

# Pell's equation

Jordan Bell

October 30, 2016

## 1 Side and diagonal numbers

Heath [17, pp. 117–118]:

$$\begin{aligned} a_1 &= 1, & d_1 &= 1. \\ a_n &= a_{n-1} + d_{n-1}, & d_n &= 2a_{n+1} + d_{n+1}. \end{aligned}$$

$$\begin{aligned} d_n^2 - 2a_n^2 &= 4a_{n-1}^2 + 4a_{n-1}d_{n-1} + d_{n-1}^2 - 2(a_{n-1}^2 + 2a_{n-1}d_{n-1} + d_{n-1}^2) \\ &= 4a_{n+1}^2 + 4a_{n+1}d_{n+1} + d_{n+1}^2 - 2a_{n-1}^2 - 4a_{n-1}d_{n-1} - 2d_{n-1}^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2). \end{aligned}$$

As  $d_1^2 - 2a_1^2 = -1$ ,

$$d_n^2 - 2a_n^2 = (-1)^n.$$

$$\begin{aligned} \begin{pmatrix} a_n \\ d_n \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ d_{n-1} \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ d_{n-1} \end{pmatrix}. \\ \begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} &= A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

$$A = PDP^{-1}.$$

$$D = \begin{pmatrix} 1 - \sqrt{2} & 0 \\ 0 & 1 + \sqrt{2} \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

$$A^n = \begin{pmatrix} \frac{(1-\sqrt{2})^n}{2} + \frac{(1+\sqrt{2})^n}{2} & -\frac{(1-\sqrt{2})^n}{2\sqrt{2}} + \frac{(1+\sqrt{2})^n}{2\sqrt{2}} \\ -\frac{(1-\sqrt{2})^n}{\sqrt{2}} + \frac{(1+\sqrt{2})^n}{\sqrt{2}} & \frac{(1-\sqrt{2})^n}{2} + \frac{(1+\sqrt{2})^n}{2} \end{pmatrix}.$$

$$\begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1-\sqrt{2})^n - \frac{(1-\sqrt{2})^n}{2\sqrt{2}} + \frac{1}{2}(1+\sqrt{2})^n + \frac{(1+\sqrt{2})^n}{2\sqrt{2}} \\ \frac{1}{2}(1-\sqrt{2})^n - \frac{(1-\sqrt{2})^n}{\sqrt{2}} + \frac{1}{2}(1+\sqrt{2})^n + \frac{(1+\sqrt{2})^n}{\sqrt{2}} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} a_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} a_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}, \quad \begin{pmatrix} a_5 \\ d_5 \end{pmatrix} = \begin{pmatrix} 29 \\ 41 \end{pmatrix}.$$

## 2 Diophantus

If  $x^2 - Ay^2 = 1$  and  $y = m(x+1)$  for rational  $m$ , then  $y^2 = m^2(x^2 + 2x + 1)$ , then  $x^2 - Am^2(x^2 + 2x + 1) = 1$ , then  $(Am^2 - 1)x^2 + 2Am^2x + Am^2 + 1 = 0$ . Write  $(x+1)(px+q) = (Am^2 - 1)x^2 + 2Am^2x + Am^2 + 1$ . Then  $px^2 + (p+q)x + q = (Am^2 - 1)x^2 + 2Am^2x + Am^2 + 1$ . Then  $p = Am^2 - 1$ ,  $q = Am^2 + 1$ , and so  $p+q = Am^2 - 1 + Am^2 + 1 = 2Am^2$ . Thus if  $x \neq -1$  then  $px+q = 0$  and hence  $(Am^2 - 1)x + Am^2 + 1 = 0$ . Hence  $(Am^2 - 1)x = -(Am^2 + 1)$  and so  $x = -\frac{Am^2+1}{Am^2-1}$ . Thus

$$y = m(x+1) = m \left( -\frac{Am^2+1}{Am^2-1} + 1 \right) \frac{m}{Am^2-1} (-Am^2-1+Am^2-1) = \frac{-2m}{Am^2-1}.$$

Therefore for rational  $m$ ,

$$x = -\frac{Am^2+1}{Am^2-1}, \quad y = \frac{-2m}{Am^2-1}$$

satisfy  $x^2 - Ay^2 = 1$ . cf. Heath [17, pp. 68–69], Nesselmann [24, p. 331].

Diophantus V.11:  $30x^2 + 1 = y^2$ . Say  $y = 5x + 1$ .  $y^2 = 25x^2 + 10x + 1$ . Then  $5x^2 - 10x = 0$ , so  $x = 0$  or  $5x - 10 = 0$ , i.e.  $x = 0$  or  $x = 2$ . Hence  $x = 0, y = 1$  and  $x = 2, y = 11$  satisfy  $30x^2 + 1 = y^2$ .

Diophantus V.14 [17, pp. 211–212].  $34y^2 + 1 = x^2$ . Say  $x = 6y - 1$ .  $x^2 = 36y^2 - 12y + 1$ . Then  $2y^2 - 12y = 0$ , i.e.  $y(y - 6) = 0$  so  $y = 0$  or  $y = 6$ . Then  $x = -1, y = 0$  and  $x = 35, y = 6$  satisfy  $34y^2 + 1 = x^2$ .

## 3 Fermat

Fermat, February 1657 [31, p. 29]:

Given any number not a square, then there are an infinite number of squares which, when multiplied by the given number, make a square when unity is added.

Example. Given 3, a nonsquare number; this number multiplied by the square number 1, and 1 being added, produces 4, which is a square.

Moreover, the same 3 multiplied by the square 16, with 1 added makes 49, which is a square.

And instead of 1 and 16, an infinite number of squares may be found showing the same property; I demand, however, a general rule, any number being given which is not a square.

It is sought, for example, to find a square which when multiplied into 149, 109, 433, etc., becomes a square when unity is added.

## 4 Wallis

Wallis [1, p. 546].  
Stedall [29]

## 5 Brouncker

Weil [32, pp. 92–99].

## 6 Ozanam

Ozanam [25, pp. 503–516], Liv. III, Quest. XXVI.

## 7 Continued fractions

Let

$$[a_0, a_1] = a_0 + \frac{1}{a_1}$$

and

$$[a_0, \dots, a_{n-1}, a_n] = \left[ a_0, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n} \right].$$

Then for  $1 \leq m < n$ ,

$$[a_0, \dots, a_n] = [a_0, \dots, a_{m-1}, [a_m, \dots, a_n]].$$

Define

$$p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

and for  $n \geq 2$ ,

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Hardy and Wright [13, p. 130, Theorem 149]. For  $n \geq 0$ ,

$$[a_0, \dots, a_n] = \frac{p_n}{q_n}.$$

For  $n \geq 1$ ,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

For  $n \geq 2$ ,

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

For  $n \geq 0$  let

$$a'_n = [a_n, a_{n+1}, \dots].$$

For  $x = [a_0, a_1, \dots]$ ,

$$x = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}, \quad n \geq 2.$$

Hardy and Wright [13, p. 144, Theorem 176]. A continued fraction  $[a_0, a_1, \dots]$  is said to be **periodic** if there is some  $L \geq 0$  and some  $k \geq 1$  such that  $a_{l+k} = a_l$  for all  $l \geq L$ .

**Theorem 1.** If  $x = [a_0, a_1, \dots]$  is a periodic continued fraction, then  $x$  is a quadratic surd.

*Proof.* Let

$$\overline{a_L, \dots, a_{L+k-1}} = [a_L, a_{L+1}, \dots] = a'_L.$$

Thus

$$\begin{aligned} [a_0, \dots, a_{L-1}, a_L, a_{L+1}, \dots] &= [a_0, \dots, a_{L-1}, a'_L] \\ &= [a_0, \dots, a_{L-1}, \overline{a_L, \dots, a_{L+k-1}}]. \end{aligned}$$

As  $a_{L+k} = a_L, a_{L+k+1} = a_{L+1}, \dots$ ,

$$\begin{aligned} a'_L &= [a_L, a_{L+1}, \dots] \\ &= [a_L, a_{L+1}, \dots, a_{L+k-1}, a_L, a_{L+1}, \dots] \\ &= [a_L, a_{L+1}, \dots, a_{L+k-1}, a'_L]. \end{aligned}$$

Let

$$\frac{p'}{q'} = [a_L, a_{L+1}, \dots, a_{L+k-1}], \quad \frac{p''}{q''} = [a_L, a_{L+1}, \dots, a_{L+k-2}].$$

For  $t = [a_L, a_{L+1}, \dots, a_{L+k-1}, a'_L]$ ,

$$t = \frac{a'_L p' + p''}{a'_L q' + q''} = \frac{p' t + p''}{q' t + q''}.$$

Hence  $q' t^2 + q'' t = p' t + p''$ , so  $q' t^2 + (q'' - p') t - p'' = 0$ . For  $x = [a_0, a_1, \dots]$ ,

$$x = \frac{a'_L p_{L-1} + p_{L-2}}{a'_L q_{L-1} + q_{L-2}}.$$

Then

$$a'_L = \frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - p_{L-1}}.$$

Thus, with  $t = a'_L$ ,

$$q' \left( \frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - p_{L-1}} \right)^2 + (q'' - p') \frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - p_{L-1}} - p'' = 0.$$

Then

$$q' (p_{L-2} - q_{L-2} x)^2 + (q'' - p') (p_{L-2} - q_{L-2} x) (q_{L-1} x - p_{L-1}) - p'' (q_{L-1} x - p_{L-1})^2 = 0.$$

Therefore there are integers  $a, b, c$  such that

$$ax^2 + bx + c = 0.$$

This means that  $x$  is a quadratic surd, as  $x$  is irrational.  $\square$

**Example.** Say  $x = [3, 2, 7, 4, \overline{5, 1, 12}]$ .  $L = 4, k = 3$ .

$$\frac{p'}{q'} = [5, 1, 12] = \frac{77}{13}, \quad \frac{p''}{q''} = [5, 1] = \frac{6}{1}.$$

$$\frac{p_{L-1}}{q_{L-1}} = \frac{p_3}{q_3} = [3, 2, 7, 4] = \frac{215}{62}, \quad \frac{p_{L-2}}{q_{L-2}} = \frac{p_2}{q_2} = [3, 2, 7] = \frac{52}{15}.$$

Then

$$\begin{aligned} & q'(p_{L-2} - q_{L-2}x)^2 + (q'' - p')(p_{L-2} - q_{L-2}x)(q_{L-1}x - p_{L-1}) - p''(q_{L-1}x - p_{L-1})^2 \\ &= 13(52 - 15x)^2 + (1 - 77)(52 - 15x)(62x - 215) - 6(62x - 215)^2 \\ &= 50541x^2 - 350444x + 607482. \end{aligned}$$

Hence  $x = [3, 2, 7, 4, \overline{5, 1, 12}]$  satisfies

$$50541x^2 - 350444x + 607482 = 0.$$

In fact,

$$x = \frac{175222 + \sqrt{1522}}{50541}.$$

Hardy and Wright [13, p. 144, Theorem 177].

**Theorem 2.** If  $x$  is a quadratic surd, then the continued fraction of  $x$  is periodic.

**Example.** Say  $x^2 = 218$ .  $14^2 = 196$ .

$$\sqrt{218} = 14 + \sqrt{218} - 14 = 14 + \frac{1}{\frac{1}{\sqrt{218} - 14}}.$$

$$(\sqrt{218} - 14)(\sqrt{218} + 14) = 218 - 196 = 22, \quad \frac{1}{\sqrt{218} - 14} = \frac{\sqrt{218} + 14}{22}.$$

We do not need to compute the decimal expansion of  $\sqrt{218}$ ; we merely have to calculate  $\lfloor \frac{\sqrt{218}+14}{22} \rfloor$ . Using  $14 < \sqrt{218} < 15$ ,

$$\frac{1}{\sqrt{218} - 14} = 1 + \frac{\sqrt{218} + 14}{22} - 1 = 1 + \frac{\sqrt{218} - 8}{22}.$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{\sqrt{218} - 8}{22}} = 14 + \frac{1}{1 + \frac{1}{\frac{22}{\sqrt{218} - 8}}}$$

$$(\sqrt{218} - 8)(\sqrt{218} + 8) = 218 - 64 = 154, \quad \frac{1}{\sqrt{218} - 8} = \frac{\sqrt{218} + 8}{154}.$$

Then

$$\frac{22}{\sqrt{218} - 8} = \frac{22\sqrt{218} + 176}{154}.$$

Using that  $14 < \sqrt{218} < 15$ ,

$$\frac{22}{\sqrt{218} - 8} = 3 + \frac{22\sqrt{218} + 176}{154} - 3 = 3 + \frac{22\sqrt{218} - 286}{154}.$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{22\sqrt{218} - 286}{154}}} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{\frac{22\sqrt{218} - 286}{154}}}}.$$

$$(22\sqrt{218} - 286)(22\sqrt{218} + 286) = 22^2 \cdot 218 - 286^2 = 23716,$$

$$\frac{1}{22\sqrt{218} - 286} = \frac{22\sqrt{218} + 286}{23716}.$$

$$\frac{154}{22\sqrt{218} - 286} = \frac{3388\sqrt{218} + 44044}{23716}.$$

Using  $14 < \sqrt{218} < 15$ ,

$$\frac{154}{22\sqrt{218} - 286} = 3 + \frac{3388\sqrt{218} + 44044}{23716} - 3 = 3 + \frac{3388\sqrt{218} - 27104}{23716}.$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{3388\sqrt{218} - 27104}{23716}}}} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{\frac{3388\sqrt{218} - 27104}{23716}}}}}.$$

$$(3388\sqrt{218} - 27104)(3388\sqrt{218} + 27104) = 3388^2 \cdot 218 - 27104^2 = 1767695776,$$

$$\frac{1}{3388\sqrt{218} - 27104} = \frac{3388\sqrt{218} + 27104}{1767695776}.$$

$$\frac{23716}{3388\sqrt{218} - 27104} = 23716 \cdot \frac{3388\sqrt{218} + 27104}{1767695776}.$$

Using  $14 < \sqrt{218} < 15$ ,

$$\begin{aligned}\frac{23716}{3388\sqrt{218} - 27104} &= 1 + 23716 \cdot \frac{3388\sqrt{218} + 27104}{1767695776} - 1 \\ &= 1 + \frac{80349808\sqrt{218} - 1124897312}{1767695776}.\end{aligned}$$

Then

$$\begin{aligned}\sqrt{218} &= 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{80349808\sqrt{218} - 1124897312}{1767695776}}}}} \\ &= 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1767695776}}}}}.\end{aligned}$$

$$(80349808\sqrt{218} - 1124897312)(80349808\sqrt{218} + 1124897312) = 142034016204011008.$$

$$\frac{1767695776}{80349808\sqrt{218} - 1124897312} = 1767695776 \cdot \frac{80349808\sqrt{218} + 1124897312}{142034016204011008}.$$

Using  $14 < \sqrt{218} < 15$ , the floor of the above quantity is 28. Hence

$$\begin{aligned}\frac{1767695776}{80349808\sqrt{218} - 1124897312} &= 28 + \frac{142034016204011008\sqrt{218} - 1988476226856154112}{142034016204011008} \\ &= 28 + \sqrt{218} - 14.\end{aligned}$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{28 + \sqrt{218} - 14}}}}}$$

Thus for  $x = \sqrt{218}$ ,

$$x - 14 = \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{28 + x - 14}}}}}.$$

Thus for  $t = x - 14$ ,

$$t = \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{28 + t}}}}}.$$

Therefore  $t = [0, \overline{1, 3, 3, 1, 28}]$ . Hence  $x = 14 + t = [14, \overline{1, 3, 3, 1, 28}]$ :

$$\sqrt{218} = [14, \overline{1, 3, 3, 1, 28}].$$

## 8 Euler

Euler, *Algebra* [10], Part II, Chapter VII.

## 9 Lagrange

Konen [20, pp. 75–77].

## 10 Chakravala

Hankel [12, pp. 200–203]

Strachey [30, pp. 36–53]. Dickson [5, pp. 349–350].

Colebrooke [3, pp. 170–184]

Colebrooke [3, pp. 363–372]

Datta and Singh [4, II, pp. 93–99]

Datta and Singh [4, II, pp. 146–161]

Datta and Singh [4, II, pp. 161–172]

Suppose that  $p_n, q_n$  are relatively prime and

$$Aq_n^2 + s_n = p_n^2.$$

If  $d$  is a common factor of  $q_n$  and  $s_n$  then  $d \mid p_n^2$ , so  $d$  is a common factor of  $p_n^2$  and  $q_n^2$ , which implies that  $p_n$  and  $q_n$  have a common factor, a contradiction.



Therefore  $q_n$  and  $s_n$  are relatively prime. Because  $q_n$  and  $s_n$  are relatively prime, by the Kuttaka algorithm there are some  $\rho_n, \rho'_n$  satisfying  $-q_n\rho_n + s_n\rho'_n = p_n$ . For  $r_n = \rho_n + k_ns_n$ ,  $r'_n = \rho'_n + k_nq_n$ ,

$$\begin{aligned} -q_nr_n + s_nr'_n &= -q_n(\rho_n + k_ns_n) + s_n(\rho'_n + k_nq_n) \\ &= -q_n\rho_n - k_nq_ns_n + s_n\rho'_n + k_nq_ns_n \\ &= -q_n\rho_n + s_n\rho'_n \\ &= p_n. \end{aligned}$$

Take  $r_n < \sqrt{A} < r_n + |s_n|$ .  $r'_n = \frac{p_n + q_nr_n}{s_n}$ . Let

$$q_{n+1} = r'_n, \quad p_{n+1} = \frac{p_nq_{n+1} - 1}{q_n}, \quad s_{n+1} = p_{n+1}^2 - Aq_{n+1}^2.$$

**Example.**  $69y^2 + 1 = x^2$ .  $A = 69$ .

$Aq_0^2 + s_0 = p_0^2$ :  $p_0 = 8, q_0 = 1, s_0 = -5$ .

$p_0 + q_0\rho_0 = \rho'_0s_0$  is equivalent to  $8 + \rho_0 = -5\rho'_0$ . It is satisfied by  $\rho_0 = -8, \rho'_0 = 0$ . Take  $r_0 = -8 - 5k_0 = 7$ .  $r'_0 = \frac{p_0 + q_0r_0}{s_0} = \frac{8+1 \cdot 7}{-5} = -3$ .

$q_1 = -3$ .

$$p_1 = \frac{p_0q_1 - 1}{q_0} = \frac{8 \cdot -3 - 1}{1} = -25.$$

$$s_1 = p_1^2 - Aq_1^2 = 4.$$

$p_1 + q_1\rho_1 = \rho'_1s_1$  is equivalent to  $-25 - 3\rho_1 = 4\rho'_1$ . This is satisfied by  $\rho_1 = 1, \rho'_1 = -7$ . Take  $r_1 = 1 + 4k_1 = 5$ . Then  $r'_1 = \frac{p_1 + q_1r_1}{s_1} = \frac{-25 - 3 \cdot 5}{4} = -10$ .

$q_2 = -10$ .

$$p_2 = \frac{p_1q_2 - 1}{q_1} = \frac{-25 \cdot -10 - 1}{-3} = -83.$$

$$s_2 = p_2^2 - Aq_2^2 = -11.$$

$p_2 + q_2\rho_2 = \rho'_2s_2$  is equivalent to  $-83 - 10\rho_2 = -11\rho'_2$ . This is satisfied by  $\rho_2 = 6, \rho'_2 = 13$ . Take  $r_2 = 6 - 11k_2 = 6$ . Then  $r'_2 = \frac{p_2 + q_2r_2}{s_2} = \frac{-83 - 10 \cdot 6}{-11} = 13$ .

$q_3 = 13$ .

$$p_3 = \frac{p_2q_3 - 1}{q_2} = \frac{-83 \cdot 13 - 1}{-10} = 108.$$

$$s_3 = p_3^2 - Aq_3^2 = 3.$$

$p_3 + q_3\rho_3 = \rho'_3s_3$  is equivalent to  $108 + 13\rho_3 = 3\rho'_3$ . This is satisfied by  $\rho_3 = 0, \rho'_3 = 36$ . Take  $r_3 = 36 + 3k_3 = 6$ . Then  $r'_3 = \frac{p_3 + q_3r_3}{s_3} = \frac{108 + 13 \cdot 6}{3} = 62$ .

$q_4 = 62$ .

$$p_4 = \frac{p_3q_4 - 1}{q_3} = \frac{108 \cdot 62 - 1}{13} = 515.$$

$$s_4 = p_4^2 - Aq_4^2 = -11.$$

$p_4 + q_4\rho_4 = \rho'_4s_4$  is equivalent to  $515 + 62\rho_4 = -11\rho'_4$ . This is satisfied by  $\rho_4 = 5, \rho'_4 = -75$ . Take  $r_4 = 5 - 11k_4 = 5$ . Then  $r'_4 = \frac{p_4 + q_4r_4}{s_4} = \frac{515 + 62 \cdot 5}{-11} = -75$ .

$$q_5 = -75.$$

$$p_5 = \frac{p_4 q_5 - 1}{q_4} = \frac{515 \cdot -75 - 1}{62} = -623.$$

$$s_5 = p_5^2 - A q_5^2 = 4.$$

$p_5 + q_5 \rho_5 = \rho'_5 s_5$  is equivalent to  $-623 - 75 \rho_5 = 4 \rho'_5$ . This is satisfied by  $\rho_5 = 3, \rho'_5 = -212$ . Take  $r_5 = 3 + 4k_5 = 7$ . Then  $r'_5 = \frac{p_5 + q_5 r_5}{s_5} = \frac{-623 - 75 \cdot 7}{4} = -287$ .

$$q_6 = -287.$$

$$p_6 = \frac{p_5 q_6 - 1}{q_5} = \frac{-623 \cdot -287 - 1}{-75} = -2384.$$

$$s_6 = p_6^2 - A q_6^2 = -5.$$

$p_6 + q_6 \rho_6 = \rho'_6 s_6$  is equivalent to  $-2384 - 287 \rho_6 = -5 \rho'_6$ . This is satisfied by  $\rho_6 = 3, \rho'_6 = 649$ . Take  $r_6 = 3 - 5k_6 = 8$ . Then  $r'_6 = \frac{p_6 + q_6 r_6}{s_6} = \frac{-2384 - 287 \cdot 8}{-5} = 936$ .  
 $q_7 = 936$ .

$$p_7 = \frac{p_6 q_7 - 1}{q_6} = \frac{-2384 \cdot 936 - 1}{-287} = 7775.$$

$$s_7 = p_7^2 - A q_7^2 = 1.$$

Therefore

$$7775^2 - 69 \cdot 936^2 = 1.$$

Thus  $\sqrt{69} \sim \frac{7775}{936}$ .

**Example.**  $91y^2 + 1 = x^2$ .  $A = 91$ .

$$A q_0^2 + s_0 = p_0^2: p_0 = 10, q_0 = 1, s_0 = 9.$$

$p_0 + q_0 \rho_0 = \rho'_0 s_0$  is equivalent to  $10 + \rho_0 = 9 \rho'_0$ . This is satisfied by  $\rho_0 = -10, \rho'_0 = 0$ . Take  $r_0 = -10 + 9k_0 = 8$ . Then  $r'_0 = \frac{p_0 + q_0 r_0}{s_0} = \frac{10 + 1 \cdot 8}{9} = 2$ .

$$q_1 = 2.$$

$$p_1 = \frac{p_0 q_1 - 1}{q_0} = \frac{10 \cdot 2 - 1}{1} = 19.$$

$$s_1 = p_1^2 - A q_1^2 = -3.$$

$p_1 + q_1 \rho_1 = \rho'_1 s_1$  is equivalent with  $19 + 2 \rho_1 = -3 \rho'_1$ . This is satisfied by  $\rho_1 = 1, \rho'_1 = -7$ . Take  $r_1 = 1 - 3k_1 = 7$ . Then  $r'_1 = \frac{p_1 + q_1 r_1}{s_1} = \frac{19 + 2 \cdot 7}{-3} = -11$ .

$$q_2 = -11.$$

$$p_2 = \frac{p_1 q_2 - 1}{q_1} = \frac{19 \cdot -11 - 1}{2} = -105.$$

$$s_2 = p_2^2 - A q_2^2 = 14.$$

$p_2 + q_2 \rho_2 = \rho'_2 s_2$  is equivalent with  $-105 - 11 \rho_2 = 14 \rho'_2$ . This is satisfied by  $\rho_2 = 7, \rho'_2 = -13$ . Take  $r_2 = 7, r'_2 = -13$ .

$$q_3 = -13.$$

$$p_3 = \frac{p_2 q_3 - 1}{q_2} = \frac{-105 \cdot -13 - 1}{-11} = -124.$$

$$s_3 = p_3^2 - A q_3^2 = -3.$$

$p_3 + q_3\rho_3 = \rho'_3 s_3$  is equivalent with  $-124 - 13\rho_3 = -3\rho'_3$ . This is satisfied by  $\rho_3 = 2, \rho'_3 = 50$ . Take  $r_3 = 2 - 3k_3 = 8$ . Then  $r'_3 = \frac{p_3 + q_3 r_3}{s_3} = \frac{-124 - 13 \cdot 8}{-3} = 76$ .  
 $q_4 = 76$ .

$$p_4 = \frac{p_3 q_4 - 1}{q_3} = \frac{-124 \cdot 76 - 1}{-13} = 725.$$

$$s_4 = p_4^2 - A q_4^2 = 9.$$

$p_4 + q_4\rho_4 = \rho'_4 s_4$  is equivalent with  $725 + 76\rho_4 = 9\rho'_4$ . This is satisfied by  $\rho_4 = 1, \rho'_4 = 89$ . Take  $r_4 = 1, r'_4 = 89$ .  
 $q_5 = 89$ .

$$p_5 = \frac{p_4 q_5 - 1}{q_4} = \frac{725 \cdot 89 - 1}{76} = 849.$$

$$s_5 = p_5^2 - A q_5^2 = -10.$$

$p_5 + q_5\rho_5 = \rho'_5 s_5$  is equivalent with  $849 + 89\rho_5 = -10\rho'_5$ . This is satisfied by  $\rho_5 = 9, \rho'_5 = -165$ . Take  $r_5 = 9, r'_5 = -165$ .  
 $q_6 = -165$ .

$$p_6 = \frac{p_5 q_6 - 1}{q_5} = \frac{849 \cdot -165 - 1}{89} = -1574.$$

$$s_6 = p_6^2 - A q_6^2 = 1.$$

Therefore

$$1574^2 - 91 \cdot 165^2 = 1.$$

Thus  $\sqrt{91} \sim \frac{1574}{165}$ .

**Example.**  $109y^2 + 1 = x^2$ .  $A = 109$ .

$Ay_0^2 + s_0 = x_0^2$ :  $x_0 = 10, y_0 = 1, s_0 = -9$ .

$x_0 + y_0\rho_0 = s_0\rho'_0$  is equivalent to  $10 + \rho_0 = -9\rho'_0$ . This is satisfied by  $\rho_0 = -10, \rho'_0 = 0$ . Take  $r_0 = -10 + 9k_0 = 8$ . Then  $r'_0 = \frac{x_0 + y_0 r_0}{s_0} = \frac{10 + 1 \cdot 8}{-9} = -2$ .  
 $y_1 = -2$ .

$$x_1 = \frac{x_0 y_1 - 1}{y_0} = \frac{10 \cdot -2 - 1}{1} = -21.$$

$$s_1 = x_1^2 - A y_1^2 = 5.$$

$x_1 + y_1\rho_1 = \rho'_1 s_1$  is equivalent with  $-21 - 2\rho_1 = 5\rho'_1$ . This is satisfied by  $\rho_1 = 2, \rho'_1 = -5$ . Take  $r_1 = 2 + 5k_1 = 7$ . Then  $r'_1 = \frac{x_1 + y_1 r_1}{s_1} = \frac{-21 - 2 \cdot 7}{5} = -7$ .  
 $y_2 = -7$ .

$$x_2 = \frac{x_1 y_2 - 1}{y_1} = \frac{-21 \cdot -7 - 1}{-2} = -73.$$

$$s_2 = x_2^2 - A y_2^2 = -12.$$

$x_2 + y_2\rho_2 = \rho'_2 s_2$  is equivalent with  $-73 - 7\rho_2 = -12\rho'_2$ . This is satisfied by  $\rho_2 = 5, \rho'_2 = 9$ . Take  $r_2 = 5, r'_2 = 9$ .  
 $y_3 = 9$ .

$$x_3 = \frac{x_2 y_3 - 1}{y_2} = \frac{-73 \cdot 9 - 1}{-7} = 94.$$

$$s_3 = x_3^2 - Ay_3^2 = 7.$$

$x_3 + y_3\rho_3 = \rho'_3 s_3$  is equivalent with  $94 + 9\rho_3 = 7\rho'_3$ . This is satisfied by  $\rho_3 = 2, \rho'_3 = 16$ . Take  $r_3 = 2 + 7k_3 = 9$ . Then  $r'_3 = \frac{x_3 + y_3 r_3}{s_3} = \frac{94 + 9 \cdot 9}{7} = 25$ .  
 $y_4 = 25$ .

$$x_4 = \frac{x_3 y_4 - 1}{y_3} = \frac{94 \cdot 25 - 1}{9} = 261.$$

$$s_4 = x_4^2 - Ay_4^2 = -4.$$

$x_4 + y_4\rho_4 = \rho'_4 s_4$  is equivalent with  $261 + 25\rho_4 = -4\rho'_4$ . This is satisfied by  $\rho_4 = 3, \rho'_4 = -84$ . Take  $r_4 = 3 - 4k_4 = 7$ . Then  $r'_4 = \frac{x_4 + y_4 r_4}{s_4} = \frac{261 + 25 \cdot 7}{-4} = -109$ .

$$y_5 = -109.$$

$$x_5 = \frac{x_4 y_5 - 1}{y_4} = \frac{261 \cdot -109 - 1}{25} = -1138.$$

$$s_5 = x_5^2 - Ay_5^2 = 15.$$

$x_5 + y_5\rho_5 = \rho'_5 s_5$  is equivalent with  $-1138 - 109\rho_5 = 15\rho'_5$ . This is satisfied by  $\rho_5 = 8, \rho'_5 = -134$ . Take  $r_5 = 8, r'_5 = -134$ .  
 $y_6 = -134$ .

$$x_6 = \frac{x_5 y_6 - 1}{y_5} = \frac{-1138 \cdot -134 - 1}{-109} = -1399.$$

$$s_6 = x_6^2 - Ay_6^2 = -3.$$

$x_6 + y_6\rho_6 = \rho'_6 s_6$  is equivalent with  $-1399 - 134\rho_6 = -3\rho'_6$ . This is satisfied by  $\rho_6 = 1, \rho'_6 = 511$ . Take  $r_6 = 1 - 3k_6 = 10$ . Then  $r'_6 = \frac{x_6 + y_6 r_6}{s_6} = \frac{-1399 - 134 \cdot 10}{-3} = 913$ .

$$y_7 = 913.$$

$$x_7 = \frac{x_6 y_7 - 1}{y_6} = \frac{-1399 \cdot 913 - 1}{-134} = 9532.$$

$$s_7 = x_7^2 - Ay_7^2 = 3.$$

$x_7 + y_7\rho_7 = \rho'_7 s_7$  is equivalent with  $9532 + 913\rho_7 = 3\rho'_7$ . This is satisfied by  $\rho_7 = 2, \rho'_7 = 3786$ . Take  $r_7 = 2 + 3k_7 = 8$ . Then  $r'_7 = \frac{x_7 + y_7 r_7}{s_7} = \frac{9532 + 913 \cdot 8}{3} = 5612$ .

$$y_8 = 5612.$$

$$x_8 = \frac{x_7 y_8 - 1}{y_7} = \frac{9532 \cdot 5612 - 1}{913} = 58591.$$

$$s_8 = x_8^2 - Ay_8^2 = -15.$$

$x_8 + y_8\rho_8 = \rho'_8 s_8$  is equivalent with  $58591 + 5612\rho_8 = -15\rho'_8$ . This is satisfied by  $\rho_8 = 7, \rho'_8 = -6525$ . Take  $r_8 = 7, r'_8 = -6525$ .

$$y_9 = -6525.$$

$$x_9 = \frac{x_8 y_9 - 1}{y_8} = \frac{58591 \cdot -6525 - 1}{5612} = -68123.$$

$$s_9 = x_9^2 - A y_9^2 = 4.$$

$x_9 + y_9 \rho_9 = \rho'_9 s_9$  is equivalent with  $-68123 - 6525 \rho_9 = 4 \rho'_9$ . This is satisfied by  $\rho_9 = 1, \rho'_9 = -18662$ . Take  $r_9 = 1 + 4k_9 = 9$ . Then  $r'_9 = \frac{x_9 + y_9 r_9}{s_9} = \frac{-68123 - 6525 \cdot 9}{4} = -31712$ .  
 $y_{10} = -31712$ .

$$x_{10} = \frac{x_9 y_{10} - 1}{y_9} = \frac{-68123 \cdot (-31712) - 1}{-6525} = -331083.$$

$$s_{10} = x_{10}^2 - A y_{10}^2 = -7.$$

$x_{10} + y_{10} \rho_{10} = \rho'_{10} s_{10}$  is equivalent with  $-331083 - 31712 \rho_{10} = -7 \rho'_{10}$ . This is satisfied by  $\rho_{10} = 5, \rho'_{10} = 69949$ . Take  $r_{10} = 5, r'_{10} = 69949$ .  
 $y_{11} = 69949$ .

$$x_{11} = \frac{x_{10} y_{11} - 1}{y_{10}} = \frac{-331083 \cdot 69949 - 1}{-31712} = 730289.$$

$$s_{11} = x_{11}^2 - A y_{11}^2 = 12.$$

$x_{11} + y_{11} \rho_{11} = \rho'_{11} s_{11}$  is equivalent with  $730289 + 69949 \rho_{11} = 12 \rho'_{11}$ . This is satisfied by  $\rho_{11} = 7, \rho'_{11} = 101661$ . Take  $r_{11} = 5, r'_{11} = 101661$ .  
 $y_{12} = 101661$ .

$$x_{12} = \frac{x_{11} y_{12} - 1}{y_{11}} = \frac{730289 \cdot 101661 - 1}{69949} = 1061372.$$

$$s_{12} = x_{12}^2 - A y_{12}^2 = -5.$$

$x_{12} + y_{12} \rho_{12} = \rho'_{12} s_{12}$  is equivalent with  $1061372 + 101661 \rho_{12} = -5 \rho'_{12}$ . This is satisfied by  $\rho_{12} = 3, \rho'_{12} = -273271$ . Take  $r_{12} = 3 - 5k_{12} = 8$ . Then  $r'_{12} = \frac{x_{12} + y_{12} r_{12}}{s_{12}} = \frac{1061372 + 101661 \cdot 8}{-5} = -374932$ .  
 $y_{13} = -374932$ .

$$x_{13} = \frac{x_{12} y_{13} - 1}{y_{12}} = \frac{1061372 \cdot (-374932) - 1}{101661} = -3914405.$$

$$s_{13} = x_{13}^2 - A y_{13}^2 = 9.$$

$x_{13} + y_{13} \rho_{13} = \rho'_{13} s_{13}$  is equivalent with  $-3914405 - 374932 \rho_{13} = 9 \rho'_{13}$ . This is satisfied by  $\rho_{13} = 1, \rho'_{13} = -476593$ . Take  $r_{13} = 1 + 9k_{13} = 10$ . Then  $r'_{13} = \frac{x_{13} + y_{13} r_{13}}{s_{13}} = \frac{-3914405 - 374932 \cdot 10}{9} = -851525$ .  
 $y_{14} = -851525$ .

$$x_{14} = \frac{x_{13} y_{14} - 1}{y_{13}} = \frac{-3914405 \cdot (-851525) - 1}{-374932} = -8890182.$$

$$s_{14} = x_{14}^2 - Ay_{14}^2 = -1.$$

$x_{14} + y_{14}\rho_{14} = \rho'_{14}s_{14}$  is equivalent with  $-8890182 - 851525\rho_{14} = -\rho'_{14}$ . This is satisfied by  $\rho_{14} = 0, \rho'_{14} = 8890182$ . Take  $r_{14} = -k_{14} = 10$ . Then  $r'_{14} = \frac{x_{14} + y_{14}r_{14}}{s_{14}} = \frac{-8890182 - 851525 \cdot 10}{-1} = 17405432$ .  
 $y_{15} = 17405432$ .

$$x_{15} = \frac{x_{14}y_{15} - 1}{y_{14}} = \frac{-8890182 \cdot 17405432 - 1}{-851525} = 181718045.$$

$$s_{15} = x_{15}^2 - Ay_{15}^2 = 9.$$

$$r_{15} = 8, r'_{15} = 35662389.$$

$$y_{16} = 35662389.$$

$$x_{16} = \frac{x_{15}y_{16} - 1}{y_{15}} = \frac{35662389 \cdot 35662389 - 1}{17405432} = 372326272.$$

$$s_{16} = x_{16}^2 - Ay_{16}^2 = -5.$$

$$\rho_{16} = 2, \rho'_{16} = -88730210. \quad r_{16} = 2 - 5k_{16} = 7. \quad \text{Then } r'_{16} = -124392599.$$

$$y_{17} = -124392599.$$

$$x_{17} = -1298696861.$$

$$s_{17} = 12.$$

$$r_{18} = 5, r'_{18} = -160054988.$$

$$y_{18} = -160054988.$$

$$x_{18} = -1671023133.$$

$$s_{18} = -7.$$

$$\rho_{19} = 2, \rho'_{19} = 284447587. \quad \text{Take } r_{19} = 2 - 7k_{19} = 9. \quad \text{Then } r'_{19} = 444502575.$$

$$y_{19} = 444502575.$$

$$x_{19} = 4640743127.$$

$$s_{19} = 4.$$

$$\rho_{20} = 3, \rho'_{20} = 1493562713. \quad \text{Take } r_{20} = 3 + 4k_{20} = 7. \quad \text{Then } r'_{20} = 1938065288.$$

$$y_{20} = 1938065288.$$

$$x_{20} = 20233995641.$$

$$s_{20} = -15.$$

$$r_{21} = 8, r'_{21} = -2382567863.$$

$$y_{21} = -2382567863.$$

$$x_{21} = -24874738768.$$

$$s_{21} = 3.$$

$$\rho_{22} = 1, \rho'_{22} = -9085768877. \quad \text{Take } r_{22} = 1 + 3k_{22} = 10. \quad \text{Then } r'_{22} = -16233472466.$$

$$y_{22} = -16233472466.$$

$$x_{22} = . - 169482428249.$$

$$s_{22} = -3.$$

$$\rho_{23} = 2, \rho'_{23} = 67316457727. \text{ Take } r_{23} = 2 - 3k_{23} = 8. \text{ Then } r'_{23} = 99783402659.$$

$$y_{23} = 99783402659.$$

$$x_{23} = 1041769308262.$$

$$s_{23} = 15.$$

$$r_{24} = 7, r'_{24} = 116016875125.$$

$$y_{24} = 116016875125.$$

$$x_{24} = 1211251736511.$$

$$s_{24} = -4.$$

$$\rho_{25} = 1, \rho'_{25} = -331817152909. \text{ Take } r_{25} = 1 - 4k_{25} = 9. \text{ Then } r'_{25} = -563850903159.$$

$$y_{25} = -563850903159.$$

$$x_{25} = . - 5886776254306.$$

$$s_{25} = 7.$$

$$r_{26} = 5, r'_{26} = -1243718681443.$$

$$y_{26} = -1243718681443.$$

$$x_{26} = -12984804245123.$$

$$s_{26} = . - 12.$$

$$r_{27} = 7, r'_{27} = 1807569584602.$$

$$y_{27} = 1807569584602.$$

$$x_{27} = 18871580499429.$$

$$s_{27} = 5.$$

$$\rho_{28} = 3, \rho'_{28} = 4858857850647. \text{ Take } r_{28} = 3 + 5k_{28} = 8. \text{ Then } r'_{28} = 6666427435249.$$

$$y_{28} = 6666427435249.$$

$$x_{28} = 69599545743410.$$

$$s_{28} = -9.$$

$$\rho_{29} = 1, \rho'_{29} = -8473997019851. \text{ Take } r_{29} = 1 + 9k_{29} = 10. \text{ Then } r'_{29} = -15140424455100.$$

$$y_{29} = -15140424455100.$$

$$x_{29} = -158070671986249.$$

$$s_{29} = 1.$$

Therefore

$$158070671986249^2 - 109 \cdot 15140424455100^2 = 1.$$

Thus  $\sqrt{109} \sim \frac{158070671986249}{15140424455100}$ .

## References

- [1] Philip Beeley and Christoph J. Scriba, editors. *The Correspondence of John Wallis, Volume II (1660–September 1668)*. Oxford University Press, 2005.
- [2] Walter Eugene Clark. *The Āryabhaṭīya of Āryabhata*. University of Chicago Press, 1930.
- [3] Henry Thomas Colebrooke. *Algebra, with arithmetic and mensuration, from the Sanscrit of Brahmeḡupta and Bhāscara*. John Murray, London, 1817.
- [4] Bibhutibhusan Datta and Avadhesh Narayan Singh. *History of Hindu Mathematics: A Source Book. Parts I and II*. Asia Publishing House, Bombay, 1962.
- [5] Leonard Eugene Dickson. *History of the theory of numbers, volume II: Diophantine analysis*. Dover Publications, 2005.
- [6] Kurt Elfering. *Die Mathematik des Āryabhata I. Text, Übersetzung aus dem Sanskrit und Kommentar*. Wilhelm Fink Verlag, München, 1975.
- [7] Leonhard Euler. De solutione problematum Diophantaeorum per numeros integros. *Commentarii academiae scientiarum imperialis Petropolitanae*, 6:175–188, 1738. E29, Leonhardi Euleri Opera omnia I.2, pp. 6–17.
- [8] Leonhard Euler. De resolutione formularum quadraticarum indeterminatarum per numeros integros. *Novi commentarii academiae scientiarum imperialis Petropolitanae*, 9:3–39, 1764. E279, Leonhardi Euleri Opera omnia I.2, pp. 576–611.
- [9] Leonhard Euler. De usu novi algorithmi in problemate Pelliano solvendo. *Novi commentarii academiae scientiarum imperialis Petropolitanae*, 11:28–66, 1767. E323, Leonhardi Euleri Opera omnia I.3, pp. 73–111.
- [10] Leonhard Euler. *Elements of Algebra*. Springer, 1984. Translated by Rev. John Hewlett.



- [11] David Fowler. *The mathematics of Plato's Academy: a new reconstruction*. Clarendon Press, Oxford, second edition, 1999.
- [12] Hermann Hankel. *Zur Geschichte der Mathematik in Alterthum und Mittelalter*. B. G. Teubner, Leipzig, 1874.
- [13] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Clarendon Press, Oxford, fifth edition, 1979.
- [14] Thomas L. Heath. *The thirteen books of Euclid's Elements, volume I: Books I-II*. Dover Publications, second edition, 1956.
- [15] Thomas L. Heath. *The thirteen books of Euclid's Elements, volume II: Books III-IX*. Dover Publications, second edition, 1956.
- [16] Thomas L. Heath. *The thirteen books of Euclid's Elements, volume III: Books X-XIII*. Dover Publications, second edition, 1956.
- [17] Thomas L. Heath. *Diophantus of Alexandria: A Study in the History of Greek Algebra*. Dover Publications, second edition, 1964.
- [18] Agathe Keller. *Expounding the Mathematical Seed. Volume 1: The Translation. A Translation of Bhāskara I on the Mathematical Chapter of the Āryabhatīya*, volume 30 of *Science Networks. Historical Studies*. Birkhäuser, 2006.
- [19] Agathe Keller. *Expounding the Mathematical Seed. Volume 2: The Supplements. A Translation of Bhāskara I on the Mathematical Chapter of the Āryabhatīya*, volume 31 of *Science Networks. Historical Studies*. Birkhäuser, 2006.
- [20] H. Konen. *Geschichte der Gleichung  $t^2 - Du^2 = 1$* . S. Hirzel, Leipzig, 1901.
- [21] Joseph-Louis Lagrange. Solution d'un Problème d'Arithmétique. *Miscellanea Taurinensia*, IV:41–97, 1766–1769. Œuvres de Lagrange, tome 1, pp. 671–731.
- [22] Joseph-Louis Lagrange. Sur la solution des Problemes indéterminés du second degré. *Mémoires de l'Académie royale des Sciences et Belles-lettres, Berlin*, XXIII:165–310, 1767. Œuvres de Lagrange, tome 2, pp. 377–535.
- [23] Joseph-Louis Lagrange. Nouvelle Méthode pour résoudre les problemes indéterminés en nombres entiers. *Mémoires de l'Académie royale des Sciences et Belles-lettres, Berlin*, XXIV:181–250, 1768. Œuvres de Lagrange, tome 2, pp. 655–726.
- [24] G. H. F. Nesselmann. *Die Algebra der Griechen*. G. Reimer, Berlin, 1842.
- [25] Jacques Ozanam. *Nouveaux Elemens d'Algebre*. George Gallet, Amsterdam, 1702.

- [26] Roshdi Rashed, editor. *Abū Kāmil. Algèbre et analyse diophantienne: édition, traduction et commentaire*, volume 9 of *Scientia Graeco-Arabica*. De Gruyter, 2012.
- [27] Clas-Olof Selenius. Rationale of the chakravāla process of Jayadeva and Bhāskara II. *Historia Math.*, 2(2):167–184, 1975.
- [28] K. S. Shukla. Acarya Jayadeva, the mathematician. *Gaṇita*, 5:1–20, 1954.
- [29] Jacqueline A. Stedall. *A Discourse Concerning Algebra: English Algebra to 1685*. Oxford University Press, 2002.
- [30] Edward Strachey. *Bija Ganita: or the Algebra of the Hindus*. W. Glendinning, London, 1813.
- [31] D. J. Struik, editor. *A Source Book in Mathematics, 1200–1800*. Harvard University Press, 1969.
- [32] André Weil. *Number Theory: An approach through history from Hammurapi to Legendre*. Birkhäuser, 1984.