The adeles

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

April 11, 2016

1 Restricted products

Let I be a nonempty set and for $i \in I$ suppose that X_i is a locally compact space and that K_i is a compact open set in K_i. A subset J of I is said to be **almost all** I if $I \setminus J$ is finite. Define the **restricted product**

$$X = \widehat{\prod_{i\in I}}^{K_i} X_i = \left\{ x \in \prod_{i \in I} X_i : x_i \in K_i \text{ for almost all } i \in I \right\}.$$

The **restricted product topology** is the topology τ on X generated by the collection $\mathscr{B}$ of sets of the form $\prod_{i \in E} U_i \times \prod_{i \notin E} K_i$ where $E \subset I$ is finite and for each $i \in E$, U_i is an open set in X_i.

Lemma 1. *$\mathscr{B}$ is a base for τ.*

Proof. For $B_1 = \prod_{i \in E_1} U_i \times \prod_{i \notin E_1} K_i$ and $B_2 = \prod_{i \in E_2} V_i \times \prod_{i \notin E_2} K_i$ in $\mathscr{B}$,

$$B_1 \cap B_2 = \prod_{i \in E_1 \setminus E_2} (U_i \cap K_i) \prod_{i \in E_2 \setminus E_1} (K_i \cap V_i) \prod_{i \in E_1 \cap E_2} (U_i \cap V_i) \prod_{i \notin E_1 \cup E_2} K_i.$$

Since K_i is open, $U_i \cap K_i$ and $K_i \cap V_i$ are open, and because $E_1 \cup E_2$ is finite we get that $B_1 \cap B_2$ belongs to $\mathscr{B}$. □

We prove that the restricted product is locally compact.[1] This is a motivation for using this object.

Lemma 2. *X is locally compact.*

Proof. For $x \in X$, let $E \subset I$ be finite with $x \in K_i$ for $i \notin E$. For $i \in E$, because X_i is locally compact there is a compact neighborhood N_i of x_i, with

[1] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 258, Lemma 13.3.1.

U_i open, $x \in U_i \subset N_i$. Then the product $\prod_{i \in E} N_i \times \prod_{i \notin E} K_i$ is compact, $\prod_{i \in E} U_i \times \prod_{i \notin E} K_i$ is open, and

$$x \in \prod_{i \in E} U_i \times \prod_{i \notin E} K_i \subset \prod_{i \in E} N_i \times \prod_{i \notin E} K_i,$$

showing that X is locally compact. □

The following is part of the machinery of restricted products.[2]

Lemma 3. *For nonempty disjoint sets $A, B \subset I$ with $I = A \cup B$, the topological spaces*

$$\widehat{\prod_{i \in I}}^{K_i} X_i$$

and

$$\left(\widehat{\prod_{i \in A}}^{K_i} X_i \right) \times \left(\widehat{\prod_{i \in B}}^{K_i} X_i \right).$$

are homeomorphic.

2 Adeles

For a nonempty set of primes S, define

$$\mathbb{A}_S = \widehat{\prod_{p \in S}}^{\mathbb{Z}_p} \mathbb{Q}_p, \qquad \mathbb{A}^S = \widehat{\prod_{p \notin S}}^{\mathbb{Z}_p} \mathbb{Q}_p.$$

Because $\mathbb{Z}_p$ is a ring, $\mathbb{A}_S$ is a ring. We prove that $\mathbb{A}_S$ is a locally compact topological ring.[3]

Lemma 4. *For any nonempty set S of primes, $\mathbb{A}_S$ is a locally compact topological ring.*

Proof. For $a, b \in \mathbb{A}_S$, let $a + b \in U = \prod_{p \in E} U_p \times \prod_{p \in S \setminus E} \mathbb{Z}_p$, for U_p open in $\mathbb{Q}_p$. But $(x, y) \mapsto x + y$ is continuous $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}$, so each for $p \in E$ there is an open neighborhood V_p of a_p in $\mathbb{Q}_p$ and an open neighborhood W_p of b_p in $\mathbb{Q}_p$ such that $x + y \in U_p$ for $(x, y) \in V_p \times W_p$. Let

$$V = \prod_{p \in E} V_p \times \prod_{p \in S \setminus E} \mathbb{Z}_p, \qquad W = \prod_{p \in E} W_p \times \prod_{p \in S \setminus E} \mathbb{Z}_p.$$

which belong to $\mathscr{B}$. $(a, b) \in V \times W$, and if $(x, y) \in V \times W$ then $x + y \in U$, which shows that $(x, y) \mapsto x + y$ is continuous $\mathbb{A}_S \times \mathbb{A}_S \to \mathbb{A}_S$.

[2] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 258, Lemma 13.3.1.

[3] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 258, Theorem 13.3.2.

Likewise, let $a \cdot b \in U = \prod_{p \in E} U_p \times \prod_{p \in S \setminus E} \mathbb{Z}_p$. Because $(x, y) \mapsto x \cdot y$ is continuous $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p$, for each $p \in E$ there are open $a_p \in V_p \subset \mathbb{Q}_p$ and $b_p \in W_p \subset \mathbb{Q}_p$ such that $x \cdot y \in U_p$ for $(x, y) \in V_p \times W_p$.

This shows that $\mathbb{A}_S$ is a topological ring. Finally, by Lemma 2, $\mathbb{A}_S$ is locally compact. □

Let $\mathbb{A}_{\text{fin}} = \widehat{\prod}^{\mathbb{Z}_p}_{p<\infty} \mathbb{Q}_p$, which is a locally compact topological ring. Finally let

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}} = \mathbb{R} \times \widehat{\prod}^{\mathbb{Z}_p}_{p<\infty} \mathbb{Q}_p,$$

which is also a locally compact topological ring, whose elements are called **adeles**.

3 Embedding the rationals in the adeles

Write $N_p = \{0, \ldots, p-1\}$. $\mathbb{Q}_p \subset \prod_{\mathbb{Z}} N_p$. For $x \in \mathbb{Q}_p$,

$$v_p(x) = \inf\{k \in \mathbb{Z} : x(k) \neq 0\}.$$

$v_p(x) = \infty$ if and only if $x = 0$.

$$|x|_p = p^{-v_p(x)}.$$

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \geq 0\} = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

For $r \in \mathbb{Q}$ write $|r|_\infty = |r|$. It is straightforward that for $r \in \mathbb{Q}$, $r \neq 0$,

$$|r|_\infty \cdot \prod_{p<\infty} |r|_p = \prod_{p \leq \infty} |r|_p = 1.$$

Let $E_r = \{p < \infty : v_p(r) < 0\} = \{p < \infty : |r|_p > 1\}$, which is finite. Thus it makes sense to define $\iota : \mathbb{Q} \to \mathbb{A}$ by $\iota(r)_p = r$ for $p \leq \infty$. It is immediate that ι is one-to-one. Assign $\mathbb{Q}$ the discrete topology, and then ι is continuous. We shall prove that $\iota : \mathbb{Q} \to \iota(\mathbb{Q})$ is a homeomorphism.

Theorem 5 (Chinese remainder theorem)**.** *Let S be a nonempty finite set of primes. For each $p \in S$ suppose e_p is a positive integer and c_p is an integer. Then there is a unique $x + \prod_{p \in S} p^{e_p}\mathbb{Z}$ such that $x + p^{e_p}\mathbb{Z} = c_p + p^{e_p}\mathbb{Z}$ for all $p \in S$.*

Proof. Let $x, y \in \mathbb{Z}$ and suppose that $x + p^{e_p}\mathbb{Z} = c_p + p^{e_p}\mathbb{Z}$ and $x + p^{e_p}\mathbb{Z} = c_p + p^{e_p}\mathbb{Z}$ for $p \in S$. This means that for each $p \in S$, p^{e_p} divides $x - y$, and for $p, q \in S$, $p \neq q$, $\gcd(p^{e_p}, q^{e_q}) = 1$ so $\prod_{p \in S} p^{e_p}$ divides $x - y$, meaning $x + \prod_{p \in S} p^{e_p}\mathbb{Z} = y + \prod_{p \in S} p^{e_p}$.

Now let $N = \prod_{p \in S} p^{e_p}$ and for $p \in S$ let $N_p = p^{-e_p}N$. Then $\gcd(N_p, p^{e_p}) = 1$ so there is some $1 \leq u_p \leq p^{e_p} - 1$ such that

$$N_p u_p \equiv 1 \pmod{p^{e_p}}.$$

Let $x = \sum_{p \in S} c_p N_p u_p \in \mathbb{Z}$. For $p, q \in S$, $q \neq p$, $N_q \equiv 0 \pmod{p^{e_p}}$, so $x \equiv c_p \pmod{p^{e_p}}$. In other words, $x + p^{e_p}\mathbb{Z} = c_p + p^{e_p}\mathbb{Z}$. □

Theorem 6 (Weak approximation theorem)**.** *Let S be a nonempty finite set of primes and for $p \in S$ let $x_p \in \mathbb{Q}_p$. For $\epsilon > 0$ there is some $r \in \mathbb{Q}$ such that*

$$|r - x_p|_p < \epsilon, \qquad p \in S.$$

Proof. Let $N = \prod_{p \in S} p$. For each $p \in S$ let $k_p > 1$ such that $p^{-k_p} N < 1$. Then define $y_p = p^{-k_p} N \in \mathbb{Q}_p$. $|y_p|_p = |p^{-k_p} p| = p^{k_p - 1}$, so

$$\left| \frac{y_p^n}{1 + y_p^n} - 1 \right|_p = \frac{1}{|1 + y_p^n|_p} \leq \frac{1}{|y_p|_p^n - 1} = \frac{1}{p^{n(k_p - 1)} - 1} \to 0,$$

and for $q \in S$ with $q \neq p$, $|y_p|_q = |q|_q = q^{-1}$, so

$$\left| \frac{y_p^n}{1 + y_p^n} \right|_q = \frac{q^{-n}}{|1 + y_p^n|_q} \leq \frac{q^{-n}}{1 - q^{-n}} \to 0.$$

For $p \in S$ take $r_p \in \mathbb{Q}$ with $|r_p - x_p|_p < \epsilon$. For $n \geq 1$ define

$$z_n = \sum_{p \in S} \frac{r_p y_p^n}{1 + y_p^n} \in \mathbb{Q}.$$

For $p \in S$, $\sum_{q \in S} \frac{r_q y_q^n}{1 + y_q^n} \to r_p$ in $\mathbb{Q}_p$. Take n_p with $|z_n - r_p|_p < \epsilon$, and for $n = \max\{n_p : p \in S\}$ and $r = z_n$, for any $p \in S$ we have

$$|r - x_p|_p = |r - r_p + r_p - x_p|_p \leq \max(|r - r_p|_p, |r_p - x_p|_p) < \epsilon.$$

$\square$

Lemma 7. *Let S be a nonempty finite set of primes and for each $p \in S$ suppose $x_p \in \mathbb{Z}_p$. For any $\epsilon > 0$ there is some $x \in \mathbb{Z}$ such that*

$$|x - x_p|_p < \epsilon, \qquad p \in S.$$

Proof. For each $p \in S$ let $y_p \in \mathbb{Z}_{\geq 0}$ with $|y_p - x_p|_p < \epsilon$. Take $p^{-n_p} < \epsilon$ and let $n = \max\{n_p : p \in S\}$. By the Chinese remainder theorem, there is some $x \in \mathbb{Z}$ such that $x + p^n\mathbb{Z} = y_p + p^n\mathbb{Z}$ for each $p \in S$. Because p^n divides $x - y_p$, $|x - y_p|_p \leq p^{-n}$. Then for any $p \in S$,

$$|x - x_p|_p = |x - y_p + y_p - x_p|_p \leq \max(|x - y_p|_p, |y_p - x_p|_p) < \epsilon.$$

$\square$

In some arguments it is more convenient to work with the following **fundamental domain** D rather than $\mathbb{A}$.[4]

[4]Dorian Goldfeld and Joseph Hundley, *Automorphic Representations and L-functions for the General Linear Group*, volume I, p. 10, Proposition 1.4.5.

Lemma 8. *Let*

$$D = [0,1) \times \prod_{p<\infty} \mathbb{Z}_p.$$

The sets $\iota(r) + D$, $r \in \mathbb{Q}$, *are pairwise disjoint and* $\mathbb{A} = \bigcup_{r\in\mathbb{Q}} (\iota(r) + D)$.

Theorem 9. *The subspace topology on* $\iota(\mathbb{Q})$ *inherited from* $\mathbb{A}$ *is discrete.*