

# The Gelfand transform, positive linear functionals, and positive-definite functions

Jordan Bell

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## 1 Introduction

In this note, unless we say otherwise every vector space or algebra we speak about is over  $\mathbb{C}$ .

If  $A$  is a Banach algebra and  $e \in A$  satisfies  $xe = x$  and  $ex = x$  for all  $x \in A$ , and also  $\|e\| = 1$ , we say that  $e$  is *unity* and that  $A$  is *unital*.

If  $A$  is a unital Banach algebra and  $x \in A$ , the *spectrum* of  $x$  is the set  $\sigma(x)$  of those  $\lambda \in \mathbb{C}$  for which  $\lambda e - x$  is not invertible. It is a fact that if  $A$  is a unital Banach algebra and  $x \in A$ , then  $\sigma(x) \neq \emptyset$ .<sup>1</sup>

If  $A$  and  $B$  are Banach algebras and  $T : A \rightarrow B$  is a map, we say that  $T$  is an *isomorphism of Banach algebras* if  $T$  is an algebra isomorphism and an isometry.

**Theorem 1** (Gelfand-Mazur). *If  $A$  is a Banach algebra and every nonzero element of  $A$  is invertible, then there is an isomorphism of Banach algebras  $A \rightarrow \mathbb{C}$ .*

*Proof.* Let  $x \in A$ .  $\sigma(x) \neq \emptyset$ . If  $\lambda_1, \lambda_2 \in \sigma(x)$ , then neither  $\lambda_1 e - x$  nor  $\lambda_2 e - x$  is invertible, so they are both 0:  $x = \lambda_1 e$  and  $x = \lambda_2 e$ , whence  $\lambda_1 = \lambda_2$ . Therefore  $\sigma(x)$  has precisely one element, which we denote by  $\lambda(x)$ , and which satisfies

$$x = \lambda(x)e.$$

If  $x, y \in A$ , then  $x + y = \lambda(x)e + \lambda(y)e = (\lambda(x) + \lambda(y))e$  and also  $x + y = \lambda(x + y)e$ , so  $\lambda(x + y) = \lambda(x) + \lambda(y)$ . If  $x \in A$  and  $\alpha \in \mathbb{C}$ , then  $\alpha x = \alpha \lambda(x)e$  and also  $\alpha x = \lambda(\alpha x)e$ , so  $\lambda(\alpha x) = \alpha \lambda(x)$ . Hence  $x \mapsto \lambda(x)$  is linear. If  $\lambda_0 \in \mathbb{C}$ , then  $\lambda(\lambda_0 e) = \lambda_0$ , showing that  $x \mapsto \lambda(x)$  is onto. If  $\lambda(x) = \lambda(y)$  then  $x = \lambda(x)e = \lambda(y)e = y$ , showing that  $x \mapsto \lambda(x)$  is one-to-one. Therefore  $x \mapsto \lambda(x)$  is a linear isomorphism  $A \rightarrow \mathbb{C}$ .

If  $x \in A$ , then  $x = \lambda(x)e$  gives

$$\|x\| = \|\lambda(x)e\| = |\lambda(x)| \|e\| = |\lambda(x)|,$$

showing that the map  $x \mapsto \lambda(x)$  is an isometry  $A \rightarrow \mathbb{C}$ . □

<sup>1</sup>Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.

## 2 Complex homomorphisms

An ideal  $J$  of an algebra  $A$  is said to be *proper* if  $J \neq A$ . An ideal is called *maximal* if it is a maximal element in the collection of proper ideals of  $A$  ordered by set inclusion.

The following theorem, which is proved using the fact that a maximal ideal is closed, the fact that a quotient of a Banach algebra with a closed ideal is a Banach algebra, and the Gelfand-Mazur theorem, states some basic facts about algebra homomorphisms from a Banach algebra to  $\mathbb{C}$ .<sup>2</sup>

**Theorem 2.** *If  $A$  is a commutative unital Banach algebra and  $\Delta$  is the set of all nonzero algebra homomorphisms  $A \rightarrow \mathbb{C}$ , then:*

1. *If  $M$  is a maximal ideal of  $A$  then there is some  $h \in \Delta$  for which  $M = \ker h$ .*
2. *If  $h \in \Delta$  then  $\ker h$  is a maximal ideal of  $A$ .*
3.  *$x \in A$  is invertible if and only if  $h(x) \neq 0$  for all  $h \in \Delta$ .*
4.  *$x \in A$  is invertible if and only if  $x$  does not belong to any proper ideal of  $A$ .*
5.  *$\lambda \in \sigma(x)$  if and only if there is some  $h \in \Delta$  for which  $h(x) = \lambda$ .*

## 3 The Gelfand transform and maximal ideals

Suppose that  $A$  is a commutative unital Banach algebra and that  $\Delta$  is the set of all nonzero algebra homomorphisms  $A \rightarrow \mathbb{C}$ . For each  $x \in A$ , we define  $\hat{x} : \Delta \rightarrow \mathbb{C}$  by

$$\hat{x}(h) = h(x), \quad h \in \Delta.$$

We call  $\hat{x}$  the *Gelfand transform of  $x$* , and we call the map  $\Gamma : A \rightarrow \mathbb{C}^\Delta$  defined by  $\Gamma(x) = \hat{x}$  the *Gelfand transform*.

We define  $\hat{A} = \{\hat{x} : x \in A\}$ , and we call the set  $\Delta$  with the initial topology for  $\hat{A}$  the *maximal ideal space of  $A$* . That is, the topology of  $\Delta$  is the coarsest topology on  $\Delta$  such that each  $\hat{x} : \Delta \rightarrow \mathbb{C}$  is continuous. If  $X$  is a topological space, we denote by  $C(X)$  the set of all continuous functions  $X \rightarrow \mathbb{C}$ .  $C(X)$  is a commutative unital algebra, although it need not be a Banach algebra.

The *radical of  $A$* , denoted  $\text{rad } A$ , is the intersection of all maximal ideals of  $A$ . If  $\text{rad } A = \{0\}$ , we say that  $A$  is *semisimple*.

The following theorem establishes some basic facts about the Gelfand transform and the maximal ideal space.<sup>3</sup>

**Theorem 3.** *If  $A$  is a commutative unital Banach algebra and  $\Delta$  is the maximal ideal space of  $A$ , then:*

<sup>2</sup>Walter Rudin, *Functional Analysis*, second ed., p. 277, Theorem 11.5.

<sup>3</sup>Walter Rudin, *Functional Analysis*, second ed., p. 280, Theorem 11.9.

1.  $\Gamma : A \rightarrow C(\Delta)$  is an algebra homomorphism with  $\ker \Gamma = \text{rad } A$ .
2. If  $x \in A$ , then  $\text{im } \hat{x} = \sigma(x)$ .
3.  $\Delta$  is a compact Hausdorff space.

*Proof.* Let  $x, y \in A$  and  $\alpha \in \mathbb{C}$ . For  $h \in \Delta$ ,

$$\Gamma(\alpha x + y)(h) = h(\alpha x + y) = \alpha h(x) + h(y) = \alpha \Gamma(x)(h) + \Gamma(y)(h) = (\Gamma(x) + \Gamma(y))(h),$$

showing that  $\Gamma(\alpha x + y) = \alpha \Gamma(x) + \Gamma(y)$ , and

$$\Gamma(xy)(h) = h(xy) = h(x)h(y) = \Gamma(x)(h)\Gamma(y)(h) = (\Gamma(x)\Gamma(y))(h),$$

showing that  $\Gamma(xy) = \Gamma(x)\Gamma(y)$ . Therefore  $\Gamma : A \rightarrow C(\Delta)$  is an algebra homomorphism.  $x \in \ker \Gamma$  is equivalent to  $h(x) = 0$  for all  $h \in \Delta$ , which is equivalent to  $x \in \ker h$  for all  $h \in \Delta$ . But by Theorem 2,  $\{\ker h : h \in \Delta\}$  is equal to the set of all maximal ideals of  $A$ , so  $x \in \ker \Gamma$  is equivalent to  $x \in \text{rad } A$ , i.e.  $\ker \Gamma = \text{rad } A$ .

Let  $x \in A$ . If  $\lambda \in \text{im } \hat{x}$  then there is some  $h \in \Delta$  for which  $\hat{x}(h) = \lambda$ , and by Theorem 2, this yields  $\lambda \in \sigma(x)$ . Hence  $\text{im } \hat{x} \subseteq \sigma(x)$ . If  $\lambda \in \sigma(x)$ , then by Theorem 2 there is some  $h \in \Delta$  for which  $h(x) = \lambda$ , i.e. there is some  $h \in \Delta$  for which  $\hat{x}(h) = \lambda$ , i.e.  $\lambda \in \text{im } \hat{x}$ . Hence  $\sigma(x) \subseteq \text{im } \hat{x}$ . Therefore,  $\text{im } \hat{x} = \sigma(x)$ .

It is straightforward to check that the topology of  $\Delta$  is the subspace topology inherited from  $A^*$  with the weak-\* topology; in particular, the topology of  $\Delta$  is Hausdorff. Therefore, to prove that  $\Delta$  is compact it suffices to prove that  $\Delta$  is a weak-\* compact subset of  $A^*$ . Let

$$K = \{\lambda \in A^* : \|\lambda\| \leq 1\}.$$

By the Banach-Alaoglu theorem,  $K$  is a weak-\* compact subset of  $A^*$ . If  $h \in \Delta$ , then because  $h$  is an algebra homomorphism  $A \rightarrow \mathbb{C}$  it follows that  $\|h\| \leq 1$ .<sup>4</sup> Thus,  $\Delta \subset K$ . Therefore, to prove that  $\Delta$  is compact it suffices to prove that  $\Delta$  is a weak-\* closed subset of  $A^*$ .

Suppose that  $h_i \in \Delta$  is a net that weak-\* converges to  $\lambda \in A^*$ . Then  $h_i(e) \rightarrow \lambda(e)$ , i.e.  $1 \rightarrow \lambda(e)$ , so  $\lambda(e) = 1$ . Thus  $\lambda \neq 0$ . Let  $x, y \in A$ . On the one hand,  $h_i(xy) \rightarrow \lambda(xy)$ , and on the other hand,  $h_i(x) \rightarrow \lambda(x)$  and  $h_i(y) \rightarrow \lambda(y)$ , so  $h_i(x)h_i(y) \rightarrow \lambda(x)\lambda(y)$  and hence  $h_i(xy) = h_i(x)h_i(y) \rightarrow \lambda(x)\lambda(y)$ . Therefore,  $\lambda(xy) = \lambda(x)\lambda(y)$ , and because  $\lambda \in A^*$  is linear, this shows that  $\lambda : A \rightarrow \mathbb{C}$  is an algebra homomorphism, and hence that  $\lambda \in \Delta$ . Therefore  $\Delta$  is a weak-\* closed subset of  $A^*$ .  $\square$

If  $A$  is a commutative unital Banach algebra, the above theorem shows that  $\Gamma : A \rightarrow \hat{A}$  is an algebra isomorphism if and only if  $\text{rad } A = \{0\}$ , i.e.,  $\Gamma$  is an algebra isomorphism if and only if  $A$  is semisimple.

<sup>4</sup>Walter Rudin, *Functional Analysis*, second ed., p. 249, Theorem 10.7.

The above theorem tells us that if  $A$  is a commutative unital Banach algebra and  $x \in A$ , then  $\text{im } \hat{x} = \sigma(x)$ . This gives us

$$\|\hat{x}\|_\infty = \rho(x), \quad (1)$$

where  $\rho(x)$  is the *spectral radius* of  $x$ , defined by

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Therefore,  $\hat{x} = 0$  is equivalent to  $\rho(x) = 0$ , and so by the above theorem,  $x \in \text{rad } A$  is equivalent to  $\rho(x) = 0$ . Moreover, it is a fact that  $\rho(x) \leq \|x\|$ .<sup>5</sup> Therefore,

$$\|\hat{x}\|_\infty \leq \|x\|. \quad (2)$$

In the proof of Theorem 3 we used the fact<sup>6</sup> that the norm of any algebra homomorphism from a Banach algebra to  $\mathbb{C}$  is  $\leq 1$ . In particular, this means that any algebra homomorphism from a Banach algebra to  $\mathbb{C}$  is continuous. The following theorem shows that any algebra homomorphism from a Banach algebra to a commutative unital semisimple Banach algebra is continuous.<sup>7</sup>

**Theorem 4.** *Suppose that  $A$  is a Banach algebra and that  $B$  is a commutative unital semisimple Banach algebra. If  $\psi : A \rightarrow B$  is an algebra homomorphism, then  $\psi$  is continuous.*

*Proof.* Because  $\psi : A \rightarrow B$  is linear, to prove that  $\psi$  is continuous, by the closed graph theorem<sup>8</sup> it suffices to prove that

$$G = \{(x, \psi(x)) : x \in A\}$$

is closed in  $A \times B$ . To prove that  $G$  is closed in  $A \times B$ , it suffices to prove that if  $(x_n, y_n) \in G$  converges to  $(x, y) \in A \times B$  then  $(x, y) \in G$ .

Let  $h \in \Delta_B$ . Then  $\phi = h \circ \psi : A \rightarrow \mathbb{C}$  is an algebra homomorphism. Because  $h : B \rightarrow \mathbb{C}$  and  $\phi : A \rightarrow \mathbb{C}$  are algebra homomorphisms with codomain  $\mathbb{C}$ , they are both continuous. Therefore,  $h(y_n) \rightarrow h(y)$  and  $\phi(x_n) \rightarrow \phi(x)$ . Therefore,

$$h(y) = \lim h(y_n) = \lim h(\psi(x_n)) = \lim (h \circ \psi)(x_n) = \lim \phi(x_n) = \phi(x) = h(\psi(x)),$$

so  $h(y - \psi(x)) = 0$ . This is true for all  $h \in \Delta_B$ , hence  $y - \psi(x) \in \text{rad } B$ . But  $B$  is semisimple, so  $y - \psi(x) = 0$ , i.e.  $y = \psi(x)$ , so  $(x, y) \in G$ .  $\square$

If  $A$  is a commutative unital Banach algebra and  $x \in A$ , we recorded in (2) that  $\|\hat{x}\|_\infty \leq \|x\|$ . The following lemma<sup>9</sup> shows that if  $\|x^2\| = \|x\|^2$  and  $x \neq 0$ , then  $\inf \frac{\|\hat{x}\|_\infty}{\|x\|} \geq 1$ , hence that  $\|\hat{x}\|_\infty = \|x\|$ . Therefore, if  $\|x^2\| = \|x\|^2$  for all  $x \in A$ , then  $\Gamma : A \rightarrow C(\Delta)$  is an isometry.

<sup>5</sup>Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.

<sup>6</sup>Walter Rudin, *Functional Analysis*, second ed., p. 249, Theorem 10.7.

<sup>7</sup>Walter Rudin, *Functional Analysis*, second ed., p. 281, Theorem 11.10.

<sup>8</sup>Walter Rudin, *Functional Analysis*, second ed., p. 51, Theorem 2.15.

<sup>9</sup>Walter Rudin, *Functional Analysis*, second ed., p. 282, Lemma 11.11.

**Lemma 5.** *Let  $A$  be a commutative unital Banach algebra. If*

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2}, \quad s = \inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|},$$

*then  $s^2 \leq r \leq s$ .*

Theorem 3 shows that if  $A$  is a commutative unital Banach algebra, then  $\Gamma : A \rightarrow C(\Delta)$  is an algebra homomorphism. Therefore  $\Gamma(A) = \hat{A}$  is a subalgebra of  $C(\Delta)$ . Moreover, Theorem 3 also shows that  $\Delta$  is a compact Hausdorff space. Therefore,  $C(\Delta)$  is a unital Banach algebra with the supremum norm. (If  $X$  is a topological space then  $C(X)$  is an algebra, but need not be a Banach algebra.) For  $\hat{A}$  to be a Banach subalgebra of  $C(\Delta)$  it is necessary and sufficient that  $\hat{A}$  be a closed subset of the Banach algebra  $C(\Delta)$ . The following theorem gives conditions under which this occurs.<sup>10</sup>

**Theorem 6.** *If  $A$  is a commutative unital Banach algebra, then  $A$  is semisimple and  $\hat{A}$  is a closed subset of  $C(\Delta)$  if and only if there exists some  $K < \infty$  such that  $\|x\|^2 \leq K \|x^2\|$  for all  $x \in A$ .*

*Proof.* Suppose that there is some  $0 < K < \infty$  such that  $x \in A$  implies that  $\|x\|^2 \leq K \|x^2\|$ . Then

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2} \geq \inf_{x \neq 0} \frac{\|x^2\|}{K \|x^2\|} = \frac{1}{K}.$$

By Lemma 5, with  $s = \inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|}$  we have

$$\frac{1}{K} \leq s,$$

hence  $\|\hat{x}\|_\infty \geq \frac{1}{K} \|x\|$ . Thus, if  $x \in A$  then  $\|\hat{x}\|_\infty \geq \frac{1}{K} \|x\|$ , from which it follows that  $\Gamma : A \rightarrow C(\Delta)$  is one-to-one. Since  $\Gamma$  is one-to-one, by Theorem 3 we get that  $A$  is semisimple. Suppose that  $\hat{x}_n \in \hat{A}$  converges to  $\hat{x} \in \hat{A}$ , i.e.  $\|\hat{x}_n - \hat{x}\|_\infty \rightarrow 0$ , i.e.  $\|\Gamma(x_n - x)\|_\infty \rightarrow 0$ . But  $\|\Gamma(x_n - x)\|_\infty \geq \frac{1}{K} \|x_n - x\|$ , so  $\|x_n - x\| \rightarrow 0$ , showing that  $\Gamma^{-1} : \hat{A} \rightarrow A$  is bounded. Therefore  $\Gamma : A \rightarrow \hat{A}$  is bilipschitz, and so  $\hat{A}$  is a complete metric space, from which it follows that  $\hat{A}$  is a closed subset of  $C(\Delta)$ .

Suppose that  $A$  is semisimple and that  $\hat{A}$  is a closed subset of  $C(\Delta)$ . The fact that  $A$  is semisimple gives us by Theorem 3 that  $\Gamma : A \rightarrow \hat{A}$  is a bijection. The fact that  $\hat{A}$  is closed means that  $\hat{A}$  is a Banach algebra. Because  $\Gamma : A \rightarrow \hat{A}$  is continuous, linear, and a bijection, by the open mapping theorem<sup>11</sup> it follows that there are positive real numbers  $a, b$  such that if  $x \in A$  then

$$a \|x\| \leq \|\Gamma x\|_\infty \leq b \|x\|.$$

Then  $\inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|} \geq a$ . By Lemma 5, it follows that  $\inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2} \geq a^2$ . Hence, for all  $x \neq 0$  we have  $\|x\|^2 \leq K \|x^2\|$ , with  $K = \frac{1}{a^2}$ .  $\square$

<sup>10</sup>Walter Rudin, *Functional Analysis*, second ed., p. 282, Theorem 11.12.

<sup>11</sup>Walter Rudin, *Functional Analysis*, second ed., p. 49, Corollary 2.12.

## 4 $L^1$

Let  $M(\mathbb{R}^n)$  denote the set of all complex Borel measures on  $\mathbb{R}^n$ , and let  $S : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by  $S(x, y) = x + y$ . For  $\mu_1, \mu_2 \in M(\mathbb{R}^n)$ , we denote by  $\mu_1 \times \mu_2$  the product measure on  $\mathbb{R}^n \times \mathbb{R}^n$ , and we define the *convolution* of  $\mu_1$  and  $\mu_2$  to be  $\mu_1 * \mu_2 = S_*(\mu_1 \times \mu_2)$ , the pushforward of  $\mu_1 \times \mu_2$  with respect to  $S$ . That is, if  $E$  is a Borel subset of  $\mathbb{R}^n$ , then

$$\begin{aligned} (\mu_1 * \mu_2)(E) &= (S_*(\mu_1 \times \mu_2))(E) \\ &= (\mu_1 \times \mu_2)(S^{-1}(E)) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_E(x + y) d\mu_1(x) d\mu_2(y). \end{aligned}$$

With convolution as multiplication,  $M(\mathbb{R}^n)$  is an algebra.

If  $\mu \in M(\mathbb{R}^n)$ , the *variation* of  $\mu$  is the measure  $|\mu| \in M(\mathbb{R}^n)$ , where for a Borel subset  $E$  of  $\mathbb{R}^n$ , we define  $|\mu|(E)$  to be the supremum of  $\sum_{A \in \pi} |\mu(A)|$  over all partitions  $\pi$  of  $E$  into finitely many disjoint Borel subsets. The *total variation* of  $\mu$  is  $\|\mu\| = |\mu|(\mathbb{R}^n)$ . One proves that  $\|\cdot\|$  is a norm on  $M(\mathbb{R}^n)$  and that with this norm,  $M(\mathbb{R}^n)$  is a Banach algebra.<sup>12</sup>

Let  $m_n$  be Lebesgue measure on  $\mathbb{R}^n$ , let  $\delta$  be the Dirac measure on  $\mathbb{R}^n$ , and let  $A$  be the set of those  $\mu \in M(\mathbb{R}^n)$  for which there is some  $f \in L^1(\mathbb{R}^n)$  and some  $\alpha \in \mathbb{C}$  with which

$$d\mu = f dm_n + \alpha d\delta.$$

One proves that  $A$  is a Banach subalgebra of  $M(\mathbb{R}^n)$ .  $A$  is a unital Banach algebra, with unity  $\delta$ . In particular,  $A$  is a unital Banach algebra that contains the Banach algebra  $L^1(\mathbb{R}^n)$ .

If  $f + \alpha\delta, g + \beta\delta \in A$  (identifying  $f \in L^1(\mathbb{R}^n)$  with the complex Borel measure whose Radon-Nikodym derivative with respect to  $m_n$  is  $f$ ), then

$$(f + \alpha\delta) * (g + \beta\delta) = (f * g + \beta f + \alpha g) + \alpha\beta\delta, \quad (3)$$

where

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y) dm_n(y).$$

If  $t \in \mathbb{R}^n$ , let  $e_t(x) = \exp(it \cdot x)$ , and if  $f \in L^1(\mathbb{R}^n)$ , define  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ , the *Fourier transform* of  $f$ , by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f e_{-t} dm_n, \quad t \in \mathbb{R}^n.$$

If  $t \in \mathbb{R}^n$ , define  $h_t : A \rightarrow \mathbb{C}$  by

$$h_t(f + \alpha\delta) = \hat{f}(t) + \alpha, \quad f + \alpha\delta \in A,$$

and define  $h_\infty : A \rightarrow \mathbb{C}$  by

$$h_\infty(f + \alpha\delta) = \alpha, \quad f + \alpha\delta \in A.$$

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<sup>12</sup>See Walter Rudin, *Real and Complex Analysis*, third ed., chapter 6.

By (3) it is apparent that for each  $t \in \mathbb{R}^n \cup \{\infty\}$ , the map  $h_t$  is a homomorphism of algebras. It can be proved that  $\Delta = \{h_t : t \in \mathbb{R}^n\} \cup \{h_\infty\}$ .<sup>13</sup> Let  $\mathbb{R}^n \cup \{\infty\}$  be the one-point compactification of  $\mathbb{R}^n$ , and define  $T : \mathbb{R}^n \cup \{\infty\} \rightarrow \Delta$  by  $T(t) = h_t$ , which is a bijection.

Suppose that  $t_k \rightarrow t$  in  $\mathbb{R}^n$ . If  $f + \alpha\delta \in A$ , then because  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$  is continuous, we have

$$T(t_k)(f + \alpha\delta) = h_{t_k}(f + \alpha\delta) = \hat{f}(t_k) + \alpha \rightarrow \hat{f}(t) + \alpha = T(t)(f + \alpha\delta).$$

Suppose that  $t_k \rightarrow \infty$ . If  $f + \alpha\delta \in A$ , then by the Riemann-Lebesgue lemma we have  $\hat{f}(t_k) \rightarrow 0$ , and hence

$$T(t_k)(f + \alpha\delta) = \hat{f}(t_k) + \alpha \rightarrow \alpha = h_\infty(f + \alpha\delta) = T(\infty)(f + \alpha\delta).$$

Therefore,  $T : \mathbb{R}^n \cup \{\infty\} \rightarrow \Delta$  is continuous.

Suppose that  $h_{t_k} \rightarrow h_t$  in  $\Delta$ ,  $t_k, t \in \mathbb{R}^n$ . If  $f + \alpha\delta \in A$ , then  $h_{t_k}(f + \alpha\delta) \rightarrow h_t(f + \alpha\delta)$ . But  $h_{t_k}(f + \alpha\delta) = \hat{f}(t_k) + \alpha$  and  $h_t(f + \alpha\delta) = \hat{f}(t) + \alpha$ , so  $\hat{f}(t_k) \rightarrow \hat{f}(t)$ . Because this is true for all  $f \in L^1(\mathbb{R}^n)$ , it follows that  $t_k \rightarrow t$ . Suppose that  $h_{t_k} \rightarrow h_\infty$  in  $\Delta$ ,  $t_k \in \mathbb{R}^n$ . If  $f + \alpha\delta \in A$ , then  $h_{t_k}(f + \alpha\delta) \rightarrow h_\infty(f + \alpha\delta)$ , i.e.  $\hat{f}(t_k) + \alpha \rightarrow \alpha$ , i.e.  $\hat{f}(t_k) \rightarrow 0$ . Because this is true for all  $f \in L^1(\mathbb{R}^n)$ , it follows that  $t_k \rightarrow \infty$ . Therefore,  $T^{-1} : \Delta \rightarrow \mathbb{R}^n \cup \{\infty\}$  is continuous, and so  $\Delta$  is homeomorphic to the one-point compactification of  $\mathbb{R}^n$ .

## 5 Involutions

If  $A$  is an algebra, an *involution* of  $A$  is a map  $*$  :  $A \rightarrow A$  satisfying

1.  $(x + y)^* = x^* + y^*$
2.  $(\alpha x)^* = \overline{\alpha}x^*$
3.  $(xy)^* = y^*x^*$
4.  $x^{**} = x$ .

We say that  $x$  is *self-adjoint* if  $x^* = x$ .

Following Rudin, if  $A$  is a Banach algebra with an involution  $*$  :  $A \rightarrow A$  satisfying

$$\|xx^*\| = \|x\|^2, \quad x \in A,$$

we say that  $A$  is a *B\*-algebra*.

The following theorem shows that a commutative unital B\*-algebra with maximal ideal space  $\Delta$  is isomorphic as a B\*-algebra to  $C(\Delta)$ .<sup>14</sup> (An *isomorphism of B\*-algebras* is an isomorphism of Banach algebras that preserves the involution; the involution on  $C(\Delta)$  is  $(x \mapsto f(x)) \mapsto (x \mapsto \overline{f(x)})$ .)

<sup>13</sup>Walter Rudin, *Functional Analysis*, second ed., p. 285.

<sup>14</sup>Walter Rudin, *Functional Analysis*, second ed., p. 289, Theorem 11.18.

**Theorem 7** (Gelfand-Naimark). *If  $A$  is a commutative unital  $B^*$ -algebra, then  $\Gamma : A \rightarrow C(\Delta)$  is an isomorphism of Banach algebras, and if  $x \in A$  then  $\Gamma(x^*) = \overline{\Gamma(x)}$ .*

*Proof.* Let  $u \in A$  be self-adjoint, let  $h \in \Delta$ , and let  $h(u) = \alpha + i\beta$ . For  $t \in \mathbb{R}$ , put  $z = u + ite$ . We have

$$h(z) = h(u) + h(ite) = \alpha + i\beta + it = \alpha + i(\beta + t),$$

and

$$zz^* = (u + ite)(u - ite) = u^2 + t^2e,$$

hence

$$\alpha^2 + (\beta + t)^2 = |h(z)|^2 \leq \|z\|^2 = \|zz^*\| = \|u^2 + t^2e\| \leq \|u\|^2 + t^2,$$

i.e.

$$\alpha^2 + \beta^2 + 2\beta t \leq \|u\|^2.$$

Because this is true for all  $t \in \mathbb{R}$ , it follows that  $\beta = 0$ . Therefore, if  $u \in A$  is self-adjoint then  $h(u) \in \mathbb{R}$ .

Furthermore, if  $x \in A$  then with  $2u = x + x^*$  and  $2v = i(x^* - x)$  we have  $x = u + iv$  with  $u$  and  $v$  self-adjoint. Then  $x^* = u - iv$ , and so

$$h(x^*) = h(u - iv) = h(u) - ih(v) = \overline{h(x)}.$$

This shows that if  $x \in A$  then  $\Gamma(x^*) = \overline{\Gamma(x)}$ . In particular,  $\widehat{A}$  is closed under complex conjugation. If  $h_1 \neq h_2$ , then there is some  $x \in A$  for which  $h_1(x) \neq h_2(x)$ , i.e.  $\widehat{x}(h_1) \neq \widehat{x}(h_2)$ , so  $\widehat{A}$  separates points in  $\Delta$ . Because  $\widehat{A}$  is a unital Banach algebra, it follows from the Stone-Weierstrass theorem that  $\widehat{A}$  is dense in  $C(\Delta)$ .

Let  $x \in A$ . With  $y = xx^*$ , we have  $y^* = (xx^*)^* = x^{**}x^* = xx^* = y$ , from which it follows that  $\|y^2\| = \|y\|^2$ . Assume by induction that  $\|y^m\| = \|y\|^m$ , for  $m = 2^n$ . Then, as  $(y^m)^* = y^m$ ,

$$\|y^{2m}\| = \|y^m y^m\| = \|y^m (y^m)^*\| = \|y^m\|^2 = (\|y\|^m)^2 = \|y\|^{2m}.$$

The spectral radius formula<sup>15</sup> gives

$$\rho(y) = \lim \|y^n\|^{1/n},$$

and because  $\|y^m\| = \|y\|^m$  for  $m = 2^n$ , we have  $\lim \|y^m\|^{1/m} = \|y\|$ . Because the limit of this subsequence is  $\|y\|$ , the limit of  $\|y^n\|^{1/n}$  is also  $\|y\|$ , so we obtain

$$\rho(y) = \|y\|.$$

But (1) tells us  $\|\widehat{y}\|_\infty = \rho(y)$ , so we have  $\|\widehat{y}\|_\infty = \|y\|$ . Because  $y = xx^*$ , using  $\Gamma(x^*) = \overline{\Gamma(x)}$  and the fact that  $\Gamma$  is an algebra homomorphism, we get

$$\Gamma(y) = \Gamma(xx^*) = \Gamma(x)\Gamma(x^*) = \Gamma(x)\overline{\Gamma(x)} = |\Gamma(x)|^2.$$

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<sup>15</sup>Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.



That is,  $\hat{y} = |\hat{x}|^2$  and with  $\|\hat{y}\|_\infty = \|y\|$  we obtain

$$\|\hat{x}\|_\infty^2 = \|\hat{y}\|_\infty = \|y\| = \|xx^*\| = \|x\|^2,$$

i.e.

$$\|\hat{x}\|_\infty = \|x\|.$$

This shows that  $\Gamma : A \rightarrow C(\Delta)$  is an isometry. In particular,  $\Gamma$  maps closed sets to closed sets, so  $\hat{A} = \Gamma(A)$  is a closed subset of  $C(\Delta)$ . We have already established that  $\hat{A}$  is dense in  $C(\Delta)$ , so  $\hat{A} = C(\Delta)$ . The fact that  $\Gamma$  is an isometry yields that  $\Gamma$  is one-to-one, and the fact that  $\hat{A} = C(\Delta)$  means that  $\Gamma$  is onto, hence  $\Gamma$  is a bijection, and therefore it is an isomorphism of algebras. Because  $\Gamma$  is an isometry, it is an isomorphism of Banach algebras.  $\square$

The following theorem states conditions under which a self-adjoint element of a unital Banach algebra with an involution has a square root.<sup>16</sup>

**Theorem 8.** *Let  $A$  be a unital Banach algebra with an involution  $*$  :  $A \rightarrow A$ . If  $x \in A$  is self-adjoint and  $\sigma(x)$  contains no real  $\lambda$  with  $\lambda \leq 0$ , then there is some self-adjoint  $y \in A$  satisfying  $y^2 = x$ .*

If  $A$  is a Banach algebra and  $x \in A$ , we say that  $x \in A$  is *normal* if  $xx^* = x^*x$ . If  $A$  is a Banach algebra with involution  $*$  :  $A \rightarrow A$ , by  $x \geq 0$  we mean that  $x$  is self-adjoint and  $\sigma(x) \subseteq [0, \infty)$ , and we say that  $x$  is *positive*. The following theorem states basic facts about the spectrum of elements of a unital  $B^*$ -algebra.<sup>17</sup>

**Theorem 9.** *If  $A$  is a unital  $B^*$ -algebra, then:*

1. *If  $x$  is self-adjoint, then  $\sigma(x) \subseteq \mathbb{R}$ .*
2. *If  $x$  is normal, then  $\rho(x) = \|x\|$ .*
3. *If  $x \in A$ , then  $\rho(xx^*) = \|x\|^2$ .*
4. *If  $x \geq 0$  and  $y \geq 0$ , then  $x + y \geq 0$ .*
5. *If  $x \in A$ , then  $xx^* \geq 0$ .*
6. *If  $x \in A$ , then  $e + xx^*$  is invertible.*

## 6 Positive linear functionals

Suppose that  $A$  is a Banach algebra with an involution  $*$  :  $A \rightarrow A$ . If  $F : A \rightarrow \mathbb{C}$  is a linear map such that  $F(xx^*)$  is real and  $\geq 0$  for all  $x \in A$ , we say that  $F$  is a *positive linear functional*. In particular, if  $h \in \Delta$  and  $x \in A$ , then from Theorem 7 we have  $h(x^*) = \overline{h(x)}$ , and so  $h(xx^*) = h(x)h(x^*) = h(x)\overline{h(x)} = |h(x)|^2 \geq 0$ . Thus, the elements of  $\Delta$  are positive linear functionals.

We shall use the following theorem to prove the theorem after it.<sup>18</sup>

<sup>16</sup>Walter Rudin, *Functional Analysis*, second ed., p. 294, Theorem 11.26.

<sup>17</sup>Walter Rudin, *Functional Analysis*, second ed., p. 294, Theorem 11.28.

<sup>18</sup>Walter Rudin, *Functional Analysis*, second ed., p. 137, Theorem 5.20.

**Theorem 10.** *If  $X$  is a real or complex Banach space,  $X_1$  and  $X_2$  are closed subspaces of  $X$ , and  $X = X_1 + X_2$ , then there is some  $\gamma < \infty$  such that for every  $x \in X$  there are  $x_1 \in X_1, x_2 \in X_2$  satisfying  $x = x_1 + x_2$  and*

$$\|x_1\| + \|x_2\| \leq \gamma \|x\|.$$

The following theorem establishes some basic properties of positive linear functionals on a unital Banach algebra with an involution.<sup>19</sup>

**Theorem 11.** *Suppose that  $A$  is a unital Banach algebra with an involution  $*$  :  $A \rightarrow A$ . If  $F : A \rightarrow \mathbb{C}$  is a positive linear functional, then:*

1.  $F(x^*) = \overline{F(x)}$ .
2.  $|F(xy^*)|^2 \leq F(xx^*)F(yy^*)$ .
3.  $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2\rho(xx^*)$ .
4. If  $x$  is normal, then  $|F(x)| \leq F(e)\rho(x)$ .
5. If  $A$  is commutative, then  $\|F\| = F(e)$ .
6. If there is some  $\beta$  such that  $\|x^*\| \leq \beta\|x\|$  for all  $x \in A$ , then  $\|F\| \leq \beta^{1/2}F(e)$ .
7.  $F$  is a bounded linear map.

*Proof.* Suppose that  $x, y \in A$ . For any  $\alpha \in \mathbb{C}$ , we have on the one hand  $F((x + \alpha y)(x + \alpha y)^*) \geq 0$ , and on the other hand

$$F((x + \alpha y)(x + \alpha y)^*) = F((x + \alpha y)(x^* + \bar{\alpha}y^*)) = F(xx^* + \bar{\alpha}xy^* + \alpha yx^* + |\alpha|^2 yy^*).$$

Therefore,

$$F(xx^*) + \bar{\alpha}F(xy^*) + \alpha F(yx^*) + |\alpha|^2 F(yy^*) \geq 0. \quad (4)$$

Applying (4) with  $\alpha = 1$  gives

$$F(xx^*) + F(xy^*) + F(yx^*) + F(yy^*) \geq 0.$$

In particular, this expression is real, and because  $F(xx^*)$  and  $F(yy^*)$  are real we get that  $F(xy^*) + F(yx^*)$  is real, so  $\text{Im } F(yx^*) = -\text{Im } F(xy^*)$ . Applying (4) with  $\alpha = i$  gives

$$F(xx^*) - iF(xy^*) + iF(yx^*) + F(yy^*) \geq 0.$$

In particular, this expression is real, and so  $-iF(xy^*) + iF(yx^*)$  is real, i.e.  $\frac{F(xy^*)}{F(yx^*)} - F(yx^*)$  is imaginary, so  $\text{Re } F(yx^*) = \text{Re } F(xy^*)$ . Therefore  $F(yx^*) = \overline{F(xy^*)}$ . Using  $y = e$  yields

$$F(x^*) = \overline{F(x)}.$$

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<sup>19</sup>Walter Rudin, *Functional Analysis*, second ed., p. 296, Theorem 11.31.

Suppose that  $x, y \in A$  and that  $F(xy^*) \neq 0$ . For any  $t \in \mathbb{R}$ , using (4) with  $\alpha = \frac{t}{|F(xy^*)|} F(xy^*)$  gives

$$F(xx^*) + \frac{t}{|F(xy^*)|} \overline{F(xy^*)} F(xy^*) + \frac{t}{|F(xy^*)|} F(xy^*) F(yx^*) + t^2 F(yy^*) \geq 0,$$

i.e.

$$F(xx^*) + t|F(xy^*)| + \frac{t}{|F(xy^*)|} F(xy^*) F(yx^*) + t^2 F(yy^*) \geq 0,$$

and as  $F(yx^*) = F((xy^*)^*) = \overline{F(xy^*)}$ , we have

$$F(xx^*) + 2t|F(xy^*)| + t^2 F(yy^*) \geq 0.$$

For  $t = -\frac{|F(xy^*)|}{F(yy^*)}$  this is

$$F(xx^*) - 2\frac{|F(xy^*)|^2}{F(yy^*)} + \frac{|F(xy^*)|^2}{F(yy^*)} \geq 0,$$

i.e.

$$|F(xy^*)|^2 \leq F(xx^*) F(yy^*).$$

Suppose that  $x \in A$ . Because  $xe^* = x$  and  $ee^* = e$ , we have

$$|F(x)|^2 \leq F(e) F(xx^*).$$

We shall prove that  $F(xx^*) \leq F(e) \rho(xx^*)$ . Let  $t > \rho(xx^*)$ . It then follows that  $\sigma(te - xx^*)$  is contained in the open right half-plane, and thus by Theorem 8 there is some self-adjoint  $u \in A$  satisfying  $u^2 = te - xx^*$ . Then

$$F(te - xx^*) = F(u^2) = F(uu^*) \geq 0,$$

so

$$F(xx^*) \leq tF(e).$$

Because this is true for all  $t > \rho(xx^*)$ , we obtain

$$F(xx^*) \leq F(e) \rho(xx^*).$$

Suppose that  $x$  is normal. It is a fact that if  $x$  and  $y$  belong to a unital Banach algebra and  $xy = yx$ , then  $\sigma(xy) \subseteq \sigma(x)\sigma(y)$ .<sup>20</sup> Thus  $\sigma(xx^*) \subseteq \sigma(x)\sigma(x^*)$ , from which we get

$$\rho(xx^*) \leq \rho(x)\rho(x^*).$$

It is a fact that  $\sigma(x^*) = \overline{\sigma(x)}$ ,<sup>21</sup> so we have  $\rho(x) = \rho(x^*)$ , and thus

$$\rho(xx^*) \leq \rho(x)^2.$$

<sup>20</sup>Walter Rudin, *Functional Analysis*, second ed., p. 293, Theorem 11.23.

<sup>21</sup>Walter Rudin, *Functional Analysis*, second ed., p. 288, Theorem 11.15.

But  $|F(x)|^2 \leq F(e)^2 \rho(xx^*)$ , so we have  $|F(x)|^2 \leq F(e)^2 \rho(x)^2$ , i.e.

$$|F(x)| \leq F(e) \rho(x).$$

Suppose that  $A$  is commutative, and let  $x \in A$ . Since  $A$  is commutative,  $x$  is normal and hence we have  $|F(x)| \leq F(e) \rho(x)$ , and as always we have  $\rho(x) \leq \|x\|$ . Therefore, for every  $x \in A$  we have

$$|F(x)| \leq F(e) \|x\|.$$

This implies that  $\|F\| \leq F(e)$ , and because the above inequality is an equality for  $x = e$ , we have  $\|F\| = F(e)$ .

Suppose that there is some  $\beta$  such that  $\|x^*\| \leq \beta \|x\|$  for all  $x \in A$ . We have  $\rho(xx^*) \leq \|xx^*\| \leq \|x\| \|x^*\| \leq \beta \|x\|^2$ . (We merely stipulated that  $A$  is a unital Banach algebra with an involution; if we had demanded that  $A$  be a  $B^*$ -algebra, then we would have  $\|xx^*\| = \|x\| \|x^*\| = \|x\|^2$ .) Using  $|F(x)|^2 \leq F(e)^2 \rho(xx^*)$  then gives us  $|F(x)|^2 \leq \beta F(e)^2 \|x\|^2$ , hence

$$|F(x)| \leq \beta^{1/2} F(e) \|x\|.$$

If  $F(e) = 0$ , then  $|F(x)|^2 \leq 0$  for all  $x \in A$ , and hence  $F = 0$ , which indeed is bounded. Otherwise,  $F(e) > 0$ , and  $F$  is bounded if and only if  $\frac{1}{F(e)} F$  is bounded. Therefore, to prove that  $F$  is bounded it suffices to prove that  $F$  is bounded in the case where  $F(e) = 1$ .

Let  $H$  be the set of all self-adjoint elements of  $A$ .  $H$  and  $iH$  are real vector spaces. For any  $x \in A$ , defining  $2u = x + x^*$  and  $2v = i(x^* - x)$ , we have  $x = u + iv$ , and  $u, v$  are self-adjoint. It follows that

$$A = H + iH.$$

Because the elements of  $H$  are self-adjoint, the restriction of  $F$  to  $H$  is a real-linear map  $H \rightarrow \mathbb{R}$ . For  $u \in H$ , because  $u$  is self-adjoint it is in particular normal, and so  $|F(u)| \leq F(e) \rho(u) \leq F(e) \|u\| = \|u\|$ , because  $F(e) = 1$ . Hence the restriction of  $F$  to  $H$  is a real-linear map  $H \rightarrow \mathbb{R}$  with norm 1, and therefore there is a unique bounded real-linear map  $\Phi : \overline{H} \rightarrow \mathbb{R}$  whose restriction to  $H$  is equal to the restriction of  $F$  to  $H$ , and  $\|\Phi\| = 1$ .

Suppose that  $y \in \overline{H} \cap i\overline{H}$ . There are  $u_n \in H$  with  $u_n \rightarrow y$  and there are  $v_n \in H$  with  $iv_n \rightarrow y$ . Then  $u_n^2 \rightarrow y^2$  and  $-v_n^2 \rightarrow y$ , or  $v_n^2 \rightarrow -y^2$ . Because  $|F(u_n)|^2 \leq F(e)F(u_n u_n^*) = F(u_n^2)$ , we have

$$|F(u_n)|^2 \leq F(u_n^2) \leq F(u_n^2 + v_n^2).$$

Because  $u_n$  and  $v_n$  are self-adjoint,  $u_n^2 + v_n^2$  is normal and hence

$$|F(u_n^2 + v_n^2)| \leq F(e) \rho(u_n^2 + v_n^2) = \rho(u_n^2 + v_n^2) \leq \|u_n^2 + v_n^2\|,$$

and so we have

$$|F(u_n)|^2 \leq \|u_n^2 + v_n^2\|.$$

But  $u_n^2 \rightarrow y$  and  $v_n^2 \rightarrow -y$ , so  $\|u_n^2 + v_n^2\| \rightarrow \|y - y\| = 0$ . Therefore,  $F(u_n) \rightarrow 0$ , and so

$$\Phi(y) = \lim F(u_n) \rightarrow 0.$$

That is, if  $y \in \overline{H} \cap i\overline{H}$ , then  $F(y) = 0$ .

Because  $A = H + iH$ , certainly  $A = \overline{H} + i\overline{H}$ , so by Theorem 10 there is some  $\gamma < \infty$  such that for all  $x \in A$ , there are  $x_1 \in \overline{H}$  and  $x_2 \in \overline{H}$  satisfying

$$x = x_1 + ix_2, \quad \|x_1\| + \|x_2\| \leq \gamma \|x\|.$$

Let  $x \in A$  and let  $x = x_1 + ix_2$ , where  $x_1, x_2$  satisfy the above, and let  $x = u + iv$  with  $u, v \in H$ , namely  $2u = x + x^*$  and  $2v = i(x^* - x)$ . Supposing that  $x_1 - u$  and  $x_2 - v \in \overline{H} \cap i\overline{H}$ , which Rudin asserts but whose truth is not apparent to me, we obtain  $F(x_1 - u) = 0$  and  $F(x_2 - v) = 0$ , or  $F(x_1) = F(u)$  and  $F(x_2) = F(v)$ . Then,

$$F(x) = F(u + iv) = F(u) + iF(v) = F(x_1) + iF(x_2) = \Phi(x_1) + i\Phi(x_2),$$

and therefore, because  $\|\Phi\| = 1$  and because  $\|x_1\| + \|x_2\| \leq \gamma \|x\|$ ,

$$|F(x)| \leq |\Phi(x_1) + i\Phi(x_2)| \leq |\Phi(x_1)| + |\Phi(x_2)| \leq \|x_1\| + \|x_2\| \leq \gamma \|x\|,$$

showing that  $\|F\| \leq \gamma$ , and in particular that  $F$  is bounded.  $\square$

## 7 The Riesz-Markov theorem and extreme points

We say that a positive Borel measure  $\mu$  on a compact Hausdorff space  $X$  is *regular* if for every Borel subset  $E$  of  $X$  we have

$$\mu(E) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq E\}$$

and

$$\mu(E) = \inf\{\mu(G) : G \text{ is open and } E \subseteq G\}.$$

We say that a complex Borel measure  $\mu$  on a compact Hausdorff space is regular if the positive Borel measure  $|\mu|$  is regular, and we write  $\|\mu\| = |\mu|(X)$ . The following is the Riesz-Markov theorem, stated for complex Borel measures on a compact Hausdorff space.<sup>22</sup>

**Theorem 12** (Riesz-Markov). *Suppose that  $X$  is a compact Hausdorff space. If  $\Lambda$  is a bounded linear functional on  $C(X)$ , then there is one and only one regular complex Borel measure  $\mu$  on  $X$  satisfying*

$$\Lambda f = \int_X f d\mu, \quad f \in C(X).$$

*This measure  $\mu$  satisfies  $\|\mu\| = \|\Lambda\|$ .*

The following theorem uses the Riesz-Markov theorem to define a correspondence between positive linear functionals on a commutative unital Banach algebra with a symmetric involution and regular positive Borel measures on its maximal ideal space.<sup>23</sup>

<sup>22</sup>Walter Rudin, *Real and Complex Analysis*, third ed., p. 130, Theorem 6.19.

<sup>23</sup>Walter Rudin, *Functional Analysis*, second ed., p. 299, Theorem 11.33.

**Theorem 13.** Suppose that  $A$  is a commutative unital Banach algebra with an involution  $*$  :  $A \rightarrow A$  satisfying

$$h(x^*) = \overline{h(x)}, \quad x \in A, h \in \Delta. \quad (5)$$

Let  $K$  be the set of all positive linear functionals  $F : A \rightarrow \mathbb{C}$  satisfying  $F(e) \leq 1$ , and let  $M$  be the set of all regular positive Borel measures  $\mu$  on  $\Delta$  satisfying  $\mu(\Delta) \leq 1$ .  $K$  and  $M$  are convex sets. If  $\mu \in M$ , then  $F : A \rightarrow \mathbb{C}$  defined by

$$F_\mu(x) = \int_\Delta \hat{x} d\mu, \quad x \in A,$$

belongs to  $K$ , and this map  $\mu \mapsto F_\mu$  is an isometric bijection  $M \rightarrow K$ .

*Proof.* If  $F_1, F_2 \in K$  and  $0 \leq t \leq 1$ , then  $(1-t)F_1 + tF_2$  is linear, and it is straightforward to check that it is positive. Moreover,  $((1-t)F_1 + tF_2)(e) = (1-t)F_1(e) + tF_2(e) \leq (1-t) + t = 1$ , so  $(1-t)F_1 + tF_2 \in K$ . Therefore  $K$  is a convex set.

Suppose that  $\mu_1, \mu_2 \in M$ , that  $a_1, a_2$  are nonnegative real numbers, and let  $\mu = a_1\mu_1 + a_2\mu_2$ . If  $E$  is a Borel subset of  $\Delta$ , then for any  $\epsilon > 0$  there are compact subsets  $F_1, F_2$  of  $\Delta$  such that  $\mu_1(E) < \mu_1(F_1) - \epsilon$  and  $\mu_2(E) < \mu_2(F_2) - \epsilon$ . With  $F = F_1 \cup F_2$ , we have

$$\begin{aligned} \mu(F) &= a_1\mu_1(F) + a_2\mu_2(F) \\ &\geq a_1\mu_1(F_1) + a_2\mu_2(F_2) \\ &\geq a_1(\mu_1(E) + \epsilon) + a_2(\mu_2(E) + \epsilon) \\ &= \mu(E) + (a_1 + a_2)\epsilon. \end{aligned}$$

It follows that  $\mu(E) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq E\}$ . If  $E$  is a Borel subset of  $\Delta$ , then for any  $\epsilon > 0$  there are open subsets  $G_1, G_2$  of  $\Delta$  such that  $\mu_1(E) > \mu_1(G_1) - \epsilon$  and  $\mu_2(E) > \mu_2(G_2) - \epsilon$ . With  $G = G_1 \cap G_2$ , we have

$$\begin{aligned} \mu(G) &= a_1\mu_1(G) + a_2\mu_2(G) \\ &\leq a_1\mu_1(G_1) + a_2\mu_2(G_2) \\ &< a_1(\mu_1(E) + \epsilon) + a_2(\mu_2(E) + \epsilon) \\ &= \mu(E) + (a_1 + a_2)\epsilon. \end{aligned}$$

It follows that  $\mu(E) = \inf\{\mu(G) : G \text{ is open and } E \subseteq G\}$ . Therefore,  $\mu = a_1\mu_1 + a_2\mu_2$  is a regular positive Borel measure. In particular, if  $0 \leq t \leq 1$  and  $a_1 = 1-t, a_2 = t$ , then  $\mu$  is a regular positive Borel measure. Finally, for  $\mu = (1-t)\mu_1 + t\mu_2, 0 \leq t \leq 1$ , we have, because  $\mu_1(\Delta) \leq 1$  and  $\mu_2(\Delta) \leq 1$ ,

$$\mu(\Delta) = (1-t)\mu_1(\Delta) + t\mu_2(\Delta) \leq (1-t) + t = 1,$$

so  $\mu \in M$ , showing that  $M$  is a convex set.

Let  $\mu \in M$ . It is apparent that  $F_\mu : A \rightarrow \mathbb{C}$  is linear. For  $x \in A$ , we have  $\Gamma(xx^*) = \Gamma(x)\Gamma(x^*)$ , and as  $\Gamma(x^*) = \overline{\Gamma(x)}$  by (5), we get  $\Gamma(xx^*) = |\Gamma(x)|^2$ . As

$|\Gamma(x)|^2(h) \geq 0$  for all  $h \in \Delta$ , we have

$$F_\mu(xx^*) = \int_\Delta \Gamma(xx^*) d\mu = \int_\Delta |\Gamma(x)|^2 d\mu \geq 0,$$

showing that  $F_\mu$  is a positive linear functional. Furthermore,  $\hat{e}(h) = h(e) = 1$  for all  $h \in \Delta$ , so

$$F_\mu(e) = \mu(\Delta) \leq 1,$$

showing that  $F_\mu \in K$ .

If  $x \in \text{rad } A$ , then  $\rho(x) = 0$  by (1), and so  $F(x) = 0$  by Theorem 11. We define  $\hat{F} : \hat{A} \rightarrow \mathbb{C}$

$$\hat{F}(\hat{x}) = F(x);$$

this makes sense because if  $\hat{x} = \hat{y}$  then  $\Gamma(x - y) = 0$ , and so by Theorem 3 we have  $x - y \in \text{rad } A$  and hence  $F(x - y) = 0$ , i.e. so  $F(x) = F(y)$ . For  $x \in A$ ,  $x$  is normal because  $A$  is commutative so we have by Theorem 11 that

$$|\hat{F}(\hat{x})| = |F(x)| \leq F(e)\rho(x)$$

and by (1) we have  $\rho(x) = \|\hat{x}\|_\infty$ , so

$$|\hat{F}(\hat{x})| \leq F(e) \|\hat{x}\|_\infty.$$

As  $\hat{F}(\hat{e}) = F(e)$ , it follows that  $\|\hat{F}\| = F(e)$ . By (5) and because  $\hat{A}$  separates points in  $\Delta$ , applying the Stone-Weierstrass we obtain that  $\hat{A}$  is dense in  $C(\Delta)$ . Because  $\hat{F}$  is a continuous linear functional on the dense subspace  $\hat{A}$  of  $C(\Delta)$ , there is a unique continuous linear functional  $\Lambda$  on  $C(\Delta)$  such that  $\Lambda = \hat{F}$  on  $\hat{A}$ , and  $\|\Lambda\| = \|\hat{F}\|$ . Applying Theorem 12, there is one and only one regular complex Borel measure  $\mu$  on  $X$  that satisfies

$$\Lambda f = \int_\Delta f d\mu, \quad f \in C(\Delta), \quad (6)$$

and  $\|\mu\| = \|\Lambda\| = \|\hat{F}\| = F(e)$ . It follows that  $\mu \mapsto F_\mu$  is one-to-one. Because  $\hat{e}(h) = 1$  for all  $h \in \Delta$ ,

$$\mu(\Delta) = \int_\Delta \chi_\Delta d\mu = \int_\Delta \hat{e} d\mu = \Lambda \hat{e} = \hat{F}(\hat{e}) = F(e) = \|\mu\| = |\mu|(\Delta).$$

The fact that  $\mu(\Delta) = |\mu|(\Delta)$  implies that  $\mu$  is a positive measure. The above equalities also state  $\mu(\Delta) = F(e)$ , and since  $F \in K$  we have  $F(e) \leq 1$ , hence  $\mu(\Delta) \leq 1$ . Therefore,  $\mu \in M$ . For  $x \in A$ , as  $\hat{x} \in C(\Delta)$  we have by (6) that

$$F_\mu(x) = \int_\Delta \hat{x} d\mu = \Lambda \hat{x} = \hat{F}(\hat{x}) = F(x),$$

showing that  $F = F_\mu$ . This shows that  $\mu \mapsto F_\mu$  is onto, and therefore  $\mu \mapsto F_\mu$  is a bijection  $M \rightarrow K$ .  $\square$

Because the map  $\mu \mapsto F_\mu$  in the above theorem is an isometric bijection  $M \rightarrow K$ , it follows that that  $\mu$  is an extreme point of  $M$  if and only if  $F_\mu$  is an extreme point of  $K$ .

It is a fact that the set of extreme points of the set of regular Borel probability measures on a compact Hausdorff space  $X$  is  $\{\delta_x : x \in X\}$ .<sup>24</sup> Given this, one proves that the set of extreme points of  $M$  is  $\{0\} \cup \{\delta_h : h \in \Delta\}$ . For  $x \in A$ ,  $F_0(x) = 0$ , i.e.  $F_0 = 0$ . For  $h \in \Delta$  and  $x \in A$ ,

$$F_{\delta_h}(x) = \int_{\Delta} \hat{x} d\delta_h = \hat{x}(h) = h(x),$$

so  $F_{\delta_h} = h$ . Therefore, the extreme points of  $K$  are  $\{0\} \cup \Delta$ , that is, the set of algebra homomorphisms  $A \rightarrow \mathbb{C}$ .

**Corollary 14.** *Suppose that  $A$  is a commutative unital Banach algebra with involution  $*$  :  $A \rightarrow A$  satisfying*

$$h(x^*) = \overline{h(x)}, \quad x \in A, h \in \Delta.$$

*If  $K$  is the set of all positive linear functionals  $F : A \rightarrow \mathbb{C}$  satisfying  $F(e) \leq 1$ , then*

$$\text{ext } K = \{0\} \cup \Delta.$$

Moreover, it is straightforward to check that the set  $K$  in the above corollary is a weak-\* closed subset of  $A^*$ : if  $F_i \in K$  is a net that weak-\* converges to  $\Lambda \in A^*$ , one checks that  $\Lambda(xx^*) \geq 0$  for all  $x \in A$  and that  $\Lambda e \leq 1$ . By the Banach-Alaoglu theorem, the set  $B = \{\Lambda \in A^* : \|\Lambda\| \leq 1\}$  is weak-\* compact, and if  $F \in K$  then  $\|F\| = F(e)$  by Theorem 11 and  $F(e) \leq 1$ , so  $K \subseteq B$ . Hence,  $K$  is a weak-\* compact subset of  $A^*$ . Therefore, the Krein-Milman theorem<sup>25</sup> tells us that  $K$  is equal to the weak-\* closure of the convex hull of the set of its extreme points, and by the above corollary this means that  $K$  is equal to the weak-\* closure of the convex hull of  $\{0\} \cup \Delta$ .

## 8 Positive definite functions

A function  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  is said to be *positive-definite* if  $r \geq 1$ ,  $x_1, \dots, x_r \in \mathbb{R}^n$ ,  $c_1, \dots, c_r \in \mathbb{C}$  imply that

$$\sum_{i,j=1}^r c_i \overline{c_j} \phi(x_i - x_j) \geq 0;$$

in particular, for  $\phi$  to be positive-definite demands that the left-hand side of this inequality is real.

A positive-definite function need not be measurable. For example,  $\mathbb{R}$  is a vector space over  $\mathbb{Q}$ , and if  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a vector space automorphism of  $\mathbb{R}$  over

<sup>24</sup>Barry Simon, *Convexity: An Analytic Viewpoint*, p. 128, Example 8.16.

<sup>25</sup>Walter Rudin, *Functional Analysis*, second ed., p. 75, Theorem 3.23.



$\mathbb{Q}$ , one proves that  $x \mapsto e^{i\psi(x)}$  is a positive-definite function  $\mathbb{R} \rightarrow \mathbb{C}$ , and that there are  $\psi$  for which  $x \mapsto e^{i\psi(x)}$  is not measurable.

The following theorem states some basic facts about positive-definite functions. More material on positive-definite functions is presented in Bogachev; for example, if  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  is a measurable positive-definite function, then there is a continuous positive-definite function  $\mathbb{R}^n \rightarrow \mathbb{C}$  that is equal to  $\phi$  almost everywhere.<sup>26</sup>

**Theorem 15.** *If  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  is positive-definite, then*

$$\phi(0) \geq 0, \quad (7)$$

and for all  $x \in \mathbb{R}^n$  we have

$$\overline{\phi(x)} = \phi(-x) \quad (8)$$

and

$$|\phi(x)| \leq \phi(0). \quad (9)$$

*Proof.* Using  $r = 1$  and  $c_1 = 1$ , we have for all  $x_1 \in \mathbb{R}^n$  that  $\phi(x_1 - x_1) \geq 0$ , i.e.  $\phi(0) \geq 0$ .

Using  $r = 2$ , for  $x_1, x_2 \in \mathbb{R}^n$  and  $c_1, c_2 \in \mathbb{C}$  we have

$$c_1 \overline{c_1} \phi(x_1 - x_1) + c_1 \overline{c_2} \phi(x_1 - x_2) + c_2 \overline{c_1} \phi(x_2 - x_1) + c_2 \overline{c_2} \phi(x_2 - x_2) \geq 0.$$

Take  $x_1 = x$  and  $x_2 = 0$ , with which

$$|c_1|^2 \phi(0) + c_1 \overline{c_2} \phi(x) + c_2 \overline{c_1} \phi(-x) + |c_2|^2 \phi(0) \geq 0;$$

in particular, the left-hand side is real, and because  $\phi(0)$  is real by (7), this implies that  $c_1 \overline{c_2} \phi(x) + c_2 \overline{c_1} \phi(-x)$  is real. That is, it is equal to its complex conjugate:

$$c_1 \overline{c_2} \phi(x) + c_2 \overline{c_1} \phi(-x) = \overline{c_1 c_2 \phi(x)} + \overline{c_2 c_1 \phi(-x)}.$$

The fact that this holds every  $c_1, c_2 \in \mathbb{C}$  implies that  $\overline{\phi(x)} = \phi(-x)$ .

Again using that

$$c_1 \overline{c_1} \phi(x_1 - x_1) + c_1 \overline{c_2} \phi(x_1 - x_2) + c_2 \overline{c_1} \phi(x_2 - x_1) + c_2 \overline{c_2} \phi(x_2 - x_2) \geq 0,$$

with  $x_1 = x, x_2 = 0$  and  $c_2 = 1$  we get

$$|c_1|^2 \phi(0) + c_1 \phi(x) + \overline{c_1} \phi(-x) + \phi(0) \geq 0.$$

Applying (8) gives

$$|c_1|^2 \phi(0) + c_1 \phi(x) + \overline{c_1 \phi(x)} + \phi(0) \geq 0.$$

For  $c_1 \in \mathbb{C}$  such that  $|c_1| = 1$ ,

$$2\phi(0) + 2\operatorname{Re}(c_1 \phi(x)) \geq 0,$$

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<sup>26</sup>Vladimir I. Bogachev, *Measure Theory*, vol. 1, p. 221, Theorem 3.10.20. See also Anthony W. Knap, *Basic Real Analysis*, p. 406.

or

$$-\operatorname{Re} (c_1 \phi(x)) \leq \phi(0).$$

Thus, taking  $c_1 \in \mathbb{C}$  such that  $|c_1| = 1$  and for which  $-\operatorname{Re} (c_1 \phi(x)) = |\phi(x)|$ , we get  $|\phi(x)| \leq \phi(0)$ .  $\square$

The following lemma about positive-definite functions follows a proof in Bogachev.<sup>27</sup>

**Lemma 16.** *If  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  is a measurable positive-definite function and  $f \in L^1(\mathbb{R}^n)$  is nonnegative, then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x-y) f(x) f(y) dm_n(x) dm_n(y) \geq 0.$$

*Proof.* For  $r \geq 2$  and for any  $x_1, \dots, x_r$  and  $c_1 = 1, \dots, c_r = 1$ , we have

$$\sum_{j,k=1}^r \phi(x_j - x_k) \geq 0,$$

or

$$r\phi(0) + \sum_{j \neq k} \phi(x_j - x_k) \geq 0.$$

By (9),  $\phi$  is bounded. It follows that we can integrate both sides of the above inequality over  $(\mathbb{R}^n)^r$  with respect to the positive measure

$$f(x_1) \cdots f(x_r) dm_n(x_1) \cdots dm_n(x_r).$$

Writing

$$I = \int_{\mathbb{R}^n} f(x) dm_n(x),$$

we obtain

$$r\phi(0)I^r + \sum_{j \neq k} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \phi(x_j - x_k) f(x_1) \cdots f(x_r) dm_n(x_1) \cdots dm_n(x_r) \geq 0,$$

and so, writing

$$J = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x-y) f(x) f(y) dm_n(x) dm_n(y),$$

we have

$$r\phi(0)I^r + \sum_{j \neq k} JI^{r-2} \geq 0,$$

or

$$r\phi(0)I^r + r(r-1)JI^{r-2} \geq 0.$$

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<sup>27</sup>Vladimir I. Bogachev, *Measure Theory*, vol. 1, p. 221, Lemma 3.10.19.

If  $I = 0$ , then because  $f$  is nonnegative it follows that  $f$  is 0 almost everywhere, in which case  $J = 0$ , so the claim is true. If  $I > 0$ , then dividing by  $r(r-1)I^{r-2}$  we obtain

$$\frac{1}{r-1}\phi(0)I^2 + J \geq 0.$$

This inequality holds for all  $r \geq 2$ , so taking  $r \rightarrow \infty$  yields

$$J \geq 0,$$

which is the claim.  $\square$

For  $f, g \in L^1(\mathbb{R}^n)$ ,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dm_n(y).$$

The *support* of a function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , denoted  $\text{supp} f$ , is the closure of the set  $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ . We denote by  $C_c(\mathbb{R}^n)$  the set of all continuous functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $\text{supp} f$  is a compact set. It is straightforward to check that an element of  $C_c(\mathbb{R}^n)$  is uniformly continuous on  $\mathbb{R}^n$ . The following theorem is similar to the previous lemma, but applies to functions that need not be nonnegative.<sup>28</sup>

**Theorem 17.** *For  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , define  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{C}$  by  $\tilde{f}(x) = \overline{f(-x)}$ . If  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  is a continuous positive-definite function, then for all  $f \in C_c(\mathbb{R}^n)$ , we have*

$$\int_{\mathbb{R}^n} (f * \tilde{f})\psi \geq 0.$$

If  $\mu$  is a complex Borel measure on  $\mathbb{R}^n$ , the *Fourier transform* of  $\mu$  is the function  $\hat{\mu} : \mathbb{R}^n \rightarrow \mathbb{C}$  defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e_{-\xi} d\mu, \quad \xi \in \mathbb{R}^n.$$

One proves using the dominated convergence theorem that  $\hat{\mu}$  is continuous.

**Theorem 18.** *If  $\mu$  is a finite positive Borel measure on  $\mathbb{R}^n$ , then  $\hat{\mu} : \mathbb{R}^n \rightarrow \mathbb{C}$  is positive-definite.*

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<sup>28</sup>Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 85, Proposition 3.35.

*Proof.* For  $\xi_1, \dots, \xi_r \in \mathbb{R}^n$  and  $c_1, \dots, c_r \in \mathbb{C}$ , we have

$$\begin{aligned}
\sum_{j,k=1}^r c_j \overline{c_k} \hat{\mu}(\xi_j - \xi_k) &= \sum_{j,k=1}^r c_j \overline{c_k} \int_{\mathbb{R}^n} e^{-i(\xi_j - \xi_k) \cdot x} d\mu(x) \\
&= \int_{\mathbb{R}^n} \sum_{j,k=1}^r c_j e^{-i\xi_j \cdot x} \overline{c_k e^{-i\xi_k \cdot x}} d\mu(x) \\
&= \int_{\mathbb{R}^n} \left( \sum_{j=1}^r c_j e^{-i\xi_j \cdot x} \right) \overline{\left( \sum_{k=1}^r c_k e^{-i\xi_k \cdot x} \right)} d\mu(x) \\
&= \int_{\mathbb{R}^n} \left| \sum_{j=1}^r c_j e^{-i\xi_j \cdot x} \right|^2 d\mu(x) \\
&\geq 0.
\end{aligned}$$

□

The following proof of Bochner's theorem follows an exercise in Rudin.<sup>29</sup>

**Theorem 19** (Bochner). *If  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  is continuous and positive-definite, then there is some finite positive Borel measure  $\nu$  on  $\mathbb{R}^n$  for which  $\phi = \hat{\nu}$ .*

*Proof.* Let  $A$  be the Banach algebra defined in §4, whose elements are those complex Borel measures  $\mu$  on  $\mathbb{R}^n$  for which there is some  $f \in L^1(\mathbb{R}^n)$  and some  $\alpha \in \mathbb{C}$  such that

$$d\mu = f dm_n + \alpha d\delta,$$

where  $m_n$  is Lebesgue measure on  $\mathbb{R}^n$ . For  $f + \alpha\delta, g + \beta\delta \in A$ , we have

$$(f + \alpha\delta) * (g + \beta\delta) = (f * g + \beta f + \alpha g) + \alpha\beta\delta;$$

we are identifying  $f \in L^1(\mathbb{R}^n)$  with the complex Borel measure whose Radon-Nikodym derivative with respect to  $m_n$  is  $f$ . The norm on  $A$  is the total variation norm of a complex measure; one checks that for  $f + \alpha\delta$  this is

$$\|f + \alpha\delta\| = \|f\| + |\alpha|,$$

where  $\|f\| = \int_{\mathbb{R}^n} |f(x)| dm_n(x)$ .

For  $f \in L^1(\mathbb{R}^n)$ , we define  $\tilde{f} \in L^1(\mathbb{R}^n)$  by  $\tilde{f}(x) = \overline{f(-x)}$ , and we define  $*$  :  $A \rightarrow A$  by

$$(f + \alpha\delta)^* = \tilde{f} + \bar{\alpha}\delta, \quad f + \alpha\delta \in A.$$

<sup>29</sup>Walter Rudin, *Functional Analysis*, second ed., p. 303, Exercise 14. Other references on Bochner's theorem are the following: Barry Simon, *Convexity: An Analytic Viewpoint*, p. 153, Theorem 9.17; Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis*, vol. II, p. 293, Theorem 33.3; Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 220, Theorem 3.9.16; Walter Rudin, *Fourier Analysis on Groups*, p. 19, Theorem 1.4.3; Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 170; Vladimir I. Bogachev, *Measure Theory*, vol. II, p. 121, Theorem 7.13.1; Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 95, Theorem 4.18.

On the one hand,

$$\begin{aligned}
((f + \alpha\delta) * (g + \beta\delta))^* &= ((f * g + \beta f + \alpha g) + \alpha\beta\delta)^* \\
&= \widetilde{f * g} + \overline{\beta f} + \overline{\alpha g} + \overline{\alpha\beta\delta} \\
&= \widetilde{f * g} + \overline{\beta f} + \overline{\alpha g} + \overline{\alpha\beta\delta},
\end{aligned}$$

and

$$\widetilde{f * g}(x) = \overline{\int_{\mathbb{R}^n} f(y)g(-x - y)dm_n(y)}.$$

On the other hand,

$$\begin{aligned}
(g + \beta\delta)^* * (f + \alpha\delta)^* &= (\widetilde{g} + \overline{\beta\delta}) * (\widetilde{f} + \overline{\alpha\delta}) \\
&= (\widetilde{g} * \widetilde{f} + \overline{\alpha\widetilde{g}} + \overline{\beta\widetilde{f}}) + \overline{\beta\alpha\delta},
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{g} * \widetilde{f})(x) &= \int_{\mathbb{R}^n} \widetilde{g}(y)\widetilde{f}(x - y)dm_n(y) \\
&= \int_{\mathbb{R}^n} \overline{g(-y)f(-x + y)}dm_n(y) \\
&= \overline{\int_{\mathbb{R}^n} g(-y - x)f(y)dm_n(y)}.
\end{aligned}$$

Therefore we have

$$((f + \alpha\delta) * (g + \beta\delta))^* = (g + \beta\delta)^* * (f + \alpha\delta)^*.$$

Thus  $*$  :  $A \rightarrow A$  is an involution (the other properties demanded of an involution are immediate).

We define  $F : A \rightarrow \mathbb{C}$  by

$$F(f + \alpha\delta) = \int_{\mathbb{R}^n} f\phi dm_n + \alpha\phi(0), \quad f + \alpha\delta \in A.$$

It is apparent that  $F$  is linear, and because  $|\phi(x)| \leq \phi(0)$  for all  $x$ ,

$$\begin{aligned}
|F(f + \alpha\delta)| &\leq \left| \int_{\mathbb{R}^n} f\phi dm_n \right| + |\alpha\phi(0)| \\
&\leq \int_{\mathbb{R}^n} |f||\phi| dm_n + |\alpha\phi(0)| \\
&\leq \phi(0) \int_{\mathbb{R}^n} |f| dm_n + |\alpha\phi(0)| \\
&= \phi(0) \|f + \alpha\delta\|,
\end{aligned}$$

from which it follows that  $\|F\| = \phi(0)$ , and in particular that  $F$  is bounded. Let  $A_0 = \{f + \alpha\delta \in A : f \in C_c(\mathbb{R}^n)\}$ . Because  $F : A \rightarrow \mathbb{C}$  is bounded and  $A_0$

is a dense subset of  $A$ , to prove that  $F$  is a positive linear functional it suffices to prove that for all  $f + \alpha\delta \in A_0$  we have  $F((f + \alpha\delta) * (f + \alpha\delta)^*) \geq 0$ .

For  $g \in C_c(\mathbb{R}^n)$ , by Theorem 17 we obtain

$$F(g * g^*) = F(g * \tilde{g}) = \int_{\mathbb{R}^n} (g * \tilde{g}) \phi dm_n \geq 0. \quad (10)$$

Define  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\eta(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

and for  $\epsilon > 0$ , define  $\eta_\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\eta_\epsilon(x) = \epsilon^{-n} \eta\left(\frac{x}{\epsilon}\right)$ . Let  $f + \alpha\delta \in A_0$  and define  $g_\epsilon = f + \alpha\eta_\epsilon \in C_c(\mathbb{R}^n)$ . From (10) we have  $F(g_\epsilon * g_\epsilon^*) \geq 0$  for any  $\epsilon > 0$ . On the other hand,

$$\begin{aligned} F(g_\epsilon * g_\epsilon^*) &= F((f + \alpha\eta_\epsilon) * (\tilde{f} + \bar{\alpha}\eta_\epsilon)) \\ &= F(f * \tilde{f} + \bar{\alpha}f * \eta_\epsilon + \alpha\eta_\epsilon * \tilde{f} + |\alpha|^2 \eta_\epsilon * \eta_\epsilon) \\ &= F(f * \tilde{f}) + \bar{\alpha} \int_{\mathbb{R}^n} (f * \eta_\epsilon) \phi dm_n + \alpha \int_{\mathbb{R}^n} (\eta_\epsilon * \tilde{f}) \phi dm_n \\ &\quad + |\alpha|^2 \int_{\mathbb{R}^n} (\eta_\epsilon * \eta_\epsilon) \phi dm_n. \end{aligned}$$

We take as granted that

$$\int_{\mathbb{R}^n} (f * \eta_\epsilon) \phi dm_n \rightarrow \int_{\mathbb{R}^n} f \phi dm_n$$

as  $\epsilon \rightarrow 0$ , that

$$\int_{\mathbb{R}^n} (\eta_\epsilon * \tilde{f}) \phi dm_n \rightarrow \int_{\mathbb{R}^n} \tilde{f} \phi dm_n$$

as  $\epsilon \rightarrow 0$ , and that

$$\int_{\mathbb{R}^n} (\eta_\epsilon * \eta_\epsilon) \phi dm_n \rightarrow \phi(0)$$

as  $\epsilon \rightarrow 0$ . Furthermore,

$$\begin{aligned} F((f + \alpha\delta) * (f + \alpha\delta)^*) &= F((f + \alpha\delta) * (\tilde{f} + \bar{\alpha}\delta)) \\ &= F(f * \tilde{f} + \bar{\alpha}f + \alpha\tilde{f} + |\alpha|^2) \end{aligned}$$

Thus

$$F(g_\epsilon * g_\epsilon^*) \rightarrow F((f + \alpha\delta) * (f + \alpha\delta)^*)$$

as  $\epsilon \rightarrow 0$ . Since  $F(g_\epsilon * g_\epsilon^*) \geq 0$  for all  $\epsilon > 0$ , it follows that

$$F((f + \alpha\delta) * (f + \alpha\delta)^*) \geq 0.$$

Therefore,  $F : A \rightarrow \mathbb{C}$  is a positive linear functional.

Because  $F$  is a positive linear functional and  $F(e) = F(\delta) = 1$ , we can apply Theorem 12, according to which there is a regular positive Borel measure  $\mu$  on  $\Delta$  satisfying

$$F(f + \alpha\delta) = \int_{\Delta} \Gamma(f + \alpha\delta) d\mu, \quad f + \alpha\delta \in A,$$

and hence, from the definition of  $F$ ,

$$\int_{\mathbb{R}^n} f \phi dm_n + \alpha\phi(0) = \int_{\Delta} \Gamma(f + \alpha\delta) d\mu, \quad f + \alpha \in A.$$

We state the following again from §4 for easy access. If  $t \in \mathbb{R}^n$ , define  $h_t : A \rightarrow \mathbb{C}$  by

$$h_t(f + \alpha\delta) = \hat{f}(t) + \alpha, \quad f + \alpha\delta \in A,$$

and also define  $h_{\infty} : A \rightarrow \mathbb{C}$  by

$$h_{\infty}(f + \alpha\delta) = \alpha, \quad f + \alpha\delta \in A.$$

Let  $\mathbb{R}^n \cup \{\infty\}$  be the one-point compactification of  $\mathbb{R}^n$ . We proved in §4 that the map  $T : \mathbb{R}^n \cup \{\infty\} \rightarrow \Delta$  defined by  $T(t) = h_t$  is a homeomorphism. With  $\nu = (T^{-1})_*\mu$  we have  $\mu = T_*\nu$ , and then

$$\begin{aligned} \int_{\Delta} \Gamma(f + \alpha\delta) d\mu &= \int_{\Delta} \Gamma(f) d\mu + \alpha \int_{\Delta} \Gamma(\delta) d\mu \\ &= \int_{\Delta} \Gamma(f) d(T_*\nu) + \alpha \int_{\Delta} \chi_{\Delta} d\mu \\ &= \int_{\mathbb{R}^n \cup \{\infty\}} \Gamma(f) \circ T d\nu + \alpha\mu(\Delta) \\ &= \int_{\mathbb{R}^n \cup \{\infty\}} \Gamma(f) \circ T(t) d\nu(t) + \alpha F(\delta) \\ &= \int_{\mathbb{R}^n \cup \{\infty\}} h_t(f) d\nu(t) + \alpha\phi(0) \\ &= \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t) + \alpha\phi(0). \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^n} f \phi dm_n + \alpha\phi(0) = \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t) + \alpha\phi(0),$$

i.e.

$$\int_{\mathbb{R}^n} f \phi dm_n = \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t).$$

As

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t) &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-it \cdot x} f(x) dm_n(x) \right) d\nu(t) \\ &= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{-ix \cdot t} d\nu(t) dm_n(x) \\ &= \int_{\mathbb{R}^n} f(x) \hat{\nu}(x) dm_n(x), \end{aligned}$$

we have

$$\int_{\mathbb{R}^n} f\phi dm_n = \int_{\mathbb{R}^n} f\hat{\nu} dm_n.$$

This is true for all  $f \in L^1(\mathbb{R}^n)$ , from which it follows that  $\phi = \hat{\nu}$ . □