# Chebyshev polynomials

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# 1 Chebyshev polynomials of first kind

On the one hand,

$$(\cos \theta + i \sin \theta)^n = \sum_{0 \le \nu \le n} i^{\nu} \binom{n}{\nu} \cos^{n-\nu} \theta \sin^{\nu} \theta$$
$$= \sum_{0 \le 2k \le n} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) \sin^{2k}(\theta)$$
$$+ i \sum_{0 \le 2k+1 \le n} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(\theta) \sin^{2k+1}(\theta).$$

On the other hand,

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta.$$

Therefore

$$\cos n\theta = \sum_{0 \le k \le n/2} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) \sin^{2k}(\theta)$$

$$= \sum_{0 \le k \le n/2} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) (1 - \cos^2 \theta)^k$$

$$= \sum_{0 \le k \le n/2} \binom{n}{2k} \cos^{n-2k}(\theta) (\cos^2 \theta - 1)^k$$

$$= \sum_{0 \le k \le n/2} \binom{n}{2k} \cos^{n-2k}(\theta) \sum_{0 \le j \le k} \binom{k}{j} \cos^{2k-2j}(\theta) (-1)^j$$

$$= \sum_{0 \le j \le n/2} (-1)^j \cos^{n-2j}(\theta) \sum_{j \le k \le n/2} \binom{n}{2k} \binom{k}{j}.$$

Now,

$$\sum_{1 \le k \le n/2} \binom{n}{2k} \binom{k}{j} = 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j}.$$

Hence

$$\cos n\theta = \sum_{0 \le j \le n/2} (-1)^j \cos^{n-2j}(\theta) 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j}.$$

For  $z \in \mathbb{C}$  let

$$T_n(z) = \sum_{0 \le j \le n/2} (-1)^j 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j} z^{n-2j}.$$
 (1)

Note

$$T_n(z)[z^n] = 2^{n-1}z^n.$$

#### Theorem 1.

$$T_n(\cos\theta) = \cos(n\theta)$$

and

$$T_m \circ T_n = T_{mn}$$
.

*Proof.* For  $\theta \in \mathbb{R}$ ,

$$T_n(\cos\theta) = \cos(n\theta).$$

Then

$$T_m(T_n(\cos\theta)) = T_m(\cos(n\theta)) = \cos(mn\theta) = T_{mn}(\theta).$$

That is, for  $z \in [-1, 1]$  we have  $T_m(T_n(z)) = T_{mn}(z)$ . Then by analytic continuation it follows that this is true for all z.

### Theorem 2.

$$T_n(z) + T_{n-2}(z) = 2zT_{n-1}(z).$$

*Proof.* Using  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ ,

$$\cos(n\theta) = \cos(\theta + (n-1)\theta) = \cos\theta\cos((n-1)\theta) - \sin\theta\sin((n-1)\theta)$$

and

$$\cos((n-2)\theta) = \cos(-\theta + (n-1)\theta) = \cos\theta\cos((n-1)\theta) + \sin\theta\sin((n-1)\theta).$$

Then

$$\cos(n\theta) + \cos((n-2)\theta) = 2\cos\theta\cos((n-1)\theta).$$

Therefore

$$T_n(\cos \theta) + T_{n-2}(\cos \theta) = \cos(n\theta) + \cos((n-2)\theta)$$
$$= 2\cos \theta \cos((n-1)\theta)$$
$$= 2\cos \theta \cdot T_{n-1}(\cos \theta).$$

That is, for  $z \in [-1, 1]$ ,

$$T_n(z) + T_{n-2}(z) = 2zT_{n-1}(z),$$

and by analytic continuation this is true for all  $z \in \mathbb{C}$ .

# 2 Chebyshev polynomials of second kind

Define

$$nU_{n-1}(z) = T'_n(z).$$
 (2)

Theorem 3.

$$U_{n-1}(\cos\theta) = \frac{\sin(n\theta)}{\sin\theta}$$

and

$$(1-z^2)T_n''(z) = nU_n(z) - n(n+1)T_n(z).$$

Proof. On the one hand,

$$(T_n(\cos\theta))' = -\sin\theta \cdot T'_n(\cos\theta).$$

On the other hand,

$$(T_n(\cos\theta))' = (\cos(n\theta))' = -n\sin(n\theta).$$

Hence

$$T'_n(\cos \theta) = n \frac{\sin(n\theta)}{\sin \theta}, \qquad U_{n-1}(\cos \theta) = \frac{\sin(n\theta)}{\sin \theta}.$$

Now,

$$(T'_n(\cos\theta))' = -\sin\theta \cdot T''_n(\cos\theta)$$

and

$$\begin{split} (T_n'(\cos\theta))' &= n \frac{n\cos(n\theta)\sin\theta - \sin(n\theta)\cos\theta}{\sin^2\theta} \\ &= -n \frac{\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta}{\sin^2\theta} + n(n+1) \frac{\cos(n\theta)\sin\theta}{\sin^2\theta} \\ &= -n \frac{\sin((n+1)\theta)}{\sin^2\theta} + n(n+1) \frac{\cos(n\theta)}{\sin\theta} \\ &= -n \frac{U_n(\cos\theta)}{\sin\theta} + n(n+1) \frac{T_n(\cos\theta)}{\sin\theta}. \end{split}$$

Hence

$$T_n''(\cos \theta) = n \frac{U_n(\cos \theta)}{\sin^2 \theta} - n(n+1) \frac{T_n(\cos \theta)}{\sin^2 \theta}$$

and then

$$T_n''(\cos\theta) = n \frac{U_n(\cos\theta)}{1 - \cos^2\theta} - n(n+1) \frac{T_n(\cos\theta)}{1 - \cos^2\theta}.$$

By analytic continuation,

$$(1-z^2)T_n''(z) = nU_n(z) - n(n+1)T_n(z).$$

#### Theorem 4.

$$T_{n+1}(z) = zT_n(z) - (1-z^2)U_{n-1}(z).$$

Proof.

$$T_{n+1}(\cos \theta) = \cos(n\theta + \theta)$$

$$= \cos(n\theta)\cos \theta - \sin(n\theta)\sin \theta$$

$$= T_n(\cos \theta)\cos \theta - U_{n-1}(\cos \theta)\sin^2 \theta$$

$$= T_n(\cos \theta)\cos \theta - U_{n-1}(\cos \theta)(1 - \cos^2 \theta).$$

Therefore by analytic continuation,

$$T_{n+1}(z) = zT_n(z) - (1-z^2)U_{n-1}(z).$$

### Theorem 5.

$$U_n(z) = T_n(z) + zU_{n-1}(z).$$

Proof.

$$U_n(\cos \theta) = \frac{\sin(n\theta + \theta)}{\sin \theta}$$

$$= \frac{\cos(n\theta)\sin \theta + \cos \theta \sin(n\theta)}{\sin \theta}$$

$$= T_n(\cos \theta) + \cos \theta \cdot U_{n-1}(\cos \theta).$$

Therefore by analytic continuation,

$$U_n(z) = T_n(z) + zU_{n-1}(z).$$

### Theorem 6.

$$U_n(z) = 2zU_{n-1}(z) + U_{n-2}(z).$$

Proof. Using Theorem 4 and Theorem 5,

$$\begin{split} U_n(z) &= T_n(z) + z U_{n-1}(z) \\ &= z T_{n-1}(z) - (1-z^2) U_{n-2}(z) + z U_{n-1}(z) \\ &= z \left[ U_{n-1}(z) - z U_{n-2}(z) \right] - (1-z^2) U_{n-2}(z) + z U_{n-1}(z) \\ &= 2z U_{n-1}(z) + U_{n-2}(z). \end{split}$$

#### Theorem 7.

$$(1 - z2)T''n(z) - zT'n(z) + n2Tn(z) = 0.$$

Proof. Using Theorem 3, and Theorem 5,

$$(1-z^2)T_n''(z) - zT_n'(z) + n^2T_n(z)$$

$$= nU_n(z) - n(n+1)T_n(z) - nzU_{n-1}(z) + n^2T_n(z)$$

$$= n(T_n(z) + zU_{n-1}(z)) - n(n+1)T_n(z) - nzU_{n-1}(z) + n^2T_n(z)$$

$$= 0.$$

From Theorem 1

$$T_n(1) = T_n(\cos 0) = \cos(n \cdot 0) = 1.$$

From Theorem 3,

$$T'_n(1) = nU_{n-1}(1) = n^2.$$

Thus,  $T_n$  is the unique solution of the initial value problem

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0,$$
  $y(1) = 1, y'(1) = n^2.$ 

Theorem 8.

$$T_n(z)^2 - (z^2 - 1)U_{n-1}(z)^2 = 1.$$

*Proof.* Using Theorem 1 and Theorem 3, for  $z = \cos \theta$ ,

$$T_n(z)^2 - (z^2 - 1)U_{n-1}(z)^2 = T_n(\cos\theta)^2 + (\sin^2\theta)U_{n-1}(\cos\theta)^2$$
$$= \cos^2(n\theta) + (\sin^2\theta)\frac{\sin^2(n\theta)}{\sin^2\theta}$$
$$= \cos^2(n\theta) + \sin^2(n\theta)$$
$$= 1$$

By analytic continuation, this is true for all z.

## 3 Inner products

For  $0 \le \theta \le \pi$  let  $y_n(\theta) = \cos(n\theta)$ .

$$y_n'' + n^2 y_n = 0,$$
  $y_n'(0) = 0, y_n'(\pi) = 0.$ 

Theorem 9.

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \int_{0}^{\pi} y_m y_n d\theta = \frac{\pi}{2} \cdot \delta_{m,n}.$$

*Proof.* Let  $W = y_m y'_n - y_n y'_m$ . We calculate

$$W' = y'_m y'_n + y_m y''_n - y'_n y'_m - y_n y''_m y_m y''_n - y_n y''_m$$
  
=  $y_m y''_n - y_n y''_m$   
=  $y_m (-n^2 y_n) - y_n (-m^2 y_m)$   
=  $(m^2 - n^2) y_m y_n$ .

Using W(0) = 0 and  $W(\pi) = 0$ ,

$$\int_0^{\pi} W'(\theta)d\theta = W(\pi) - W(0) = 0.$$

Then

$$\int_0^{\pi} (m^2 - n^2) y_m y_n d\theta = 0.$$

Doing the substitution  $\phi = n\theta$ ,

$$\int_0^{\pi} y_n^2 d\theta = \int_0^{\pi} \cos^2(n\theta) d\theta$$
$$= \int_0^{\pi} \frac{1 + \cos(2n\theta)}{2} d\theta$$
$$= \frac{\pi}{2}.$$

Therefore

$$\int_0^{\pi} y_m y_n d\theta = \frac{\pi}{2} \cdot \delta_{m,n}.$$

For  $0 \le \theta \le \pi$ ,  $\sqrt{1 - \cos^2 \theta} = \sin \theta$ . Then doing the substitution  $x = \cos \theta$ ,  $dx = -\sin \theta d\theta$ ,

$$\int_0^{\pi} y_m y_n d\theta = \int_0^{\pi} \cos(m\theta) \cos(n\theta) d\theta$$

$$= \int_0^{\pi} \cos(m\theta) \cos(n\theta) \frac{-\sin\theta d\theta}{-\sin\theta}$$

$$= \int_0^{\pi} \frac{\cos(m\theta) \cos(n\theta)}{-\sqrt{1 - \cos^2\theta}} (-\sin\theta) d\theta$$

$$= \int_1^{-1} \frac{T_m(x) T_n(x)}{-\sqrt{1 - x^2}} dx$$

$$= \int_{-1}^1 \frac{T_m(x) T_n(x)}{\sqrt{1 - x^2}} dx.$$