

L^p norms of trigonometric polynomials

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1 Introduction

A *trigonometric polynomial of degree n* is an expression of the form

$$\sum_{k=-n}^n c_k e^{ikt}, \quad c_k \in \mathbb{C}.$$

Using the identity $e^{it} = \cos t + i \sin t$, we can write a trigonometric polynomial of degree n in the form

$$a_0 + \sum_{k=1}^n a_k \cos kt + \sum_{k=1}^n b_k \sin kt, \quad a_k, b_k \in \mathbb{C}.$$

For $1 \leq p < \infty$ and for a 2π -periodic function f , we define the L^p norm of f by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right)^{1/p}.$$

For a continuous 2π -periodic function f , we define the L^∞ norm of f by

$$\|f\|_\infty = \max_{0 \leq t \leq 2\pi} |f(t)|.$$

If f is a continuous 2π -periodic function, then there is a sequence of trigonometric polynomials f_n such that $\|f - f_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ [31, p. 54, Corollary 5.4].

If $1 \leq p < \infty$ and f is a continuous 2π -periodic function, then

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right)^{1/p} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|f\|_\infty^p dt \right)^{1/p} = \|f\|_\infty.$$

Jensen's inequality [16, p. 44, Theorem 2.2] (cf. [30, p. 113, Problem 7.5]) tells us that if $\phi : [0, \infty) \rightarrow \mathbb{R}$ is convex, then for any function $h : [0, 2\pi] \rightarrow [0, \infty)$ we have

$$\phi \left(\frac{1}{2\pi} \int_0^{2\pi} h(t) dt \right) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(h(t)) dt.$$

If $1 \leq p < q < \infty$, then $\phi : [0, \infty) \rightarrow \mathbb{R}$ defined by $\phi(x) = x^{q/p}$ is convex. Hence, if $1 \leq p < q < \infty$ then for any 2π -periodic function f ,

$$\begin{aligned}\|f\|_p &= (\phi(\|f\|_p^p))^{1/q} \\ &= \left(\phi \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right) \right)^{1/q} \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} \phi(|f(t)|^p) dt \right)^{1/q} \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^q dt \right)^{1/q} \\ &= \|f\|_q.\end{aligned}$$

The *Dirichlet kernel* D_n is defined by

$$D_n(t) = \sum_{k=-n}^n e^{ikt} = 1 + 2 \sum_{k=1}^n \cos kt.$$

One can show [14, p. 71, Exercise 1.1] that

$$\|D_n\|_1 = \frac{4}{\pi^2} \cdot \log n + O(1).$$

(On the other hand, it can quickly be seen that $\|D_n\|_\infty = 2n + 1$, and it follows from Parseval's identity that $\|D_n\|_2 = \sqrt{2n + 1}$.)

Pólya and Szegő [27, Part VI] present various problems about trigonometric polynomials together with solutions to them. A result on L^∞ norms of trigonometric polynomials that Pólya and Szegő present is for the sum $A_n(t) = \sum_{k=1}^n \frac{\sin kt}{k}$. The local maxima and local minima of A_n can be explicitly determined [27, p. 74, no. 23], and it can be shown that [27, p. 74, no. 25]

$$\|A_n\|_\infty \sim \int_0^\pi \frac{\sin t}{t} dt.$$

2 L^p norms

If $1 \leq p < q < \infty$, then [14, p. 123, Exercise 1.8] (cf. [7, p. 102, Theorem 2.6]) there is some $C(p, q)$ such that for any trigonometric polynomial f of degree n , we have

$$\|f\|_q \leq C(p, q) n^{\frac{1}{p} - \frac{1}{q}} \|f\|_p.$$

This inequality is sharp [33, p. 230]: for $1 \leq p < q < \infty$ there is some $C'(p, q)$ such that if $F_n(t) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(t)$ (F_n is called the *Fejér kernel*) then

$$\|F_n\|_q > C'(p, q) n^{\frac{1}{p} - \frac{1}{q}} \|F_n\|_p.$$

Theorem 1. Let $1 \leq p \leq q \leq \infty$. If $\hat{f}(j) = 0$ for $|j| > n + 1$ then

$$\|f\|_q \leq 5(n+1)^{\frac{1}{p}-\frac{1}{q}} \|f\|_p.$$

Proof. Let $K_n(t) = \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) e^{ijt}$, the Fejér kernel. From this expression we get $|K_n(t)| \leq K_n(0) = n+1$. It's straightforward to show that $K_n(t) = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2}t}{\sin \frac{1}{2}t} \right)^2$. Since $\sin \frac{t}{2} > \frac{t}{\pi}$ for $0 < t < \pi$, we get $|K_n(t)| \leq \frac{\pi^2}{(n+1)t^2}$, and thus we obtain

$$|K_n(t)| \leq \min \left(n+1, \frac{\pi^2}{(n+1)t^2} \right).$$

Then, for any $r \geq 1$,

$$\begin{aligned} \|K_n\|_r^r &= \frac{1}{2\pi} \int_0^{2\pi} |K_n(t)|^r dt \\ &\leq \frac{1}{2\pi} \int_0^{\frac{\pi}{n+1}} (n+1)^r dt + \frac{1}{2\pi} \int_{\frac{\pi}{n+1}}^{2\pi} \left(\frac{\pi^2}{(n+1)t^2} \right)^r dt \\ &= \frac{(n+1)^{r-1}}{2} + \frac{1}{2} \frac{1}{(n+1)^r} \frac{1}{2r-1} \left((n+1)^{2r-1} - \frac{1}{2^{2r-1}} \right) \\ &\leq \frac{(n+1)^{r-1}}{2} + \frac{1}{2} \frac{1}{(n+1)^r} \frac{1}{2r-1} (n+1)^{2r-1} \\ &\leq (n+1)^{r-1}. \end{aligned}$$

Hence $\|K_n\|_r \leq (n+1)^{1-\frac{1}{r}}$.

Let $V_n(t) = 2K_{2n+1}(t) - K_n(t)$, the **de la Vallée Poussin kernel**. Then

$$\|V_n\|_r \leq 2\|K_{2n+1}\|_r + \|K_n\|_r \leq 2(2n+2)^{1-\frac{1}{r}} + (n+1)^{1-\frac{1}{r}} \leq 5(n+1)^{1-\frac{1}{r}}.$$

For $|j| \leq n+1$ we have $\widehat{V_n}(j) = 1$, and one thus checks that $V_n * f = f$. Take $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$. By Young's inequality we have

$$\|f\|_q = \|V_n * f\|_q \leq \|V_n\|_r \|f\|_p \leq 5(n+1)^{\frac{1}{p}-\frac{1}{q}} \|f\|_p.$$

□

Let $X_n = \{a_0 + \sum_{k=1}^n a_k \cos kt + b_k \sin kt : a_k, b_k \in \mathbb{R}\}$, the real vector space of real valued trigonometric polynomials of degree n , have norm

$$\|f\|_{X_n} = \max\{|a_0|, |a_1|, \dots, |a_n|, |b_1|, \dots, |b_n|\}.$$

Let $Y_{n,p}$ be the same vector space with the L^p norm. Ash and Ganzburg [1] give upper and lower bounds on the operator norm of the map $i : X_n \rightarrow Y_{n,p}$ defined by $i(f) = f$.

Bernstein's inequality [14, p. 50, Exercise 7.16] states that for $1 \leq p \leq \infty$, if f is a trigonometric polynomial of degree n , then

$$\|f'\|_p \leq n\|f\|_p.$$

In the other direction, if $f \in C^1$ then

$$\begin{aligned}
& \frac{1}{2\pi} \int_0^{2\pi} f(s) ds + \frac{1}{2\pi} \int_0^t s f'(s) ds + \frac{1}{2\pi} \int_t^{2\pi} (s - 2\pi) f'(s) ds \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(s) ds + \frac{1}{2\pi} \int_0^{2\pi} s f'(s) ds - \int_t^{2\pi} f'(s) ds \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(s) ds + \frac{1}{2\pi} s f(s) \Big|_0^{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} f(s) ds - f(s) \Big|_t^{2\pi} \\
&= f(t).
\end{aligned}$$

Hence

$$\begin{aligned}
|f(t)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(s)| ds + \frac{1}{2\pi} \int_0^t s |f'(s)| ds + \frac{1}{2\pi} \int_t^{2\pi} (2\pi - s) |f'(s)| ds \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(s)| ds + \int_0^t |f'(s)| ds + \int_t^{2\pi} |f'(s)| ds \\
&= \|f\|_1 + 2\pi \|f'\|_1,
\end{aligned}$$

so

$$\|f\|_\infty \leq \|f\|_1 + 2\pi \|f'\|_1.$$

This is an instance of the Sobolev inequality [26].

It turns out that for a trigonometric polynomial the mass cannot be too concentrated. More precisely, the number of nonzero terms of a trigonometric polynomial restricts how concentrated its mass can be. Let $d\mu = \frac{dt}{2\pi}$. Thus $\mu([0, 2\pi]) = 1$. A result of Turán [20, p. 89, Lemma 1] states that if $\lambda_1, \dots, \lambda_N \in \mathbb{Z}$ and $T(t) = \sum_{n=1}^N b_n e^{i\lambda_n t}$, $b_n \in \mathbb{C}$, then for any closed arc $I \subset [0, 2\pi]$,

$$\|T\|_\infty \leq \left(\frac{2e}{\mu(I)} \right)^{N-1} \max_{t \in I} |T(t)|.$$

Nazarov [11, p. 452] shows that there is some constant A such that if E is a closed subset of $[0, 2\pi]$ (not necessarily an arc), then

$$\|\hat{T}\|_1 \leq \left(\frac{A}{\mu(E)} \right)^N \max_{t \in E} |f(T)|.$$

Nazarov [23] proves that there exists some constant C such that if $0 \leq q \leq 2$ and $\mu(E) \geq \frac{1}{3}$, then

$$\|T\|_q \leq e^{C(N-1)(1-\frac{\mu(E)}{2\pi})} \left(\frac{1}{2\pi} \int_E |T(t)|^q dt \right)^{1/q}.$$

These results of Turán and Nazarov are examples of the *uncertainty principle* [9], which is the general principle that a constrain on the support of the Fourier transform of a function constrains the support of the function itself.

In [10], Hardy and Littlewood present inequalities for norms of 2π -periodic functions in terms of certain series formed from their Fourier coefficients. Let $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$, be such that $c_k \rightarrow 0$ as $k \rightarrow \pm\infty$, and define $c_0^*, c_1^*, c_{-1}^*, c_2^*, c_{-2}^*, \dots$ to be the absolute values of the c_k ordered in decreasing magnitude. For real $r > 1$, define

$$S_r^*(c) = \left(\sum_{k=-\infty}^{\infty} c_k^{*r} (|k| + 1)^{r-2} \right)^{1/r}.$$

For instance, if $c_k = 1$ for $-N \leq k \leq N$ and $c_k = 0$ for $|k| > N$, then $S_r^*(c) = \left(1 + 2 \sum_{k=2}^{N+1} k^{r-2} \right)^{1/r}$. Hardy and Littlewood state the result [10, p. 164, Theorem 2] that if $1 < p \leq 2$ then there is some constant $A(p)$ such that for any sequence c , with $c_k \rightarrow 0$ as $k \rightarrow \pm\infty$, if $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ and $\|f\|_p < \infty$ then

$$S_p^*(c) \leq A(p) \|f\|_p.$$

A proof of this is given in Zygmund [35, vol. II, p. 128, chap. XII, Theorem 6.3]. Asking if this inequality holds for $p = 1$ suggests the following question that Hardy and Littlewood pose at the end of their paper [10, p. 168]: Is there a constant A such that for all distinct positive integers $m_k, k = 1, \dots, N$, we have

$$\left\| \sum_{k=1}^N \cos m_k t \right\|_1 > A \log N?$$

McGehee, Pigno and Smith [18] prove that there is some K such that for all N , if n_1, \dots, n_N are distinct integers and $c_1, \dots, c_N \in \mathbb{C}$ satisfy $|c_k| \geq 1$, then

$$\left\| \sum_{k=1}^N c_k e^{in_k t} \right\|_1 > K \log N.$$

Thus

$$\left\| \sum_{k=1}^N \cos m_k t \right\|_1 = \frac{1}{2} \cdot \left\| \sum_{k=1}^N e^{im_k t} + e^{-im_k t} \right\|_1 \geq \frac{1}{2} \cdot K \log(2N).$$

For $k \geq 2$, define $T_N(t) = \sum_{n=1}^N e^{in^k t}$. Since $\|T_N\|_{\infty} = N$, for each $p \geq 1$ we have $\|T_N\|_p \leq N$. Hua's lemma [22, p. 116, Theorem 4.6] states that if $\epsilon > 0$, then

$$\|T_N\|_{2^k} = O\left(N^{1 - \frac{k}{2^k} + \epsilon}\right).$$

Hua's lemma is used in additive number theory. The number of sets of integer solutions of the equation

$$f(x_1, \dots, x_n) = N, \quad a_r \leq x_r \leq b_r$$

is equal to (cf. [12, p. 151])

$$\sum_{a_1 \leq x_1 \leq b_1} \dots \sum_{a_n \leq x_n \leq b_n} \int_0^1 e^{2\pi i(f(x_1, \dots, x_n) - N)t} dt.$$

Borwein and Lockhart [4]: what is the expected L^p norm of a trigonometric polynomial of order n ? Kahane [13, Chapter 6] also presents material on random trigonometric polynomials.

Nursultanov and Tikhonov [25]: the sup on a subset of \mathbb{T} of a trigonometric polynomial f of degree n being lower bounded in terms of $\|f\|_\infty$, n , and the measure of the subset.

3 ℓ^p norms

For a 2π -periodic function f , we define $\hat{f} : \mathbb{Z} \rightarrow \mathbb{C}$ by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt.$$

For $1 \leq p < \infty$, we define the ℓ^p norm of \hat{f} by

$$\|\hat{f}\|_p = \left(\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^p \right)^{1/p},$$

and we define the ℓ^∞ norm of \hat{f} by

$$\|\hat{f}\|_\infty = \max_{k \in \mathbb{Z}} |\hat{f}(k)|.$$

Parseval's identity [31, p. 80, Theorem 1.3] states that $\|f\|_2 = \|\hat{f}\|_2$. If $1 \leq p < \infty$, then

$$\|\hat{f}\|_\infty \leq \left(\cdots + \|\hat{f}\|_\infty^p + \cdots \right)^{1/p} = \|\hat{f}\|_p.$$

If $1 \leq p < q < \infty$, then, since for each k , $\frac{|\hat{f}(k)|}{\|\hat{f}\|_q} \leq 1$,

$$1 = \left(\sum_{k=-\infty}^{\infty} \left(\frac{|\hat{f}(k)|}{\|\hat{f}\|_q} \right)^q \right)^{1/q} \leq \left(\sum_{k=-\infty}^{\infty} \left(\frac{|\hat{f}(k)|}{\|\hat{f}\|_q} \right)^p \right)^{1/q} = \frac{\|\hat{f}\|_p^{p/q}}{\|\hat{f}\|_q^{p/q}}.$$

Hence for $1 \leq p < q < \infty$,

$$\|\hat{f}\|_q \leq \|\hat{f}\|_p.$$

For $1 \leq p < \infty$, if f is a trigonometric polynomial of degree n then

$$\|\hat{f}\|_p = \left(\sum_{k=-n}^n |\hat{f}(k)|^p \right)^{1/p} \leq \left(\sum_{k=-n}^n \|\hat{f}\|_\infty^p \right)^{1/p} = (2n+1)^{1/p} \|\hat{f}\|_\infty.$$

For $1 \leq p < q < \infty$, we have [30, p. 123, Problem 8.3] (this is Jensen's inequality for sums)

$$\left(\sum_{k=-n}^n \frac{1}{2n+1} |\hat{f}(k)|^p \right)^{1/p} \leq \left(\sum_{k=-n}^n \frac{1}{2n+1} |\hat{f}(k)|^q \right)^{1/q},$$

i.e.

$$(2n+1)^{-1/p} \|\hat{f}\|_p \leq (2n+1)^{-1/q} \|\hat{f}\|_q.$$

Hence for $1 < p < q < \infty$,

$$\|\hat{f}\|_p \leq (2n+1)^{\frac{1}{p}-\frac{1}{q}} \|\hat{f}\|_q.$$

For any t ,

$$|f(t)| = \left| \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt} \right| \leq \sum_{k=-\infty}^{\infty} |\hat{f}(k) e^{ikt}| = \sum_{k=-\infty}^{\infty} |\hat{f}(k)| = \|\hat{f}\|_1.$$

Hence

$$\|f\|_{\infty} \leq \|\hat{f}\|_1.$$

For any $k \in \mathbb{Z}$,

$$|\hat{f}(k)| = \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt = \|f\|_1.$$

Hence

$$\|\hat{f}\|_{\infty} \leq \|f\|_1.$$

The Hausdorff-Young inequality [32, p. 57, Corollary 2.4] states that for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p$ then

$$\|\hat{f}\|_q \leq \|f\|_p.$$

The dual Hausdorff-Young inequality [32, p. 58, Corollary 2.5] states that for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^q$ then

$$\|f\|_q \leq \|\hat{f}\|_q.$$

A survey on the Hausdorff-Young inequality is given in [6])

For $M+1 \leq k \leq M+N$, let $a_k \in \mathbb{C}$ and let $S(t) = \sum_{k=M+1}^{M+N} a_k e^{ikt}$. Let $t_1, \dots, t_R \in \mathbb{R}$, and let δ be such that if $r \neq s$ then

$$\|t_r - t_s\| \geq \delta,$$

where $\|t\| = \min_k |t - k|$ is the distance from t to a nearest integer. *The large sieve* [19] is an inequality of the form

$$\sum_{r=1}^R |S(2\pi t_r)|^2 \leq \Delta(N, \delta) \sum_{k=M+1}^{M+N} |a_k|^2.$$

A result of Selberg [19, p. 559, Theorem 3] shows that the large sieve is valid for $\Delta = N - 1 + \delta^{-1}$.

Kristiansen [15]

Boas [2]

For $F : \mathbb{Z}/n \rightarrow \mathbb{C}$, its Fourier transform $\hat{F} : \mathbb{Z}/n \rightarrow \mathbb{C}$ (called the *discrete Fourier transform*) is defined by

$$\hat{F}(k) = \frac{1}{n} \sum_{j=0}^{n-1} F(j) e^{-2\pi i j k / n}, \quad 0 \leq k \leq n-1,$$

and one can prove [31, p. 223, Theorem 1.2] that

$$F(j) = \sum_{k=0}^{n-1} \hat{F}(k) e^{2\pi i k j / n}, \quad 0 \leq j \leq n-1.$$

One can also prove Parseval's identity for the Fourier transform on \mathbb{Z}/n [31, p. 223, Theorem 1.2]. It states

$$\sum_{k=0}^{n-1} |\hat{F}(k)|^2 = \frac{1}{n} \sum_{j=0}^{n-1} |F(j)|^2.$$

Let $P(t) = \sum_{k=0}^{n-1} a_k e^{ikt}$. Define $F : \mathbb{Z}/n \rightarrow \mathbb{C}$ by

$$F(j) = \sum_{k=0}^{n-1} a_k e^{2\pi i k j / n}, \quad 0 \leq j \leq n-1.$$

(That is, $\hat{F}(k) = a_k$.) We then have

$$\sum_{k=0}^{n-1} |a_k|^2 = \frac{1}{n} \sum_{j=0}^{n-1} |F(j)|^2 = \frac{1}{n} \sum_{j=0}^{n-1} \left| P\left(\frac{2\pi j}{n}\right) \right|^2.$$

Thus

$$\|P\|_2 = \left(\frac{1}{n} \sum_{j=0}^{n-1} \left| P\left(\frac{2\pi j}{n}\right) \right|^2 \right)^{1/2}.$$

The Marcinkiewicz-Zygmund inequalities [35, vol. II, p. 28, chap. X, Theorem 7.5] state that there is a constant A such that for $1 \leq p \leq \infty$, if f is a trigonometric polynomial of degree n then

$$\left(\frac{1}{2n+1} \sum_{k=0}^{2n} \left| f\left(\frac{2\pi k}{2n+1}\right) \right|^p \right)^{1/p} \leq A(2\pi)^{1/p} \|f\|_p,$$

and for each $1 < p < \infty$ there exists some A_p such that if f is a trigonometric polynomial of degree n then

$$\|f\|_p \leq A_p \left(\frac{1}{2n+1} \sum_{k=0}^{2n} \left| f\left(\frac{2\pi k}{2n+1}\right) \right|^p \right)^{1/p}.$$

Máté and Nevai [17, p. 148, Theorem 6] prove that for $p > 0$, if S_n is a trigonometric polynomial of degree n then

$$\|S_n\|_\infty \leq \left(\frac{(1+np)e}{2} \right)^{1/p} \|S_n\|_p.$$

Máté and Nevai [17] prove a version of Bernstein's inequality for $0 < p < 1$, and their result can be sharpened to the following [34]: For $0 < p < 1$, if T_n is a trigonometric polynomial of order n then

$$\|T'_n\|_p \leq n\|T_n\|_p.$$

Let $\text{supp } \hat{f} = \{k \in \mathbb{Z} : \hat{f}(k) \neq 0\}$. A subset Λ of \mathbb{Z} is called a *Sidon set* [28, p. 121, §5.7.2] if there exists a constant B such that for every trigonometric polynomial f with $\text{supp } \hat{f} \subseteq \Lambda$ we have

$$\|\hat{f}\|_1 \leq B\|f\|_\infty.$$

Let $B(\Lambda)$ be the least such B . A sequence of positive integers λ_k is said to be *lacunary* if there is a constant ρ such that $\lambda_{k+1} > \rho\lambda_k$ for all k . If λ_k is a lacunary sequence, then $\{\lambda_k\}$ is a Sidon set [21, p. 154, Corollary 6.17]. If $\Lambda \subset \mathbb{Z}$ is a Sidon set, then [28, p. 128, Theorem 5.7.7] (cf. [21, p. 157, Corollary 6.19]) for any $2 < p < \infty$, for every trigonometric polynomial f with $\text{supp } \hat{f} \subseteq \Lambda$ we have

$$\|f\|_p \leq B(\Lambda)\sqrt{p}\|f\|_2,$$

and

$$\|f\|_2 \leq 2B(\Lambda)\|f\|_1.$$

Let $0 < p < \infty$. A subset E of \mathbb{Z} is called a $\Lambda(p)$ -set if for every $0 < r < p$ there is some $A(E, p)$ such that for all trigonometric polynomials f with $\text{supp } \hat{f} \subset E$ we have

$$\|f\|_p \leq A(E, p)\|f\|_2.$$

$\Lambda(p)$ sets were introduced by Rudin, and he discusses them in his autobiography [29, Chapter 28]. A modern survey of $\Lambda(p)$ -sets is given by Bourgain [5].

Bochkarev [3] proves various lower bounds on the L^1 norms of certain trigonometric polynomials. Let $c_k \in \mathbb{C}$, $k \geq 1$. If there are constants A and B such that

$$A \frac{(\log k)^s}{\sqrt{k}} \leq |c_k| \leq B \frac{(\log k)^s}{\sqrt{k}}, \quad k \geq 1,$$

then [3, p. 58, Theorem 19]

$$\left\| \sum_{k=1}^n c_k e^{ik^2 t} \right\|_1 \gg \begin{cases} (\log n)^{s-\frac{1}{2}}, & s > \frac{1}{2}, \\ \log \log n, & s = \frac{1}{2}. \end{cases}$$

If $P(t) = \sum_{k=0}^n a_k e^{ikt}$ with $a_k \in \{-1, 1\}$, then by the Cauchy-Schwarz inequality and Parseval's identity we have

$$\|P\|_1 = \frac{1}{2\pi} \int_0^{2\pi} 1 \cdot |P(t)| dt \leq \|1\|_2 \cdot \|P\|_2 = 1 \cdot \|\hat{P}\|_2 = \sqrt{n+1}.$$

Newman [24] shows that in fact we can do better than what we get using the Cauchy-Schwarz inequality and Parseval's identity:

$$\|P\|_1 < \sqrt{n + 0.97}.$$

A *Fekete polynomial* is a polynomial of the form $\sum_{k=1}^{l-1} \left(\frac{k}{l}\right) z^k$, l prime, where $\left(\frac{k}{l}\right)$ is the Legendre symbol. Let $P_l(t) = \sum_{k=1}^{l-1} \left(\frac{k}{l}\right) e^{ikt}$. Erdélyi [8] proves upper and lower bounds on $\left(\frac{1}{|I|} \int_I |P_l(t)|^q dt\right)^{1/q}$, $q > 0$, where I is an arc in $[0, 2\pi]$.

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