

# The Kolmogorov continuity theorem, Hölder continuity, and the Kolmogorov-Chentsov theorem

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## 1 Modifications

Let  $(\Omega, \mathcal{F}, P)$  be a probability space, let  $I$  be a nonempty set, and let  $(E, \mathcal{E})$  be a measurable space. A **stochastic process with index set  $I$  and state space  $E$**  is a family  $(X_t)_{t \in I}$  of random variables  $X_t : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{E})$ . If  $X$  and  $Y$  are stochastic processes, we say that  $X$  is a **modification of  $Y$**  if for each  $t \in I$ ,

$$P\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} = 1.$$

**Lemma 1.** *If  $X$  is a modification of  $Y$ , then  $X$  and  $Y$  have the same finite-dimensional distributions.*

*Proof.* For  $t_1, \dots, t_n \in I$ , let  $A_i \in \mathcal{E}$  for each  $1 \leq i \leq n$ , and let

$$A = \bigcap_{i=1}^n X_{t_i}^{-1}(A_i) \in \mathcal{F}, \quad B = \bigcap_{i=1}^n Y_{t_i}^{-1}(A_i) \in \mathcal{F}.$$

If  $\omega \in A \setminus B$  then there is some  $i$  for which  $\omega \notin Y_{t_i}^{-1}(A_i)$ , and  $\omega \in X_{t_i}^{-1}(A_i)$  so  $X_{t_i}(\omega) \neq Y_{t_i}(\omega)$ . Therefore

$$A \triangle B \subset \bigcup_{i=1}^n \{\omega \in \Omega : X_{t_i}(\omega) \neq Y_{t_i}(\omega)\}.$$

Because  $X$  is a modification of  $Y$ , the right-hand side is a union of finitely many  $P$ -null sets, hence is itself a  $P$ -null set.  $A$  and  $B$  each belong to  $\mathcal{F}$ , so  $P(A \triangle B) = 0$ .<sup>1</sup> Because  $P(A \triangle B) = 0$ ,  $P(A) = P(B)$ , i.e.

$$P(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) = P(Y_{t_1} \in A_1, \dots, Y_{t_n} \in A_n).$$

This implies that

$$P_*(X_{t_1} \otimes \dots \otimes X_{t_n}) = P_*(Y_{t_1} \otimes \dots \otimes Y_{t_n}),$$

namely,  $X$  and  $Y$  have the same finite-dimensional distributions. □

<sup>1</sup>We have not assumed that  $(\Omega, \mathcal{F}, P)$  is a complete measure space, so we must verify that a set is measurable before speaking about its measure.

## 2 Continuous modifications

Let  $E$  be a Polish space with Borel  $\sigma$ -algebra  $\mathcal{E}$ . A stochastic process  $(X_t)_{t \in \mathbb{R}_{\geq 0}}$  is called **continuous** if for each  $\omega \in \Omega$ , the **path**  $t \mapsto X_t(\omega)$  is continuous  $\mathbb{R}_{\geq 0} \rightarrow E$ .

A **dyadic rational** is an element of

$$D = \bigcup_{i=0}^{\infty} 2^{-i} \mathbb{Z}.$$

The **Kolmogorov continuity theorem** gives conditions under which a stochastic process whose state space is a Polish space has a continuous modification.<sup>2</sup> This is like the **Sobolev lemma**,<sup>3</sup> which states that if  $f \in H^s(\mathbb{R}^d)$  and  $s > k + \frac{d}{2}$ , then there is some  $\phi \in C^k(\mathbb{R}^d)$  such that  $f = \phi$  almost everywhere. It does not make sense to say that an element of a Sobolev space is itself  $C^k$ , because elements of Sobolev spaces are equivalence classes of functions, but it does make sense to say that there is a  $C^k$  version of this element.

**Theorem 2** (Kolmogorov continuity theorem). *Suppose that  $(\Omega, \mathcal{F}, P, (X_t)_{t \in \mathbb{R}_{\geq 0}})$  is a stochastic process with state space  $\mathbb{R}^d$ . If there are  $\alpha, \beta, c > 0$  such that*

$$E(|X_t - X_s|^\alpha) \leq c|t - s|^{1+\beta}, \quad s, t \in \mathbb{R}_{\geq 0}, \quad (1)$$

*then the stochastic process has a continuous modification that itself satisfies (1).*

*Proof.* Let  $0 < \gamma < \frac{\beta}{\alpha}$  and let

$$\delta = \beta - \alpha\gamma > 0.$$

For  $m \geq 1$ , let  $S_m$  be the set of all pairs  $(s, t)$  with

$$s, t \in \{j2^{-m} : 0 \leq j \leq 2^m\},$$

and  $|s - t| = 2^{-m}$ . There are  $2 \cdot 2^m$  such pairs, i.e.  $|S_m| = 2 \cdot 2^m$ . Let

$$A_m = \bigcup_{(s,t) \in S_m} \{|X_s - X_t| \geq 2^{-\gamma m}\} \in \mathcal{F}.$$

For  $(s, t) \in S_m$ , using Chebyshev's inequality and (1) we get

$$\begin{aligned} P(|X_t - X_s| \geq 2^{-\gamma m}) &\leq (2^{\gamma m})^\alpha E(|X_t - X_s|^\alpha) \\ &\leq 2^{\alpha\gamma m} \cdot c|t - s|^{1+\beta} \\ &= c2^{\alpha\gamma m} 2^{-m(1+\beta)} \\ &< c2^{-m-\delta m}. \end{aligned}$$

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<sup>2</sup>Heinz Bauer, *Probability Theory*, p. 335, Theorem 39.3. It was only after working through the proof given by Bauer that I realized that the statement is true when the state space is a Polish space rather than merely  $\mathbb{R}^d$ . In the proof I do not use that  $|\cdot|$  is a norm on  $\mathbb{R}^d$ , and only use that  $d(x, y) = |x - y|$  is a metric on  $\mathbb{R}^d$ , so it is straightforward to rewrite the proof.

<sup>3</sup>Walter Rudin, *Functional Analysis*, second ed., p. 202, Theorem 7.25.

Hence

$$P(A_m) \leq \sum_{(s,t) \in S_m} P\{|X_s - X_t| \geq 2^{-\gamma m}\} < \sum_{(s,t) \in S_m} c 2^{-m-\delta m} = 2c \cdot 2^{-\delta m}.$$

Because  $\sum_m P(A_m) < \infty$ , the Borel-Cantelli lemma tells us that

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = P(N_0) = 0,$$

where for each  $\omega \in \Omega \setminus N_0$  there is some  $m_0(\omega)$  such that  $\omega \notin A_m$  when  $m \geq m_0(\omega)$ . That is, for  $\omega \in \Omega \setminus N_0$  there is some  $m_0(\omega)$  such that

$$|X_t(\omega) - X_s(\omega)| < 2^{-\gamma m}, \quad m \geq m_0(\omega), \quad (s, t) \in S_m. \quad (2)$$

Now let  $\omega \in \Omega \setminus N_0$  and let  $s, t \in [0, 1]$  be dyadic rationals satisfying

$$0 < |s - t| \leq 2^{-m_0(\omega)}.$$

Let  $m = m(s, t)$  be the greatest integer such that  $|s - t| \leq 2^{-m}$ :

$$2^{-m-1} < |s - t| \leq 2^{-m}, \quad (3)$$

which implies that  $m \geq m_0(\omega)$ . There are some  $i_0, j_0 \in \{0, 1, 2, 3, \dots, 2^m\}$  such that

$$s_0 = i_0 2^{-m} \leq s < (i_0 + 1) 2^{-m}, \quad t_0 = j_0 2^{-m} \leq t < (j_0 + 1) 2^{-m}.$$

As  $0 \leq s - s_0 < 2^{-m}$  and  $0 \leq t - t_0 < 2^{-m}$ , there are sequences  $\sigma_j, \tau_j \in \{0, 1\}$ ,  $j > m$ , each of which have cofinitely many zero entries, such that

$$s = s_0 + \sum_{j>m} \sigma_j 2^{-j}, \quad t = t_0 + \sum_{j>m} \tau_j 2^{-j}.$$

Because  $0 \leq s - s_0 < 2^{-m}$  and  $0 \leq t - t_0 < 2^{-m}$ ,

$$2^{-m} > |(s - s_0) - (t - t_0)| = |(s - t) - (s_0 - t_0)| \geq |s_0 - t_0| - |s - t|,$$

and with (3),

$$|s_0 - t_0| < 2^{-m} + |s - t| \leq 2^{-m} + 2^{-m} = 2^{-m+1}.$$

Thus  $|i_0 - j_0| < 2$ , so  $|i_0 - j_0| \in \{0, 1\}$  and so either  $s_0 = t_0$  or  $(s_0, t_0) \in S_m$ . In the first case,  $|X_{t_0}(\omega) - X_{s_0}(\omega)| = 0$ . In the second case, since  $m \geq m_0(\omega)$ , by (2) we have

$$|X_{t_0}(\omega) - X_{s_0}(\omega)| < 2^{-\gamma m}. \quad (4)$$

Define by induction

$$s_n = s_{n-1} + \sigma_{m+n} 2^{-(m+n)}, \quad n \geq 1,$$

i.e.

$$s_n = s_0 + \sum_{m < j \leq m+n} \sigma_j 2^{-j}.$$

For each  $n \geq 1$ ,  $s_n - s_{n-1} \in \{0, 2^{-(m+n)}\}$ , so either  $s_n = s_{n-1}$  or  $(s_{n-1}, s_n) \in S_{m+n}$ , and because  $m+n \geq m+1 > m_0(\omega)$ , applying (2) yields

$$|X_{s_n}(\omega) - X_{s_{n-1}}(\omega)| < 2^{-\gamma(m+n)}.$$

Because the sequence  $\sigma_j$  is eventually equal to 0, the sequence  $s_n$  is eventually equal to  $s$ . Thus

$$\sum_{n=1}^{\infty} (X_{s_n}(\omega) - X_{s_{n-1}}(\omega)) = X_s(\omega) - X_{s_0}(\omega),$$

whence

$$|X_s(\omega) - X_{s_0}(\omega)| \leq \sum_{n=1}^{\infty} |X_{s_n}(\omega) - X_{s_{n-1}}(\omega)| < \sum_{n=1}^{\infty} 2^{-\gamma(m+n)} = \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}}.$$

By the same reasoning we get

$$|X_t(\omega) - X_{t_0}(\omega)| < \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}}.$$

Using these and (4) yields

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_{t_0}(\omega)| + |X_{t_0}(\omega) - X_{s_0}(\omega)| + |X_s(\omega) - X_{s_0}(\omega)| \\ &< \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}} + 2^{-\gamma m} + \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}} \\ &= C \cdot 2^{-\gamma(m+1)}, \end{aligned}$$

for  $C = 2^\gamma + \frac{2}{1-2^{-\gamma}}$ . By (3),  $2^{-(m+1)} < |t - s|$ , hence

$$|X_t(\omega) - X_s(\omega)| \leq C|t - s|^\gamma. \quad (5)$$

This is true for all dyadic rationals  $s, t \in [0, 1]$  with  $|s - t| \leq 2^{-m_0(\omega)}$ ; when  $|s - t| = 0$  it is immediate.

For  $k \geq 1$ , let  $X_t^k = X_{k+t}$ , which satisfies (1). By what we have worked out, there is a  $P$ -null set  $N_1' \in \mathcal{F}$  such that for each  $\omega \in \Omega \setminus N_1'$  there is some  $m_1'(\omega)$  such that  $m \geq m_1'(\omega)$  and  $(s, t) \in S_m$  imply that  $|X_t^1(\omega) - X_s^1(\omega)| < 2^{-\gamma m}$ . Let  $N_1 = N_0 \cup N_1'$ , which is  $P$ -null, and for  $\omega \in \Omega \setminus N_1$  let  $m_1(\omega) = \max\{m_0(\omega), m_1'(\omega)\}$ . For  $s, t \in D \cap [0, 1]$  with  $|s - t| \leq 2^{-m_1(\omega)}$ , what we have worked out yields

$$|X_t(\omega) - X_s(\omega)| \leq C|t - s|^\gamma, \quad |X_t^1(\omega) - X_s^1(\omega)| \leq C|t - s|^\gamma.$$

By induction, we get that for each  $k \geq 1$  there are  $P$ -null sets  $N_0 \subset N_1 \subset \dots \subset N_k$  and for each  $\omega \in \Omega \setminus N_k$  there is some  $m_k(\omega)$  such that for  $s, t \in D \cap [0, 1]$  with  $|s - t| \leq 2^{-m_k(\omega)}$ ,

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq C|t - s|^\gamma \\ |X_t^1(\omega) - X_s^1(\omega)| &\leq C|t - s|^\gamma \\ &\dots \\ |X_t^k(\omega) - X_s^k(\omega)| &\leq C|t - s|^\gamma. \end{aligned}$$

Let

$$N_\gamma = \bigcup_{k \geq 1} N_k,$$

which is an increasing sequence of sets whose union is  $P$ -null. For  $\omega \in \Omega \setminus N_\gamma$ , there is a nondecreasing sequence  $m_k(\omega)$  such that when  $0 \leq j \leq k$  and  $s, t \in D \cap [j, j+1]$  with  $|s - t| \leq 2^{-m_k(\omega)}$ , it is the case that  $|X_t(\omega) - X_s(\omega)| \leq C|t - s|^\gamma$ . For  $s, t \in D \cap [0, k+1]$  with  $|s - t| \leq 2^{-m_k(\omega)}$ , because  $|s - t| \leq \frac{1}{2}$ , either there is some  $0 \leq j \leq k$  for which  $s, t \in [j, j+1]$  or there is some  $1 \leq j \leq k$  for which, say,  $s < j < t$ . In the first case,  $|X_t(\omega) - X_s(\omega)| \leq C|t - s|^\gamma$ . In the second case, because  $|j - s| < |t - s| \leq 2^{-m_k(\omega)}$  and  $|t - j| < |t - s| \leq 2^{-m_k(\omega)}$ , we get, because  $s, j \in D \cap [j-1, j]$  and  $j, t \in D \cap [j, j+1]$ ,

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_j(\omega)| + |X_j(\omega) - X_s(\omega)| \\ &\leq C|t - j|^\gamma + C|j - s|^\gamma \\ &< 2C|t - s|^\gamma. \end{aligned}$$

Thus for

$$C_\gamma = 2C = 2^{\gamma+1} + \frac{4}{1 - 2^{-\gamma}},$$

we have established that for  $\omega \in \Omega \setminus N_\gamma$ ,  $k \geq 1$ , and  $s, t \in D \cap [0, k+1]$  satisfying  $|t - s| \leq 2^{-m_k(\omega)}$ , it is the case that

$$|X_t(\omega) - X_s(\omega)| \leq C_\gamma |t - s|^\gamma. \quad (6)$$

This implies that for each  $\omega \in \Omega \setminus N_\gamma$  and for  $k \geq 1$ , the mapping  $t \mapsto X_t(\omega)$  is uniformly continuous on  $D \cap [0, k+1]$ . For  $t \in \mathbb{R}_{\geq 0}$  and  $\omega \in \Omega \setminus N_\gamma$ , define

$$Y_t(\omega) = \lim_{\substack{s \rightarrow t \\ s \in D}} X_s(\omega). \quad (7)$$

For each  $k \geq 0$ , because  $t \mapsto X_t(\omega)$  is uniformly continuous  $D \cap [0, k+1] \rightarrow \mathbb{R}^d$ , where  $D \cap [0, k+1]$  is dense in  $[0, k+1]$  and  $\mathbb{R}^d$  is a complete metric space, the map  $t \mapsto Y_t(\omega)$  is uniformly continuous  $[0, k+1] \rightarrow \mathbb{R}^d$ .<sup>4</sup> Then  $t \mapsto Y_t(\omega)$  is continuous  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ . For  $\omega \in N_\gamma$ , we define

$$Y_t(\omega) = 0, \quad t \in \mathbb{R}_{\geq 0}.$$

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<sup>4</sup>Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 77, Lemma 3.11.

Then for each  $\omega \in \Omega$ ,  $t \mapsto Y_t(\omega)$  is continuous  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ . For  $t \in \mathbb{R}_{\geq 0}$ ,  $\omega \mapsto Y_t(\omega)$  is the pointwise limit of the sequence of mappings  $\omega \mapsto X_s(\omega)$  as  $s \rightarrow t$ ,  $s \in D$ . For each  $s \in D$ ,  $\omega \mapsto X_s(\omega)$  is measurable  $\mathcal{F} \rightarrow \mathcal{B}_{\mathbb{R}^d}$ , which implies that  $\omega \mapsto Y_t(\omega)$  is itself measurable  $\mathcal{F} \rightarrow \mathcal{B}_{\mathbb{R}^d}$ .<sup>5</sup> Namely,  $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$  is a continuous stochastic process.

We must show that  $Y$  is a modification of  $X$ . For  $s \in D$ , for all  $\omega \in \Omega \setminus N_\gamma$  we have  $Y_s(\omega) = X_s(\omega)$ . For  $t \in \mathbb{R}_{\geq 0}$ , there is a sequence  $s_n \in D$  tending to  $t$ , and then for all  $\omega \in \Omega \setminus N_\gamma$  by (7) we have  $X_{s_n}(\omega) \rightarrow Y_t(\omega)$ .  $P(N_\gamma) = 0$ , namely,  $X_{s_n}$  converges to  $Y_t$  almost surely. Because  $X_{s_n}$  converges to  $Y_t$  almost surely and  $P$  is a probability measure,  $X_{s_n}$  converges in measure to  $Y_t$ .<sup>6</sup> On the other hand, for  $\eta > 0$ , by Chebyshev's inequality and (1),

$$P\{|X_{s_n} - X_t| \geq \eta\} \leq \eta^{-\alpha} E(|X_{s_n} - X_t|^\alpha) \leq \eta^{-\alpha} \cdot c|s_n - t|^{1+\beta},$$

and because this is true for each  $\eta > 0$ , this shows that  $X_{s_n}$  converges in measure to  $X_t$ . Hence, the limits  $Y_t$  and  $X_t$  are equal as equivalence classes of measurable functions  $\Omega \rightarrow \mathbb{R}^d$ . That is,  $P\{Y_t = X_t\} = 1$ . This is true for each  $t \in \mathbb{R}_{\geq 0}$ , showing that  $Y$  is a modification of  $X$ , completing the proof.  $\square$

### 3 Hölder continuity

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces, let  $0 < \gamma < 1$ , and let  $\phi : X \rightarrow Y$  be a function. For  $x_0 \in X$ , we say that  $\phi$  is  **$\gamma$ -Hölder continuous at  $x_0$**  if there is some  $0 < \epsilon_{x_0} < 1$  and some  $C_{x_0}$  such that when  $d(x, x_0) < \epsilon_{x_0}$ ,

$$\rho(\phi(x), \phi(x_0)) \leq C_{x_0} d(x, x_0)^\gamma.$$

We say that  $\phi$  is **locally  $\gamma$ -Hölder continuous** if for each  $x_0 \in X$  there is some  $0 < \epsilon_{x_0} < 1$  and some  $C_{x_0}$  such that when  $d(x, x_0) < \epsilon_{x_0}$  and  $d(y, x_0) < \epsilon_{x_0}$ ,

$$\rho(\phi(x), \phi(y)) \leq C_{x_0} d(x, y)^\gamma.$$

We say that  $\phi$  is **uniformly  $\gamma$ -Hölder continuous** if there is some  $C$  such that for all  $x, y \in X$ ,

$$\rho(\phi(x), \phi(y)) \leq C d(x, y)^\gamma.$$

We establish properties of Hölder continuous functions in the following.<sup>7</sup>

**Lemma 3.** *Let  $V$  be a nonempty subset of  $\mathbb{R}_{\geq 0}$ , let  $0 < \gamma < 1$ , and let  $f : V \rightarrow \mathbb{R}^d$  be locally  $\gamma$ -Hölder continuous.*

1. *If  $0 < \gamma' < \gamma$  then  $f$  is locally  $\gamma'$ -Hölder continuous.*
2. *If  $V$  is compact, then  $f$  is uniformly  $\gamma$ -Hölder continuous.*

<sup>5</sup>Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 142, Lemma 4.29.

<sup>6</sup>Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 479, Theorem 13.37.

<sup>7</sup>Achim Klenke, *Probability Theory: A Comprehensive Course*, p. 448, Lemma 21.3.

3. If  $V$  is an interval of length  $T > 0$  and there is some  $\epsilon > 0$  and some  $C$  such that for all  $s, t \in V$  with  $|t - s| \leq \epsilon$  we have

$$|f(t) - f(s)| \leq C|t - s|^\gamma, \quad (8)$$

then

$$|f(t) - f(s)| \leq C \left[ \frac{T}{\epsilon} \right]^{1-\gamma} |t - s|^\gamma, \quad s, t \in V.$$

*Proof.* For  $t_0 \in \mathbb{R}_{\geq 0}$ , there is some  $0 < \epsilon_{t_0} < 1$  and some  $C_{t_0}$  such that when  $|t - t_0| < \epsilon_{t_0}$ ,

$$|f(t) - f(t_0)| \leq C_{t_0}|t - t_0|^\gamma \leq C_{t_0}|t - t_0|^{\gamma'},$$

showing that  $f$  is locally  $\gamma'$ -Hölder continuous.

With the metric inherited from  $\mathbb{R}_{\geq 0}$ ,  $V$  is a compact metric space. For  $t \in V$  and  $\epsilon > 0$ , write

$$B_\epsilon(t) = \{v \in V : |v - t| < \epsilon\},$$

which is an open subset of  $V$ . Because  $f$  is locally  $\gamma$ -Hölder continuous, for each  $t \in V$  there is some  $0 < \epsilon_t < 1$  and some  $C_t$  such that for all  $u, v \in B_{\epsilon_t}(t)$ ,

$$|f(u) - f(v)| \leq C_t|u - v|^\gamma. \quad (9)$$

Write  $U_t = B_{\epsilon_t}(t)$ . Because  $t \in U_t$ ,  $\{U_t : t \in V\}$  is an open cover of  $V$ , and because  $V$  is compact there are  $t_1, \dots, t_n \in V$  such that  $\mathfrak{U} = \{U_{t_1}, \dots, U_{t_n}\}$  is an open cover of  $V$ . Because  $V$  is a compact metric space, there is a **Lebesgue number**  $\delta > 0$  of the open cover  $\mathfrak{U}$ :<sup>8</sup> for each  $t \in V$ , there is some  $1 \leq i \leq n$  such that  $B_\delta(t) \subset U_{t_i}$ . Let

$$C = \max\{C_{t_1}, \dots, C_{t_n}, 2\|f\|_u \delta^{-\gamma}\},$$

For  $s, t \in V$  with  $|t - s| < \delta$ , i.e.  $s \in B_\delta(t)$ , there is some  $1 \leq i \leq n$  with  $s, t \in U_{t_i}$ . By (9),

$$|f(s) - f(t)| \leq C_{t_i}|s - t|^\gamma \leq C|s - t|^\gamma.$$

On the other hand, for  $s, t \in V$  with  $|t - s| \geq \delta$ ,

$$|f(s) - f(t)| \leq 2\|f\|_u \leq 2\|f\|_u \left( \frac{|s - t|}{\delta} \right)^\gamma = 2\|f\|_u \delta^{-\gamma} |s - t|^\gamma \leq C|s - t|^\gamma.$$

Thus, for all  $s, t \in V$ ,

$$|f(s) - f(t)| \leq C|s - t|^\gamma,$$

showing that  $f$  is uniformly  $\gamma$ -Hölder continuous.

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<sup>8</sup>Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 85, Lemma 3.27.

Let  $n = \lceil \frac{T}{\epsilon} \rceil$ . For  $s, t \in V$ , because  $V$  is an interval of length  $T$ ,  $|s - t| \leq T \leq \epsilon n$ , and then applying (8), because  $\frac{|t-s|}{n} \leq \epsilon$ ,

$$\begin{aligned} |f(t) - f(s)| &= \left| \sum_{k=1}^n f\left(s + (t-s)\frac{k}{n}\right) - f\left(s + (t-s)\frac{k-1}{n}\right) \right| \\ &\leq \sum_{k=1}^n \left| f\left(s + (t-s)\frac{k}{n}\right) - f\left(s + (t-s)\frac{k-1}{n}\right) \right| \\ &\leq \sum_{k=1}^n C \left| \frac{t-s}{n} \right|^\gamma \\ &= C n^{1-\gamma} |t-s|^\gamma. \end{aligned}$$

□

The following theorem does not speak about a version of a stochastic process. Rather, it shows what can be said about a stochastic process that satisfies (1) when almost all of its sample paths are continuous.<sup>9</sup>

**Theorem 4.** *If a stochastic process  $(X_t)_{t \in \mathbb{R}_{\geq 0}}$  with state space  $\mathbb{R}^d$  satisfies (1) and for almost every  $\omega \in \Omega$  the map  $t \mapsto X_t(\omega)$  is continuous  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ , then for almost every  $\omega \in \Omega$ , for every  $0 < \gamma < \frac{\beta}{\alpha}$ , the map  $t \mapsto X_t(\omega)$  is locally  $\gamma$ -Hölder continuous.*

*Proof.* There is a  $P$ -null set  $N \in \mathcal{F}$  such that for  $\omega \in \Omega \setminus N$ , the map  $t \mapsto X_t(\omega)$  is continuous  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ . For each  $0 < \gamma < \frac{\beta}{\alpha}$ , we have established in (6) that there is a  $P$ -null set  $N_\gamma \in \mathcal{F}$  such that for  $k \geq 1$  there is some  $m_k(\omega)$  such that when  $s, t \in D \cap [0, k+1]$  and  $|t-s| \leq 2^{-m_k(\omega)}$ ,

$$|X_t(\omega) - X_s(\omega)| \leq C_\gamma |t-s|^\gamma, \quad (10)$$

where  $C_\gamma = 2^{\gamma+1} + \frac{4}{1-2^{-\gamma}}$ . Write  $\delta(k, \omega) = 2^{-m_k(\omega)}$ , and let  $M_\gamma = N_\gamma \cup N$ . For  $\omega \in \Omega \setminus M_\gamma$ , the map  $t \mapsto X_t(\omega)$  is continuous  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ . For  $k \geq 1$  and for  $s, t \in [0, k+1]$  satisfying  $|s-t| \leq \delta(k, \omega)$ , say with  $s \leq t$ , let  $m = \frac{t-s}{2}$  and let  $s \leq s_n \leq t$  be a sequence of dyadic rationals decreasing to  $s$  and let  $s \leq t_n \leq t$  be a sequence of dyadic rationals increasing to  $t$ . Then  $s_n, t_n \in D \cap [0, k+1]$  and  $|s_n - t_n| \leq |s-t| \leq \delta(k, \omega)$ , so by (10),

$$|X_{t_n}(\omega) - X_{s_n}(\omega)| \leq C_\gamma |t_n - s_n|^\gamma.$$

Because  $\omega \in \Omega \setminus N$ ,  $X_{t_n}(\omega) \rightarrow X_t(\omega)$  and  $X_{s_n}(\omega) \rightarrow X_s(\omega)$ , so

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_{t_n}(\omega)| + |X_{t_n}(\omega) - X_{s_n}(\omega)| + |X_{s_n}(\omega) - X_s(\omega)| \\ &\leq |X_t(\omega) - X_{t_n}(\omega)| + C_\gamma |t_n - s_n|^\gamma + |X_{s_n}(\omega) - X_s(\omega)| \\ &\downarrow C_\gamma |t-s|^\gamma, \end{aligned}$$

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<sup>9</sup>Heinz Bauer, *Probability Theory*, p. 338, Theorem 39.4.



thus

$$|X_t(\omega) - X_s(\omega)| \leq C_\gamma |t - s|^\gamma,$$

showing that for  $0 < \gamma < \frac{\beta}{\alpha}$  and  $\omega \in \Omega \setminus M_\gamma$ , the map  $t \mapsto X_t(\omega)$  is locally  $\gamma$ -Hölder continuous.

Let  $0 < \gamma_n < \frac{\beta}{\alpha}$  be a sequence increasing to  $\frac{\beta}{\alpha}$  and let

$$M = \bigcup_{n \geq 1} M_{\gamma_n},$$

which is a  $P$ -null set. Let  $0 < \gamma < \frac{\beta}{\alpha}$  and let  $n$  be such that  $\gamma_n \geq \gamma$ . For  $\omega \in \Omega \setminus M$ , the map  $t \mapsto X_t(\omega)$  is locally  $\gamma_n$ -Hölder continuous, and because  $\gamma \leq \gamma_n$  this implies that the map is locally  $\gamma$ -Hölder continuous, completing the proof.  $\square$

Bauer attributes the following theorem to Kolmogorov and Chentsov.<sup>10</sup> It does not merely state that for any  $0 < \gamma < \frac{\beta}{\alpha}$  there is a modification that is locally  $\gamma$ -Hölder continuous, but that there is a modification that for all  $0 < \gamma < \frac{\beta}{\alpha}$  is locally  $\gamma$ -Hölder continuous.<sup>11</sup>

**Theorem 5** (Kolmogorov-Chentsov theorem). *If a stochastic process  $(X_t)_{t \in \mathbb{R}_{\geq 0}}$  with state space  $\mathbb{R}^d$  satisfies (1), then  $X$  has a modification  $Y$  such that for all  $\omega \in \Omega$  and  $0 < \gamma < \frac{\beta}{\alpha}$ , the path  $t \mapsto Y_t(\omega)$  is locally  $\gamma$ -Hölder continuous.*

*Proof.* Applying the Kolmogorov continuity theorem, there is a continuous modification  $Z$  of  $X$  that also satisfies (1). By Theorem 4, there is a  $P$ -null set  $M$  such that for  $\omega \in \Omega \setminus M$  and  $0 < \gamma < \frac{\beta}{\alpha}$ , the map  $t \mapsto Z_t(\omega)$  is locally  $\gamma$ -Hölder continuous. For  $t \in \mathbb{R}_{\geq 0}$ , define

$$Y_t(\omega) = \begin{cases} Z_t(\omega) & \omega \in \Omega \setminus M \\ 0 & \omega \in M, \end{cases}$$

i.e.  $Y_t = 1_{\Omega \setminus M} Z_t$ , which is measurable  $\mathcal{F} \rightarrow \mathcal{B}_{\mathbb{R}^d}$ , and so  $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$  is a stochastic process. For every  $\omega \in \Omega$  and  $0 < \gamma < \frac{\beta}{\alpha}$ , the map  $t \mapsto Y_t(\omega)$  is locally  $\gamma$ -Hölder continuous. For  $t \in \mathbb{R}_{\geq 0}$ ,

$$\{X_t \neq Y_t\} = \{X_t \neq Y_t, X_t = Z_t\} \cup \{X_t \neq Y_t, X_t \neq Z_t\} \subset \{Y_t \neq Z_t\} \cup \{X_t \neq Z_t\}.$$

Because  $P(Y_t \neq Z_t) = P(M) = 0$  and  $P(X_t \neq Z_t) = 0$ , since  $Z$  is a modification of  $X$ , we get  $P(X_t \neq Y_t) = 0$ , namely,  $Y$  is a modification of  $X$ .  $\square$

<sup>10</sup>Nikolai Nikolaevich Chentsov, 1930–1993, obituary in Russian Math. Surveys **48** (1993), no. 2, 161–166.

<sup>11</sup>Heinz Bauer, *Probability Theory*, p. 339, Corollary 39.5.