Alternating multilinear forms

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1 Permutations

We follow Cartan [2] and Abraham and Marsden [1].

Let E be a real vector space. Let $\mathscr{L}_p(E;\mathbb{R})$ be the set of multilinear maps $E^p \to \mathbb{R}$.

Definition 1. A map $f \in \mathscr{L}_p(E;\mathbb{R})$ is called **alternating** if $(x_1,\ldots,x_p) \in E^p$ with $x_i = x_{i+1}$ for some $1 \leq i < p$ implies $f(x_1,\ldots,x_p) = 0$. Let $\mathscr{A}_p(E;\mathbb{R})$ be the set of alternating elements of $\mathscr{L}_p(E;\mathbb{R})$.

For a set X, let S_X be the group of bijections $X \to X$, and let $S_p = S_{\{1,\ldots,p\}}$. For $\sigma,\tau \in S_X$, write $\sigma\tau = \sigma \circ \tau$.

Definition 2. For a function $f : E^p \to \mathbb{R}$ and a permutation $\sigma \in S_p$, define the function $\sigma f : E^p \to \mathbb{R}$ by

$$(\sigma f)(x_1,\ldots,x_p) = f(x_{\sigma(1)},\ldots,x_{\sigma(p)}), \qquad (x_1,\ldots,x_p) \in E^p.$$

Theorem 3. For a function $f : E^p \to \mathbb{R}$ and for $\sigma,\tau \in S_p$,

$$\tau(\sigma f) = (\tau\sigma)f.$$

Proof. Define $g = \sigma f$. For $(x_1,\ldots,x_p) \in E^p$ and for $y_i = x_{\tau(i)}$, we have

$$\begin{aligned}
\tau(\sigma f)(x_1,\ldots,x_p) &= \tau(g)(x_1,\ldots,x_p) \\
&= g(x_{\tau(1)},\ldots,x_{\tau(p)}) \\
&= g(y_1,\ldots,y_p) \\
&= (\sigma f)(y_1,\ldots,y_p) \\
&= f(y_{\sigma(1)},\ldots,y_{\sigma(p)}) \\
&= f(x_{\tau(\sigma(1))},\ldots,x_{\tau(\sigma(p))}) \\
&= (\tau\sigma)(f)(x_1,\ldots,x_p).
\end{aligned}$$

Thus

$$\tau(\sigma f) = (\tau\sigma)f.$$

□

For $1 \le i, j \le p$, define $(i,j) \in S_p$ by

$$(i,j)(k) = \begin{cases} j & k = i, \\ i & k = j, \\ k & k \neq i, j, \end{cases}$$

called a **transposition**. Define

$$\tau_i = (i, i+1),$$

called an **adjacent transposition**. We can write a transposition (i,j), $i < j$, as a product of $2j - 2i - 1$ adjacent transpositions:

$$\begin{aligned}(i,j) &= (j-1,j)(j-2,j-1)\cdots(i+1,i+2)(i,i+1)(i+1,i+2)\cdots(j-1,j) \\ &= \tau_{j-1}\cdots\tau_{i+1}\tau_i\tau_{i+1}\cdots\tau_{j-1}.\end{aligned}$$

Theorem 4. For $\sigma, \tau \in S_p$,

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

Theorem 5. Let $f \in \mathscr{L}_p(E;\mathbb{R})$. $f \in \mathscr{A}_p(E;\mathbb{R})$ if and only if $\sigma f = (\operatorname{sgn}\sigma) f$ for all $\sigma \in S_p$.

Proof. (i) Suppose that $f \in \mathscr{A}_p(E;\mathbb{R})$ and let $\sigma \in S_p$; we have to show that $\sigma f = (\operatorname{sgn}\sigma) f$. Let $(x_1, \dots, x_p) \in E^p$ and for $1 \le i < p$ define $g_i : E^2 \to \mathbb{R}$ by

$$g_i(y_1, y_2) = f(x_1, \dots, \underbrace{y_1}_{i}, \underbrace{y_2}_{i+1}, \dots, x_p), \qquad (y_1, y_2) \in E^2.$$

Because f is multilinear and alternating, on the one hand

$$g_i(x_i + x_{i+1}, x_i + x_{i+1}) = 0,$$

and on the other hand

$$\begin{aligned} g_i(x_i + x_{i+1}, x_i + x_{i+1}) &= g_i(x_i, x_i) + g_i(x_i, x_{i+1}) + g_i(x_{i+1}, x_i) + g_i(x_{i+1}, x_{i+1}) \\ &= g_i(x_i, x_{i+1}) + g_i(x_{i+1}, x_i). \end{aligned}$$

Therefore

$$g_i(x_{i+1}, x_i) = -g_i(x_i, x_{i+1}),$$

that is,

$$f(x_1, \dots, x_p) = -f(x_1, \dots, x_p).$$

Thus, as $\operatorname{sgn}\tau_i = -1$,

$$\tau_i f = (\operatorname{sgn}\tau_i) f.$$

Because σ is equal to a product of adjacent transpositions, it then follows from Theorem 3 and Theorem 4 that $\sigma f = (\operatorname{sgn}\sigma) f$.

(ii) Suppose that $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_p$. Let $(x_1, \ldots, x_p) \in E^p$ with $x_i = x_{i+1}$ for some $1 \leq i < p$; we have to show that $f(x_1, \ldots, x_p) = 0$. On the one hand,

$$\tau_i f(x_1, \ldots, x_p) = (\operatorname{sgn} \tau_i) f(x_1, \ldots, x_p) = -f(x_1, \ldots, x_p).$$

On the other hand, using that $x_i = x_{i+1}$,

$$\begin{aligned}
\tau_i f(x_1, \ldots, x_p) &= f(x_{\tau_i(1)}, \ldots, x_{\tau_i(i)}, x_{\tau_i(i+1)}, \ldots, x_{\tau_i(p)}) \\
&= f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_p) \\
&= f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_p).
\end{aligned}$$

Hence

$$-f(x_1, \ldots, x_p) = f(x_1, \ldots, x_p),$$

which implies that $f(x_1, \ldots, x_p) = 0$. This shows that $f \in \mathscr{A}_p(E; \mathbb{R})$. $\square$

Theorem 6. Let $f \in \mathscr{A}_p(E; \mathbb{R})$. If $(x_1, \ldots, x_p) \in E^p$ with $x_i = x_j$ for some $i \neq j$, then $f(x_1, \ldots, x_p) = 0$.

Proof. Check that there is some $\sigma \in S_p$ satisfying $\sigma(1) = i$ and $\sigma(2) = j$. For this σ,

$$\begin{aligned}
(\sigma f)(x_1, \ldots, x_p) &= f(x_i, x_j, x_{\sigma(3)}, \ldots, x_{\sigma(p)}) \\
&= f(x_i, x_i, x_{\sigma(3)}, \ldots, x_{\sigma(p)}) \\
&= 0.
\end{aligned}$$

But $(\sigma f) = (\operatorname{sgn} \sigma) f$, so $(\operatorname{sgn} \sigma) f(x_1, \ldots, x_p) = 0$. Therefore $f(x_1, \ldots, x_p) = 0$. $\square$

Definition 7. For $f \in \mathscr{L}_p(E; \mathbb{R})$, define

$$A_p f = \frac{1}{p!} \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) \sigma f.$$

Lemma 8. A_p is a linear map $\mathscr{L}_p(E; \mathbb{R}) \to \mathscr{A}_p(E; \mathbb{R})$.

Proof. Let $f \in \mathscr{L}_p(E; \mathbb{R})$. For $\sigma \in S_p$, $\sigma f \in \mathscr{L}_p(E; \mathbb{R})$, hence $A_p f \in \mathscr{L}_p(E; \mathbb{R})$. Namely, $A_p f$ is multilinear. It remains to show that it is alternating.

For $\sigma \in S_p$, as $\tau \mapsto \sigma\tau$ is a bijection $S_p \to S_p$,

$$\begin{aligned}
\sigma(A_p f) &= \frac{1}{p!} \sum_{\tau \in S_p} (\operatorname{sgn} \tau) \sigma\tau f \\
&= (\operatorname{sgn} \sigma) \frac{1}{p!} \sum_{\tau \in S_p} (\operatorname{sgn} \tau) \tau f \\
&= (\operatorname{sgn} \sigma) A f,
\end{aligned}$$

showing that $A_p f$ is alternating by Theorem 5, so $A_p f \in \mathscr{A}_p(E; \mathbb{R})$. $\square$

Theorem 9. Let $f \in \mathscr{L}_p(E; \mathbb{R})$. $f \in \mathscr{A}_p(E; \mathbb{R})$ if and only if $A_p f = f$.

Proof. Suppose $f \in \mathscr{A}_p(E; \mathbb{R})$. Then $\sigma f = (\operatorname{sgn} \sigma) f$ for each $\sigma \in S_p$, by Theorem 5. Then

$$A_p f = \frac{1}{p!} \sum_{\sigma \in S_p} \sigma f = \frac{1}{p!} \sum_{\sigma \in S_p} f = f.$$

Suppose $A_p f = f$. Lemma 8 tells us $A_p f \in \mathscr{A}_p(E; \mathbb{R})$, hence $f \in \mathscr{A}_p(E; \mathbb{R})$. $\square$

2 Wedge products

A permutation $\sigma \in S_{p+q}$ is called a (p,q)**-riffle shuffle** if

$$\sigma(1) < \cdots < \sigma(p), \qquad \sigma(p+1) < \cdots < \sigma(p+q).$$

Denote by $S_{p,q}$ those elements of S_{p+q} that are (p,q)-riffle shuffles.

Lemma 10. $|S_{p,q}| = \binom{p+q}{p} = \frac{(p+q)!}{p!q!}$.

Let $\mathscr{A}_{p,q}(E; \mathbb{R})$ be the set of those $h \in \mathscr{L}_{p+q}(E; \mathbb{R})$ such that (i) for each $(y_1, \ldots, y_q) \in E^q$, the map

$$(x_1, \ldots, x_p) \mapsto h(x_1, \ldots, x_p, y_1, \ldots, y_q), \qquad E^p \to \mathbb{R},$$

belongs to $\mathscr{A}_p(E; \mathbb{R})$, and (ii) for $(x_1, \ldots, x_p) \in E^p$, the map

$$(y_1, \ldots, y_q) \mapsto h(x_1, \ldots, x_p, y_1, \ldots, y_q), \qquad E^q \to \mathbb{R},$$

belongs to $\mathscr{A}_q(E; \mathbb{R})$.

Definition 11. For $h \in \mathscr{A}_{p,q}(E; \mathbb{R})$ define

$$\phi_{p,q}(h) = \sum_{\sigma \in S_{p,q}} (\operatorname{sgn} \sigma)(\sigma h).$$

Theorem 12. $\phi_{p,q}$ is a linear map $\mathscr{A}_{p,q}(E; \mathbb{R}) \to \mathscr{A}_{p+q}(E; \mathbb{R})$.

Proof. Let $h \in \mathscr{A}_{p,q}(E; \mathbb{R})$, and say $(x_1, \ldots, x_{p+q}) \in E^{p+q}$ with $x_k = x_{k+1}$ for some $1 \leq k < p$.

Let A_1 be those $\sigma \in S_{p,q}$ such that $i = \sigma^{-1}(k), j = \sigma^{-1}(k+1) \leq p$. For $\sigma \in A_1$, by Theorem 6,[1]

$$(\sigma h)(x_1, \ldots, x_{p+q}) = h(x_{\sigma(1)}, \ldots, x_{\sigma(p)}, \ldots, x_{\sigma(p+q)}) = 0.$$

Let A_2 be those $\sigma \in S_{p,q}$ such that $\sigma^{-1}(k), \sigma^{-1}(k+1) \geq p+1$. For $\sigma \in A_2$, by Theorem 6,

$$(\sigma h)(x_1, \ldots, x_{p+q}) = h(x_{\sigma(1)}, \ldots, x_{\sigma(p)}, \ldots, x_{\sigma(p+q)}) = 0.$$

[1] i, j are distinct and $1 \leq i, j \leq p$; they need not be adjacent.

Thus

$$\sum_{\sigma \in A_1} (\operatorname{sgn} \sigma)(\sigma h)(x_1, \ldots, x_{p+q}) = 0$$

and

$$\sum_{\sigma \in A_2} (\operatorname{sgn} \sigma)(\sigma h)(x_1, \ldots, x_{p+q}) = 0.$$

Let A_3 be those $\sigma \in S_{p,q}$ for which $\sigma^{-1}(k) < p$ and $\sigma^{-1}(k+1) \geq p+1$ and let A_4 be those $\sigma \in S_{p,q}$ for which $\sigma^{-1}(k) \geq p+1$ and $\sigma^{-1}(k+1) \leq p$. If $\sigma \in A_3$ then

$$(\tau_k \sigma)^{-1}(k) = \sigma^{-1} \tau_k^{-1}(k) = \sigma^{-1}(k+1) \geq p+1$$

and

$$(\tau_k \sigma)^{-1}(k+1) = \sigma^{-1} \tau_k^{-1}(k+1) = \sigma^{-1}(k) < p,$$

so $\tau_k \sigma \in A_4$. Likewise, if $\sigma \in A_4$ then $\tau_k \sigma \in A_3$. Thus $A_4 = \tau_k A_3$. For $\sigma \in A_3$, let $i = \sigma^{-1}(k)$ and $j = \sigma^{-1}(k+1)$, for which $i < p$ and $j \geq p+1$. Then, as $x_k = x_{k+1}$,

$$\begin{aligned}
&(\operatorname{sgn} \sigma)(\sigma h)(x_1, \ldots, x_{p+q}) + (\operatorname{sgn} \tau_k \sigma)(\tau_k \sigma h)(x_1, \ldots, x_{p+q}) \\
=&(\operatorname{sgn} \sigma) h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)}) - (\operatorname{sgn} \sigma) h(x_{\tau_k \sigma(1)}, \ldots, x_{\tau_k \sigma(p+q)}) \\
=&(\operatorname{sgn} \sigma)\big(h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)}) - h(x_{\tau_k \sigma(1)}, \ldots, x_{\tau_k \sigma(i)}, \ldots, x_{\tau_k \sigma(j)}, \ldots, x_{\tau_k \sigma(p+q)})\big) \\
=&(\operatorname{sgn} \sigma)\big(h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)}) - h(x_{\tau_k \sigma(1)}, \ldots, x_{\tau_k(k)}, \ldots, x_{\tau_k(k+1)}, \ldots, x_{\tau_k \sigma(p+q)})\big) \\
=&(\operatorname{sgn} \sigma)\big(h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)}) - h(x_{\sigma(1)}, \ldots, x_{k+1}, \ldots, x_k, \ldots, x_{\sigma(p+q)})\big) \\
=&(\operatorname{sgn} \sigma)\big(h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)}) - h(x_{\sigma(1)}, \ldots, x_k, \ldots, x_{k+1}, \ldots, x_{\sigma(p+q)})\big) \\
=&(\operatorname{sgn} \sigma)\big(h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)}) - h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)})\big) \\
=&0.
\end{aligned}$$

Therefore

$$\sum_{\sigma \in A_3 \cup A_4} (\operatorname{sgn} \sigma)(\sigma h)(x_1, \ldots, x_{p+q}) = 0.$$

But $S_{p,q} = A_1 \cup A_2 \cup A_3 \cup A_4$, so

$$\phi_{p,q}(h)(x_1, \ldots, x_{p+q}) = 0.$$

Thus $\phi_{p,q}(h) \in \mathscr{A}_{p+q}(E; \mathbb{R})$. □

Definition 13. For $f \in \mathscr{L}_p(E; \mathbb{R})$ and $g \in \mathscr{L}_q(E; \mathbb{R})$, define the **tensor product** $f \otimes_{p,q} g \in \mathscr{L}_{p+q}(E; \mathbb{R})$ by

$$(f \otimes_{p,q} g)(x_1, \ldots, x_{p+q}) = f(x_1, \ldots, x_p) g(x_{p+1}, \ldots, x_{p+q}).$$

It is apparent that

$$(f \otimes_{p,q} g) \otimes_{p+q,r} h = f \otimes_{p,q+r} (g \otimes_{q,r} h),$$

and thus it makes sense to write the tensor product without indices.

Definition 14. Define the **wedge product**

$$\wedge_{p,q} : \mathscr{A}_p(E;\mathbb{R}) \times \mathscr{A}_q(E;\mathbb{R}) \to \mathscr{A}_{p+q}(E;\mathbb{R})$$

by, for $f \in \mathscr{A}_p(E;\mathbb{R}), g \in \mathscr{A}_q(E;\mathbb{R})$,

$$f \wedge_{p,q} g = \phi_{p,q}(f \otimes g),$$

i.e., for $h = f \otimes g$,

$$\begin{aligned}(f \wedge_{p,q} g)(x_1, \ldots, x_{p+q}) &= \sum_{\sigma \in S_{p,q}} (\operatorname{sgn}\sigma)(\sigma h)\\ &= \sum_{\sigma \in S_{p,q}} (\operatorname{sgn}\sigma) h(x_{\sigma(1)}, \ldots, x_{\sigma(p+q)})\\ &= \sum_{\sigma \in S_{p,q}} (\operatorname{sgn}\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) g(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}).\end{aligned}$$

Theorem 15. For $f \in \mathscr{A}_p(E;\mathbb{R})$ and $g \in \mathscr{A}_q(E;\mathbb{R})$,

$$f \wedge_{p,q} g = \frac{(p+q)!}{p!q!} A_{p+q}(f \otimes g).$$

Proof. For $\sigma \in S_{p,q}$,

$$\sigma(1) < \cdots < \sigma(p), \qquad \sigma(p+1) < \cdots < \sigma(p+q).$$

Let $I_\sigma = \{\sigma(i) : 1 \le i \le p\}$ and $J_\sigma = \{\sigma(i) : p+1 \le i \le p+q\}$.

$$f \wedge_{p,q} g =$$

□

Theorem 16. For $f \in \mathscr{A}_p(E;\mathbb{R})$ and $g \in \mathscr{A}_q(E;\mathbb{R})$,

$$g \wedge_{q,p} f = (-1)^{pq} f \wedge_{p,q} g.$$

Proof. Define $\alpha \in S_{p,q}$ by

$$\alpha(i) = q + i, \quad 1 \le i \le p, \qquad \alpha(p+i) = i, \quad 1 \le i \le q.$$

Then[2]

$$\alpha = \prod_{1\le i\le p}\prod_{1\le j\le q}(i+q-j, i+q-j+1).$$

Thus

$$\operatorname{sgn}\alpha = \prod_{1\le i\le p}\prod_{1\le j\le q}(-1) = (-1)^{pq}.$$

Let $\tau \in S_{q,p}$, then for $1 \le i \le p$,

$$(\tau\alpha)(i) = \tau(q+i)$$

and for $1 \le i \le q$,

$$(\tau\alpha)(p+i) = \tau(i),$$

But $\tau \in S_{q,p}$ so

$$\tau(1) < \cdots < \tau(q), \qquad \tau(q+1) < \cdots < \tau(q+p),$$

thus

$$(\tau\alpha)(1) < \cdots < (\tau\alpha)(p), \qquad (\tau\alpha)(p+1) < \cdots < (\tau\alpha)(p+q),$$

which means that $\tau\alpha \in S_{p,q}$. Likewise, if $\sigma \in S_{p,q}$ then

$$(\sigma\alpha^{-1})(1) = \sigma(q+1), \ldots, (\sigma\alpha^{-1})(q) = \sigma(p+q)$$

and

$$(\sigma\alpha^{-1})(q+1) = \sigma(1), \ldots, (\sigma\alpha^{-1})(q+p) = \sigma(p),$$

and because $\sigma \in S_{p,q}$ it follows that $\sigma\alpha^{-1} \in S_{q,p}$.

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[2]For example, take $p = 3$ and $q = 2$. Then

$$\alpha(1) = 3, \alpha(2) = 4, \alpha(3) = 5, \alpha(4) = 1, \alpha(5) = 2.$$

Here

$$\begin{aligned}\prod_{1\le i\le p}\prod_{1\le j\le q}(i+q-j, i+q-j+1) &= \prod_{1\le i\le 3}\prod_{1\le j\le 2}(i-j+2, i-j+3)\\ &= \prod_{1\le i\le 3}(i+1, i+2)(i, i+1)\\ &= (2,3)(1,2)(3,4)(2,3)(4,5)(3,4)\\ &= \alpha.\end{aligned}$$

Hence for $(x_1, \dots, x_{p+q}) \in E^{p+q}$,

$$
\begin{aligned}
&(g \wedge_{q,p} f)(x_1, \dots, x_{p+q})\\
=&\sum_{\tau \in S_{q,p}} (\operatorname{sgn} \tau) g(x_{\tau(1)}, \dots, x_{\tau(q)}) f(x_{\tau(q+1)}, \dots, x_{\tau(q+p)})\\
=&\sum_{\sigma \in S_{p,q}} (\operatorname{sgn} \sigma\alpha^{-1}) g(x_{(\sigma\alpha^{-1})(1)}, \dots, x_{(\sigma\alpha^{-1})(q)}) f(x_{(\sigma\alpha^{-1})(q+1)}, \dots, x_{(\sigma\alpha^{-1})(q+p)})\\
=&(\operatorname{sgn} \alpha^{-1}) \sum_{\sigma \in S_{p,q}} (\operatorname{sgn} \sigma) g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}) f(x_{\sigma(1)}, \dots, x_{\sigma(p)})\\
=&(-1)^{pq} \sum_{\sigma \in S_{p,q}} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(p)}) g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)})\\
=&(-1)^{pq} (f \wedge_{p,q} g)(x_1, \dots, x_{p+q}).
\end{aligned}
$$

Thus

$$g \wedge_{q,p} f = (-1)^{pq} f \wedge_{p,q} g.$$

□

Let $\mathscr{A}_{p,q,r}(E;\mathbb{R})$ be the set of those $u \in \mathscr{L}_{p+q+r}(E;\mathbb{R})$ such that (i) for each $(y_1, \dots, y_q, z_1, \dots, z_r) \in E^{q+r}$, the map

$$(x_1, \dots, x_p) \mapsto u(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r), \qquad E^p \to \mathbb{R},$$

belongs to $\mathscr{A}_p(E;\mathbb{R})$, (ii) for $(x_1, \dots, x_p, z_1, \dots, z_r) \in E^{p+r}$, the map

$$(y_1, \dots, y_q) \mapsto u(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r), \qquad E^q \to \mathbb{R},$$

belongs to $\mathscr{A}_q(E;\mathbb{R})$, and (iii) for $(x_1, \dots, x_p, y_1, \dots, y_q) \in E^{p+q}$, the map

$$(z_1, \dots, z_r) \mapsto u(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r), \qquad E^r \to \mathbb{R},$$

belongs to $\mathscr{A}_r(E;\mathbb{R})$.

Let $S_{p,q,\overline{r}}$ be those $\sigma \in S_{p+q+r}$ such that

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q), \quad \sigma(p+q+i) = p+q+i, 1 \le i \le r.$$

Let $S_{\overline{p},q,r}$ be those $\sigma \in S_{p+q+r}$ such that

$$\sigma(i) = i, 1 \le i \le p, \qquad \sigma(p+1) < \dots < \sigma(p+q), \quad \sigma(p+q+1) < \dots < \sigma(p+q+r).$$

Let $S_{p,q,r}$ be those $\sigma \in S_{p+q+r}$ such that

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q), \quad \sigma(p+q+1) < \dots < \sigma(p+q+r).$$

Lemma 17.

$$S_{p+q,r} S_{p,q,\overline{r}} = S_{p,q,r}$$

and

$$S_{p,q+r} S_{\overline{p},q,r} = S_{p,q,r}.$$

Proof. Let $\sigma \in S_{p+q,r}$ and $\tau \in S_{p,q,\overline{r}}$. Then

$$\sigma(1) < \cdots < \sigma(p+q), \quad \sigma(p+q+1) < \cdots < \sigma(p+q+r)$$

and

$$\tau(1) < \cdots < \tau(p), \quad \tau(p+1) < \cdots < \tau(p+q), \quad \tau(p+q+i) = p+q+i, 1 \leq i \leq r.$$

It follows that

$$(\sigma\tau)(1) < \cdots < (\sigma\tau)(p)$$

and

$$(\sigma\tau)(p+1) < \cdots < (\sigma\tau)(p+q)$$

and for $1 \leq i \leq r$, $(\sigma\tau)(p+q+i) = \sigma(p+q+i)$, so

$$(\sigma\tau)(p+q+1) < \cdots < \sigma(p+q+r).$$

Thus $\sigma\tau \in S_{p,q,r}$. $\square$

Define $\phi_{p,q,\overline{r}} : \mathscr{A}_{p,q,r}(E;\mathbb{R}) \to \mathscr{A}_{p+q,r}(E;\mathbb{R})$ by

$$\phi_{p,q,\overline{r}}(u) = \sum_{\sigma \in S_{p,q,\overline{r}}} (\operatorname{sgn}\sigma)(\sigma u), \qquad u \in \mathscr{A}_{p,q,r}(E;\mathbb{R})$$

and define $\phi_{\overline{p},q,r} : \mathscr{A}_{p,q,r}(E;\mathbb{R}) \to \mathscr{A}_{p,q+r}(E;\mathbb{R})$ by

$$\phi_{\overline{p},q,r}(u) = \sum_{\sigma \in S_{\overline{p},q,r}} (\operatorname{sgn}\sigma)(\sigma u), \qquad u \in \mathscr{A}_{p,q,r}(E;\mathbb{R})$$

Lemma 18. For $u \in \mathscr{A}_{p,q,r}(E;\mathbb{R})$,

$$(\phi_{p+q,r} \circ \phi_{p,q,\overline{r}})u = \sum_{\rho \in S_{p,q,r}} (\operatorname{sgn}\rho)\rho u$$

and

$$(\phi_{p,q+r} \circ \phi_{\overline{p},q,r})u = \sum_{\rho \in S_{p,q,r}} (\operatorname{sgn}\rho)\rho u,$$

and so

$$\phi_{p+q,r} \circ \phi_{p,q,\overline{r}} = \phi_{p,q+r} \circ \phi_{\overline{p},q,r}.$$

Proof. Applying Lemma 17 we get

$$\begin{aligned}
(\phi_{p+q,r} \circ \phi_{p,q,\overline{r}})u &= \sum_{\sigma \in S_{p+q,r}} (\operatorname{sgn}\sigma)\sigma\phi_{p,q,\overline{r}}(u) \\
&= \sum_{\sigma \in S_{p+q,r}} (\operatorname{sgn}\sigma)\sigma \sum_{\tau \in S_{p,q,\overline{r}}} (\operatorname{sgn}\tau)(\tau u) \\
&= \sum_{\sigma \in S_{p+q,r}} (\operatorname{sgn}\sigma\tau) \sum_{\tau \in S_{p,q,\overline{r}}} \sigma\tau u \\
&= \sum_{\rho \in S_{p,q,r}} (\operatorname{sgn}\rho)\rho u
\end{aligned}$$

and similarly

$$\begin{aligned}(\phi_{p,q+r} \circ \phi_{\overline{p},q,r})u &= \sum_{\sigma \in S_{p,q+r}} (\operatorname{sgn} \sigma)\sigma\phi_{\overline{p},q,r}(u) \\ &= \sum_{\sigma \in S_{p,q+r}} (\operatorname{sgn} \sigma)\sigma \sum_{\tau \in S_{\overline{p},q,r}} (\operatorname{sgn} \tau)(\tau u) \\ &= \sum_{\sigma \in S_{p,q+r}} (\operatorname{sgn} \sigma\tau) \sum_{\tau \in S_{\overline{p},q,r}} \sigma\tau u \\ &= \sum_{\rho \in S_{p,q,r}} (\operatorname{sgn} \rho)\rho u.\end{aligned}$$

Thus

$$(\phi_{p+q,r} \circ \phi_{p,q,\overline{r}})u = (\phi_{p,q+r} \circ \phi_{\overline{p},q,r})u,$$

from which the claim follows. □

Theorem 19. If $f \in \mathscr{A}_p(E;\mathbb{R})$, $g \in \mathscr{A}_q(E;\mathbb{R})$, and $h \in \mathscr{A}_r(E;\mathbb{R})$, then

$$(f \wedge_{p,q} g) \wedge_{p+q,r} h = f \wedge_{p,q+r} (g \wedge_{q,r} h).$$

Proof. On the one hand,

$$\begin{aligned}(\phi_{p+q,r} \circ \phi_{p,q,\overline{r}})(f \otimes g \otimes h) &= \phi_{p+q,r}(\phi_{p,q,\overline{r}}((f \otimes g) \otimes h) \\ &= \phi_{p+q,r}((f \wedge_{p,q} g) \otimes h) \\ &= (f \wedge_{p,q} g) \wedge_{p+q,r} h.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\phi_{p,q+r} \circ \phi_{\overline{p},q,r})(f \otimes g \otimes h) &= \phi_{p,q+r}(\phi_{\overline{p},q,r})(f \otimes (g \otimes h)) \\ &= \phi_{p,q+r}(f \otimes (g \wedge_{q,r} h)) \\ &= f \wedge_{p,q+r} (g \wedge_{q,r} h).\end{aligned}$$

But by Lemma 18,

$$\phi_{p+q,r} \circ \phi_{p,q,\overline{r}} = \phi_{p,q+r} \circ \phi_{\overline{p},q,r},$$

hence

$$(f \wedge_{p,q} g) \wedge_{p+q,r} h = f \wedge_{p,q+r} (g \wedge_{q,r} h).$$

□

3 Linear forms

Let $E^* = \mathscr{L}_1(E;\mathbb{R})$, the **dual space of** E, whose elements we call **linear forms**. It is immediate that $\mathscr{A}_1(E;\mathbb{R}) = \mathscr{L}_1(E;\mathbb{R}) = E^*$.

Theorem 20. If $f_1, \dots, f_n \in E^*$ then for $(x_1, \dots, x_n) \in E^n$,

$$(f_1 \wedge \cdots \wedge f_n)(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) f_1(x_{\sigma(1)}) \cdots f_n(x_{\sigma(n)}).$$

Proof. For $n = 1$ the claim is immediate. For $n = 2$, on the one hand, using the definition of the wedge product,

$$(f_1 \wedge f_2)(x_1, x_2) = \sum_{\sigma \in S_{1,1}} (\operatorname{sgn} \sigma) f_1(x_{\sigma(1)}) f_2(x_{\sigma(2)}),$$

and as $S_{1,1} = S_2$ the claim is true for $n = 2$. Suppose the claim is true for some $n \geq 2$ and let $(f_1, \dots, f_n, f_{n+1}) \in E^*$ and $(x_1, \dots, x_n, x_{n+1}) \in E^{n+1}$. Then, setting $u = f_1 \wedge \cdots \wedge f_n \in \mathscr{A}_n(E; \mathbb{R})$, we have

$$\begin{aligned}
&(f_1 \wedge \cdots \wedge f_n \wedge f_{n+1})(x_1, \dots, x_n, x_{n+1}) \\
=&(u \wedge_{n,1} f_{n+1})(x_1, \dots, x_n, x_{n+1}) \\
=& \sum_{\sigma \in S_{n,1}} (\operatorname{sgn} \sigma) u(x_{\sigma(1)}, \dots, x_{\sigma(n)}) f_{n+1}(x_{\sigma(n+1)}) \\
=& \sum_{\sigma \in S_{n,1}} (\operatorname{sgn} \sigma) \left(\sum_{\tau \in S_n} (\operatorname{sgn} \tau) f_1(x_{(\sigma\tau)(1)}) \cdots f_n(x_{(\sigma\tau)(n)}) \right) f_{n+1}(x_{\sigma(n+1)}) \\
=& \sum_{\rho \in S_{n+1}} (\operatorname{sgn} \rho) f_1(x_{\rho(1)}) \cdots f_n(x_{\rho(n)}) f_{n+1}(x_{\rho(n+1)}),
\end{aligned}$$

thus the claim is true for $n + 1$. $\square$

Let $f_1, \dots, f_n \in E^*$ and $x_1, \dots, x_n \in E$ and put

$$a_{i,j} = f_i(x_j), \qquad 1 \leq i, j \leq n;$$

$a \in \mathrm{Mat}_n(\mathbb{R})$. The Leibniz formula for the determinant of an $n \times n$ matrix tells us

$$\det a = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} f_i(x_{\sigma(j)}).$$

Then Theorem 20 gives

$$\det(f_i(x_j))_{1 \leq i,j \leq n} = (f_1 \wedge \cdots \wedge f_n)(x_1, \dots, x_n).$$

Lemma 21. If $f_1, \dots, f_n \in E^*$ are linearly independent then there are $x_1, \dots, x_n \in E$ such that

$$f_i(x_j) = \delta_{i,j}, \qquad 1 \leq i, j \leq n.$$

Theorem 22. $f_1, \dots, f_n \in E^*$ are linearly dependent if and only if

$$f_1 \wedge \cdots \wedge f_n = 0.$$

Proof. Suppose $f_1, \dots, f_n$ are linearly dependent, say, for some $\lambda_i \in \mathbb{R}$, $i \neq k$,

$$f_k = \sum_{i \neq k} \lambda_i f_i.$$

Then, as $f_i \wedge f_i = 0$,

$$f_1 \wedge \cdots \wedge f_n = 0.$$

Suppose that $f_1, \dots, f_n \in E^*$ are linearly independent. By Lemma 21, there are $x_1, \dots, x_n \in E$ such that

$$f_i(x_j) = \delta_{i,j}, \qquad 1 \leq i, j \leq n.$$

Then $\det(f_i(x_j)) = 1$, and hence

$$(f_1 \wedge \cdots \wedge f_n)(x_1, \dots, x_n) = 1,$$

so $f_1 \wedge \cdots \wedge f_n$ is not identically 0. □

4 *k*-tuples

We now take $E = \mathbb{R}^k$. For $1 \leq i \leq k$ define $\xi_i \in (\mathbb{R}^k)^*$ by

$$\xi_i(x_1, \dots, x_k) = x_i, \qquad (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Let $e_i = (0, \dots, \underbrace{1}_{i}, \dots, 0) \in \mathbb{R}^k$ for $1 \leq i \leq k$, in other words,

$$\xi_i(e_j) = \delta_{i,j}, \qquad 1 \leq i, j \leq k.$$

For $x \in \mathbb{R}^k$,

$$x = \sum_{1 \leq i \leq k} \xi_i(x) e_i.$$

Theorem 23. (i) If $f \in \mathscr{L}_p(\mathbb{R}^k; \mathbb{R})$ then for $(x_1, \dots, x_k) \in (\mathbb{R}^k)^p$,

$$f(x_1, \dots, x_p) = \sum_{1 \leq i_1, \dots, i_p \leq k} f(e_{i_1}, \dots, e_{i_p}) \xi_{i_1}(x_1) \cdots \xi_{i_p}(x_p).$$

(ii) If $f \in \mathscr{A}_p(\mathbb{R}^k; \mathbb{R})$ then

$$f = \sum_{1 \leq i_1 < \cdots < i_p \leq k} f(e_{i_1}, \dots, e_{i_p}) \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}.$$

(iii)

$$\dim \mathscr{A}_p(\mathbb{R}^k; \mathbb{R}) = \binom{k}{p}.$$

(iv) If $f \in \mathscr{A}_k(\mathbb{R}^k; \mathbb{R})$ then

$$f = f(e_1, \dots, e_n) \xi_1 \wedge \cdots \wedge \xi_k.$$

Proof. (i) Let $f \in \mathscr{L}_p(\mathbb{R}^k;\mathbb{R})$. For $(x_1,\dots,x_p) \in (\mathbb{R}^k)^p$, because $f : (\mathbb{R}^k)^p \to \mathbb{R}$ is multilinear,

$$\begin{aligned} f(x_1,\dots,x_p) &= f\left(\sum_{1\le i_1\le k} \xi_{i_1}(x_1)e_{i_1},\dots,\sum_{1\le i_p\le k} \xi_{i_p}(x_p)e_{i_p}\right) \\ &= \sum_{1\le i_1,\dots,i_p\le k} \xi_{i_1}(x_1)\cdots\xi_{i_p}(x_p)f(e_{i_1},\dots,e_{i_p}). \end{aligned}$$

(ii) Let $f \in \mathscr{A}_p(\mathbb{R}^k;\mathbb{R})$. Then $f = A_p f$ (Theorem 9),

$$f = \frac{1}{p!}\sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\sigma f,$$

so for $(x_1,\dots,x_p) \in (\mathbb{R}^k)^p$, applying Theorem 20,

$$\begin{aligned} f(x_1,\dots,x_p) &= \frac{1}{p!}\sum_{\sigma\in S_p} (\operatorname{sgn}\sigma) f(x_{\sigma(1)},\dots,x_{\sigma(p)}) \\ &= \frac{1}{p!}\sum_{\sigma\in S_p} (\operatorname{sgn}\sigma) \sum_{1\le i_1,\dots,i_p\le k} \xi_{i_1}(x_{\sigma(1)})\cdots\xi_{i_p}(x_{\sigma(p)})f(e_{i_1},\dots,e_{i_p}) \\ &= \frac{1}{p!}\sum_{1\le i_1,\dots,i_p\le k} f(e_{i_1},\dots,e_{i_p}) \sum_{\sigma\in S_p} (\operatorname{sgn}\sigma)\xi_{i_1}(x_{\sigma(1)})\cdots\xi_{i_p}(x_{\sigma(p)}) \\ &= \frac{1}{p!}\sum_{1\le i_1,\dots,i_p\le k} f(e_{i_1},\dots,e_{i_p})(\xi_{i_1}\wedge\cdots\wedge\xi_{i_p})(x_1,\dots,x_p). \end{aligned}$$

Since f is alternating, $i_r = i_s$ for $r \neq s$ implies $f(e_{i_1},\dots,e_{i_p}) = 0$. Let

$$\mathscr{I}_{p,k} = \{I \subset \{1,\dots,k\} : |I| = p\};$$

For $I \in \mathscr{I}_{p,k}$, define $I_1,\dots,I_p$ by $I = \{I_1,\dots,I_p\}$ and $I_1 < \cdots < I_p$. Then, applying Theorem 16, as $|S_I| = p!$,

$$\begin{aligned} f &= \frac{1}{p!}\sum_{I\in\mathscr{I}_{p,k}}\sum_{\tau\in S_I} f(e_{\tau(I_1)},\dots,e_{\tau(I_p)})\xi_{\tau(I_1)}\wedge\cdots\wedge\xi_{\tau(I_p)} \\ &= \frac{1}{p!}\sum_{I\in\mathscr{I}_{p,k}}\sum_{\tau\in S_I} (\operatorname{sgn}\tau)f(e_{I_1},\dots,e_{I_p})(\operatorname{sgn}\tau)\xi_{I_1}\wedge\cdots\wedge\xi_{I_p} \\ &= \sum_{I\in\mathscr{I}_{p,k}} f(e_{I_1},\dots,e_{I_p})\xi_{I_1}\wedge\cdots\wedge\xi_{I_p}. \end{aligned}$$

proving the claim.

(iii) $|\mathscr{I}_{p,k}| = \binom{k}{p}$.

(iv) This follows from (ii) and the fact that $|\mathscr{I}_{p,k}| = 1$ with $\mathscr{I}_{p,k} = \{\{1,\dots,k\}\}$. $\square$

References

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