Hermite functions

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1 Locally convex spaces

If V is a vector space and $\{p_\alpha : \alpha \in A\}$ is a separating family of seminorms on V, then there is a unique topology with which V is a locally convex space and such that the collection of finite intersections of sets of the form

$$\{v \in V : p_\alpha(v) < \epsilon\}, \qquad \alpha \in A, \quad \epsilon > 0$$

is a local base at 0.[1] We call this the **topology induced by the family of seminorms**. If $\{p_n : n \geq 0\}$ is a separating family of seminorms, then

$$d(v, w) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(v-w)}{1 + p_n(v-w)}, \qquad v, w \in V,$$

is a metric on V that induces the same topology as the family of seminorms. If d is a complete metric, then V is called a **Fréchet space**.

2 Schwartz functions

For $\phi \in C^\infty(\mathbb{R}, \mathbb{C})$ and $n \geq 0$, let

$$p_n(\phi) = \sup_{0 \leq k \leq n} \sup_{u \in \mathbb{R}} (1 + u^2)^{n/2} |\phi^{(k)}(u)|.$$

We define $\mathscr{S}$ to be the set of those $\phi \in C^\infty(\mathbb{R}, \mathbb{C})$ such that $p_n(\phi) < \infty$ for all $n \geq 0$. $\mathscr{S}$ is a complex vector space and each p_n is a norm, and because each p_n is a norm, a fortiori $\{p_n : n \geq 0\}$ is a separating family of seminorms. With the topology induced by this family of seminorms, $\mathscr{S}$ is a Fréchet space.[2] As well, $D : \mathscr{S} \to \mathscr{S}$ defined by

$$(D\phi)(x) = \phi'(x), \qquad x \in \mathbb{R}$$

and $M : \mathscr{S} \to \mathscr{S}$ defined by

$$(M\phi)(x) = x\phi(x), \qquad x \in \mathbb{R}$$

are continuous linear maps.

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[1] http://individual.utoronto.ca/jordanbell/notes/holomorphic.pdf, Theorem 1 and Theorem 4.

[2] Walter Rudin, *Functional Analysis*, second ed., p. 184, Theorem 7.4.

3 Hermite functions

Let λ be Lebesgue measure on $\mathbb{R}$ and let

$$(f, g)_{L^2} = \int_{\mathbb{R}} f\overline{g} d\lambda.$$

With this inner product, $L^2(\lambda)$ is a separable Hilbert space. We write

$$|f|_{L^2}^2 = (f, f)_{L^2} = \int_{\mathbb{R}} |f|^2 d\lambda.$$

For $n \geq 0$, define $H_n : \mathbb{R} \to \mathbb{R}$ by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2},$$

which is a polynomial of degree n. H_n are called **Hermite polynomials**. It can be shown that

$$\exp(2zx - z^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^n, \qquad z \in \mathbb{C}. \tag{1}$$

For $m, n \geq 0$,

$$\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} d\lambda(x) = 2^n n! \sqrt{\pi} \delta_{m,n}.$$

For $n \geq 0$, define $h_n : \mathbb{R} \to \mathbb{R}$ by

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} D^n e^{-x^2}.$$

h_n are called **Hermite functions**. Then for $m, n \geq 0$,

$$(h_m, h_n)_{L^2} = \int_{\mathbb{R}} h_m(x) h_n(x) d\lambda(x) = \delta_{m,n}.$$

One proves that $\{h_n : n \geq 0\}$ is an orthonormal basis for $(L^2(\lambda), (\cdot, \cdot)_{L^2})$.[3]

We remind ourselves that for $x \in \mathbb{R}$,[4]

$$e^{-x^2} = 2^{-1}\pi^{-1/2} \int_{\mathbb{R}} e^{-y^2/4} e^{-ixy} dy,$$

and by the dominated convergence theorem this yields

$$D^n e^{-x^2} = 2^{-1}\pi^{-1/2} \int_{\mathbb{R}} (-iy)^n e^{-y^2/4} e^{-ixy} dy,$$

and so

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \cdot 2^{-1}\pi^{-1/2} \int_{\mathbb{R}} (iy)^n e^{-y^2/4} e^{-ixy} dy. \tag{2}$$

[3] `http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf`, Theorem 8.

[4] `http://individual.utoronto.ca/jordanbell/notes/completelymonotone.pdf`, Lemma 5.

4 Mehler's formula

We now prove **Mehler's formula** for the Hermite functions.[5]

Theorem 1 (Mehler's formula)**.** *For $z \in \mathbb{C}$ with $|z| < 1$ and for $x, y \in \mathbb{R}$,*

$$\sum_{n=0}^{\infty} h_n(x)h_n(y)z^n = \pi^{-1/2}(1-z^2)^{-1/2} \exp\left(-\frac{1}{2} \cdot \frac{1+z^2}{1-z^2}(x^2+y^2) + \frac{2z}{1-z^2}xy\right).$$

Proof. Using (2),

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x)h_n(y)z^n \\
=&\sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{2^n n!} e^{(x^2+y^2)/2} z^n \left(\int_{\mathbb{R}} (2\pi i\xi)^n e^{-\pi^2\xi^2} e^{-2\pi i x\xi} d\xi\right) \left(\int_{\mathbb{R}} (2\pi i\zeta)^n e^{-\pi^2\zeta^2} e^{-2\pi i y\zeta} d\zeta\right) \\
&= \sqrt{\pi} e^{(x^2+y^2)/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi^2\xi^2 - \pi^2\zeta^2 - 2\pi i x\xi - 2\pi i\zeta y} \sum_{n=0}^{\infty} \frac{(-2\pi^2\xi\zeta z)^n}{n!} d\xi d\zeta \\
&= \sqrt{\pi} e^{(x^2+y^2)/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi^2\xi^2 - \pi^2\zeta^2 - 2\pi i x\xi - 2\pi i\zeta y} e^{-2\pi^2\xi\zeta z} d\xi d\zeta.
\end{aligned}$$

Now, writing $a = \frac{iy}{\pi} + \xi z$, we calculate

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\pi^2\zeta^2 - 2\pi i\zeta y - 2\pi^2\xi\zeta z} d\zeta &= \int_{\mathbb{R}} e^{-\pi^2(\zeta+a)^2 + \pi^2 a^2} d\zeta \\
&= \frac{1}{\sqrt{\pi}} e^{\pi^2 a^2} \\
&= \frac{1}{\sqrt{\pi}} \exp\left(-y^2 + 2\pi i y\xi z + \pi^2\xi^2 z^2\right).
\end{aligned}$$

[5] Sundaram Thangavelu, *An Introduction to the Uncertainty Principle: Hardy's Theorem on Lie Groups*, p. 8, Proposition 1.2.1.

Then, for $\alpha = (1-z^2)\pi^2$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x)h_n(y)z^n \\
=&e^{(x^2+y^2)/2}\int_{\mathbb{R}} e^{-\pi^2\xi^2-2\pi ix\xi-y^2+2\pi iy\xi z+\pi^2\xi^2z^2}d\xi \\
=&e^{(x^2-y^2)/2}\int_{\mathbb{R}} e^{-\alpha\xi^2-2\pi i(x-yz)\xi}d\xi \\
=&e^{(x^2-y^2)/2}\sqrt{\frac{\pi}{\alpha}}\exp\left(-\frac{\pi^2}{\alpha}(x-yz)^2\right) \\
=&\pi^{-1/2}e^{(x^2-y^2)/2}(1-z^2)^{-1/2}\exp\left(-\frac{(x-yz)^2}{1-z^2}\right) \\
=&\pi^{-1/2}(1-z^2)^{-1/2}\exp\left(-\frac{x^2}{1-z^2}+\frac{2xyz}{1-z^2}-\frac{y^2z^2}{1-z^2}+\frac{x^2}{2}-\frac{y^2}{2}\right) \\
=&\pi^{-1/2}(1-z^2)^{-1/2}\exp\left(-\frac{1}{2}\frac{1+z^2}{1-z^2}(x^2+y^2)+\frac{2z}{1-z^2}xy\right).
\end{aligned}$$

□

5 The Hermite operator

We define $A : \mathscr{S} \to \mathscr{S}$ by

$$(A\phi)(x) = -\phi''(x) + (x^2+1)\phi(x), \qquad x \in \mathbb{R},$$

i.e.,

$$A = -D^2 + M^2 + 1,$$

which is a continuous linear map $\mathscr{S} \to \mathscr{S}$, which we call the **Hermite operator**. $\mathscr{S}$ is a dense linear subspace of the Hilbert space $L^2(\lambda)$, and $A : \mathscr{S} \to \mathscr{S}$ is a linear map, so A is a densely defined operator in $L^2(\lambda)$. For $\phi, \psi \in \mathscr{S}$, integrating by parts,

$$\begin{aligned}
(A\phi,\psi)_{L^2} &= \int_{\mathbb{R}} (-\phi''(x)+(x^2+1)\phi(x))\overline{\psi(x)}d\lambda(x) \\
&= \int_{\mathbb{R}} -\phi''(x)\overline{\psi(x)}d\lambda(x) + \int_{\mathbb{R}} (x^2+1)\phi(x)\overline{\psi(x)}d\lambda(x) \\
&= \int_{\mathbb{R}} -\phi(x)\overline{\psi''(x)}d\lambda(x) + \int_{\mathbb{R}} (x^2+1)\phi(x)\overline{\psi(x)}d\lambda(x) \\
&= (\phi, A\psi)_{L^2},
\end{aligned}$$

showing that $A : \mathscr{S} \to \mathscr{S}$ is symmetric. Furthermore, also integrating by parts,

$$(A\phi,\phi)_{L^2} = \int_{\mathbb{R}} (\phi'(x)\overline{\phi'(x)} + (x^2+1)\phi(x)\overline{\phi(x)})d\lambda(x) \geq 0,$$

so A is a positive operator.

It is straightforward to check that each h_n belongs to $\mathscr{S}$. For $n \geq 0$, we calculate that

$$h_n''(x) + (2n+1-x^2)h_n(x) = 0,$$

and hence

$$(Ah_n)(x) = (2n+1-x^2)h_n(x) + x^2 h_n(x) + h_n(x) = (2n+2)h_n(x),$$

i.e.

$$Ah_n = (2n+2)h_n.$$

Therefore, for each h_n, $A^{-1}h_n = \frac{1}{2n+2}h_n$, and it follows that there is a unique bounded linear operator $T : L^2(\lambda) \to L^2(\lambda)$ such that[6]

$$Th_n = A^{-1}h_n = (2n+2)^{-1}h_n, \qquad n \geq 0. \tag{3}$$

The operator norm of T is

$$\|T\| = \sup_{n\geq 0} \frac{1}{2n+2} = \frac{1}{2}.$$

The Hermite functions are an orthonormal basis for $L^2(\lambda)$, so for $f \in L^2(\lambda)$,

$$f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} h_n.$$

For $f, g \in L^2(\lambda)$,

$$\begin{aligned}
(Tf, g)_{L^2} &= \left(\sum_{n=0}^{\infty} (f, h_n)_{L^2} Th_n, \sum_{n=0}^{\infty} (g, h_n)_{L^2} h_n \right)_{L^2} \\
&= \left(\sum_{n=0}^{\infty} (f, h_n)_{L^2} (2n+2)^{-1} h_n, \sum_{n=0}^{\infty} (g, h_n)_{L^2} h_n \right)_{L^2} \\
&= \sum_{n=0}^{\infty} (2n+2)^{-1} (f, h_n)_{L^2} \overline{(g, h_n)_{L^2}},
\end{aligned}$$

from which it is immediate that T is self-adjoint.

For $p \geq 0$,

$$|T^p h_n|_{L^2}^2 = |(2n+2)^{-p} h_n|_{L^2}^2 = (2n+2)^{-2p} |h_n|_{L^2}^2 = (2n+2)^{-2p}.$$

Therefore for $p \geq 1$,

$$\sum_{n=0}^{\infty} |T^p h_n|_{L^2}^2 = \sum_{n=0}^{\infty} (2n+2)^{-2p} = 2^{-2p} \sum_{m=1}^{\infty} m^{-2p} = 2^{-2p} \zeta(2p).$$

This means that for $p \geq 1$, T^p is a **Hilbert-Schmidt operator** with **Hilbert-Schmidt norm**[7]

$$\|T^p\|_{\mathrm{HS}} = 2^{-p} \sqrt{\zeta(2p)}.$$

[6] `http://individual.utoronto.ca/jordanbell/notes/traceclass.pdf`, Theorem 11.

[7] `http://individual.utoronto.ca/jordanbell/notes/traceclass.pdf`, §7.

6 Creation and annihilation operators

Taking the derivative of (1) with respect to x gives

$$2\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)z^{n+1}=\sum_{n=0}^{\infty}\frac{1}{n!}H_n'(x)z^n,$$

so $H_0'=0$ and for $n\geq 1$, $\frac{1}{n!}H_n'(x)=\frac{1}{(n-1)!}2H_{n-1}(x)$, i.e.

$$H_n'=2nH_{n-1},$$

and so

$$h_n'(x)=(2n)^{1/2}h_{n-1}(x)-xh_n(x),$$

i.e.

$$Dh_n=(2n)^{1/2}h_{n-1}-Mh_n.$$

Furthermore, from its definition we calculate

$$h_n'(x)=xh_n(x)-(2n+2)^{1/2}h_{n+1}(x),$$

i.e.

$$Dh_n=Mh_n-(2n+2)^{1/2}h_{n+1}.$$

We define $B:\mathscr{S}\to\mathscr{S}$, called the **annihilation operator**, by

$$(B\phi)(x)=\phi'(x)+x\phi(x),\qquad x\in\mathbb{R},$$

i.e.

$$B=D+M,$$

which is a continuous linear map $\mathscr{S}\to\mathscr{S}$. For $n\geq 1$, we calculate

$$Bh_n=(2n)^{1/2}h_{n-1},$$

and $h_0(x)=\pi^{-1/4}e^{-x^2/2}$, so $Bh_0=0$.

We define $C:\mathscr{S}\to\mathscr{S}$, called the **creation operator**, by

$$(C\phi)(x)=-\phi'(x)+x\phi(x),\qquad x\in\mathbb{R},$$

i.e.

$$C=-D+M,$$

which is a continuous linear map $\mathscr{S}\to\mathscr{S}$. For $n\geq 0$, we calculate

$$Ch_n=(2n+2)^{1/2}h_{n+1}.$$

Thus,

$$h_n=(2^n n!)^{-1/2}C^n h_0=\pi^{-1/4}(2^n n!)^{-1/2}C^n(e^{-x^2/2}). \tag{4}$$

For $\phi\in\mathscr{S}$,

$$B-C=2D.$$

Furthermore,

$$BC=-D^2+M^2+1=A$$

and

$$CB=-D^2+M^2-1=A-2.$$

7 The Fourier transform

Define $\mathscr{F} : \mathscr{S} \to \mathscr{S}$, for $\phi \in \mathscr{S}$, by

$$(\mathscr{F}\phi)(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-i\xi x} \frac{dx}{(2\pi)^{1/2}}, \qquad \xi \in \mathbb{R}.$$

For $\xi \in \mathbb{R}$, by the dominated convergence theorem we have

$$\lim_{h \to 0} \frac{\hat{\phi}(\xi + h) - \hat{\phi}(\xi)}{h} = \int_{\mathbb{R}} (-ix)\phi(x) e^{-i\xi x} \frac{dx}{(2\pi)^{1/2}},,$$

i.e.

$$\widehat{x\phi(x)}(\xi) = -i^{-1} D\hat{\phi}(\xi) = iD\hat{\phi}(\xi),$$

in other words,

$$\mathscr{F}(M\phi) = iD(\mathscr{F}\phi). \tag{5}$$

Also, by the dominated convergence theorem we obtain

$$\widehat{D\phi}(\xi) = i\xi\hat{\phi}(\xi),$$

in other words,

$$\mathscr{F}(D\phi) = iM(\mathscr{F}\phi). \tag{6}$$

For $\phi \in \mathscr{S}$,

$$\phi(x) = \int_{\mathbb{R}} \hat{\phi}(\xi) e^{ix\xi} \frac{d\xi}{(2\pi)^{1/2}}, \qquad x \in \mathbb{R}. \tag{7}$$

$\phi \mapsto \hat{\phi}$ is an isomorphism of locally convex spaces $\mathscr{S} \to \mathscr{S}$.[8] Using (7) and the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\phi\|_\infty &\le \int_{\mathbb{R}} (1+\xi^2)^{1/2}(1+\xi^2)^{-1/2} |\hat{\phi}(\xi)| \frac{d\xi}{(2\pi)^{1/2}} \\
&\le (2\pi)^{-1/2} \left(\int_{\mathbb{R}} (1+\xi^2)^{-1} d\xi \right)^{1/2} \left(\int_{\mathbb{R}} (1+\xi^2) |\hat{\phi}(\xi)|^2 d\xi \right)^{1/2} \\
&= 2^{-1/2} \left(\int_{\mathbb{R}} (1+\xi^2) |\hat{\phi}(\xi)|^2 d\xi \right)^{1/2},
\end{aligned}$$

and using (6) and the fact that $|\hat{\phi}|_{L^2} = |\phi|_{L^2}$,

$$\begin{aligned}
\|\phi\|_\infty^2 &\le 2^{-1} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 d\xi + 2^{-1} \int_{\mathbb{R}} \xi^2 |\hat{\phi}(\xi)|^2 d\xi \\
&= 2^{-1} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 d\xi + 2^{-1} \int_{\mathbb{R}} |(\mathscr{F}\phi')(\xi)|^2 d\xi \\
&= 2^{-1} |\phi|_{L^2}^2 + 2^{-1} |\phi'|_{L^2}^2,
\end{aligned}$$

[8] Walter Rudin, *Functional Analysis*, second ed., p. 186, Theorem 7.7.

and therefore

$$\|\phi\|_\infty \le 2^{-1/2}(|\phi|_{L^2} + |\phi'|_{L^2}). \tag{8}$$

We remind ourselves that

$$A = -D^2 + M^2 + 1, \qquad B = D + M, \qquad C = -D + M.$$

Using

$$\mathscr{F} D = iM\mathscr{F}, \qquad D\mathscr{F} = \frac{1}{i}\mathscr{F} M,$$

we get

$$\begin{aligned}\mathscr{F} A &= \mathscr{F}(-D^2 + M^2 + 1)\\ &= -(iM\mathscr{F})D + (iD\mathscr{F})M + \mathscr{F}\\ &= -iM(iM\mathscr{F}) + iD(iD\mathscr{F}) + \mathscr{F}\\ &= M^2\mathscr{F} - D^2\mathscr{F} + \mathscr{F}\\ &= A\mathscr{F},\end{aligned}$$

and

$$\mathscr{F} B = \mathscr{F}(D + M) = iM\mathscr{F} + iD\mathscr{F} = iB\mathscr{F}$$

and

$$\mathscr{F} C = \mathscr{F}(-D + M) = -iM\mathscr{F} + iD\mathscr{F} = -iC\mathscr{F}.$$

We now determine the Fourier transform of the Hermite functions.

Theorem 2. *For $n \ge 0$,*

$$\mathscr{F} h_n = (-i)^n h_n.$$

Proof. For $n \ge 0$, by induction, from $\mathscr{F} C = -iC\mathscr{F}$ we get

$$\mathscr{F} C^n = (-iC)^n \mathscr{F}.$$

From (4),

$$h_n = \pi^{-1/4}(2^n n!)^{-1/2} C^n(e^{-x^2/2}).$$

Writing $g(x) = e^{-x^2/2}$, it is a fact that

$$\mathscr{F} g = g,$$

and using this with the above yields

$$\begin{aligned}\mathscr{F} h_n &= \pi^{-1/4}(2^n n!)^{-1/2}\mathscr{F} C^n g\\ &= \pi^{-1/4}(2^n n!)^{-1/2}(-iC)^n \mathscr{F} g\\ &= \pi^{-1/4}(2^n n!)^{-1/2}(-iC)^n g\\ &= \pi^{-1/4}(2^n n!)^{-1/2}(-i)^n \cdot \pi^{1/4}(2^n n!)^{1/2} h_n\\ &= (-i)^n h_n.\end{aligned}$$

□

There is a unique Hilbert space isomorphism $\mathscr{F} : L^2(\lambda) \to L^2(\lambda)$ such that $\mathscr{F} f = \hat{f}$ for all $f \in \mathscr{S}$.[9] For $f \in L^2(\lambda)$,

$$f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} h_n,$$

and then

$$\mathscr{F} f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} \mathscr{F} h_n = \sum_{n=0}^{\infty} (f, h_n)_{L^2} (-i)^n h_n.$$

8 Asymptotics

For $x = 0$, (1) reads

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(0) z^n = \exp(-z^2) = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!},$$

thus

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \qquad H_{2n+1}(0) = 0.$$

Similarly, taking the derivative of (1) with respect to x yields

$$H'_{2n}(0) = 0, \qquad H'_{2n+1}(0) = 2(-1)^n \frac{(2n+1)!}{n!}.$$

For $u(x) = e^{-x^2/2} H_n(x)$,[10]

$$u'(x) = -xu + e^{-x^2/2} H'_n(x), \qquad u''(x) = -u - xu' - xe^{-x^2/2} H'_n(x) + e^{-x^2/2} H''_n(x).$$

Using

$$H'_n(x) = 2xH_n(x) - H_{n+1}(x), \qquad H'_n(x) = 2nH_{n-1}(x)$$

we get

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0,$$

and thence

$$u'' = -u + x^2 u - 2nu.$$

Thus, writing $f(x) = x^2 u(x)$, u satisfies the initial value problem

$$v'' + (2n+1)v = f, \qquad v(0) = H_n(0), \qquad v'(0) = H'_n(0). \tag{9}$$

Now, for $\lambda > 0$, two linearly independent solutions of $v'' + \lambda v = 0$ are $v_1(x) = \cos(\lambda^{1/2} x)$ and $v_2(x) = \sin(\lambda^{1/2} x)$. The Wronskian of (v_1, v_2) is $W = \lambda^{1/2}$, and

[9] Walter Rudin, *Functional Analysis*, second ed., p. 188, Theorem 7.9.
[10] N. N. Lebedev, *Special Functions and Their Applications*, p. 66, §4.14.

using variation of parameters, if v satisfies $v''+\lambda v=g$ then there are c_1, c_2 such that

$$v(x)=c_1v_1+c_2v_2+Av_1+Bv_2,$$

where

$$A(x)=-\int_0^x \frac{1}{W}v_2(t)g(t)dt, \qquad B(x)=\int_0^x \frac{1}{W}v_1(t)g(t)dt.$$

We calculate that the unique solution of the initial value problem $v''+\lambda v=g$, $v(0)=a$, $v'(0)=b$, is

$$\begin{aligned}
v(x)&=av_1(x)+b\lambda^{-1/2}v_2(x)\\
&\quad-\lambda^{-1/2}v_1(x)\int_0^x v_2(t)g(t)dt+\lambda^{-1/2}v_2(x)\int_0^x v_1(t)g(t)dt\\
&=a\cos(\lambda^{1/2}x)+b\lambda^{-1/2}\sin(\lambda^{1/2}x)\\
&\quad+\lambda^{-1/2}\int_0^x(\cos(\lambda^{1/2}t)\sin(\lambda^{1/2}x)-\sin(\lambda^{1/2}t)\cos(\lambda^{1/2}x))g(t)dt\\
&=a\cos(\lambda^{1/2}x)+b\lambda^{-1/2}\sin(\lambda^{1/2}x)+\lambda^{-1/2}\int_0^x\sin(\lambda^{1/2}(x-t))g(t)dt.
\end{aligned}$$

Therefore the unique solution of the initial value problem (9) is

$$\begin{aligned}
v(x)&=H_n(0)\cos((2n+1)^{1/2}x)+H_n'(0)(2n+1)^{-1/2}\sin((2n+1)^{1/2}x)\\
&\quad+(2n+1)^{-1/2}\int_0^x\sin((2n+1)^{1/2}(x-t))\cdot t^2u(t)dt,
\end{aligned}$$

where $u(x)=e^{-x^2/2}H_n(x)$. If $n=2k$ then

$$\begin{aligned}
v(x)&=(-1)^k\frac{(2k)!}{k!}\cos((4k+1)^{1/2}x)\\
&\quad+(4k+1)^{-1/2}\int_0^x\sin((4k+1)^{1/2}(x-t))\cdot t^2u(t)dt\\
&=(-1)^k\frac{(2k)!}{k!}\cos((4k+1)^{1/2}x)+(4k+1)^{-1/2}r_{2k}(x).
\end{aligned}$$

We calculate

$$\begin{aligned}
|r_{2k}(x)|^2&\le\left(\int_0^{|x|}t^4dt\right)\left(\int_0^{|x|}|u(t)|^2dt\right)\\
&\le\frac{|x|^5}{10}\cdot\int_{\mathbb{R}}e^{-t^2}|H_{2k}(t)|^2dt\\
&=\frac{|x|^5}{10}\cdot 2^{2k}(2k)!\sqrt{\pi},
\end{aligned}$$

i.e.

$$|r_{2k}(x)|\le\pi^{1/4}\frac{|x|^{5/2}}{\sqrt{10}}2^k\sqrt{(2k)!}.$$

By Stirling's approximation,

$$\frac{2^k\sqrt{(2k)!}}{\frac{(2k)!}{k!}} = \frac{2^k k!}{\sqrt{(2k)!}} \sim \frac{2^k(2\pi k)^{1/2}k^k e^{-k}}{((4\pi k)^{1/2}(2k)^{2k}e^{-2k})^{1/2}} = \pi^{1/4}k^{1/4}.$$

Thus for $\alpha_{2k} = \frac{(2k)!}{k!}$,

$$\frac{|r_{2k}(x)|}{\alpha_{2k}} = O(|x|^{5/2}\cdot k^{1/4}\cdot k^{-1/2}) = O(|x|^{5/2}k^{-1/4}).$$

Thangavelu states the following inequality and asymptotics without proof, and refers to Szegő and Muckenhoupt.[11]

Lemma 3. *There are $\gamma, C, \epsilon > 0$ such that for $N = 2n+1$,*

$$\begin{aligned} |h_n(x)| &\le C(N^{1/3} + |x^2 - N|)^{-1/4}, \qquad x^2 \le 2N \\ &\le Ce^{-\gamma x^2}, \qquad x^2 > 2N, \end{aligned}$$

and

$$|h_n(x)| \le N^{-1/8}(x - N^{1/2})^{-1/4}e^{-\epsilon N^{1/4}(x-N^{1/2})^{3/2}}$$

for $N^{1/2} + N^{-1/6} \le x \le (2N)^{1/2}$.

Lemma 4. *For $N = 2n+1$, $0 \le x \le N^{\frac{1}{2}} - N^{-\frac{1}{6}}$, and $\theta = \arccos(xN^{-\frac{1}{2}})$,*

$$h_n(x) = \left(\frac{2}{\pi}\right)^{1/2}(N-x^2)^{-1/4}\cos\left(\frac{N(2\theta - \sin\theta) - \pi}{4}\right) + O(N^{1/2}(N-x^2)^{-7/4}).$$

Theorem 5.
1. *$\|h_n\|_p \asymp n^{\frac{1}{2p}-\frac{1}{4}}$ for $1 \le p < 4$.*
2. *$\|h_n\|_p \asymp n^{-\frac{1}{8}}\log n$ for $p = 4$.*
3. *$\|h_n\|_p \asymp n^{-\frac{1}{6p}-\frac{1}{12}}$ for $4 < p \le \infty$.*

Rather than taking the pth power of h_n, one can instead take the pth power of H_n and integrate this with respect to Gaussian measure. Writing $d\gamma(x) = (2\pi)^{-1/2}e^{-x^2/2}dx$ and taking H_n to be the Hermite polynomial that is monic, now write

$$\|H_n\|_p^p = \int_{\mathbb{R}} |H_n|^p d\gamma.$$

Larsson-Cohn[12] proves that for $0 < p < 2$ there is an explicit $c(p)$ such that

$$\|H_n\|_p = \frac{c(p)}{n^{1/4}}\sqrt{n!}(1 + O(n^{-1})),$$

[11] Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, pp. 26–27, Lemma 1.5.1 and Lemma 1.5.2; Gábor Szegő, *Orthogonal Polynomials*; Benjamin Muckenhoupt, *Mean convergence of Hermite and Laguerre series. II*, Trans. Amer. Math. Soc. **147** (1970), 433–470, Lemma 15.

[12] Lars Larsson-Cohn, *L^p-norms of Hermite polynomials and an extremal problem on Wiener chaos*, Ark. Mat. **40** (2002), 134–144.

and for $2 < p < \infty$ there is an explicit $c(p)$ such that

$$\|H_n\|_p = \frac{c(p)}{n^{1/4}} \sqrt{n!}(p-1)^{n/2}(1 + O(n^{-1})).$$

This uses the asymptotic expansion of Plancherel and Rotach.[13]

[13] M. Plancherel and W. Rotach, *Sur les valeurs asymptotiques des polynomes d'Hermite*, *Commentarii mathematici Helvetici* **1** (1929), 227–254.