\sin^2\theta} \\
&= -n\frac{\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta}{\sin^2\theta} + n(n+1)\frac{\cos(n\theta)\sin\theta}{\sin^2\theta} \\
&= -n\frac{\sin((n+1)\theta)}{\sin^2\theta} + n(n+1)\frac{\cos(n\theta)}{\sin\theta} \\
&= -n\frac{U_n(\cos\theta)}{\sin\theta} + n(n+1)\frac{T_n(\cos\theta)}{\sin\theta}.
\end{aligned}$$

Hence

$$T_n''(\cos\theta) = n\frac{U_n(\cos\theta)}{\sin^2\theta} - n(n+1)\frac{T_n(\cos\theta)}{\sin^2\theta}$$

and then

$$T_n''(\cos\theta) = n\frac{U_n(\cos\theta)}{1-\cos^2\theta} - n(n+1)\frac{T_n(\cos\theta)}{1-\cos^2\theta}.$$

By analytic continuation,

$$(1-z^2)T_n''(z) = nU_n(z) - n(n+1)T_n(z).$$

□

Theorem 4.
$$T_{n+1}(z) = zT_n(z) - (1-z^2)U_{n-1}(z).$$

Proof.
$$\begin{aligned}
T_{n+1}(\cos\theta) &= \cos(n\theta+\theta) \\
&= \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta \\
&= T_n(\cos\theta)\cos\theta - U_{n-1}(\cos\theta)\sin^2\theta \\
&= T_n(\cos\theta)\cos\theta - U_{n-1}(\cos\theta)(1-\cos^2\theta).
\end{aligned}$$

Therefore by analytic continuation,
$$T_{n+1}(z) = zT_n(z) - (1-z^2)U_{n-1}(z).$$

□

Theorem 5.
$$U_n(z) = T_n(z) + zU_{n-1}(z).$$

Proof.
$$\begin{aligned}
U_n(\cos\theta) &= \frac{\sin(n\theta+\theta)}{\sin\theta} \\
&= \frac{\cos(n\theta)\sin\theta + \cos\theta\sin(n\theta)}{\sin\theta} \\
&= T_n(\cos\theta) + \cos\theta\cdot U_{n-1}(\cos\theta).
\end{aligned}$$

Therefore by analytic continuation,
$$U_n(z) = T_n(z) + zU_{n-1}(z).$$

□

Theorem 6.
$$U_n(z) = 2zU_{n-1}(z) + U_{n-2}(z).$$

Proof. Using Theorem 4 and Theorem 5,
$$\begin{aligned}
U_n(z) &= T_n(z) + zU_{n-1}(z) \\
&= zT_{n-1}(z) - (1-z^2)U_{n-2}(z) + zU_{n-1}(z) \\
&= z\Big[U_{n-1}(z) - zU_{n-2}(z)\Big] - (1-z^2)U_{n-2}(z) + zU_{n-1}(z) \\
&= 2zU_{n-1}(z) + U_{n-2}(z).
\end{aligned}$$

□

Theorem 7.
$$(1-z^2)T_n''(z) - zT_n'(z) + n^2 T_n(z) = 0.$$

Proof. Using Theorem 3, and Theorem 5,

$$\begin{aligned}
&(1-z^2)T_n''(z) - zT_n'(z) + n^2T_n(z) \\
=&nU_n(z) - n(n+1)T_n(z) - nzU_{n-1}(z) + n^2T_n(z) \\
=&n(T_n(z) + zU_{n-1}(z)) - n(n+1)T_n(z) - nzU_{n-1}(z) + n^2T_n(z) \\
=&0.
\end{aligned}$$

□

From Theorem 1

$$T_n(1) = T_n(\cos 0) = \cos(n \cdot 0) = 1.$$

From Theorem 3,
$$T_n'(1) = nU_{n-1}(1) = n^2.$$

Thus, T_n is the unique solution of the initial value problem

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0, \qquad y(1) = 1, y'(1) = n^2.$$

Theorem 8.
$$T_n(z)^2 - (z^2-1)U_{n-1}(z)^2 = 1.$$

Proof. Using Theorem 1 and Theorem 3, for $z = \cos\theta$,

$$\begin{aligned}
T_n(z)^2 - (z^2-1)U_{n-1}(z)^2 &= T_n(\cos\theta)^2 + (\sin^2\theta)U_{n-1}(\cos\theta)^2 \\
&= \cos^2(n\theta) + (\sin^2\theta)\frac{\sin^2(n\theta)}{\sin^2\theta} \\
&= \cos^2(n\theta) + \sin^2(n\theta) \\
&= 1.
\end{aligned}$$

By analytic continuation, this is true for all z. □

3 Inner products

For $0 \le \theta \le \pi$ let $y_n(\theta) = \cos(n\theta)$.

$$y_n'' + n^2 y_n = 0, \qquad y_n'(0) = 0, y_n'(\pi) = 0.$$

Theorem 9.

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \int_0^\pi y_m y_n d\theta = \frac{\pi}{2} \cdot \delta_{m,n}.$$

Proof. Let $W = y_m y_n' - y_n y_m'$. We calculate

$$\begin{aligned} W' &= y_m' y_n' + y_m y_n'' - y_n' y_m' - y_n y_m'' y_m y_n'' - y_n y_m'' \\ &= y_m y_n'' - y_n y_m'' \\ &= y_m(-n^2 y_n) - y_n(-m^2 y_m) \\ &= (m^2 - n^2) y_m y_n. \end{aligned}$$

Using $W(0) = 0$ and $W(\pi) = 0$,

$$\int_0^\pi W'(\theta) d\theta = W(\pi) - W(0) = 0.$$

Then

$$\int_0^\pi (m^2 - n^2) y_m y_n d\theta = 0.$$

Doing the substitution $\phi = n\theta$,

$$\begin{aligned} \int_0^\pi y_n^2 d\theta &= \int_0^\pi \cos^2(n\theta) d\theta \\ &= \int_0^\pi \frac{1 + \cos(2n\theta)}{2} d\theta \\ &= \frac{\pi}{2}. \end{aligned}$$

Therefore

$$\int_0^\pi y_m y_n d\theta = \frac{\pi}{2} \cdot \delta_{m,n}.$$

For $0 \leq \theta \leq \pi$, $\sqrt{1 - \cos^2\theta} = \sin\theta$. Then doing the substitution $x = \cos\theta$, $dx = -\sin\theta d\theta$,

$$\begin{aligned} \int_0^\pi y_m y_n d\theta &= \int_0^\pi \cos(m\theta)\cos(n\theta) d\theta \\ &= \int_0^\pi \cos(m\theta)\cos(n\theta) \frac{-\sin\theta d\theta}{-\sin\theta} \\ &= \int_0^\pi \frac{\cos(m\theta)\cos(n\theta)}{-\sqrt{1 - \cos^2\theta}} (-\sin\theta) d\theta \\ &= \int_1^{-1} \frac{T_m(x) T_n(x)}{-\sqrt{1 - x^2}} dx \\ &= \int_{-1}^1 \frac{T_m(x) T_n(x)}{\sqrt{1 - x^2}} dx. \end{aligned}$$

□