

Convolution semigroups, canonical processes, and Brownian motion

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1 Convolution semigroups, projective families, and canonical processes

Let

$$E = \mathbb{R}^d$$

and let $\mathcal{E} = \mathcal{B}_{\mathbb{R}^d}$, the Borel σ -algebra of \mathbb{R}^d , and let $\mathcal{P}(E)$ be the collection of Borel probability measures on \mathbb{R}^d . With the **narrow topology**, $\mathcal{P}(E)$ is a Polish space. For a nonempty set J , we write

$$\mathcal{E}^J = \bigotimes_{t \in J} \mathcal{E},$$

the product σ -algebra.

Let $A : E \times E \rightarrow E$ be $A(x_1, x_2) = x_1 + x_2$. For $\nu_1, \nu_2 \in \mathcal{P}(E)$, the **convolution of ν_1 and ν_2** is the pushforward of the product measure $\nu_1 \times \nu_2$ by A :

$$\nu_1 * \nu_2 = A_*(\nu_1 \times \nu_2).$$

The convolution $\nu_1 * \nu_2$ is an element of $\mathcal{P}(E)$.

Let

$$I = \mathbb{R}_{\geq 0}.$$

A **convolution semigroup** is a family $(\nu_t)_{t \in I}$ of elements of $\mathcal{P}(E)$ such that for $s, t \in I$,

$$\nu_{s+t} = \nu_s * \nu_t.$$

From this, it turns out that $\mu_0 = \delta_0$. A convolution semigroup is called **continuous** when the map $t \mapsto \nu_t$ is continuous $I \rightarrow \mathcal{P}(E)$.

For $\nu \in \mathcal{P}(E)$ and $x \in E$, and for $B \in \mathcal{E}$,

$$(\nu * \delta_x)(B) = \int_E \left(\int_E 1_B(x_1 + x_2) d\delta_x(x_1) \right) d\nu(x_2) = \nu(B - x),$$

and we define $\nu^x \in \mathcal{P}(E)$ by

$$\nu^x = \nu * \delta_x.$$

For $\nu \in \mathcal{P}(E)$ and for a Borel measurable function $f : E \rightarrow [0, \infty]$, write

$$\nu f = \int_E f d\nu.$$

For $x \in E$, using the change of variables formula¹ and Fubini's theorem,

$$\begin{aligned} \nu^x f &= \int_E f d(\nu * \delta_x) \\ &= \int_{E \times E} f \circ \text{Ad}(\nu \times \delta_x) \\ &= \int_E \left(\int_E f(x_1 + x_2) d\delta_x(x_2) \right) d\nu(x_1) \\ &= \int_E f(x_1 + x) d\nu(x_1). \end{aligned}$$

That is, for $\nu \in \mathcal{P}(E)$, for $f : E \rightarrow [0, \infty]$ Borel measurable, and for $x \in E$,

$$\nu^x f = \int_E f d\nu^x = \int_E f(x + y) d\nu(y). \quad (1)$$

For nonempty subsets J and K of I with $J \subset K$, let

$$\pi_{K,J} : E^K \rightarrow E^J$$

be the projection map. Let $\mathcal{K} = \mathcal{K}(I)$ be the collection of finite nonempty subsets of I . Let $(\Omega, \mathcal{F}, P, (X_t)_{t \in I})$ be a stochastic process with state space E . For $J \in \mathcal{K}$, with elements $t_1 < \dots < t_n$, we define

$$X_J = X_{t_1} \otimes \dots \otimes X_{t_n},$$

which is measurable $\mathcal{F} \rightarrow \mathcal{E}^J$. The **joint distribution** P_J of the family of random variables $(X_t)_{t \in J}$ is the distribution of X_J , i.e.

$$P_J = X_{J*} P.$$

The **family of finite-dimensional distributions** of X is the family $(P_J)_{J \in \mathcal{K}}$. For $J, K \in \mathcal{K}$ with $J \subset K$,

$$X_J = \pi_{K,J} \circ X_K,$$

from which

$$(\pi_{K,J})_* P_K = P_J. \quad (2)$$

Forgetting the stochastic process X , a family of probability measures P_J on \mathcal{E}^J , for $J \in \mathcal{K}$, is called a **projective family** when (2) is true. The **Kolmogorov**

¹Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 484, Theorem 13.46.

extension theorem tells us that if $(P_J)_{J \in \mathcal{K}}$ is a projective family, then there is a unique probability measure P_I on \mathcal{E}^I such that for any $J \in \mathcal{K}$,

$$(\pi_{I,J})_* P_I = P_J. \quad (3)$$

Then for $\Omega = E^I$ and $\mathcal{F} = \mathcal{E}^I$, $(\Omega, \mathcal{F}, P_I)$ is a probability space, and for $t \in I$ we define $X_t : \Omega \rightarrow E$ by

$$X_t(\omega) = \pi_{I,\{t\}}(\omega) = \omega(t), \quad (4)$$

which is measurable $\mathcal{F} \rightarrow \mathcal{E}$, and thus the family $(X_t)_{t \in I}$ is a stochastic process with state space E . For $J \in \mathcal{K}$ it is immediate that

$$X_J = \pi_{I,J}.$$

For $B \in \mathcal{E}^J$, applying (3) gives

$$(X_{J*} P_I)(B) = ((\pi_{I,J})_* P_I)(B) = P_J(B),$$

which means that $X_{J*} P_I = P_J$, namely, $(P_J)_{J \in \mathcal{K}}$ is the family of finite-dimensional distributions of the stochastic process $(X_t)_{t \in I}$. We call the stochastic process (4) the **canonical process associated with the projective family** $(P_J)_{J \in \mathcal{K}}$.

Let $(\nu_t)_{t \in I}$ be a convolution semigroup and let $\mu \in \mathcal{P}(E)$. For $J \in \mathcal{K}$, with elements $t_1 < \dots < t_n$, and for $B \in \mathcal{E}^J$, define

$$\begin{aligned} & P_J(B) \\ &= \int_E \int_E \cdots \int_E 1_B(x_1, \dots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0). \end{aligned} \quad (5)$$

We say that $(P_J)_{J \in \mathcal{K}}$ **is the family of measures induced by the convolution semigroup** $(\nu_t)_{t \in I}$. It is proved that $(P_J)_{J \in \mathcal{K}}$ is a projective family. Therefore, from the Kolmogorov extension theorem it follows that there is a unique probability measure P^μ on \mathcal{E}^I such that

$$(\pi_{I,J})_* P^\mu = P_J. \quad (6)$$

For $\Omega = E^I$ and $\mathcal{F} = \mathcal{E}^I$, $(\Omega, \mathcal{F}, P^\mu)$ is a probability space. For $t \in I$ define $X_t : \Omega \rightarrow E$ by

$$X_t(\omega) = \pi_{I,\{t\}}(\omega) = \omega(t).$$

$(X_t)_{t \in I}$ is a stochastic processes whose family of finite-dimensional distributions is $(P_J)_{J \in \mathcal{K}}$, i.e. for $J \in \mathcal{K}$ with elements $t_1 < \dots < t_n$ and for $B \in \mathcal{E}^J$,

$$\begin{aligned} & ((X_{t_1} \otimes \cdots \otimes X_{t_n})_* P^\mu)(B) \\ &= \int_E \int_E \cdots \int_E 1_B(x_1, \dots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0). \end{aligned}$$

Applying this with $\mu = \delta_x$ yields

$$((X_{t_1} \otimes \cdots \otimes X_{t_n})_* P^{\delta_x})(B) = \int_E \cdots \int_E 1_B(x_1, \dots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^x(x_1),$$

and thus, for any $\mu \in \mathcal{P}(E)$,

$$\begin{aligned} & \int_E \int_E \cdots \int_E 1_B(x_1, \dots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x) \\ &= \int_E ((X_{t_1} \otimes \cdots \otimes X_{t_n})_* P^{\delta_x})(B) d\mu(x). \end{aligned}$$

That is, for $\mu \in \mathcal{P}(E)$, for $J \in \mathcal{K}$, and $B \in \mathcal{E}^J$,

$$(X_{J*} P^\mu)(B) = \int_E (X_{J*} P^{\delta_x})(B) d\mu(x). \quad (7)$$

For $J \in \mathcal{K}$, $A_t \in \mathcal{E}$ for $t \in J$, and $A = \prod_{t \in J} A_t \times \prod_{t \in I \setminus J} E \in \mathcal{F} = \mathcal{E}^I$, namely A is a **cylinder set**, let $B = \pi_{I,J}(A) = \prod_{t \in J} A_t \in \mathcal{E}^J$,

$$X_J^{-1}(B) = \pi_{I,J}^{-1}(B) = A,$$

so by (7),

$$P^\mu(A) = \int_E P^{\delta_x}(A) d\mu(x). \quad (8)$$

Because this is true for all cylinder sets in the product σ -algebra \mathcal{E}^I and \mathcal{E}^I is generated by the collection of cylinder sets, (8) is true for all $A \in \mathcal{F}$.

Let $J \in \mathcal{K}$, with elements $t_1 < \cdots < t_n$, and let $\sigma_n : E^{n+1} \rightarrow E^n$ be

$$\sigma_n(x_0, x_1, \dots, x_n) = (x_0 + x_1, x_0 + x_1 + x_2, \dots, x_0 + x_1 + x_2 + \cdots + x_n).$$

For $B \in \mathcal{E}^n$ using (1) we obtain by induction

$$\begin{aligned} & \int_E \int_E \cdots \int_E 1_B(x_1, \dots, x_{n-1}, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0) \\ &= \int_E \int_E \cdots \int_E 1_B(x_1, \dots, x_{n-1}, x_n + x_{n-1}) d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0) \\ &= \cdots \\ &= \int_E \int_E \cdots \int_E 1_B \circ \sigma_n d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0). \end{aligned}$$

Thus, with P_J the probability measure on \mathcal{E}^J defined in (5),

$$\int_{E^J} 1_B dP_J = P_J(B) = \int_E \int_E \cdots \int_E 1_B \circ \sigma_n d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0).$$

For $f : E^n \rightarrow [0, \infty]$ a Borel measurable function, there is a sequence of measurable simple functions pointwise increasing to f , and applying the monotone convergence theorem yields

$$\int_{E^n} f dP_J = \int_E \int_E \cdots \int_E f \circ \sigma_n d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0). \quad (9)$$

2 Increments

Let $(\Omega, \mathcal{F}, P, (X_t)_{t \in I})$ be a stochastic process with state space E . X is said to have **stationary increments** if there is a family $(\nu_t)_{t \in I}$ of probability measures on \mathcal{E} such that for all $s, t \in I$ with $s \leq t$,

$$P_*(X_t - X_s) = \nu_{t-s}.$$

In particular, for $s = t$ this implies that $P_*(0) = \nu_0$, hence $\nu_0 = \delta_0$.

A stochastic process is said to have **independent increments** if for any $J \in \mathcal{H}$, with elements $0 = t_0 < t_1 < \dots < t_n$, the random variables

$$X_{t_0}, \quad X_{t_1} - X_{t_0}, \quad \dots, \quad X_{t_n} - X_{t_{n-1}}$$

are independent.

We now prove that the canonical process associated with the projective family of probability measures induced by a convolution semigroup and any initial distribution has stationary and independent increments.²

Theorem 1. *Let $(\nu_t)_{t \in I}$ be a convolution semigroup, let $(P_J)_{J \in \mathcal{H}}$ be the family of measures induced by this convolution semigroup, let $\mu \in \mathcal{P}(E)$, and let $(\Omega, \mathcal{F}, P^\mu, (X_t)_{t \in I})$, $\Omega = E^I$ and $\mathcal{F} = \mathcal{E}^I$, be the associated canonical process. X has stationary increments,*

$$(X_t - X_s)_* P^\mu = \nu_{t-s}, \quad s \leq t, \quad (10)$$

and has independent increments.

Proof. $\nu_0 = \delta_0$, so (10) is immediate when $s = t$. When $s < t$, let

$$Y = X_s \otimes X_t = X_{\{s,t\}} = \pi_{I, \{s,t\}},$$

which is measurable $\mathcal{F} \rightarrow \mathcal{E} \otimes \mathcal{E}$, and let $q : E \times E \rightarrow E$ be $(x_1, x_2) \mapsto x_2 - x_1$, which is continuous and hence Borel measurable. Then $q \circ Y$ is measurable $\mathcal{F} \rightarrow \mathcal{E}$, and for $B \in \mathcal{E}$,

$$\begin{aligned} (q \circ Y)^{-1}(B) &= \{\omega \in \Omega : (q \circ Y)(\omega) \in B\} \\ &= \{\omega \in \Omega : X_t(\omega) - X_s(\omega) \in B\} \\ &= (X_t - X_s)^{-1}(B), \end{aligned}$$

and thus

$$(q \circ Y)_* P^\mu = (X_t - X_s)_* P^\mu. \quad (11)$$

Now, according to (6),

$$Y_* P^\mu = (\pi_{I, \{s,t\}})_* P^\mu = P_{\{s,t\}}.$$

²Heinz Bauer, *Probability Theory*, p. 321, Theorem 37.2.

Therefore, using that $x_2 - x_1 \in B$ if and only if $x_2 \in x_1 + B$ and also using $\nu_{t-s}^{x_1}(x_1 + B) = \nu_{t-s}(B)$,

$$\begin{aligned}
(X_t - X_s)_* P^\mu(B) &= (q \circ Y)_* P^\mu(B) \\
&= Y_* P^\mu(q^{-1}(B)) \\
&= P_{\{s,t\}*}(q^{-1}(B)) \\
&= \int_E \int_E \int_E 1_{q^{-1}(B)}(x_1, x_2) d\nu_{t-s}^{x_1}(x_2) d\nu_s^x(x_1) d\mu(x) \\
&= \int_E \int_E \int_E 1_{x_1+B}(x_2) d\nu_{t-s}^{x_1}(x_2) d\nu_s^x(x_1) d\mu(x) \\
&= \nu_{t-s}(B) \int_E \int_E d\nu_s^x(x_1) d\mu(x) \\
&= \nu_{t-s}(B) \int_E \nu_s^x(E) d\mu(x) \\
&= \nu_{t-s}(B) \int_E d\mu(x) \\
&= \nu_{t-s}(B),
\end{aligned}$$

which shows that

$$(X_t - X_s)_* P^\mu = \nu_{t-s},$$

and thus that X has stationary increments.

Let $0 = t_0 < t_1 < \dots < t_n$, let $J = \{t_0, t_1, \dots, t_n\} \in \mathcal{H}$, write $X_{t_{-1}} = 0$, and let

$$Y_0 = X_{t_0} - X_{t_{-1}}, \quad Y_1 = X_{t_1} - X_{t_0}, \quad \dots, \quad Y_n = X_{t_n} - X_{t_{n-1}}.$$

For the random variables Y_0, \dots, Y_n to be independent means for their joint distribution to be equal to the product of the distributions of each, i.e. to prove that X has independent increments, writing

$$Z = Y_0 \otimes \dots \otimes Y_n = \tau_n \circ (X_{t_0} \otimes \dots \otimes X_{t_n}) = \tau_n \circ X_J = \tau_n \circ \pi_{I,J},$$

with $\tau_n : E^{n+1} \rightarrow E^n$ defined by

$$\tau_n(x_0, x_1, \dots, x_n) = (x_0, x_1 - x_0, \dots, x_n - x_{n-1}),$$

we have to prove that

$$Z_* P^\mu = \prod_{j=0}^n Y_{j*} P^\mu.$$

To prove this, it suffices (because the collection of cylinder sets generates the product σ -algebra) to prove that for any $A_0, \dots, A_n \in \mathcal{E}$ and for $A = \prod_{j=0}^n A_j \in \mathcal{E}^{n+1}$,

$$(Z_* P^\mu)(A) = \left(\prod_{j=0}^n Y_{j*} P^\mu \right)(A),$$

i.e. that

$$(Z_*P^\mu)(A) = \prod_{j=0}^n (Y_{j*}P^\mu)(A_j).$$

We now prove this. Using the change of variables theorem and (6),

$$\begin{aligned} (Z_*P^\mu)(A) &= \int_{E^{n+1}} 1_A d(Z_*P^\mu) \\ &= \int_{\Omega} 1_A \circ Z dP^\mu \\ &= \int_{\Omega} 1_A \circ \tau_n \circ (X_{t_0} \otimes \cdots \otimes X_{t_n}) dP^\mu \\ &= \int_{E^J} 1_A \circ \tau_n d(X_{J*}P^\mu) \\ &= \int_{E^J} 1_A \circ \tau_n dP_J. \end{aligned}$$

Then applying (9) with $f = 1_A \circ \tau_n$,

$$\begin{aligned} &\int_{E^J} 1_A \circ \tau_n dP_J \\ &= \int_E \int_E \cdots \int_E 1_A \circ \tau_n \circ \sigma_{n+1} d\nu_{t_n-t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x_{-1}) \\ &= \int_E \int_E \cdots \int_E 1_A(x_{-1} + x_0, x_1, \dots, x_n) d\nu_{t_n-t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x_{-1}) \\ &= \int_E \int_E \cdots \int_E 1_{A_0}(x_{-1} + x_0) 1_{A_1}(x_1) \cdots 1_{A_n}(x_n) \\ &\quad d\nu_{t_n-t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x_{-1}) \\ &= \prod_{j=1}^n \nu_{t_j-t_{j-1}}(A_j) \int_E \int_E 1_{A_0}(x_{-1} + x_0) d\nu_{t_0}(x_0) d\mu(x_{-1}), \end{aligned}$$

and because $t_0 = 0$ and $\nu_0 = \delta_0$,

$$\begin{aligned} \int_E \int_E 1_{A_0}(x_{-1} + x_0) d\nu_{t_0}(x_0) d\mu(x_{-1}) &= \int_E \int_E 1_{A_0}(x_{-1} + x_0) d\delta_0(x_0) d\mu(x_{-1}) \\ &= \int_E 1_{A_0}(x_{-1}) d\mu(x_{-1}) \\ &= \mu(A_0), \end{aligned}$$

and therefore

$$(Z_*P^\mu)(A) = \mu(A_0) \cdot \prod_{j=1}^n \nu_{t_j-t_{j-1}}(A_j).$$

But we have already proved that (10), which tells us that for each j ,

$$Y_{j*}P^\mu = (X_{t_j} - X_{t_{j-1}})_*P^\mu = \nu_{t_j-t_{j-1}},$$

and thus

$$(Z_*P^\mu)(A) = \mu(A_0) \cdot \prod_{j=1}^n (Y_{j*}P^\mu)(A_j).$$

But $Y_{0*}P^\mu = X_{0*}P^\mu$ and from (7) we have

$$(X_{0*}P^\mu)(A_0) = \int_E (X_{0*}P^{\delta_x})(A_0) d\mu(x) = \int_E (\pi_{0*}P^{\delta_x})(A_0) d\mu(x),$$

and, from (5),

$$\begin{aligned} (\pi_{0*}P^{\delta_x})(A_0) &= \int_E \int_E 1_{A_0}(x_0) d\nu_0^y(x_0) d\delta_x(y) \\ &= \int_E \int_E 1_{A_0}(x_0) d\delta_y(x_0) d\delta_x(y) \\ &= \int_E 1_{A_0}(y) d\delta_x(y) \\ &= 1_{A_0}(x), \end{aligned}$$

thus

$$(X_{0*}P^\mu)(A_0) = \int_E 1_{A_0}(x) d\mu(x) = \mu(A_0).$$

Therefore

$$(Z_*P^\mu)(A) = (X_{0*}P^\mu)(A_0) \cdot \prod_{j=1}^n (Y_{j*}P^\mu)(A_j) = \prod_{j=0}^n (Y_{j*}P^\mu)(A_j),$$

which completes the proof that X has independent increments. \square

3 The Brownian convolution semigroup and Brownian motion

For $a \in \mathbb{R}$ and $\sigma > 0$, let γ_{a,σ^2} be the **Gaussian measure** on \mathbb{R} , the probability measure on \mathbb{R} whose density with respect to Lebesgue measure is

$$p(x, a, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right).$$

For $\sigma = 0$, let

$$\gamma_{a,0} = \delta_a.$$

Define for $t \in I$,

$$\nu_t = \prod_{k=1}^d \gamma_{0,t},$$

which is an element of $\mathcal{P}(E)$. For $s, t \in I$, we calculate

$$\nu_s * \mu_t = \left(\prod_{k=1}^d \gamma_{0,s} \right) * \left(\prod_{k=1}^d \gamma_{0,t} \right) = \prod_{k=1}^d (\gamma_{0,s} * \gamma_{0,t}) = \prod_{k=1}^d \gamma_{0,s+t} = \nu_{s+t},$$

showing that $(\nu_t)_{t \in I}$ is a convolution semigroup. It is proved using Lévy's continuity theorem that $t \mapsto \nu_t$ is continuous $I \rightarrow \mathcal{P}(E)$, showing that $(\nu_t)_{t \in I}$ is a continuous convolution semigroup.

We first prove a lemma (which is made explicit in **Isserlis's theorem**) about the moments of random variables with Gaussian distributions.³

Lemma 2. *If $Z : \Omega \rightarrow E$ is a random variable with Gaussian distribution ν_τ , $\tau > 0$, then for each n there is some $C_n > 0$ such that*

$$E(|Z|^{2n}) = C_n \tau^n.$$

In particular, $C_2 = d$ and $C_4 = d(d+2)$.

Proof. That Z has distribution ν_τ means that

$$Z_* P = \nu_\tau = \prod_{j=1}^d \gamma_{0,\tau}.$$

Write $Z = Z_1 \otimes \cdots \otimes Z_d$, each of which has distribution $\gamma_{0,\tau}$, and $Z_* P = \prod_{j=1}^d Z_{j*} P$, which means that Z_1, \dots, Z_d independent. Let $U_j = \tau^{-1/2} Z_j$ for $j = 1, \dots, d$, and then U_1, \dots, U_d are independent random variables each with distribution $\gamma_{0,1}$. Then using the multinomial formula,

$$\begin{aligned} E(|Z|^{2n}) &= E((Z_1^2 + \cdots + Z_d^2)^n) \\ &= \tau^n \cdot E((U_1^2 + \cdots + U_d^2)^n) \\ &= \tau^n \cdot E \left(\sum_{k_1 + \cdots + k_d = n} \frac{n!}{k_1! \cdots k_d!} \prod_{1 \leq i \leq d} U_i^{2k_i} \right) \\ &= \tau^n \cdot \sum_{k_1 + \cdots + k_d = n} \frac{n!}{k_1! \cdots k_d!} E \left(\prod_{1 \leq i \leq d} U_i^{2k_i} \right). \end{aligned}$$

For $n = 2$, since $E(U_i U_j) = E(U_i) E(U_j) = 0$ for $i \neq j$,

$$\tau^2 \cdot \sum_{k_1 + \cdots + k_d = 2} \frac{2!}{k_1! \cdots k_d!} E \left(\prod_{1 \leq i \leq d} U_i^{2k_i} \right) = \tau^2 \cdot \sum_{j=1}^d E(U_j^2) = \tau^2 \cdot \sum_{j=1}^d 1 = d\tau^2,$$

showing that $C_2 = d$. □

³Heinz Bauer, *Probability Theory*, p. 341, Lemma 40.2.

A stochastic process $(\Omega, \mathcal{F}, P, (X_t)_{t \in I})$ with state space E is called a **d -dimensional Brownian motion** when:

1. For $s \leq t$,

$$(X_t - X_s)_*P = \nu_{t-s},$$

and thus X has stationary increments.

2. X has independent increments.

3. For almost all $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ is continuous $I \rightarrow E$.

We call $X_{0*}P$ the **initial distribution** of the Brownian motion. When $X_{0*}P = \delta_x$ for some $x \in E$, we say that x is the **starting point of the Brownian motion**. We now prove that for any Borel probability measure on \mathcal{E} , in particular δ_x , there is a d -dimensional Brownian motion which has this as its initial distribution.⁴

Theorem 3 (Brownian motion). *For any $\mu \in \mathcal{P}(E)$, there is a d -dimensional Brownian motion with initial distribution μ .*

Proof. Let $(P_J)_{J \in \mathcal{K}}$ be the family of measures induced by the Brownian convolution semigroup

$$\nu_t = \prod_{k=1}^d \gamma_{0,t}, \quad t \in I,$$

and let $(\Omega, \mathcal{F}, P^\mu, (X_t)_{t \in I})$, $\Omega = E^I$ and $\mathcal{F} = \mathcal{E}^I$, be the associated canonical process. Theorem 1 tells us that X has stationary increments,

$$(X_t - X_s)_*P^\mu = \nu_{t-s}, \quad s \leq t, \quad (12)$$

and has independent increments. For $\tau = t - s > 0$, by (12) and Lemma 2,

$$E(|X_t - X_s|^4) = d(d+2)\tau^2 = d(d+2)|t-s|^2.$$

Because $E(|X_t - X_t|^4) = E(0) = 0$, we have that for any $s, t \in I$,

$$E(|X_t - X_s|^4) = d(d+2)|t-s|^2.$$

The initial distribution of X is $X_{0*}P^\mu = \mu$. For $\alpha = 4, \beta = 1, c = d(d+2)$, the **Kolmogorov continuity theorem** tells us that there is a continuous modification B of X . That is, there is a stochastic process $(B_t)_{t \in I}$ such that for each $\omega \in \Omega$, the path $t \mapsto B_t(\omega)$ is continuous $I \rightarrow E$, namely, B is a **continuous stochastic process**, and for each $t \in I$,

$$P(X_t = B_t) = 1,$$

namely, B is a **modification** of X . Because B is a modification of X , B has the same finite-dimensional distributions as X , from which it follows that B satisfies

⁴Heinz Bauer, *Probability Theory*, p. 342, Theorem 40.3.

(12) and has independent increments. For $A \in \mathcal{E}$, because B is a modification of X ,

$$(B_{0*}P^\mu)(A) = P^\mu(B_0 \in A) = P^\mu(X_0 \in A) = (X_{0*}P^\mu)(A),$$

thus $B_{0*}P^\mu = X_{0*}P^\mu = \mu$, namely, B has initial distribution μ . Therefore, B is a Brownian motion (indeed, all the paths of B are continuous, not merely almost all of them) that has initial distribution μ , proving the claim. \square

For $\mu \in \mathcal{P}(E)$, let $(\Omega, \mathcal{F}, P^\mu, (B_t)_{t \in I})$ be the d -dimensional Brownian motion with initial distribution μ constructed in Theorem 3; we are not merely speaking about some d -dimensional Brownian motion but about this construction, for which $\Omega = E^I$, all whose paths are continuous rather than merely almost all whose paths are continuous. For a measurable space (A, \mathcal{A}) and topological spaces X and Y , a function $f : X \times A \rightarrow Y$ is called a **Carathéodory function** if for each $x \in X$, the map $a \mapsto f(x, a)$ is measurable $\mathcal{A} \rightarrow \mathcal{B}_Y$, and for each $a \in A$, the map $x \mapsto f(x, a)$ is continuous $X \rightarrow Y$. It is a fact⁵ that if X is a separable metrizable space and Y is a metrizable space, then any Carathéodory function $f : X \times A \rightarrow Y$ is measurable $\mathcal{B}_X \otimes \mathcal{A} \rightarrow \mathcal{B}_Y$, namely it is **jointly measurable**. $B : I \times \Omega \rightarrow E$ is a Carathéodory function. $I = \mathbb{R}_{\geq 0}$, with the subspace topology inherited from \mathbb{R} , is a separable metrizable space, and $E = \mathbb{R}^d$ is a metrizable space, and therefore the d -dimensional Brownian motion B is jointly measurable.

The **Kolmogorov-Chentsov theorem** says that if a stochastic process $(X_t)_{t \in I}$ with state space E satisfies, for $\alpha, \beta, c > 0$,

$$E(|X_s - X_t|^\alpha) \leq c|t - s|^{1+\beta}, \quad s, t \in I,$$

and almost every path of X is continuous, then for almost every $\omega \in \Omega$, for every $0 < \gamma < \frac{\beta}{\alpha}$ the map $t \mapsto X_t(\omega)$ is **locally γ -Hölder continuous**: for each $t_0 \in I$ there is some $0 < \epsilon_{t_0} < 1$ and some C_{t_0} such that

$$|X_t(\omega) - X_s(\omega)| \leq C_{t_0}|t - s|^\gamma, \quad |s - t_0| < \epsilon_{t_0}, |t - t_0| < \epsilon_{t_0}.$$

For $\mu \in \mathcal{P}(E)$, let $(\Omega, \mathcal{F}, P^\mu, (B_t)_{t \in I})$ be the d -dimensional Brownian motion with initial distribution μ formed in Theorem 3. For $s \leq t$, $(B_t - B_s)_*P^\mu = \nu_{t-s}$, and thus Lemma 2 tells us that for each $n \geq 1$ there is some C_n with which $E(|B_t - B_s|^{2n}) = C_n(t - s)^n$ for all $s < t$. Then $E(|B_t - B_s|^{2n}) \leq C_n|t - s|^n$ for all $s, t \in I$. For $n > 1$ and for $\alpha_n = 2n$ and $\beta_n = n - 1$,

$$\frac{\beta_n}{\alpha_n} = \frac{n - 1}{2n} = \frac{1}{2} - \frac{1}{2n},$$

and for $n > 2$, take some $\frac{\beta_{n-1}}{\alpha_{n-1}} < \gamma_n < \frac{\beta_n}{\alpha_n}$. Let N_n be the set of those $\omega \in \Omega$ for which $t \mapsto B_t(\omega)$ is not locally γ_n -Hölder continuous. Then the Kolmogorov-Chentsov theorem yields $P^\mu(N_n) = 0$. Let $N = \bigcup_{n \geq 2} N_n$, which is a P^μ -null

⁵Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 153, Lemma 4.51.

set. For $\omega \in \Omega \setminus N$ and for any $0 < \gamma < \frac{1}{2}$, there is some γ_n satisfying $\gamma \leq \gamma_n < \frac{1}{2}$, and hence the map $t \mapsto B_t(\omega)$ is locally γ_n -Hölder continuous, which implies that this map is locally γ -Hölder continuous. We summarize what we have just said in the following theorem.

Theorem 4. *Let $\mu \in \mathcal{P}(E)$ and let $(\Omega, \mathcal{F}, P^\mu, (B_t)_{t \in I})$ be the d -dimensional Brownian motion with initial distribution μ formed in Theorem 3. For almost all $\omega \in \Omega$, for all $0 < \gamma < \frac{1}{2}$, the map $t \mapsto B_t(\omega)$ is locally γ -Hölder continuous.*

4 Lévy processes

A stochastic process $(X_t)_{t \in I}$ with state space E is called a **Lévy process**⁶ if (i) $X_0 = 0$ almost surely, (ii) X has stationary and independent increments, and (iii) for any $a > 0$,

$$\lim_{t \downarrow 0} P(|X_t| \geq \epsilon) = 0.$$

Because $X_0 = 0$ almost surely and X has stationary increments, (iii) yields for any $t \in I$,

$$\lim_{s \rightarrow t} P(|X_s - X_t| \geq \epsilon) = 0. \quad (13)$$

In any case, (13) is sufficient for (iii) to be true. Moreover, (iii) means that $X_s \rightarrow X_t$ in the **topology of convergence in probability** as $s \rightarrow t$, and if $X_s \rightarrow X_t$ almost surely then $X_s \rightarrow X_t$ in the topology of convergence in probability; this is proved using Egorov's theorem. Thus, a d -dimensional Brownian motion with starting point 0 is a Lévy process; we do not merely assert that the Brownian motion formed in Theorem 3 is a Lévy process. There is much that can be said generally about Lévy processes, and thus the fact that any d -dimensional Brownian motion with starting point 0 is a Lévy process lets us work in a more general setting in which some results may be more naturally proved: if we work merely with a Lévy process we know less about the process and thus have less open moves.

⁶See David Applebaum, *Lévy Processes and Stochastic Calculus*, p. 39, §1.3.