LIOUVILLE'S THEOREM AND GIBBS MEASURES

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Let M be a symplectic manifold with symplectic form ω. Define $\omega^\sharp : TM \to T^*M$ by

$$\omega^\sharp(X)Y = \omega(X, Y), \qquad Y \in C^\infty(M, TM),$$

in other words,

$$(\omega^\sharp(X))_x v = \omega_x(X_x, v), \qquad x \in M, v \in T_xM.$$

$\omega^\sharp : TM \to T^*M$ is a vector bundle isomorphism.

Let $H \in C^\infty(M, \mathbb{R})$. We define

$$X_H = (\omega^\sharp)^{-1}(dH),$$

i.e.,

$$X_H(x) = (\omega^\sharp)^{-1}(dH(x)), \qquad x \in M.$$

Thus, X_H is the unique element of $C^\infty(M, TM)$ such that

$$\omega(X_H, Y) = dH(Y), \qquad Y \in C^\infty(M, TM).$$

We call $X_H \in C^\infty(M, TM)$ the Hamiltonian vector field of H, or the symplectic gradient $\nabla_\omega H$ of H.

If $\omega = \sum_{i=1}^n dq^i \wedge dp_i$, define $X \in C^\infty(M, TM)$ by

$$X = \sum_{i=1}^n \frac{\partial H}{dp_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{dq^i}\frac{\partial}{\partial p_i}.$$

We have

$$i_X dq^i = \frac{\partial H}{\partial dp_i}$$

and

$$i_X dp_i = -\frac{\partial H}{dq^i},$$

hence, as $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta)$ for $\alpha \in \Omega^k$ [1, p. 115, Theorem 2.4.13],

$$\begin{aligned}
i_X\omega &= \sum_{i=1}^n i_X(dq^i \wedge dp_i) \\
&= \sum_{i=1}^n (i_X dq^i) \wedge dp_i - dq^i \wedge (i_X dp_i) \\
&= \sum_{i=1}^n \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i \\
&= dH.
\end{aligned}$$

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Hence $X = X_H$.

For a vector field X, the Lie derivative $L_X\omega$ of ω is defined by,

$$L_X\omega = (F_t^*)^{-1}\frac{d}{dt}F_t^*\omega,$$

which one checks is independent of t, where $F_t^*\omega$ is the pull-back of ω by F_t.

Let F_t be the flow of X_H, for $t \in I$ where I is some open interval with $0 \in I$. For $t \in I$, we have by [1, p. 115, Theorem 2.3.13],

$$\begin{aligned}
\frac{d}{dt}(F_t^*\omega) &= F_t^*(L_{X_H}\omega) \\
&= F_t^*(i_X d\omega + d(i_{X_H}\omega)) \\
&= F_t^*(i_X 0 + ddH) \\
&= F_t^*(0+0) \\
&= 0.
\end{aligned}$$

Thus, for $t \in I$ we have $F_t^*\omega = F_0^*\omega = \omega$. So for each $t \in I$, the map $F_t : M \to M$ is a symplectomorphism.

Let

$$\mu = \frac{\omega^n}{n!} = \frac{1}{n!}\underbrace{\omega \wedge \cdots \wedge \omega}_{n}.$$

μ is equal to the degree $2n$ term in

$$\exp(\omega).$$

We have, as F_t^* is a homomorphism of differential algebras [1, p. 113, Theorem 2.4.9] and as F_t is a symplectomorphism,

$$\begin{aligned}
F_t^*\mu &= \frac{1}{n!}(F_t^*\omega) \wedge \cdots (F_t^*\omega) \\
&= \frac{1}{n!}\omega \wedge \cdots \wedge \omega \\
&= \mu.
\end{aligned}$$

If $\omega = \sum_{i=1}^n dq^i \wedge dp_i$ then

$$\omega^n = (-1)^{\frac{n(n-1)}{2}} n! dq^1 \wedge \cdots dq^n \wedge dp_1 \wedge \cdots \wedge dp_n;$$

the sign comes up getting all the q^i's together; since we have to reorder both the q^i's and the p_i's the signs we get from doing those cancel.

If $f \in C^\infty(\mathbb{R}, \mathbb{R})$, then, as $H \circ F_t = H$ for all $t \in I$,

$$\begin{aligned}
F_t^*((f \circ H)\mu) &= (f \circ H \circ F_t)F_t^*\mu \\
&= (f \circ H)\mu.
\end{aligned}$$

Let $\mu_\beta = e^{-\beta H}\mu$. We call $\mu_\beta \in \Omega^{2n}(M)$ a Gibbs measure on M.

One can motivate the choice of $e^{-\beta H}$ as a function by which to multiply μ (rather than any other function invariant under the Hamiltonian flow F_t) through equivariant cohomology. See [2, pp. 197–198]. Let z be a formal variable. An equivariant differential form (for the Hamiltonian flow of H) is a finite sum $\alpha = \sum_n \alpha_n z^n$, where α_n is a differential form on M such that $L_{X_H}\alpha_n = 0$. We define the equivariant differential D (for the Hamiltonian flow of H) by

$$D\alpha = d\alpha - z i_{X_H}\alpha = \sum_n d(\alpha_n)z^n - z\sum_n i_X(\alpha_n)z^n.$$

But

$$\begin{aligned} D^2\alpha &= d^2\alpha - zdi_{X_H}\alpha - zi_{X_H}d\alpha + z^2 i_{X_H} i_{X_H}\alpha \\ &= -z\sum_n \left(d(i_{X_H}\alpha_n) + i_{X_H}(d\alpha_n)\right) z^n + z^2 \sum_n i_{X_H} i_{X_H} \alpha_n z^n \\ &= -z\sum_n L_{X_H}\alpha_n z^n + 0 \\ &= 0. \end{aligned}$$

Thus $D^2 = 0$. If $L_{X_H}\alpha = 0$, then $L_{X_H}(D\alpha) = 0$, while the differential of a regular differential form that is invariant under a Hamiltonian flow is not necessarily itself invariant under the Hamiltonian flow.

$D\omega = d\omega - zi_{X_H}\omega = -(dH)z$. As $i_{X_H} f = 0$ for a function f, we have $D(\omega + zH) = 0$; thus while ω is closed under the usual differential d, $\omega + zH$ is closed under the equivariant differential D. The degree $2n$ term of $\exp(\omega + zH)$ is

$$e^{zH}\frac{\omega^n}{n!} = e^{zH}\mu.$$

Taking $z = -\beta$ gives us the Gibbs measure μ_β.

References

1. Ralph Abraham and Jerrold E. Marsden, *Foundations of mechanics*, second ed., AMS Chelsea Publishing, Providence, RI, 2008.
2. Ana Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer, 2001.

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