Gaussian integrals

Jordan Bell

August 29, 2015

1 One dimensional Gaussian integrals

For $p \in \mathbb{C}$, let[1]

$$h(p) = \int_{\mathbb{R}} e^{-x^2/2} e^{-ipx} dx.$$

Then we check that

$$h'(p) = -i \int_{\mathbb{R}} x e^{-x^2/2} e^{-ipx} dx = i \int_{\mathbb{R}} \frac{d}{dx} \left(e^{-x^2/2} \right) e^{-ipx} dx.$$

Integrating by parts yields

$$h'(p) = -p \int_{\mathbb{R}} e^{-x^2/2} e^{-ipx} dx = -ph(p).$$

Since $h'(p) = -ph(p)$,[2]

$$h(p) = h(0) e^{-p^2/2}.$$

Now, using Fubini's theorem and then polar coordinates,

$$\begin{aligned} h(0)^2 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-x^2/2} dx \right) e^{-y^2/2} dy \\ &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy \\ &= \int_0^\infty \left(\int_{S^1} e^{-r^2/2} e^{-r^2/2} d\sigma(\theta) \right) r dr \\ &= 2\pi \int_0^\infty r e^{-r^2/2} dr \\ &= 2\pi, \end{aligned}$$

so

$$h(p) = (2\pi)^{1/2} e^{-p^2/2}.$$

[1]Eberhard Zeidler, *Quantum Field Theory I: Basics in Mathematics and Physics*, p. 493, Problem 7.1.

[2]cf. Einar Hille, *Ordinary Differential Equations in the Complex Domain*.

For $a > 0$ and $p \in \mathbb{C}$, doing the change of variable $y = a^{1/2}x$,

$$\begin{aligned}(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ax^2/2}e^{-ipx}dx &= (2\pi)^{-1/2}a^{-1/2}\int_{\mathbb{R}} e^{-y^2/2}e^{-ipa^{-1/2}y}dy\\ &= (2\pi)^{-1/2}a^{-1/2}h(pa^{-1/2})\\ &= a^{-1/2}e^{-a^{-1}p^2/2}.\end{aligned}$$

For $t > 0$ and $m \in \mathbb{R}$, doing the change of variable $y = x - m$, and using the above with $a = t^{-2}$ and $p = 0$,

$$\begin{aligned}(2\pi t^2)^{-1/2}\int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2t^2}\right)dx &= (2\pi)^{-1/2}t^{-1}\int_{\mathbb{R}} e^{-ay^2/2}dx\\ &= t^{-1}\cdot a^{-1/2}\\ &= 1.\end{aligned}$$

Theorem 1. *For $a > 0$ and $p \in \mathbb{C}$,*

$$(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ax^2/2}e^{-ipx}dx = a^{-1/2}e^{-a^{-1}p^2/2}.$$

For $t > 0$ and $m \in \mathbb{R}$,

$$(2\pi t^2)^{-1/2}\int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2t^2}\right)dx = 1.$$

For $t > 0$ and $x \in \mathbb{R}$, let

$$p_t(x) = (2\pi t^2)^{-1/2}\exp\left(-\frac{x^2}{2t^2}\right).$$

For $\phi \in \mathscr{S}(\mathbb{R})$, doing the change of variable $x = ty$,

$$\int_{\mathbb{R}} \phi(x)p_t(x)dx = (2\pi)^{-1/2}\int_{\mathbb{R}} \phi(ty)e^{-y^2/2}dy = \int_{\mathbb{R}} \phi(tx)p_1(x)dx.$$

Then as $t \downarrow 0$, using the dominated convergence theorem,

$$\int_{\mathbb{R}} \phi(x)p_t(x)dx \to \int_{\mathbb{R}} \phi(0)p_1(x)dx = \phi(0).$$

For $\phi \in L^1(\mathbb{R}^N)$, let

$$\widehat{\phi}(\xi) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{-i\langle \xi, x\rangle}\phi(x)dx, \qquad \xi \in \mathbb{R}^N.$$

By Theorem 1, with $a = t^{-2}$,

$$\begin{aligned}\widehat{p_t}(\xi) &= (2\pi t^2)^{-1/2}\cdot(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ax^2/2}e^{-i\xi x}dx\\ &= (2\pi t^2)^{-1/2}a^{-1/2}e^{-a^{-1}\xi^2/2}\\ &= (2\pi)^{-1/2}e^{-t^2\xi^2/2}.\end{aligned}$$

2 Moments

For $a > 0$, define for $Z \in \mathbb{C}$,

$$Z(J) = a^{1/2}(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{iJx} dx.$$

By Theorem 1,

$$Z(J) = e^{-a^{-1}J^2/2}. \tag{1}$$

By the dominated convergence theorem,

$$Z^{(n)}(J) = a^{1/2}(2\pi)^{-1/2} i^n \int_{\mathbb{R}} x^n e^{-ax^2/2} e^{iJx} dx,$$

and so

$$a^{1/2}(2\pi)^{-1/2} \int_{\mathbb{R}} x^n e^{-ax^2/2} dx = i^{-n} \frac{dZ}{dJ}(0).$$

From (1) we calculate

$$Z'(J) = -a^{-1} J Z(J), \quad Z''(J) = -a^{-1} Z(J) + a^{-2} J^2 Z(J),$$

so $Z''(0) = -a^{-1} Z(0) = -a^{-1}$, and thus for $t > 0$ and $a = t^{-1}$,

$$(2\pi t)^{-1/2} \int_{\mathbb{R}} x^2 e^{-t^{-2}x^2/2} dx = t,$$

i.e.

$$\int_{\mathbb{R}} x^2 p_t(x) dx = t.$$

3 *N*-dimensional Gaussian integrals

Let $S(x) = \frac{\langle x,x \rangle}{2}$ for $x \in \mathbb{R}^N$. For $\chi \in \mathscr{D}(\mathbb{R}^N)$ and $t > 0$, **Laplace's method** tells us that

$$\int_{\mathbb{R}^N} e^{-tS(x)} \chi(x) dx = (2\pi t^{-1})^{N/2} (\det \operatorname{Hess} S(0))^{-1/2} e^{-tS(0)} \chi(0)(1 + O(t^{-1}))$$

as $t \to \infty$. Here, $\operatorname{Hess} S(x) = I$ for all x and $S(0) = 0$, so

$$\int_{\mathbb{R}^N} e^{-t\frac{\langle x,x \rangle}{2}} \chi(x) dx = (2\pi t^{-1})^{N/2} \chi(0)(1 + O(t^{-1}))$$

as $t \to \infty$.

For A an $N \times N$ matrix, we write $A > 0$ if A is symmetric and has positive eigenvalues. It is proved that

$$\begin{aligned}&\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\langle Ax, x\rangle - i\langle \xi, x\rangle\right) dx \\ =&(\det A)^{-1/2}(2\pi)^{N/2} \exp\left(-\frac{1}{2}\langle A^{-1}\xi, \xi\rangle\right)\end{aligned}$$

for all $\xi \in \mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle\right) dx = (\det A)^{-1/2}(2\pi)^{N/2} \exp\left(\frac{1}{2}\langle A^{-1}b, b\rangle\right).$$

for all $b \in \mathbb{R}^N$. Let

$$Z_A = \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Ax,x\rangle} dx = (\det A)^{-1/2}(2\pi)^{N/2}.$$

Let λ_N be Lebesgue measure on $\mathbb{R}^N$ and let μ_A be the following Borel probability measure on $\mathbb{R}^N$:

$$d\mu_A(x) = \frac{1}{Z_A} e^{-\frac{1}{2}\langle Ax,x\rangle} d\lambda_N(x) = (\det A)^{1/2}(2\pi)^{-N/2} e^{-\frac{1}{2}\langle Ax,x\rangle} d\lambda_N(x).$$

For $\xi \in \mathbb{R}^N$,

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{-i\langle \xi,x\rangle} d\mu_A(x) &= (\det A)^{1/2}(2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Ax,x\rangle} e^{-i\langle \xi,x\rangle} d\lambda_N(x) \\
&= (\det A)^{1/2}(2\pi)^{-N/2} \cdot (\det A)^{-1/2}(2\pi)^{N/2} e^{-\frac{1}{2}\langle A^{-1}\xi,\xi\rangle} \\
&= e^{-\frac{1}{2}\langle A^{-1}\xi,\xi\rangle},
\end{aligned}$$

and for $b \in \mathbb{R}^N$,

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{\langle b,x\rangle} d\mu_A(x) &= (\det A)^{1/2}(2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\langle b,x\rangle} e^{-\frac{1}{2}\langle Ax,x\rangle} d\lambda_N(x) \\
&= (\det A)^{1/2}(2\pi)^{-N/2} \cdot (\det A)^{-1/2}(2\pi)^{N/2} e^{\frac{1}{2}\langle A^{-1}b,b\rangle} \\
&= e^{\frac{1}{2}\langle A^{-1}b,b\rangle}.
\end{aligned}$$

Theorem 2. *For $\xi \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} e^{-i\langle \xi,x\rangle} d\mu_A(x) = e^{-\frac{1}{2}\langle A^{-1}\xi,\xi\rangle},$$

and for $b \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} e^{\langle b,x\rangle} d\mu_A(x) = e^{\frac{1}{2}\langle A^{-1}b,b\rangle}.$$

Let[3]

$$L = L^A = \sum_{j,k=1}^{N} A^{-1}_{j,k} \partial_j \partial_k.$$

We work out the semigroup whose infinitesimal generator is $L/2$.

[3] See http://www.math.ucsd.edu/~bdriver/247A-Winter2012/

Theorem 3. *For $f \in C^1(\mathbb{R}^N)$ that is μ_A-integrable and for $t > 0$,*

$$(e^{tL/2}f)(x) = \int_{\mathbb{R}^N} f(x - t^{1/2}y)d\mu_A(y), \qquad x \in \mathbb{R}^N.$$

Proof. For $\xi \in \mathbb{R}^N$ define $f(x) = e^{\langle \xi, x\rangle} = e^{\xi_1 x_1 + \cdots + \xi_N x_N}$. On the one hand,

$$Lf = \sum_{j,k=1}^{n} A^{-1}_{j,k}\xi_j\xi_k f = \left\langle A^{-1}\xi, \xi\right\rangle f.$$

Then

$$\exp(tL/2)f = \exp\left(\frac{1}{2}t\left\langle A^{-1}\xi, \xi\right\rangle\right) f.$$

On the other hand, for $x \in \mathbb{R}^N$, applying Theorem 2,

$$\begin{aligned}
\int_{\mathbb{R}^N} f(x - t^{1/2}y)d\mu_A(y) &= \int_{\mathbb{R}^N} e^{\left\langle \xi, x - t^{1/2}y\right\rangle} d\mu_A(y) \\
&= e^{\langle \lambda, x\rangle} \int_{\mathbb{R}^N} e^{\left\langle -t^{1/2}\xi, y\right\rangle} d\mu_A(y) \\
&= e^{\langle \lambda, x\rangle} e^{\frac{1}{2}\left\langle A^{-1}(-t^{1/2}\xi), (-t^{1/2}\xi)\right\rangle} \\
&= e^{\frac{1}{2}t\left\langle A^{-1}\xi, \xi\right\rangle} f.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} f(x - t^{1/2}y)d\mu_A(y) = e^{tL/2}f.$$

□

4 Concentration of measure

Let γ_N be the Borel probability measure on $\mathbb{R}^N$ defined by

$$d\gamma_N(x) = (2\pi)^{-N/2}e^{-\frac{1}{2}\langle x, x\rangle}d\lambda_N(x).$$

We estimate the mass γ_N assigns to a spherical shell about the sphere of radius $N^{1/2}$.[4]

Theorem 4. *For $\delta \geq 0$,*

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq \left(\frac{N}{N+\delta}\right)^{-N/2} e^{-\delta/2},$$

and for $0 < \delta \leq N$,

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \leq N - \delta\} \leq \left(\frac{N}{N-\delta}\right)^{-N/2} e^{\delta/2}.$$

[4] Alexander Barvinok, *Measure Concentration*, `http://www.math.lsa.umich.edu/~barvinok/total710.pdf`, p. 5, Proposition 2.2.

Proof. For $0 < \lambda < 1$, if $\|x\|^2 \geq N + \delta$ then $\lambda \|x\|^2 / 2 \geq \lambda(N+\delta)/2$ and then $e^{\lambda\|x\|^2/2} \geq e^{\lambda(N+\delta)/2}$. Hence

$$\begin{aligned}
\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N+\delta\} &= e^{-\lambda(N+\delta)/2} \int_{\|x\|^2 \geq N+\delta} e^{\lambda(N+\delta)/2} d\gamma_N(x) \\
&\leq e^{-\lambda(N+\delta)/2} \int_{\|x\|^2 \geq N+\delta} e^{\lambda\|x\|^2/2} d\gamma_N(x) \\
&\leq e^{-\lambda(N+\delta)/2} \int_{\mathbb{R}^N} e^{\lambda\|x\|^2/2} d\gamma_N(x) \\
&= e^{-\lambda(N+\delta)/2} \cdot (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\lambda\|x\|^2/2} e^{-\frac{1}{2}\|x\|^2} d\lambda_N(x) \\
&= e^{-\lambda(N+\delta)/2} \cdot \prod_{k=1}^{N} (2\pi)^{-1/2} \int_{\mathbb{R}} e^{(\lambda-1)u^2/2} du.
\end{aligned}$$

For $a = -\lambda + 1 > 0$, we have by Theorem 1

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-au^2/2} du = a^{-1/2},$$

so

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N+\delta\} \leq e^{-\lambda(N+\delta)/2} a^{-N/2} = e^{-\lambda(N+\delta)/2}(1-\lambda)^{-N/2}.$$

For $\lambda = \frac{\delta}{N+\delta}$ this is

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N+\delta\} \leq e^{-\delta/2} \left(\frac{N}{N+\delta}\right)^{-N/2}.$$

□

Let $\Sigma_N = \{x \in \mathbb{R}^N : \|x\| = N^{1/2}\}$, and let μ_N be the unique $SO(N)$-invariant Borel probability measure on S^{N-1} (any Borel probability measure on a metric space is regular so we need not explicitly demand this to ensure uniqueness). Let $\pi_N : \Sigma_N \to \mathbb{R}$ be the projection

$$\pi_N(x) = \pi_N(x_1, \ldots, x_N) = x_1,$$

and let $\nu_N = (\pi_N)_* \mu_N$, the pushforward measure which is itself a Borel probability measure on $\mathbb{R}$. The following theorem states that the measures ν_N converges strongly to the standard Gaussian measure γ_1.[5]

Theorem 5. *For A a Borel set in $\mathbb{R}$,*

$$\nu_N(A) \to \gamma_1(A)$$

as $N \to \infty$.

[5] Alexander Barvinok, *Measure Concentration*, `http://www.math.lsa.umich.edu/~barvinok/total710.pdf`, p. 54, Theorem 13.2.

5 Zeta functions

Let $A > 0$, with eigenvalues $\lambda_1, \ldots, \lambda_N$, counted according to multiplicity. For $s \in \mathbb{C}$, define[6]

$$\zeta_A(s) = \sum_{k=1}^{N} \lambda_k^{-s} = \sum_{k=1}^{N} e^{-s \log \lambda_k}.$$

The derivative of ζ_A is

$$\zeta_A'(s) = \sum_{k=1}^{N} -\log \lambda_k \cdot \lambda_k^{-s},$$

so

$$\zeta_A'(0) = -\sum_{k=1}^{N} \log \lambda_k,$$

hence

$$e^{-\zeta_A'(0)} = \prod_{k=1}^{N} \lambda_k = \det A.$$

Theorem 6. *For $\xi \in \mathbb{R}^N$,*

$$(2\pi)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\langle Ax, x\rangle - i\langle \xi, x\rangle\right) dx = e^{\zeta_A'(0)/2} \exp\left(-\frac{1}{2}\left\langle A^{-1}\xi, \xi\right\rangle\right).$$

Let $\lambda_k > 0$, $k \geq 1$, and let $Ae_k = \lambda_k e_k$, and if it makes sense let

$$\det A = \prod_{k=1}^{\infty} \lambda_k.$$

For those complex s for which the expression makes sense, let

$$\zeta_A(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} = \sum_{k=1}^{\infty} e^{-s \log \lambda_k}.$$

Then, if the above makes sense in a neighborhood of $s = 0$,

$$\zeta_A'(0) = -\sum_{k=1}^{\infty} \log \lambda_k,$$

so

$$e^{-\zeta_A'(0)} = \det A.$$

[6] Eberhard Zeidler, *Quantum Field Theory I: Basics in Mathematics and Physics*, p. 434, §7.23.3.

We calculate, doing the change of variables $t = \lambda_k u$,

$$\begin{aligned}
\Gamma(s)\zeta_A(s) &= \int_0^\infty t^{s-1} e^{-t} dt \cdot \sum_{k=1}^\infty \lambda_k^{-s} \\
&= \sum_{k=1}^\infty \lambda_k^{-s} \int_0^\infty t^{s-1} e^{-t} dt \\
&= \sum_{k=1}^\infty \lambda_k^{-s} \int_0^\infty (\lambda_k u)^{s-1} e^{-\lambda_k u} \lambda_k du \\
&= \sum_{k=1}^\infty \int_0^\infty u^{s-1} e^{-\lambda_k u} du.
\end{aligned}$$

Thus

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \sum_{k=1}^\infty e^{-\lambda_k u} du.$$

For $\gamma > 0$, the eigenvalues of γA are $\gamma \lambda_k$, and doing the change of variables $v = \gamma u$,

$$\begin{aligned}
\zeta_{\gamma A}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \sum_{k=1}^\infty e^{-\gamma \lambda_k u} du \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \gamma^{-s} v^{s-1} \sum_{k=1}^\infty e^{-\lambda_k v} dv \\
&= \gamma^{-s} \zeta_A(s).
\end{aligned}$$

Taking the derivative,

$$\zeta'_{\gamma A}(s) = -\log \gamma \cdot \gamma^{-s} \cdot \zeta_A(s) + \gamma^{-s} \gamma'_A(s),$$

and then

$$\zeta'_{\gamma A}(0) = -\log \gamma \cdot \zeta_A(0) + \zeta'_A(0).$$

Then

$$\det(\gamma A) = e^{-\zeta'_{\gamma A}(0)} = e^{\log \gamma \cdot \zeta_A(0) - \zeta'_A(0)} = \gamma^{\zeta_A(0)} \det A.$$