

The Heisenberg group and Hermite functions

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1 The Heisenberg group

For $(z, t), (w, s) \in \mathbb{C}^n \times \mathbb{R}$, define the operation

$$(z, t)(w, s) = \left(z + w, t + s + \frac{1}{2} \operatorname{Im}(z \cdot \bar{w}) \right),$$

which satisfies

$$(z, t)(0, 0) = (z, t),$$

and because $\operatorname{Im}(z \cdot \bar{z}) = 0$,

$$(z, t)^{-1} = (-z, -t).$$

We denote $\mathbb{C}^n \times \mathbb{R}$ with this operation by H^n . This is a Lie group of dimension $2n + 1$, called the **Heisenberg group**.

Writing $z = x + iy$ define

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad 1 \leq j \leq n,$$

and

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad 1 \leq j \leq n,$$

and

$$T = \frac{\partial}{\partial t}.$$

We calculate the Lie brackets of these vector fields. For X_j and X_k ,

$$\begin{aligned} X_j X_k &= \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x_k} - \frac{1}{2} y_k \frac{\partial}{\partial t} \right) \\ &= \frac{\partial^2}{\partial x_j \partial x_k} - \frac{1}{2} y_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} - \frac{1}{2} y_j \frac{\partial}{\partial t} \frac{\partial}{\partial x_k} + \frac{1}{4} y_j y_k \frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[X_j, X_k] = X_j X_k - X_k X_j = 0.$$

For Y_j and Y_k ,

$$\begin{aligned} Y_j Y_k &= \left(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial y_k} + \frac{1}{2} x_k \frac{\partial}{\partial t} \right) \\ &= \frac{\partial^2}{\partial y_j \partial y_k} + \frac{1}{2} x_k \frac{\partial}{\partial y_j} \frac{\partial}{\partial t} + \frac{1}{2} x_j \frac{\partial}{\partial t} \frac{\partial}{\partial y_k} + \frac{1}{4} x_j x_k \frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[Y_j, Y_k] = Y_j Y_k - Y_k Y_j = 0.$$

For X_j and Y_j ,

$$\begin{aligned} X_j Y_j &= \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \right) \\ &= \frac{\partial^2}{\partial x_j \partial y_j} + \frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} x_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} - \frac{1}{2} y_j \frac{\partial}{\partial t} \frac{\partial}{\partial y_j} - \frac{1}{4} y_j x_j \frac{\partial^2}{\partial t^2}, \end{aligned}$$

and

$$\begin{aligned} Y_j X_j &= \left(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \right) \\ &= \frac{\partial^2}{\partial y_j \partial x_j} - \frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} y_j \frac{\partial}{\partial y_j} \frac{\partial}{\partial t} + \frac{1}{2} x_j \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} - \frac{1}{4} x_j y_j \frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[X_j, Y_j] = X_j Y_j - Y_j X_j = \frac{\partial}{\partial t} = T.$$

For X_j and Y_k with $j \neq k$,

$$\begin{aligned} X_j Y_k &= \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial y_k} + \frac{1}{2} x_k \frac{\partial}{\partial t} \right) \\ &= \frac{\partial^2}{\partial x_j \partial y_k} + \frac{1}{2} x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} - \frac{1}{2} y_j \frac{\partial}{\partial t} \frac{\partial}{\partial y_k} - \frac{1}{4} y_j x_k \frac{\partial^2}{\partial t^2} \end{aligned}$$

and

$$\begin{aligned} Y_k X_j &= \left(\frac{\partial}{\partial y_k} + \frac{1}{2} x_k \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \right) \\ &= \frac{\partial^2}{\partial y_k \partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial y_k} \frac{\partial}{\partial t} + \frac{1}{2} x_k \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} - \frac{1}{4} x_k y_j \frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[X_j, Y_k] = 0.$$

For X_j and T ,

$$X_j T = \left(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \right) \frac{\partial}{\partial t} = \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} - \frac{1}{2} y_j \frac{\partial^2}{\partial t^2} = T X_j,$$

yielding

$$[X_j, T] = 0.$$

For Y_j and T ,

$$Y_j T = \left(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \right) \frac{\partial}{\partial t} = \frac{\partial}{\partial y_j} \frac{\partial}{\partial t} + \frac{1}{2} x_j \frac{\partial^2}{\partial t^2} = T Y_j,$$

yielding

$$[Y_j, T] = 0.$$

We summarize the above calculations in the following theorem.

Theorem 1. The Lie brackets of the vector fields $X_j, Y_j, 1 \leq j \leq n$, and T are:

- $[X_j, X_k] = 0$
- $[Y_j, Y_k] = 0$
- $[X_j, Y_j] = T$
- $[X_j, Y_k] = 0$ for $j \neq k$
- $[X_j, T] = 0$
- $[Y_j, T] = 0$

The Lie algebra of the H^n is called the **Heisenberg Lie algebra** and is denoted \mathfrak{h}^n . The above vector fields are left-invariant and are a basis for \mathfrak{h}^n .¹

2 Representation theory

For a Hilbert space H , we denote by $\mathcal{B}(H)$ the set of bounded linear operators $H \rightarrow H$, which is a Banach algebra with the operator norm. We denote by $\mathcal{B}_0(H)$ the set of compact operators $H \rightarrow H$, which is a closed ideal of the Banach algebra $\mathcal{B}(H)$. We denote by $\mathcal{B}_{\text{HS}}(H)$ the collection of Hilbert-Schmidt operators $H \rightarrow H$: if $\{e_i : i \in I\}$ is an orthonormal basis of H , a linear map $A : H \rightarrow H$ is called a **Hilbert-Schmidt operator** if

$$\|A\|_{\text{HS}}^2 = \sum_{i \in I} \|Ae_i\|^2 < \infty.$$

This satisfies $\|A\| \leq \|A\|_{\text{HS}}$. A Hilbert-Schmidt operator is a compact operator. A linear map $U : H \rightarrow H$ is called a **unitary operator** if it is a bijection and satisfies

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \quad x, y \in H.$$

We denote the set of unitary operators $H \rightarrow H$ by $\mathcal{U}(H)$.

¹Sundaram Thangavelu, *An Introduction to the Uncertainty Principle: Hardy's Theorem on Lie Groups*, p. 47, §2.1.

For $\lambda \in \mathbb{R}, \lambda \neq 0$, for $(x + iy, t) \in H^n$, and for $f \in L^2(\mathbb{R}^n)$, define

$$\pi_\lambda(x + iy, t)f(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} f(\xi + y), \quad \xi \in \mathbb{R}^n.$$

It is apparent that $\pi_\lambda(z, t)$ is a linear map $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$.

For $(x + iy, t), (u + iv, s) \in H^n$ we calculate

$$\begin{aligned} \pi_\lambda(x + iy, t)\pi_\lambda(u + iv, s)f(\xi) &= \pi_\lambda(x + iy, t)e^{i\lambda s} e^{i\lambda(u \cdot \xi + \frac{1}{2}u \cdot v)} f(\xi + v) \\ &= e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} e^{i\lambda s} e^{i\lambda(u \cdot (\xi + y) + \frac{1}{2}u \cdot v)} f(\xi + y + v) \\ &= e^{i\lambda(t+s)} e^{i\lambda((x+u) \cdot \xi + \frac{1}{2}x \cdot y + u \cdot y + \frac{1}{2}u \cdot v)} f(\xi + y + v). \end{aligned}$$

On the other hand, with $z = x + iy$ and $w = u + iv$,

$$\begin{aligned} (z, t)(w, s) &= \left(z + w, t + s + \frac{1}{2}\text{Im}(z \cdot \bar{w}) \right) \\ &= \left(x + iy + u + iv, t + s + \frac{1}{2}\text{Im}((x + iy) \cdot (u - iv)) \right) \\ &= \left(x + u + i(y + v), t + s + \frac{1}{2}\text{Im}(x \cdot u - ix \cdot v + iy \cdot u + y \cdot v) \right) \\ &= \left(x + u + i(y + v), t + s - \frac{1}{2}x \cdot v + \frac{1}{2}y \cdot u \right), \end{aligned}$$

for which

$$\begin{aligned} \pi_\lambda((z, t)(w, s))f(\xi) &= e^{i\lambda(t+s - \frac{1}{2}x \cdot v + \frac{1}{2}y \cdot u)} e^{i\lambda((x+u) \cdot \xi + \frac{1}{2}(x+u) \cdot (y+v))} f(\xi + y + v) \\ &= e^{i\lambda(t+s)} e^{i\lambda((x+u) \cdot \xi + \frac{1}{2}x \cdot y + y \cdot u + \frac{1}{2}u \cdot v)} f(\xi + y + v), \end{aligned}$$

and therefore

$$\pi_\lambda(x + iy, t)\pi_\lambda(u + iv, s) = \pi_\lambda((z, t)(w, s)).$$

We calculate

$$\pi_\lambda(0, 0)f(\xi) = f(\xi)$$

and

$$\pi_\lambda(x + iy, t)\pi_\lambda((x + iy, t)^{-1})f = \pi_\lambda(0, 0)f = f.$$

For $f, g \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} &\langle \pi_\lambda(x + iy, t)f, \pi_\lambda(x + iy, t)g \rangle \\ &= \int_{\mathbb{R}^n} \pi_\lambda(x + iy, t)f(\xi) \overline{\pi_\lambda(x + iy, t)g(\xi)} d\xi \\ &= \int_{\mathbb{R}^n} e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} f(\xi + y) e^{-i\lambda t} e^{-i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} \overline{g(\xi + y)} d\xi \\ &= \int_{\mathbb{R}^n} f(\xi + y) \overline{g(\xi + y)} d\xi \\ &= \langle f, g \rangle. \end{aligned}$$

Therefore $\pi_\lambda(z, t)$ is a unitary operator $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, and

$$\pi_\lambda : H^n \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$$

is a group homomorphism, namely, π_λ is a unitary representation of H^n on $L^2(\mathbb{R}^n)$.² Furthermore, using that $y \mapsto f(\cdot + y)$ is continuous $\mathbb{R}^n \rightarrow L^2(\mathbb{R}^n)$,

$$\|\pi_\lambda(x + iy, t)f - f\|^2 = \int_{\mathbb{R}^n} |e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} f(\xi + y) - f(\xi)|^2 d\xi \rightarrow 0$$

as $(z, t) \rightarrow 0$, showing that $\pi_\lambda : H^n \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$ is **strongly continuous**. (That is, it is continuous when $\mathcal{U}(L^2(\mathbb{R}^n))$ is assigned the strong operator topology.)

Theorem 2. For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the map π_λ defined by

$$\pi_\lambda(x + iy, t)f(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} f(\xi + y),$$

for $(x + iy, t) \in H^n$, $f \in L^2(\mathbb{R}^n)$, and $\xi \in \mathbb{R}^n$, is a strongly continuous unitary representation of H^n on $L^2(\mathbb{R}^n)$.

We call π_1 the **Schrödinger representation**. Its kernel is

$$\Gamma = \{(0, 2\pi k) : k \in \mathbb{Z}\}.$$

For $f \in L^1(H^n/\Gamma)$ we define

$$\pi_1(f) = \int_{H^n/\Gamma} f(z, t) \pi_1(z, t) dz dt.$$

For $f, g \in L^1(H^n/\Gamma)$,

$$(f * g)(z, t) = \int_{H^n/\Gamma} f((z, t) \cdot (w, s)^{-1}) g(w, s) dw ds, \quad (z, t) \in H^n/\Gamma.$$

It is a fact that Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ is a bi-invariant Haar measure on H^n , and using this we calculate

$$\begin{aligned} & \pi_1(f * g) \\ &= \int_{H^n/\Gamma} \left(\int_{H^n/\Gamma} f((z, t) \cdot (w, s)^{-1}) g(w, s) dw ds \right) \pi_1(z, t) dz dt \\ &= \int_{H^n/\Gamma} g(w, s) \left(\int_{H^n/\Gamma} f((z, t) \cdot (w, s)^{-1}) \pi_1((z, t) \cdot (w, s)^{-1}) dz dt \right) \pi_1(w, s) dw ds \\ &= \int_{H^n/\Gamma} g(w, s) \pi_1(f) dw ds \\ &= \pi_1(f) \pi_1(g). \end{aligned}$$

²cf. <https://www.math.ubc.ca/~cass/research/pdf/Unitary.pdf>

Lemma 3. For $f, g \in L^1(H^n/\Gamma)$,

$$\pi_1(f * g) = \pi_1(f)\pi_1(g).$$

We define

$$W(z) = \pi_1(z, 0),$$

with which

$$\pi_1(z, t) = e^{it}W(z).$$

Define

$$f_1(z) = (2\pi)^{-1/2} \int_0^{2\pi} f(z, t)e^{it}dt.$$

Then

$$\begin{aligned} \pi_1(f) &= \int_{H^n/\Gamma} f(z, t)e^{it}W(z)dzdt \\ &= \int_{\mathbb{C}^n} W(z) \left(\int_0^{2\pi} f(z, t)e^{it}dt \right) dz \\ &= (2\pi)^{1/2} \int_{\mathbb{C}^n} f_1(z)W(z)dz. \end{aligned}$$

For $f \in L^1(\mathbb{C}^n)$, define

$$f^\#(z, t) = (2\pi)^{-1}e^{-it}f(z).$$

$f^\# \in L^1(H^n/\Gamma)$, and

$$f_1^\#(z) = (2\pi)^{-1/2} \int_0^{2\pi} f^\#(z, t)e^{it}dt = (2\pi)^{-1/2}f(z),$$

thus

$$\pi_1(f^\#) = (2\pi)^{1/2} \int_{\mathbb{C}^n} f^\#(z)W(z)dz = \int_{\mathbb{C}^n} f(z)W(z)dz.$$

We define $W : L^1(\mathbb{C}^n) \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$ by

$$W(f) = \pi_1(f^\#),$$

called the **Weyl transform**.

For $f, g \in L^1(\mathbb{C}^n)$ and for $(z, t) \in H^n/\Gamma$,

$$\begin{aligned}
& (f^\# * g^\#)(z, t) \\
&= \int_{H^n/\Gamma} f^\#((z, t) \cdot (w, s)^{-1}) g^\#(w, s) dw ds \\
&= \int_{H^n/\Gamma} f^\#((z, t) \cdot (-w, -s)) g^\#(w, s) dw ds \\
&= \int_{H^n/\Gamma} f^\# \left(z - w, t - s - \frac{1}{2} \operatorname{Im}(z \cdot \bar{w}) \right) g^\#(w, s) dw ds \\
&= \int_{H^n/\Gamma} (2\pi)^{-2} e^{-i(t-s-\frac{1}{2}\operatorname{Im}(z \cdot \bar{w}))} f(z-w) e^{-is} g(w) dw ds \\
&= (2\pi)^{-1} e^{-it} \int_{\mathbb{C}^n} f(z-w) g(w) e^{\frac{i}{2}\operatorname{Im}(z \cdot \bar{w})} dw \\
&= (f \times g)^\#(z, t),
\end{aligned}$$

for

$$(f \times g)(z) = \int_{\mathbb{C}^n} f(z-w) g(w) e^{\frac{i}{2}\operatorname{Im}(z \cdot \bar{w})} dw,$$

called the **twisted convolution**. Using what we have established so far gives the following.

Lemma 4. For $f, g \in L^1(\mathbb{C}^n)$,

$$W(f \times g) = \pi_1((f \times g)^\#) = \pi_1(f^\# * g^\#) = \pi_1(f^\#) \pi_1(g^\#) = W(f)W(g)$$

For $\phi \in L^1(\mathbb{C}^n)$, we define

$$K_\phi(\xi, \eta) = \int_{\mathbb{R}^n} \phi(x + i(\eta - \xi)) e^{\frac{i}{2}(\xi + \eta) \cdot x} dx, \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n,$$

which satisfies, for $f \in L^2(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
W(\phi)f(\xi) &= \int_{\mathbb{C}^n} \phi(z) W(z) f(\xi) dz \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x + iy) e^{i(x \cdot \xi + \frac{1}{2}x \cdot y)} f(\xi + y) dy dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x + i(y - \xi)) e^{\frac{i}{2}(x \cdot \xi + x \cdot y)} f(y) dy dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi(x + i(y - \xi)) e^{\frac{i}{2}(\xi + y) \cdot x} dx \right) f(y) dy \\
&= \int_{\mathbb{R}^n} K_\phi(\xi, y) f(y) dy.
\end{aligned}$$

Thus K_ϕ is an integral kernel for the operator $W(\phi)$.

We show in the following theorem that the Weyl transform sends elements of $L^1(\mathbb{C}^n)$ to compact operators on $L^2(\mathbb{R}^n)$, and that it sends square integrable functions to Hilbert-Schmidt operators.³

Theorem 5. $W : L^1(\mathbb{C}^n) \rightarrow \mathcal{B}_0(L^2(\mathbb{R}^n))$, and for $\phi \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$ we have $W(\phi) \in \mathcal{B}_{\text{HS}}(L^2(\mathbb{R}^n))$ and

$$\|\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \|W(\phi)\|_{\text{HS}}.$$

Proof. First take $\phi \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$. It follows from this that $K_\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, and because K_ϕ is the integral kernel of $W(\phi)$ this implies⁴ that $W(\phi) \in \mathcal{B}_{\text{HS}}(L^2(\mathbb{R}^n))$ and

$$\|W(\phi)\|_{\text{HS}}^2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} |K(\xi, \eta)|^2 d\xi d\eta.$$

□

3 Hermite functions

For $\phi \in \mathcal{S}(\mathbb{R}^n)$, define

$$\widehat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.$$

$\mathcal{S}(\mathbb{R}^n)$ is a dense linear subspace of $L^2(\mathbb{R}^n)$, and the Fourier transform extends to a unique Hilbert space isomorphism $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. For $f, g \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx.$$

For $\phi \in \mathcal{S}(\mathbb{R})$, let

$$(D\phi)(x) = \phi'(x), \quad (M\phi)(x) = x\phi(x), \quad x \in \mathbb{R},$$

and let

$$A = -D + M, \quad B = D + M.$$

Let

$$H = \sum_{j=1}^n (-D_j^2 + M_j^2) = \frac{1}{2} \sum_{j=1}^n (A_j B_j + B_j A_j),$$

which satisfies

$$(H\phi)(x) = -(\Delta\phi)(x) + |x|^2 \phi(x),$$

³Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, p. 13, Theorem 1.2.1.

⁴Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged edition, p. 210, Theorem VI.23.

called the **Hermite operator**.

For $k \geq 0$, define

$$H_k(x) = (-1)^k e^{x^2} D^k e^{-x^2}$$

and

$$h_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_k(x).$$

The Hermite functions are an orthonormal basis for $L^2(\mathbb{R})$. Let \mathbb{N} be the non-negative integers, and for $\alpha \in \mathbb{N}^n$ let

$$\Phi_\alpha = h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_n},$$

which are an orthonormal basis for $L^2(\mathbb{R}^n)$. It is a fact that

$$A_j \Phi_\alpha = (2\alpha_j + 2)^{1/2} \Phi_{\alpha+e_j}, \quad B_j \Phi_\alpha = (2\alpha_j)^{1/2} \Phi_{\alpha-e_j}$$

and

$$H \Phi_\alpha = (2|\alpha| + n) \Phi_\alpha.$$

It is a fact that

$$\widehat{h}_k = (-i)^k h_k,$$

whence

$$\widehat{\Phi}_\alpha = (-i)^{|\alpha|} \Phi_\alpha.$$

Because $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, for $f \in L^2(\mathbb{R}^n)$,

$$f = \sum_{\alpha} \langle f, \Phi_\alpha \rangle \Phi_\alpha.$$

and then

$$\widehat{f} = \sum_{\alpha} \langle f, \Phi_\alpha \rangle (-i)^{|\alpha|} \Phi_\alpha.$$

Let E_k be the linear span of $\{\Phi_\alpha : |\alpha| = k\}$, which has dimension $\binom{k+n-1}{k}$. For $f \in E_k$, $Hf = (2k + n)f$. Let $P_k : L^2(\mathbb{R}^n) \rightarrow E_k$ be the projection:

$$P_k f = \sum_{|\alpha|=k} \langle f, \Phi_\alpha \rangle \Phi_\alpha, \quad f \in L^2(\mathbb{R}^n).$$

Let

$$\Phi_k(x, y) = \sum_{|\alpha|=k} \Phi_\alpha(x) \Phi_\alpha(y), \quad x, y \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$ we calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi_k(x, y) f(y) dy &= \sum_{|\alpha|=k} \Phi_\alpha(x) \int_{\mathbb{R}^n} f(y) \Phi_\alpha(y) dy \\ &= \sum_{|\alpha|=k} \Phi_\alpha(x) \langle f, \Phi_\alpha \rangle \\ &= (P_k f)(x), \end{aligned}$$

thus Φ_k is a kernel for the projection operator P_k .

Using the 1-dimensional Mehler's formula we obtain the n -dimensional Mehler's formula:

$$\sum_{\alpha} r^{|\alpha|} \Phi_{\alpha}(x) \Phi_{\alpha}(y) = \pi^{-\frac{n}{2}} (1-r^2)^{-\frac{n}{2}} \exp \left(-\frac{1}{2} \frac{1+r^2}{1-r^2} (|x|^2 + |y|^2) + \frac{2r}{1-r^2} x \cdot y \right).$$

4 Special Hermite functions

We first define the **Fourier-Wigner transform**. For $f, g \in L^2(\mathbb{R}^n)$ and $z = x + iy \in \mathbb{C}^n$,

$$V(f, g)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f \left(\xi + \frac{1}{2}y \right) \overline{g \left(\xi - \frac{1}{2}y \right)} d\xi.$$

The following theorem relates the inner product on $L^2(\mathbb{R}^n)$ and the inner product on $L^2(\mathbb{C}^n)$.⁵

Theorem 6. For $f, g, \phi, \psi \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{C}^n} V(f, g)(z) \overline{V(\phi, \psi)(z)} dz = \langle f, \phi \rangle \langle \psi, g \rangle.$$

We now define the **special Hermite functions** on \mathbb{C}^n . For $\alpha, \beta \in \mathbb{N}^n$, let

$$\Phi_{\alpha\beta}(z) = V(\Phi_{\alpha}, \Phi_{\beta})(z).$$

We calculate

$$\begin{aligned} \langle W(z) \Phi_{\alpha}, \Phi_{\beta} \rangle &= \int_{\mathbb{R}^n} W(z) \Phi_{\alpha}(\xi) \Phi_{\beta}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \frac{1}{2}x \cdot y)} \Phi_{\alpha}(\xi + y) \Phi_{\beta}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} e^{ix \cdot \xi} \Phi_{\alpha} \left(\xi + \frac{1}{2}y \right) \Phi_{\beta} \left(\xi - \frac{1}{2}y \right) d\xi \\ &= (2\pi)^{n/2} V(\Phi_{\alpha}, \Phi_{\beta}). \end{aligned}$$

Lemma 7. For $\alpha, \beta \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$,

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-n/2} \langle W(z) \Phi_{\alpha}, \Phi_{\beta} \rangle.$$

Using that the Hermite functions Φ_{α} are an orthonormal basis for $L^2(\mathbb{R}^n)$, it is proved that the special Hermite functions $\Phi_{\alpha\beta}$ are an orthonormal basis for $L^2(\mathbb{C}^n)$.⁶

⁵Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, p. 14, Proposition 1.3.1.

⁶Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, p. 16, Theorem 1.3.2.