Gibbs measures and the Ising model

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Let Λ be a finite subset of $\mathbb{Z}^2$ and let $\Lambda' = \mathbb{Z}^2 \setminus \Lambda$. Let $\sigma' \in \{-1,+1\}^{\Lambda'}$, a fixed configuration of spins outside Λ. Let $\Omega = \{-1,+1\}^{\Lambda}$; Ω is the space of all configurations of spins on Λ. We define a Hamiltonian $H_\Lambda(\cdot|\sigma') : \Omega \to \mathbb{R}$ (depending on the fixed external configuration σ') by

$$H_\Lambda(\sigma|\sigma') = -\sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} \sigma(x)\sigma(y) - \sum_{\substack{x\in\Lambda, y\in\Lambda' \\ |x-y|=1}} \sigma(x)\sigma'(y).$$

$H_\Lambda(\cdot|\sigma')$ gives the energy of a configuration $\sigma \in \Omega$, conditioned on the external configuration σ'.

For a parameter $\beta > 0$ (called the *inverse temperature*), we define the *partition function* by

$$Z(\beta,\Lambda,\sigma') = \sum_{\sigma\in\Omega} \exp(-\beta H_\Lambda(\sigma|\sigma')).$$

Then we define the *Gibbs distribution* for the configuration space Ω, depending on the external configuration σ', by

$$P_{\beta,\Lambda}(\sigma|\sigma') = \frac{1}{Z(\beta,\Lambda,\sigma')} \exp(-\beta H(\sigma|\sigma')).$$

The purpose of the partition function is to normalize the above expression to be a probability measure on the configuration space Ω.

For example, let Λ be a square of side length 3 centred at the origin, and take σ' to be an external configuration of all negative spins. Define $\sigma \in \Omega$ by

$$\begin{array}{lll} \sigma(-1,1) = +1 & \sigma(0,1) = +1 & \sigma(1,1) = -1 \\ \sigma(-1,0) = -1 & \sigma(0,0) = +1 & \sigma(1,0) = -1 \\ \sigma(-1,-1) = -1 & \sigma(0,-1) = -1 & \sigma(1,-1) = +1. \end{array}$$

We show this configuration in Figure 1. We calculate that the energy of this configuration is $H_\Lambda(\sigma|\sigma') = 0$. We can calculate the energy of this configuration in a different way, using line segments separating lattice points with different spins, as follows. For an $n \times n$ square, there are $2n(n+1)$ nearest neighbor interactions. Put a line segment between every two lattice points with different spins; let $B(\sigma|\sigma')$ be the set of these line segments. We show this in Figure 2.

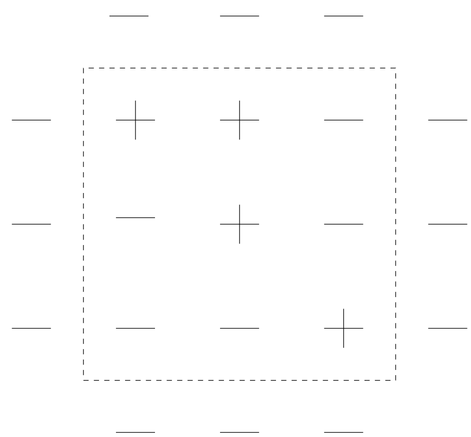

Figure 1: An example of a configuration (and negative external spins)

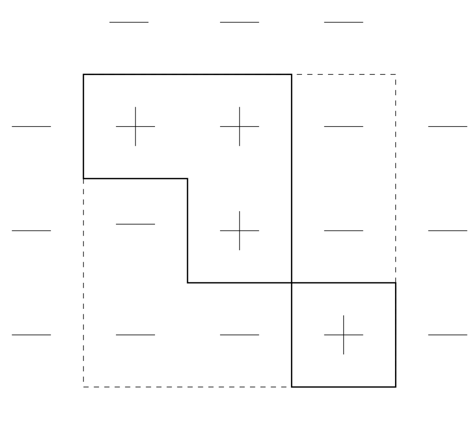

Figure 2: Calculating energy using contours

Generally, if Λ is an $n \times n$ square then we have

$$H_\Lambda(\sigma|\sigma') = -2n(n+1) + 2|B(\sigma|\sigma')|.$$

Indeed, in our above example, $n = 3$ and $|B(\sigma|\sigma')| = 12$, so the above expression is $-24 + 2 \cdot 12 = 0$, and we have already calculated that $H_\Lambda(\sigma|\sigma') = 0$. What matters is that if we know the external configuration, then to describe the configuration inside a region Λ it suffices to know the edges that separate opposite spins. And since the energy of any configuration has the term $-2n(n+1)$ and this appears in the numerator and denominator of the expression for the Gibbs distribution, we can omit it to calculate the Gibbs distribution. By a *contour* we mean a closed path of edges that does not intersect itself. We can express the Gibbs distribution in terms of contours as

$$P_{\beta,\Lambda}(\sigma|\sigma') = \frac{\prod_{\gamma \in \Gamma(\sigma,\sigma')} \exp(-2|\gamma|)}{\sum_\Gamma \prod_{\gamma \in \Gamma} \exp(-2\beta|\gamma|)};$$

$\Gamma(\sigma, \sigma')$ is the set of contours corresponding to the configuration σ with the external configuration σ', and the summation is over all sets Γ of nonintersecting contours.

We are not in fact interested in the Gibbs distribution on the configurations on a finite subset Λ of $\mathbb{Z}^2$, but instead limits of Gibbs distributions with $\Lambda_n \to \mathbb{Z}^2$. A Gibbs distribution $P_{\beta,\Lambda}(\cdot|\sigma')$ on Ω is in fact a probability measure on $\{+1,-1\}^{\mathbb{Z}^2}$: for $\sigma \in \{+1,-1\}^{\mathbb{Z}^2}$, a configuration on the plane, we define

$$\widetilde{P}_{\beta,\Lambda}(\sigma|\sigma') = \begin{cases} 0 & \sigma|\Lambda' \neq \sigma' \\ P_{\beta,\Lambda}((\sigma|\Lambda)|\sigma') & \sigma|\Lambda' = \sigma'. \end{cases}$$

Fix some β. Let Λ_n be a sequence of $n \times n$ squares centred at the origin, let $\sigma'_{n,+}$ be a sequence of external configurations where all lattice points outside Λ_n have positive spins, and let $\sigma'_{n,-}$ be a sequence of external configurations where all lattice points outside Λ_n have negative spins. Let $P_{n,+}$ be the sequence of Gibbs distributions corresponding to the positive external spins, and let $P_{n,-}$ be the sequence of Gibbs distributions corresponding to the negative external spins. These extend to probability measures $\widetilde{P}_{n,+}$ and $\widetilde{P}_{n,-}$ on $\{+1,-1\}^{\mathbb{Z}^2}$. Since $\{+1,-1\}$ is a compact metrizable space, the product $\{+1,-1\}^{\mathbb{Z}^2}$ is a compact metrizable space and thus the space of probability measures on it is compact. Hence the sequence $\widetilde{P}_{n,+}$ has at least one limit point, say P_+, and the sequence $\widetilde{P}_{n,-}$ has at least one limit point, say P_-. We shall show that $P_+ \neq P_-$, namely that there is not a unique limit Gibbs measure on the set of all configurations on $\mathbb{Z}^2$.

Let $V_+ = \{\sigma \in \{+1,-1\}^{\mathbb{Z}^2} : \sigma(0) = +1\}$ and $V_- = \{\sigma \in \{+1,-1\}^{\mathbb{Z}^2} : \sigma(0) = -1\}$. Suppose that for all n we had $\widetilde{P}_{n,+}(V_-) < \frac{1}{3}$. Taking limits we have that $P_+(V_-) \leq \frac{1}{3}$ and so $P_+(V_+) \geq \frac{2}{3}$ (since the events V_+ and V_- are disjoint and their union is the set of all configurations on $\mathbb{Z}^2$). But $\widetilde{P}_{n,+}(V_-) = \widetilde{P}_{n,-}(V_+)$, so taking limits we also get $P_-(V_+) \leq \frac{1}{3}$. Therefore the measures P_+ and P_-

give different measures to the set V_+, so they are distinct. Thus to show that the measures P_+ and P_- are distinct it suffices to show that for all n we have $\widetilde{P}_{n,+}(V_-) < \frac{1}{3}$.

We have

$$\begin{aligned}
\widetilde{P}_{n,+}(V_-) &\leq \operatorname{Prob}\left(\text{there exists a contour } \gamma \subset B(\sigma|\sigma'), 0 \in \operatorname{Int}(\gamma)\right) \\
&\leq \sum_{\substack{\gamma \\ 0 \in \operatorname{Int}(\gamma)}} \operatorname{Prob}(\gamma \subset B(\sigma|\sigma')) \\
&\leq \sum_{\substack{\gamma \\ 0 \in \operatorname{Int}(\gamma)}} \exp(-2\beta|\gamma|).
\end{aligned}$$

The above sum is over all contours such that the origin lies in their interior. We can write the set of all contours around the origin as a union of the set of all contours of length k around the origin, $k \geq 4$. There are at most $\left(\frac{k}{4}\right)^2 4^k$ contours of length k around the origin. Therefore

$$\widetilde{P}_{n,+}(V_-) \leq \sum_{k=4}^{\infty} \frac{k^2}{16} \cdot 4^k \exp(-2\beta k).$$

As $\beta \to \infty$, this is $O(\exp(-8\beta))$. In particular there is some β_0 such that if $\beta \geq \beta_0$ then for all n we have $\widetilde{P}_{n,+}(V_-) < \frac{1}{3}$. This shows that the limit Gibbs measures gives different measures to the set V_+, hence they are distinct.

Further reading

Minlos [4], Sinai [6], Cipra [1], Simon [5], Le Ny [3], Kadanoff [2].

References

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