Hamiltonian flows, cotangent lifts, and momentum maps

Jordan Bell

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1 Symplectic manifolds

Let (M, ω) and (N, η) be symplectic manifolds. A *symplectomorphism* $F : M \to N$ is a diffeomorphism such that $\omega = F^*\eta$. Recall that for $x \in M$ and $v_1, v_2 \in T_xM$,

$$(F^*\eta)_x(v_1, v_2) = \eta_{F(x)}((T_xF)v_1, (T_xF)v_2);$$

$T_xF : T_xM \to T_{F(x)}N$. (A tangent vector at $x \in M$ is *pushed forward* to a tangent vector at $F(x) \in N$, while a differential 2-form on N is *pulled back* to a differential 2-form on M.) In these notes the only symplectomorphisms in which we are interested are those from a symplectic manifold to itself.[1]

2 Symplectic gradient

If (M, ω) is a symplectic manifold and $H \in C^\infty(M)$, using the nondegeneracy of the symplectic form ω one can prove that there is a unique vector field $X_H \in \Gamma^\infty(M)$ such that, for all $x \in M, v \in T_xM$,

$$\omega_x(X_H(x), v) = (dH)_x(v).$$

This can also be written as

$$i_{X_H}\omega = dH,$$

where

$$(i_X\omega)(Y) = (X \lrcorner \omega)(Y) = \omega(X, Y).$$

We call X_H the *symplectic gradient* of H. If $X \in \Gamma^\infty(M)$ and $X = X_H$ for some $H \in C^\infty(M)$, we say that X is a *Hamiltonian vector field*.[2]

[1] I am interested in flows on a phase space and this phase space is a symplectic manifold. For some motivation for why we want phase space to be a symplectic manifold, read:
http://research.microsoft.com/en-us/um/people/cohn/thoughts/symplectic.html

[2] On a Riemannian manifold, a vector field that is the gradient of a smooth function is called a *gradient vector field* or a *conservative vector field*.

Let's check that

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

We have, because $dq_i \frac{\partial}{\partial q_j} = \delta_{ij}$, $dp_i \frac{\partial}{\partial p_j} = \delta_{ij}$, $dq_i \frac{\partial}{\partial p_j} = 0$ and $dp_i \frac{\partial}{\partial q_j} = 0$, and because $dq_j \wedge dp_j = -dp_j \wedge dq_j$,

$$\begin{aligned} i_{X_H}\omega &= \sum_{i=1}^n dq_i \wedge dp_i \sum_{j=1}^n \left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} \right) \\ &= \sum_{i=1}^n dq_i \wedge dp_i \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \right) \\ &= \sum_{i=1}^n \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \\ &= dH. \end{aligned}$$

3 Flows

Let M be a smooth manifold. Let D be an open subset of $M \times \mathbb{R}$, and for each $x \in M$ suppose that

$$D^x = \{t \in \mathbb{R} : (x,t) \in D\}$$

is an open interval including 0. A *flow* on M is a smooth map $\phi : D \to M$ such that if $x \in M$ then $\phi_0(x) = x$ and such that if $x \in M$, $s \in D^x, t \in D^{\phi_s(x)}$ and $s + t \in D^x$, then

$$\phi_t(\phi_s(x)) = \phi_{s+t}(x).$$

For $x \in M$, define $\phi^x : D^x \to M$ by $\phi^x(t) = \phi_t(x)$. The *infinitesimal generator* of a flow ϕ is the vector field V on M defined for $x \in M$ by

$$V_x = \frac{d}{dt}\Big|_{t=0} \phi^x(t).$$

It is a fact that every vector field on M is the infinitesimal generator of a flow on M, and furthermore that there is a unique flow whose domain is maximal that has that vector field as its infinitesimal generator, and we thus speak of *the* flow of a vector field.

We say that a vector field is *complete* if it is the infinitesimal generator of a flow whose domain is $\mathbb{R} \times M$, in other words if it is the infinitesimal generator of a *global flow*. It is a fact that if V is a vector field on a compact smooth manifold then V is complete.

4 Hamiltonian flows

Let (M, ω) be a symplectic manifold. We say that a vector field X on M is *symplectic* if

$$\mathcal{L}_X \omega = 0,$$

where $\mathcal{L}_X\omega$ is the Lie derivative of ω along the flow of X. A *Hamiltonian flow* is the flow of a Hamiltonian vector field.[3] If X is a complete symplectic vector field and $\phi : M \times \mathbb{R}$ is the flow of X, then for all $t \in \mathbb{R}$, the map $\phi_t : M \to M$ is a symplectomorphism.

Let $H \in C^\infty(M)$, and let ϕ be the flow of the vector field X_H. If (x, s) is in the domain of the flow ϕ, we have

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=s} H(\phi^x(t)) &= (d_{\phi^x(s)}H)((\phi^x)'(s)) \\
&= (d_{\phi^x(s)}H)(X_H(\phi^x(s))) \\
&= \omega_{\phi^x(s)}(X_H(\phi^x(s)), X_H(\phi^x(s))) \\
&= 0.
\end{aligned}$$

Thus a Hamiltonian vector field is symplectic: H does not change along the flow of X_H. We can also write this as

$$\begin{aligned}
\frac{d}{dt}(H \circ \phi_t) &= \frac{d}{dt}(\phi_t^* H) \\
&= \phi_t^*(\mathcal{L}_{X_H} H) \\
&= \phi_t^*((i_{X_H}\omega)(X_H)) \\
&= \phi_t^*(\omega(X_H, X_H)) \\
&= \phi_t^*(0) \\
&= 0.
\end{aligned}$$

It is a fact that if $H^1_{\mathrm{dR}}(M) = \{0\}$ (i.e. if α is a 1-form on M and $d\alpha = 0$ then there is some $f \in C^\infty(M)$ such that $\alpha = df$) then every symplectic vector field on M is Hamiltonian. In particular, if M is simply connected then $H^1_{\mathrm{dR}}(M) = \{0\}$, and hence if M is simply connected then every symplectic vector field on M is Hamiltonian.

5 Poisson bracket

For $f, g \in C^\infty(M)$, we define $\{f, g\} \in C^\infty(M)$ for $x \in M$ by

$$\{f, g\}(x) = \omega_x(X_f(x), X_g(x)).$$

This is called the *Poisson bracket* of f and g. We write

$$\{f, g\} = \omega(X_f, X_g).$$

We have

$$\{f, g\} = X_f g = (df)X_g.$$

We say that f and g *Poisson commute* if $\{f, g\} = 0$. The Poisson bracket of f and g tells us how f changes along the Hamiltonian flow of g. If f and g Poisson commute then f does not change along the flow of X_g.

[3] cf. gradient flow.

We have

$$\begin{aligned}
\{f,g\} &= \omega(X_f, X_g) \\
&= \sum_{i=1}^n (dq_i \wedge dp_i) \sum_{j=1}^n \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right) \sum_{k=1}^n \left(\frac{\partial g}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial g}{\partial q_k} \frac{\partial}{\partial p_k} \right) \\
&= \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} dp_i + \frac{\partial f}{\partial q_i} dq_i \right) \sum_{k=1}^n \left(\frac{\partial g}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial g}{\partial q_k} \frac{\partial}{\partial p_k} \right) \\
&= \sum_{i=1}^n - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}.
\end{aligned}$$

If $x \in M$ and $v \in T_x M$, then vf is the directional derivative in the direction v. If $v = \sum_{i=1}^n a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i}$ and $f \in C^\infty(M)$ then

$$vf = \sum_{i=1}^n a_i \frac{\partial f}{\partial q_i} + b_i \frac{\partial f}{\partial p_i}.$$

If X is a vector field on M then $Xf \in C^\infty(M)$, defined for $x \in M$ by

$$(Xf)(x) = X_x f.$$

If τ is a covariant tensor field and X is a vector field, the Lie derivative of τ along the flow of X is defined as follows: if ϕ is the flow of X, then

$$(\mathcal{L}_X \tau)(x) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* \tau)(x),$$

and so if τ is a function $f \in C^\infty(M)$, then

$$(\mathcal{L}_X f)(x) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* f)(x) = \frac{d}{dt}\Big|_{t=0} f(\phi_t(x)) = X_x f = (Xf)(x).$$

Thus if X is a vector field and $f \in C^\infty(M)$, then $\mathcal{L}_X f = Xf$.

For $f, g \in C^\infty(M)$,

$$\begin{aligned}
X_{\{f,g\}} \lrcorner \omega &= d\{f,g\} \\
&= d(X_g f) \\
&= d(\mathcal{L}_{X_g} f) \\
&= \mathcal{L}_{X_g}(df) \\
&= \mathcal{L}_{X_g}(X_f \lrcorner \omega) \\
&= (\mathcal{L}_{X_g} X_f) \lrcorner \omega + X_f \lrcorner \mathcal{L}_{X_g} \omega \\
&= [X_g, X_f] \lrcorner \omega + X_f \lrcorner 0 \\
&= [X_g, X_f] \lrcorner \omega \\
&= -[X_f, X_g] \lrcorner \omega.
\end{aligned}$$

Since the symplectic form ω is nondegenerate, if $X \lrcorner \omega = Y \lrcorner \omega$ then $X = Y$, so

$$X_{\{f,g\}} = -[X_f, X_g].$$

It follows that $C^\infty(M)$ is a Lie algebra using the Poisson bracket as the Lie bracket.

The set $\Gamma^\infty(M)$ of vector fields on M are a Lie algebra using the vector field commutator $[\cdot, \cdot]$. The symplectic vector fields are a Lie subalgebra: it is clear that they are a linear subspace of the Lie algebra of vector fields, and one shows that the commutator of two symplectic vector fields is itself a symplectic vector field. One can further show that the set of Hamiltonian vector fields is a Lie subalgebra of the Lie algebra of symplectic vector fields. It is a fact that the vector space quotient of the vector space of symplectic vector fields modulo the vector space of Hamiltonian vector fields is isomorphic to the vector space $H^1_{\mathrm{dR}}(M)$; this is why if $H^1_{\mathrm{dR}}(M) = \{0\}$ (in particular if M is simply connected) then any symplectic vector field on M is Hamiltonian.

6 Tautological 1-form

Let Q be a smooth manifold and let $\pi : T^*Q \to Q$, $\pi(q, p) = q$. For $x = (q, p) \in T^*Q$, we have

$$d_x\pi : T_xT^*Q \to T_qQ.$$

Let

$$\theta_x = (d_x\pi)^*(p) = p \circ d_x\pi : T_xT^*Q \to \mathbb{R}.$$

Thus $\theta : T^*Q \to T^*T^*Q$. θ is called the *tautological 1-form* on T^*Q.

If $(Q_1, \ldots, Q_n)$ are coordinates on an open subset U of Q, $Q_i : U \to \mathbb{R}$, then for each $q \in U$ we have that $d_qQ_i \in T^*_qU = T^*_qQ$, $1 \leq i \leq n$, are a basis for T^*_qQ and $\frac{\partial}{\partial Q_i}\big|_q$, $1 \leq i \leq n$, are a basis for T_qQ. For each $p \in T^*_qQ$,

$$p = \sum_{i=1}^n p\left(\frac{\partial}{\partial Q_i}\bigg|_q\right) d_qQ_i.$$

On T^*U, define coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ by

$$q_i(q, p) = Q_i(q),$$

and

$$p_i(q, p) = p\left(\frac{\partial}{\partial Q_i}\bigg|_q\right).$$

On T^*U we can write θ using these coordinates: for $x = (q, p) \in T^*Q$,

$$\theta_x = p \circ d_x\pi = \sum_{i=1}^n p_i(x) d_xq_i.$$

Thus, on T^*U,

$$\theta = \sum_{i=1}^{n} p_i dq_i.$$

Let $\omega = -d\theta$. We have, on T^*U,

$$\begin{aligned}
\omega &= -d\sum_{i=1}^{n} p_i dq_i \\
&= -\sum_{i=1}^{n} (dp_i \wedge dq_i + p_i d(dq_i)) \\
&= -\sum_{i=1}^{n} dp_i \wedge dq_i \\
&= \sum_{i=1}^{n} dq_i \wedge dp_i.
\end{aligned}$$

T^*Q is a symplectic manifold with the symplectic form ω.

7 Cotangent lifts

Let Q be a smooth manifold and let $F : Q \to Q$ be a diffeomorphism. Define

$$F^\sharp : T^*Q \to T^*Q$$

for $x = (q, p)$ by

$$F^\sharp(q, p) = (F(q), (d_{F(q)}(F^{-1}))^*(p)).$$

We call $F^\sharp : T^*Q \to T^*Q$ the *cotangent lift* of $F : Q \to Q$. It is a fact that it is a diffeomorphism. It is apparent that the diagram

$$\begin{tikzcd}
T^*Q \arrow[r, "F^\sharp"] \arrow[d, "\pi"] & T^*Q \arrow[d, "\pi"] \\
Q \arrow[r, "F"] & Q
\end{tikzcd}$$

commutes.

The pull-back of θ by $F^\sharp$ satisfies, for $x = (q, p) \in T^*Q$ and $(\zeta, \eta) =$

$F^\sharp(q,p) \in T^*Q$,

$$\begin{aligned}((F^\sharp)^*\theta)_x &= (d_xF^\sharp)^*(\theta_{F^\sharp(x)})\\ &= (d_xF^\sharp)^*((d_{F^\sharp(x)}\pi)^*(\eta))\\ &= (d_x(\pi\circ F^\sharp))^*(\eta)\\ &= (d_x(F\circ\pi))^*(\eta)\\ &= (d_x\pi)^*((d_{\pi(x)}F)^*(\eta))\\ &= (d_x\pi)^*((d_qF)^*(\eta))\\ &= (d_x\pi)^*(p)\\ &= \theta_x.\end{aligned}$$

Thus $(F\sharp)^*\theta = \theta$, i.e. $F^\sharp$ pulls back θ to θ. The "naturality of the exterior derivative"[4] is the statement that if G is a smooth map and η is a differential form then $G^*(d\eta) = d(G^*\eta)$. Hence, with $\omega = d\theta$,

$$(F^\sharp)^*\omega = (F^\sharp)^*(d\theta) = d((F^\sharp)^*\theta) = d(\theta) = \omega,$$

so $F^\sharp$ pulls back the symplectic form ω to itself. Thus $F^\sharp : T^*M \to T^*M$ is a symplectomorphism.

Let $\mathrm{Diff}(Q)$ be the set of diffeomorphisms $Q \to Q$. $\mathrm{Diff}(Q)$ is a group. Let G be a group and let $\tau : G \to \mathrm{Diff}(Q)$ be a homomorphism. Define $\tau^\sharp : G \to \mathrm{Diff}(T^*Q)$ by $(\tau^\sharp)_g = (\tau_g)^\sharp : T^*Q \to T^*Q$. $\tau^\sharp : G \to \mathrm{Diff}(T^*Q)$ is a homomorphism, and for each $g \in G$, $(\tau^\sharp)_g : T^*Q \to T^*Q$ is a symplectomorphism. In words, if a group acts by diffeomorphisms on a smooth manifold, then the cotangent lift of the action is an action by symplectomorphisms on the cotangent bundle.

8 Lie groups

Recall that if $F : M \to N$ then $TF : TM \to TN$ satisfies, for $X \in \Gamma^\infty(M)$ and $f \in C^\infty(N)$,[5]

$$((TF)X)(f) = X(f\circ F),$$

i.e. for $x \in M$ and $v \in T_xM$,

$$((T_xF)v)(f) = v(f\circ F),$$

the directional derivative of $f\circ F \in C^\infty(M)$ in the direction of the tangent vector v.

Let G be a Lie group and for $g \in G$ define $L_g : G \to G$ by $L_gh = gh$. If X is a vector field on G, we say that X is *left-invariant* if

$$(T_hL_g)(X_h) = X_{gh}$$

[4] For each k, Ω^k is a contravariant functor, and if $f : M \to N$, then the functor Ω^k sends f to $f^* : \Omega^k(N) \to \Omega^k(M)$. d is a natural transformation from the contravariant functor Ω^k to the contravariant functor Ω^{k+1}.

[5] In words: TF pushes forward a vector field on M to a vector field on N.

for all $g, h \in G$. That is, X is left-invariant if

$$(TL_g)(X) = X$$

for all $g \in G$.

If X and Y are left-invariant vector fields on G then so is $[X, Y]$. This is because, for $F : G \to G$,

$$(TF)[X, Y] = [(TF)X, (TF)Y].$$

Thus the set of left-invariant vector fields on G is a Lie subalgebra of the Lie algebra of vector fields on G.

Define $\epsilon : \mathrm{Lie}(G) \to T_e G$ by $\epsilon(X) = X_e$, where $e \in G$ is the identity element. It can be shown that this is a linear isomorphism. Hence, if $v \in T_e G$ then there is a unique left-invariant vector field X on G such that, for all $g \in G$,

$$V_g = (T_e L_g)(v).$$

It is a fact that every left-invariant vector field on a Lie group G is complete, i.e. that its flow has domain $G \times \mathbb{R}$. For $X \in \mathrm{Lie}(G)$, we call the unique integral curve of X that passes through e the *one-parameter subgroup generated by* X. Thus, for any $v \in T_e G$ there is a unique one-parameter subgroup $\gamma : \mathbb{R} \to G$ such that

$$\gamma(0) = e, \qquad \gamma'(0) = v.$$

We define $\exp : \mathrm{Lie}(G) \to G$ by $\exp(X) = \gamma(1)$, where γ is the one-parameter subgroup generated by X. This is called the *exponential map*. Thus $t \mapsto \exp(tX)$ is the one-parameter subgroup generated by X.

Fact: If $(TF)X = Y$ and X has flow ϕ and Y has flow η, then

$$\eta_t \circ F = F \circ \phi_t$$

for all t in the domain of ϕ. Hence

$$L_g \circ \phi_t = \phi_t \circ L_g.$$

Hence the flow ϕ of a left-invariant vector field X satisfies

$$\begin{aligned} g \exp(tX) &= L_g \exp(tX) \\ &= L_g(\phi_t e) \\ &= \phi_t(L_g e) \\ &= \phi_t(g). \end{aligned}$$

9 Coadjoint action

First we'll define the *adjoint action* of G on $\mathfrak{g} = T_{\mathrm{id}_G} G$. For $g \in G$, define $\Psi_g : G \to G$ by $\Psi_g(h) = ghg^{-1}$; Ψ_g is an automorphism of Lie groups. Define

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

by

$$\mathrm{Ad}_g = T_{\mathrm{id}_G}\Psi_g;$$

since Ψ_g is an automorphism of Lie groups, it follows that Ad_g is an automorphism of Lie algebras. We can also write Ad_g as

$$\mathrm{Ad}_g(\xi) = \frac{d}{dt}\Big|_{t=0}(g\exp(t\xi)g^{-1}).$$

The *adjoint action* of G on $\mathfrak{g}$ is

$$g \cdot \xi = \mathrm{Ad}_g(\xi).$$

For each $g \in G$, one proves that there is a unique map $\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ such that for all $l \in \mathfrak{g}^*, \xi \in \mathfrak{g}$,

$$(\mathrm{Ad}_g^* l)(\xi) = l(\mathrm{Ad}_g(\xi)).$$

The *coadjoint action* of G on $\mathfrak{g}^*$ is

$$g \cdot l = \mathrm{Ad}_{g^{-1}}^*(l).$$

10 Momentum map

Let (M, ω) be a symplectic manifold, let G be a Lie group, and let $\sigma : G \to \mathrm{Diff}(M)$ be a homomorphism such that for each g in G, σ_g is a symplectomorphism.

Let $\mathfrak{g} = T_{\mathrm{id}_G}G$, and define $\rho : \mathfrak{g} \to \Gamma^\infty(M)$ by

$$\rho(\xi)(x) = \frac{d}{dt}\Big|_{t=0}\sigma_{\exp(t\xi)}(x) \in T_xM, \qquad \xi \in \mathfrak{g}, x \in M;$$

$t \mapsto \sigma_{\exp(t\xi)}(x)$ is $\mathbb{R} \to M$ and at $t = 0$ the curve passes through x, so indeed $\rho(\xi)(x) \in T_x(M)$. ρ is called the *infinitesimal action* of $\mathfrak{g}$ on M. Each element of G acts on M as a symplectomorphism, each element of $\mathfrak{g}$ acts on M as a vector field.

A *momentum map* for the action of G on (M, ω) is a map $\mu : M \to \mathfrak{g}^*$ such that, for $x \in M$, $v \in T_xM$ and $\xi \in \mathfrak{g}$,

$$((T_x\mu)v)\xi = \omega_x(\rho(\xi)(x), v), \tag{1}$$

where

$$T_x\mu : T_xM \to T_{\mu(x)}\mathfrak{g}^* = \mathfrak{g}^*,$$

and such that if $g \in G$ and $x \in M$ then

$$\mu(\sigma_g(x)) = g \cdot \mu(x), \tag{2}$$

where $g \cdot \mu(x)$ is the *coadjoint action* of G on $\mathfrak{g}^*$, defined in section §9; we say that μ is *equivariant* with respect to the coadjoint action of G on $\mathfrak{g}^*$.

11 Angular momentum

Let $G = \mathrm{SO}(3) = \{A \in \mathbb{R}^{3\times 3} : A^T A = I, \det(A) = 1\}$. The Lie algebra of SO(3) is

$$\mathfrak{g} = \mathfrak{so}(3) = \{a \in \mathbb{R}^{3\times 3} : a + a^T = 0\}.$$

Let $Q = \mathbb{R}^3$, and define $\tau : G \to \mathrm{Diff}(Q)$ by $\tau_g(q) = gq$.

Let θ be the tautological 1-form on T^*Q and let $\omega = -d\theta$. (T^*Q, ω) is a symplectic manifold and $\tau^\sharp : G \to \mathrm{Diff}(T^*Q)$ is a homomorphism such that for each $g \in G$, $(\tau^\sharp)_g$ is a symplectomorphism. For $g \in G$, $(q,p) \in T^*Q$,

$$\begin{aligned}
(\tau^\sharp)_g(q,p) &= (\tau_g)^\sharp(q,p) \\
&= (\tau_g q, (d_{\tau_g q}(\tau_g^{-1}))^* p) \\
&= (\tau_g q, (d_{\tau_g q}(\tau_{g^{-1}}))^* p) \\
&= (\tau_g q, p \circ (d_{\tau_g q}\tau_{g^{-1}})) \\
&= (\tau_g q, p \circ \tau_{g^{-1}}) \\
&= (gq, pg^{-1}) \\
&= (gq, pg^T).
\end{aligned}$$

Hence for $\xi \in \mathfrak{g}$ and $(q,p) \in T^*Q$,

$$\begin{aligned}
\rho(\xi)(q,p) &= \left.\frac{d}{dt}\right|_{t=0} (\tau^\sharp)_{\exp(t\xi)}(q,p) \\
&= \left.\frac{d}{dt}\right|_{t=0} (\exp(t\xi)q, p\exp(t\xi^T)) \\
&= (\xi q, p\xi^T) \\
&= (\xi q, -p\xi).
\end{aligned}$$

Define $V : \mathfrak{g} \to \mathbb{R}^3$ by

$$V\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.$$

One checks that $\xi q = V(\xi) \times q$ and $p\xi = p^T \times V(\xi)$.

For $(q,p) \in T^*Q$, $(v,w) \in T_{(p,q)}T^*Q$, and $\xi \in \mathfrak{g}$, we have

$$\begin{aligned}
\omega_{(q,p)}(\rho(\xi)(q,p), (v,w)) &= \omega_{(q,p)}((\xi q, -p\xi), (v,w)) \\
&= \sum_{j=1}^3 dq_j \wedge dp_j((\xi q, -p\xi), (v,w)) \\
&= \sum_{j=1}^3 ((\xi q)_j dp_j + (p\xi)_j dq_j)(v,w) \\
&= \sum_{j=1}^3 w_j(\xi q)_j + v_j(p\xi)_j \\
&= w \cdot (V(\xi) \times q) + v \cdot (p^T \times V(\xi)).
\end{aligned}$$

Define $\mu : T^*Q \to \mathfrak{g}^*$ by $\mu(q,p)(\xi) = (q \times p^T) \cdot V(\xi)$. I claim that μ satisfies (1) and (2). We have just calculated the right-hand side of (1), so it remains to calculate the left-hand side. I find the left-hand side unwieldly to calculate in a clean and precise way, so I will merely claim that it is equal to the right-hand side. I have convinced myself that it is true by symbol pushing.

For $g \in G$ and $\xi \in \mathfrak{g}$, $\mathrm{Ad}_g \xi = g\xi g^{-1}$, and hence, for $(q,p) \in T^*Q$,

$$
\begin{aligned}
(g \cdot \mu(q,p))\xi &= \left(\mathrm{Ad}^*_{g^{-1}} \mu(q,p)\right) \xi \\
&= \mu(q,p)(\mathrm{Ad}_{g^{-1}} \xi) \\
&= \mu(q,p)(g^{-1}\xi g) \\
&= (q \times p^T) \cdot V(g^{-1}\xi g).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mu((\tau^\sharp)_g(q,p))\xi &= \mu(gq, pg^T)\xi \\
&= ((gq) \times (pg^T)^T) \cdot V(\xi) \\
&= ((gq) \times (gp^T)) \cdot V(\xi) \\
&= (g(q \times p^T)) \cdot V(\xi) \\
&= (q \times p^T) \cdot (g^T V(\xi)) \\
&= (q \times p^T) \cdot (g^{-1} V(\xi)).
\end{aligned}
$$