# Notes on the history of Liouville's theorem

Jordan Bell

April 10, 2015

## 1 Introduction

We denote by $\mathscr{B}(\mathbb{R}^n)$ the set of all linear maps $\mathbb{R}^n \to \mathbb{R}^n$. We take it as known that with the operator norm

$$\|A\| = \sup\{\|Av\| : v \in \mathbb{R}^n, \|v\| \leq 1\}, \qquad A \in \mathscr{B}(\mathbb{R}^n),$$

$\mathscr{B}(\mathbb{R}^n)$ is a Banach space.

## 2 Autonomous differential equations

**Lemma 1.** *If $A \in \mathscr{B}(\mathbb{R}^n)$, then*

$$\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon trA + o(\epsilon)$$

*as $\epsilon \to 0$.*

*Proof.* Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$, repeated according to algebraic multiplicity. For $\epsilon > 0$, the eigenvalues of $I + \epsilon A + o(\epsilon)$ repeated according to algebraic multiplicity are

$$1 + \epsilon\lambda_1 + o(\epsilon), \ldots, 1 + \epsilon\lambda_n + o(\epsilon),$$

as $\epsilon \to 0$. The determinant of a linear map $\mathbb{R}^n \to \mathbb{R}^n$ is the product of its eigenvalues according to algebraic multiplicity, so

$$\det(I + \epsilon A + o(\epsilon)) = \prod_{k=1}^{n} (1 + \epsilon\lambda_k + o(\epsilon)),$$

as $\epsilon \to 0$. But

$$\prod_{k=1}^{n} (1 + \epsilon\lambda_k + o(\epsilon)) = 1 + \epsilon \sum_{k=1}^{n} \lambda_k + o(\epsilon) = 1 + \epsilon \mathrm{tr} A + o(\epsilon)$$

as $\epsilon \to 0$. $\square$

**Theorem 2.** *If $A \in \mathscr{B}(\mathbb{R}^n)$, then*

$$\det e^A = e^{trA}$$

*Proof.* We have

$$e^A = \lim_{m\to\infty} \left( I + \frac{A}{m} \right)^m .$$

As $\det : \mathscr{B}(\mathbb{R}^n) \to \mathbb{R}$ is continuous, we have

$$\det e^A = \lim_{m\to\infty} \det \left( I + \frac{A}{m} \right)^m .$$

Then, using Lemma 1,

$$\begin{aligned} \det e^A &= \lim_{m\to\infty} \left( \det \left( I + \frac{A}{m} \right) \right)^m \\ &= \lim_{m\to\infty} \left( 1 + \frac{1}{m} \mathrm{tr} A + o\left(\frac{1}{m}\right) \right)^m \\ &= e^{\mathrm{tr} A}. \end{aligned}$$

□

If $A \in \mathscr{B}(\mathbb{R}^n)$, then the flow of the vector field $A$ is

$$(t, x) \mapsto e^{tA} x.$$

For each $t$ we have $e^{tA} \in \mathscr{B}(\mathbb{R}^n)$, and by Theorem 2 we have

$$\det(e^{tA}) = e^{\mathrm{tr}(tA)} = e^{t\mathrm{tr}A}.$$

Let $\lambda$ be Lebesgue measure on $\mathbb{R}^n$. If $U$ is an open subset of $\mathbb{R}^n$, then

$$\lambda(e^{tA}U) = \int_{e^{tA}U} dy = \int_U |\det(De^{tA})(x)| dx = \int_U |\det(e^{tA})| dx = e^{t\mathrm{tr}A} \lambda(U).$$

Therefore, $\lambda$ is an invariant measure for the flow if and only if $\mathrm{tr}A = 0$, namely, if and only if $A$ is skew-symmetric.

## 3 Nonautonomous differential equations

Suppose that $I$ is an open interval and $A \in C(I, \mathscr{B}(\mathbb{R}^n))$. The set $X$ of all functions $x : I \to \mathbb{R}^n$ that satisfy the differential equation

$$\dot{x}(t) = A(t)x(t)$$

is a vector space. For each $t \in I$ we define $B_t : X \to \mathbb{R}^n$ by $B_t(x) = x(t)$. It is apparent that for each $t \in I$ the map $B_t$ is linear. For each $x_0 \in \mathbb{R}^n$, by the existence and uniqueness theorem for ordinary differential equations there is a unique $x \in X$ for which $B_0(x) = x_0$, hence for each $t \in I$ we get that $B_t$ is a bijection, and hence a linear isomorphism.

Suppose that $\phi_1, \dots, \phi_n \in X$, and for each $t \in I$ let $\Phi(t) \in \mathscr{B}(\mathbb{R}^n)$ be defined by $\Phi(t)e_i = \phi_i(t)$. Then

$$\dot{\Phi}(t)e_i = \frac{d}{dt}(\Phi(t)e_i) = \dot{\phi}_i(t) = A(t)\phi_i(t), \qquad t \in I,$$

and hence

$$\dot{\Phi}(t) = A(t)\Phi(t), \qquad t \in I. \tag{1}$$

The *Wronskian* $W = W(\phi_1, \dots \phi_n)$ of the ordered set $\phi_1, \dots, \phi_n$ is the function that assigns to each $t \in I$ the oriented volume of the parallelepiped spanned by $\phi_1(t), \dots, \phi_n(t)$. That is,

$$W(t) = \det \Phi(t), \qquad t \in I.$$

**Theorem 3.** *Suppose that $I$ is an open interval and that $A \in C(I, \mathscr{B}(\mathbb{R}^n))$. If $\phi_1, \dots, \phi_n \in X$, then the Wronskian $W = W(\phi_1, \dots, \phi_n)$ satisfies*

$$\dot{W}(t) = (trA(t))W(t), \qquad t \in I.$$

*Proof.* By (1), for each $t \in I$ we have

$$\begin{aligned}
\Phi(t+\Delta) &= \Phi(t) + \dot{\Phi}(t)\Delta + o(\Delta) \\
&= \Phi(t) + A(t)\Phi(t)\Delta + o(\Delta) \\
&= \Phi(t) + A(t)\Phi(t)\Delta + o(\Phi(t)\Delta) \\
&= \Phi(t)(I + A(t)\Delta + o(\Delta)) \\
&= \Phi(t)(I + A(t)\Delta + o(\Delta))
\end{aligned}$$

as $\Delta \to 0$. Using Lemma 1 we get

$$\begin{aligned}
W(t+\Delta) &= \det \Phi(t+\Delta) \\
&= \det \Phi(t) \det(I + A(t)\Delta + o(\Delta)) \\
&= \det \Phi(t)(1 + \mathrm{tr}A(t)\Delta + o(\Delta)) \\
&= \det \Phi(t) + \det \Phi(t)\mathrm{tr}A(t)\Delta + o\left(\det \Phi(t)\Delta\right) \\
&= \det \Phi(t) + \det \Phi(t)\mathrm{tr}A(t)\Delta + o(\Delta)
\end{aligned}$$

as $\Delta \to 0$, i.e.,

$$W(t+\Delta) = W(t) + W(t)\mathrm{tr}A(t)\Delta + o(\Delta),$$

which gives us

$$\dot{W}(t) = (\mathrm{tr}A(t))W(t).$$

□

One checks that for any $t_0 \in I$,

$$t \mapsto W(t_0) \exp\left(\int_{t_0}^t \mathrm{tr} A(\tau) d\tau\right), \qquad t \in I$$

is a solution of the differential equation in Theorem 3. For each $t \in I$ we have that $v \mapsto (\mathrm{tr} A(t))v$ is linear, and in particular is locally Lipschitz, so by the existence and uniqueness theorem it follows that

$$W(t) = W(t_0) \exp\left(\int_{t_0}^t \mathrm{tr} A(\tau) d\tau\right), \qquad t \in I.$$

# 4 Jacobi's formula

Let $\Omega$ be the volume form on $\mathbb{R}^n$, and let $A \in \mathscr{B}(\mathbb{R}^n)$. One checks that

$$\Omega(x_1, \dots, x_n) \det A = \Omega(Ax_1, \dots, Ax_n), \qquad x_1, \dots, x_n \in \mathbb{R}^n. \tag{2}$$

If $\omega$ is an $(n-1)$-form on $\mathbb{R}^n$, then there is a unique $x_\omega \in \mathbb{R}^n$ such that for all $x_1, \dots, x_{n-1} \in \mathbb{R}^n$,

$$\omega(x_1, \dots, x_{n-1}) = \Omega(x_\omega, x_1, \dots, x_{n-1}).$$

Thus, if $x_0 \in \mathbb{R}^n$ and we define an $(n-1)$-form $\omega$ by

$$\omega(x_1, \dots, x_{n-1}) = \Omega(x_0, Ax_1, \dots, Ax_n),$$

then there is some $x_\omega$ with which

$$\Omega(x_0, Ax_1, \dots, Ax_n) = \Omega(x_\omega, x_1, \dots, x_{n-1}).$$

We define $\operatorname{adj} A \in \mathscr{B}(\mathbb{R}^n)$ by $(\operatorname{adj} A)(x_0) = x_\omega$. Thus, for $x_1, \dots, x_n \in \mathbb{R}^n$, we have

$$\Omega(x_1, Ax_2, \dots, Ax_n) = \Omega((\operatorname{adj} A)x_1, x_2, \dots, x_n). \tag{3}$$

Hence

$$\Omega(Ax_1, Ax_2, \dots, Ax_n) = \Omega((\operatorname{adj} A)Ax_1, x_2, \dots, x_n),$$

therefore

$$\Omega((\operatorname{adj} A)Ax_1, x_2, \dots, x_n) = \Omega(x_1, \dots, x_n) \det A,$$

and because this holds for all $x_1, \dots, x_n \in \mathbb{R}^n$, it follows that

$$(\operatorname{adj} A)A = (\det A)I.$$

Furthermore, if $A \in \mathscr{B}(\mathbb{R}^n)$, then one checks that for all $x_1, \dots, x_n \in \mathbb{R}^n$,

$$\Omega(x_1, \dots, x_n) \mathrm{tr} A = \sum_{i=1}^n \Omega(x_1, \dots, Ax_i, \dots, x_n). \tag{4}$$

If $I$ is an open interval and $A \in C^1(I, \mathscr{B}(\mathbb{R}^n))$, we have, using (2) for the first equality, (3) for the third equality, and (4) for the fourth equality,

$$\begin{aligned}
\frac{d}{dt}\Big(\Omega(e_1,\ldots,e_n)\det A(t)\Big) &= \frac{d}{dt}\Omega(A(t)e_1,\ldots,A(t)e_n)\\
&= \sum_{i=1}^n \Omega(A(t)e_1,\ldots,\dot{A}(t)e_i,\ldots,A(t)e_n)\\
&= \sum_{i=1}^n \Omega(e_1,\ldots,(\operatorname{adj}A(t))\dot{A}(t)e_i,\ldots,e_n)\\
&= \Omega(e_1,\ldots,e_n)\operatorname{tr}((\operatorname{adj}A(t))\dot{A}(t)),
\end{aligned}$$

that is,

$$\frac{d}{dt}\det A(t) = \operatorname{tr}((\operatorname{adj}A(t))\dot{A}(t)).$$

Kline p. 798, Jacobi [40], [41, §17], Felix Klein, *19th century*, chapter V

# 5 Reynolds transport theorem

If $V$ is a vector field with flow $\phi$ and $U$ is a bounded open subset of $\mathbb{R}^n$ with piecewise smooth boundary and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth, then with $U_t = \phi_t(U)$,

$$\int_{U_t} f(y,t)dy = \int_U f(\phi_t(x),t)\det(D\phi_t)(x)dx;$$

this presumes that $\det(D\phi_t)(x) > 0$. Write $\frac{D}{Dt} = \frac{\partial}{\partial t} + V\cdot D$. We then have

$$\begin{aligned}
\frac{d}{dt}\int_{U_t} f(y,t)dy &= \frac{d}{dt}\int_U f(\phi_t(x),t)\det(D\phi_t)(x)dx\\
&= \int_U (Df)(\phi_t(x),t)\dot{\phi}_t(x)\det(D\phi_t)(x)\\
&\quad + \frac{\partial f}{\partial t}(\phi_t(x),t)\det(D\phi_t)(x) + f(\phi_t(x),t)\frac{d}{dt}\det(D\phi_t)(x)dx\\
&= \int_U \frac{Df}{Dt}(\phi_t(x),t)\det(D\phi_t)(x) + f(\phi_t(x),t)\frac{d}{dt}\det(D\phi_t)(x)dx.
\end{aligned}$$

Writing $J_t(x) = \det(D\phi_t)(x)$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{U_t} f(y,t)dy &= \int_U \frac{Df}{Dt}(\phi_t(x),t)J_t(x) + f(\phi_t(x),t)\frac{d}{dt}J_t(x)dx\\
&= \int_U \frac{D(fJ)}{Dt}
\end{aligned}$$

Reynolds [73, pp. 12–13, art. 14]
Amann and Escher [4, p. 425, Theorem 2.11]

# 6 Symplectic geometry

# 7 Geodesic flow

Invariance of a kinematic measure on the unit tangent bundle.

# 8 References

Jacobi [42, p. 93]
Truesdell [87, pp. 101, 105, 351]
Whittaker [90, p. 323, §148]
Hartman [35, p. 91]
Barrow-Green [6, p. 83]
Goroff [70, p. I79]
Gray [32, p. 380]
Ostrogradskii [51, pp. 122–123]
Cajori [13, vol. II, p. 101, §464]
Gibbs [30, Chapter XII]
Sklar [77, p. 130] on phase space
Boltzmann [10, pp. 274–290, 443]
Jeans [43, p. 258, §206]
Kac [45, p. 63]
Lenzen [55, p. 129]

# References


[1] Niels Henrik Abel, *Ueber einige bestimmte Integrale*, J. Reine Angew. Math. **2** (1827), 22–30, Œuvres complétes, tome premier, second ed., pp. 251–262.

[2] Ilka Agricola and Thomas Friedrich, *Global analysis: differential forms in analysis, geometry and physics*, Graduate Studies in Mathematics, vol. 52, American Mathematical Society, 2002, Translated from the German by Andreas Nestke.

[3] Peter M. Ainsworth, *What chains does Liouville's theorem put on Maxwell's demon?*, Philosophy of Science **78** (2011), no. 1, 149–164.

[4] Herbert Amann and Joachim Escher, *Analysis III*, Birkhäuser, 2001.

[5] V. I. Arnold, *Ordinary differential equations*, MIT Press, 1973, Translated from the Russian by Richard A. Silverman.

[6] June Barrow-Green, *Poincaré and the three body problem*, History of Mathematics, vol. 11, American Mathematical Society, Providence, RI, 1997.

[7] Jean-Louis Basdevant, *Variational principles in physics*, Springer, 2007.

[8] David Betounes, *Differential equations: theory and applications*, second ed., Springer, 2010.

[9] L. Boltzmann and J. Nabl, *Kinetische Theorie der Materie*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band V, 1. Teil (Arnold Sommerfeld, ed.), B. G. Teubner, Leipzig, 1903–1921, pp. 493–557.

[10] Ludwig Boltzmann, *Lectures on gas theory*, Dover Publications, 1995, Translated from the German by Stephen G. Brush.

[11] Stephen G. Brush, *The development of the kinetic theory of gases: VIII. Randomness and irreversibility*, Arch. Hist. Exact Sci. **12** (1974), no. 1, 1–88.

[12] Jeremy Butterfield, *On symplectic reduction in classical mechanics*, Philosophy of Physics, Part A (Jeremy Butterfield and John Earman, eds.), Handbook of the Philosophy of Science, North-Holland, Amsterdam, 2007, pp. 1–131.

[13] Florian Cajori, *A history of mathematical notations, two volumes bound as one*, Dover Publications, 1993.

[14] Ernst Cassirer, *Determinism and indeterminism in modern physics: Historical and systematic studies of the problem of causality*, Yale University Press, 1956, Translated from the German by O. T. Benfey.

[15] Arthur Cayley, *Report on the recent progress of theoretical dynamics*, Report of the Twenty-seventh Meeting of the British Association for the Advancement of Science; held at Dublin in August and September 1857, John Murray, London, 1858, Collected Mathematical Papers, volume III, pp. 156–204, pp. 1–42.

[16] Carlo Cercignani, *The Boltzmann equation and its applications*, Applied Mathematical Sciences, vol. 67, Springer, 1987.

[17] Isaac Chavel, *Riemannian geometry: A modern introduction*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, 2006.

[18] Peter Clark, *Determinism, probability and randomness in classical statistical physics*, Imre Lakatos and Theories of Scientific Change (Kostas Gavroglu, Yorgos Goudaroulis, and Pantelis Nicolacopoulos, eds.), Boston Studies in the Philosophy of Science, vol. 111, Kluwer, 1989, pp. 95–110.

[19] Alex D. D. Craik, *"continuity and change": representing mass conservation in fluid mechanics*, Arch. Hist. Exact Sci. **67** (2013), no. 1, 43–80.

[20] Olivier Darrigol and Jürgen Renn, *The emergence of statistical mechanics*, The Oxford Handbook of the History of Physics (Jed Z. Buchwald and Robert Fox, eds.), Oxford University Press, 2013, pp. 765–788.

[21] K. G. Denbigh and J. S. Denbigh, *Entropy in relation to incomplete knowledge*, Cambridge University Press, 1985.

[22] Emmanuele DiBenedetto, *Classical mechanics: Theory and mathematical modeling*, Birkhäuser, 2011.

[23] Heinz-Dieter Ebbinghaus, *Ernst Zermelo: An approach to his life and work*, Springer, 2007.

[24] Paul Ehrenfest and Tatiana Ehrenfest, *Begriffliche Grundlagen der statistischen Auffassung in der Mechanik*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band IV, 4. Teilband, Heft 6 (Felix Klein and Conrad Müller, eds.), B. G. Teubner, Leipzig, 1907–1914, pp. 3–90.

[25] Gérard G. Emch, *Mathematical and conceptual foundations of 20th-century physics*, North-Holland Mathematics Studies, vol. 100, North-Holland, Amsterdam, 1984.

[26] Gérard G. Emch and Chuang Liu, *The logic of thermostatistical physics*, Springer, 2002.

[27] Ian E. Farquhar, *Ergodic theory in statistical mechanics*, Interscience, 1964.

[28] Harley Flanders, *Differentiation under the integral sign*, Amer. Math. Monthly **80** (1973), no. 6, 615–627.

[29] Giovanni Gallavotti, *Statistical mechanics: A short treatise*, Springer, 1999.

[30] J. Willard Gibbs, *Elementary principles of statistical mechanics*, Charles Scribner's Sons, New York, 1902.

[31] Walter T. Grandy Jr., *Entropy and the time evolution of macroscopic systems*, International Series of Monographs on Physics, vol. 141, Oxford University Press, 2012.

[32] Jeremy Gray, *Henri Poincaré: A scientific biography*, Princeton University Press, 2013.

[33] Louis N. Hand and Janet D. Finch, *Analytical mechanics*, Cambridge University Press, 1998.

[34] Stewart Harris, *An introduction to the theory of the Boltzmann equation*, Dover Publications, 2004.

[35] Philip Hartman, *Ordinary differential equations*, second ed., Classics in Applied Mathematics, vol. 38, Society for Industrial and Applied Mathematics, 2002.

[36] Meir Hemmo and Orly R. Shenker, *The road to Maxwell's Demon: conceptual foundations of statistical mechanics*, Cambridge University Press, 2012.

[37] Helmut Hofer and Eduard Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts, Birkhäuser, 1994.

[38] Nikolai V. Ivanov, *A differential forms perspective on the Lax proof of the change of variables formula*, Amer. Math. Monthly **112** (2005), no. 9, 799–806.

[39] C. G. J. Jacobi, *De determinantibus functionalibus*, J. Reine Angew. Math. **22** (1841), 319–359, Gesammelte Werke, Band III, pp. 393–438.

[40] ______, *De formatione et proprietatibus determinatium*, J. Reine Angew. Math. **22** (1841), 285–318, Gesammelte Werke, Band III, pp. 355–392.

[41] ______, *Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi*, J. Reine Angew. Math. **29** (1845), 333–376, Gesammelte Werke, Band IV, pp. 317–509.

[42] ______, *Vorlesungen über Dynamik*, Georg Reimer, Berlin, 1866, Edited by A. Clebsch.

[43] James Jeans, *An introduction to the kinetic theory of gases*, Cambridge University Press, 1940.

[44] Jürgen Jost and Xianqing Li-Jost, *Calculus of variations*, Cambridge Studies in Advanced Mathematics, vol. 64, Cambridge University Press, 1998.

[45] Mark Kac, *Probability and related topics in physical sciences*, Interscience Publishers, London, 1959.

[46] Victor J. Katz, *Change of variables in multiple integrals: Euler to Cartan*, Math. Mag. **55** (1982), no. 1, 3–11.

[47] A. I. Khinchin, *Mathematical foundations of statistical mechanics*, Dover Publications, 1949, Translated from the Russian by George Gamow.

[48] M. J Klein, *Paul ehrenfest: The making of a theoretical physicist*, Elsevier, 1970.

[49] M. J. Klein, *The physics of J. Willard Gibbs in his time*, Proceedings of the Gibbs Symposium, Yale University, May 15–17, 1989 (Providence, RI) (D. G. Caldi and G. D. Mostow, eds.), American Mathematical Society, 1990, pp. 1–21.

[50] Eberhard Knobloch, *From Gauss to Weierstrass: determinant theory and its historical evaluations*, The Intersection of History and Mathematics (Sasaki Chikara, Sugiura Mitsuo, and Joseph W. Dauben, eds.), Science Networks. Historical Studies, vol. 15, Birkhäuser, 1994, pp. 51–66.

[51] A. N. Kolmogorov and A. P. Yushkevich (eds.), *Mathematics of the 19th century, volume 3*, Birkhäuser, 1998, Translated from the Russian by Roger Cooke.

[52] Cornelius Lanczos, *The variational principles of mechanics*, fourth ed., Dover Publications, 1986.

[53] L. D. Landau and E. M. Lifshitz, *Statistical physics, part 1*, third ed., Course of Theoretical Physics, vol. 5, Pergamon Press, 1980, Translated from the Russian by J. B. Sykes and M. J. Kearsley.

[54] Joel L. Lebowitz and Oliver Penrose, *Modern ergodic theory*, Physics Today **26** (1973), no. 2, 23–29.

[55] Victor F. Lenzen, *Statistical mechanics and its applications to physics*, Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability (Jerzy Neyman, ed.), University of California Press, Berkeley and Los Angeles, 1949, pp. 125–142.

[56] Joseph Liouville, *Sur la Théorie de la Variation des constantes arbitraires*, J. Math. Pures Appl. **3** (1838), 342–349.

[57] Anthony Lo Bello, *On the origin and history of ergodic theory*, Boll. Storia Sci. Mat. **3** (1983), no. 1, 37–75.

[58] Jesper Lützen, *Joseph Liouville 1809–1882: master of pure and applied mathematics*, Studies in the History of Mathematics and Physical Sciences, vol. 15, Springer, 1990.

[59] ———, *The interaction of physics, mechanics and mathematics in Joseph Liouville's research*, The Dialectic Relation Between Physics and Mathematics in the XIXth Century (Evelyne Barbin and Raffaele Pisano, eds.), History of Mechanism and Machine Science, vol. 16, Springer, 2013, pp. 79–97.

[60] Michael C. Mackey, *Time's arrow: The origins of thermodynamic behavior*, Dover Publications, 2003.

[61] Joseph L. McCauley, *Classical mechanics: Transformations, flows, integrable and chaotic dynamics*, Cambridge University Press, 1997.

[62] Leif Mejlbro, *Solution of linear ordinary differential equations by means of the method of variation of arbitrary constants*, Internat. J. Math. Ed. Sci. Tech. **28** (1997), no. 3, 321–331.

[63] Thomas Muir, *The theory of determinants in the historical order of development, volume one and volume two*, Dover Publications, 1960.

[64] Luis Navarro, *Gibbs, Einstein and the foundations of statistical mechanics*, Arch. Hist. Exact Sci. **53** (1998), no. 2, 147–180.

[65] Eugen Netto, *Rationale Funktionen mehrerer Veränderlichen*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band I, 1. Teil (Wilhelm Franz Meyer, ed.), B. G. Teubner, Leipzig, 1898–1904, pp. 255–282.

[66] David D. Nolte, *The tangled tale of phase space*, Physics Today **63** (2010), no. 4, 33–38.

[67] M. V. Ostrogradskii, *Polnoe sobranie trudov, volume 3*, Izdatel'stvo Akademii Nauk Ukrainskoi SSR, Kiev, 1959–1961.

[68] Marco Pettini, *Geometry and topology in Hamiltonian dynamics and statistical mechanics*, Springer, 2007.

[69] George D. J. Phillies, *Elementary lectures in statistical mechanics*, Graduate Texts in Contemporary Physics, Springer, 2000.

[70] Henri Poincaré, *New methods of celestial mechanics: 1. Periodic and asymptotic solutions*, History of Modern Physics and Astronomy, vol. 13, American Institute of Physics, 1993, Edited by Daniel L. Goroff.

[71] Georg Prange, *Die allgemeinen Integrationsmethoden der analytischen Mechanik*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band IV, 2. Teilband (Felix Klein and Conrad Müller, eds.), B. G. Teubner, Leipzig, 1904–1935, pp. 509–804.

[72] Hans Reichenbach, *The direction of time*, Dover Publications, 1999.

[73] Osborne Reynolds, *Papers on mechanical and physical subjects, volume III: The sub-mechanics of the universe*, Cambridge University Press, 1903.

[74] Tatiana Roque, *Stability of trajectories from Poincaré to Birkhoff: approaching a qualitative definition*, Arch. Hist. Exact Sci. **65** (2011), no. 3, 295–342.

[75] Florian Scheck, *Mechanics: From Newton's laws to deterministic chaos*, fifth ed., Springer, 2010.

[76] Lawrence Sklar, *Physics and chance: Philosophical issues in the foundations of statistical mechanics*, Cambridge University Press, 1993.

[77] ______, *Philosophy and the foundations of dynamics*, Cambridge University Press, 2013.

[78] Arnold Sommerfeld, *Thermodynamics and statistical mechanics*, Lectures on Theoretical Physics, vol. V, Academic Press, 1956, Translated from the German by J. Kestin.

[79] Jean-Marie Souriau, *Structure of dynamical systems: a symplectic view of physics*, Progress in Mathematics, vol. 149, Birkhäuser, 1997, Translated from the French by C. H. Cushman-de Vries.

[80] ______, *On geometric mechanics*, Discrete Contin. Dyn. Syst. **19** (2007), no. 3, 595–607.

[81] Michael Spivak, *Physics for mathematicians: Mechanics I*, Publish or Perish, 2010.

[82] Gerald Jay Sussman and Jack Wisdom, *Structure and interpretation of classical mechanics*, MIT Press, 2001.

[83] John L. Synge, *Classical dynamics*, Encyclopedia of Physics, volume III/1: Principles of Classical Mechanics and Field Theory (S. Flügge, ed.), Springer, 1960, pp. 1–225.

[84] Gerald Teschl, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics, vol. 140, American Mathematical Society, Providence, RI, 2012.

[85] Richard C. Tolman, *The principles of statistical mechanics*, Dover Publications, 1979.

[86] Clifford Truesdell, *The ergodic problem in classical statistical mechanics*, Rendiconti della Scuola internazionale di fisica 'Enrico Fermi'. XIV corso, a cura di P. Caldirola, direttore del corso. Varenna sul Lago di Como, Villa Monastero, 23–31 maggio 1960. Teorie ergodiche (Piero Caldirola, ed.), Academic Press, 1962, pp. 21–56.

[87] ———, *A first course in rational continuum mechanics, volume 1*, second ed., Academic Press, 1991.

[88] Ernest Vessiot, *Gewöhnliche Differentialgleichungen; elementare Integrationsmethoden*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band II, 1. Teil, 1. Hälfte (H. Burkhardt, W. Wirtinger, and R. Fricke, eds.), B. G. Teubner, Leipzig, 1899–1916, pp. 230–293.

[89] Eduard Von Weber, *Partielle Differentialgleichungen*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band II, 1. Teil, 1. Hälfte (H. Burkhardt, W. Wirtinger, and R. Fricke, eds.), B. G. Teubner, Leipzig, 1899–1916, pp. 294–399.

[90] E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, second ed., Cambridge University Press, 1917.

[91] A. S. Wightman, *On the prescience of J. Willard Gibbs*, Proceedings of the Gibbs Symposium, Yale University, May 15–17, 1989 (Providence, RI) (D. G. Caldi and G. D. Mostow, eds.), American Mathematical Society, 1990, pp. 23–38.

[92] C. H. Wind, *Über den dem Liouville'schen Satze entsprechenden Satz der Gastheorie*, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften, Abteilung 2a **106** (1897), 21–32.

[93] Eduard Zehnder, *Lectures on dynamical systems: Hamiltonian vector fields and symplectic capacities*, EMS Textbooks in Mathematics, European Mathematical Society, 2010.