The p-adic solenoid

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1 Definition

We shall be speaking about locally compact abelian groups, and unless we say otherwise, by **morphism** we mean a continuous group homomorphism.

For p prime and $n \in \mathbb{Z}_{\geq 0}$, $p^n\mathbb{Z}$ is a closed subgroup of the locally compact abelian group $\mathbb{R}$, and the quotient $\mathbb{R}/p^n\mathbb{Z}$ is a compact abelian group. For $n \geq m$, let $\phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$ be the projection map, which is a morphism. The compact abelian groups $\mathbb{R}/p^n\mathbb{Z}$ and the morphisms $\phi_{n,m}$ are an inverse system, and the inverse limit is a compact abelian group denoted $\mathbb{T}_p$, called the **p-adic solenoid**, with morphisms $\phi_n : \mathbb{T}_p \to \mathbb{R}/p^n\mathbb{Z}$. Because the maps $\phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$ are surjective, the maps $\phi_n : \mathbb{T}_p \to \mathbb{R}/p^n\mathbb{Z}$ are surjective.[1]

Let $\pi_n : \mathbb{R} \to \mathbb{R}/p^n\mathbb{Z}$ be the projection map, which is a morphism. The projection maps π_n are compatible with the inverse system $\phi_{n,m}$, so there is a unique morphism $\pi : \mathbb{R} \to \mathbb{T}_p$ such that $\phi_n \circ \pi = \pi_n$ for all $n \in \mathbb{Z}_{\geq 0}$. If $x, y \in \mathbb{R}$ are distinct, then for sufficiently large n we have $\pi_n(x) \neq \pi_n(y)$. If $\pi(x) = \pi(y)$ then $\pi_n(x) = \phi_n(\pi(x)) = \phi_n(\pi(y)) = \pi_n(y)$, a contradiction. Therefore $\pi : \mathbb{R} \to \mathbb{T}_p$ is injective. Furthermore, the maps $\pi_n : \mathbb{R} \to \mathbb{R}/p^n\mathbb{Z}$ being surjective implies that the image $\pi(\mathbb{R})$ is dense in $\mathbb{T}_p$.[2]

2 Pontryagin dual

If G is a locally compact abelian group, we denote by G^* the collection of morphisms $G \to S^1$. We assign G^* the coarsest topology such that for all $g \in G$, the map $\gamma \mapsto \gamma(x)$ is continuous $G^* \to S^1$, and with this topology, G^* is a locally compact abelian group, called the **Pontryagin dual of** G.

If $\phi : G \to H$ is a morpism of locally compact abelian groups, then $\phi^* : H^* \to G^*$ defined by

$$\phi^*(\theta)(g) = \theta(\phi(g)), \qquad \theta \in H^*, \quad g \in G,$$

[1] Alain M. Robert, *A Course in p-adic Analysis*, Chapter 1, §4, p. 29.
[2] Luis Ribes and Pavel Zalesskii, *Profinite Groups*, p. 7, Lemma 1.1.7.

is a morphism. Say ϕ is surjective, and $\phi^*(\theta_1) = \phi^*(\theta_2)$ but that $\theta_1 \neq \theta_2$. Then there is some $h \in H$ such that $\theta_1(h) \neq \theta_2(h)$. Since $\phi : G \to H$ is surjective, there is some $g \in G$ such that $\phi(g) = h$. But then

$$\theta_1(h) = \theta_2(\phi(g)) = \phi^*(\theta_1)(g) = \phi^*(\theta_2)(g) = \theta_2(\phi(g)) = \theta_2(h),$$

contradicting $\theta_1(h) \neq \theta_2(h)$. Therefore, if $\phi : G \to H$ is surjective then $\phi^* : H^* \to G^*$ is injective.

Let

$$\frac{1}{p^n}\mathbb{Z} = \left\{ \frac{j}{p^n} : j \in \mathbb{Z} \right\} \subset \mathbb{Q},$$

which with the discrete topology is a discrete abelian group.

Theorem 1. *For prime p and $n \in \mathbb{Z}_{\geq 0}$, the map $\Phi_n : \frac{1}{p^n}\mathbb{Z} \to (\mathbb{R}/p^n\mathbb{Z})^*$ defined by*

$$\Phi_n(a)(x + p^n\mathbb{Z}) = e^{2\pi i a x}, \qquad a \in \frac{1}{p^n}\mathbb{Z}, \quad x + p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z},$$

is an isomorphism of topological groups.

Proof. Write $a = \frac{j}{p^k}$, $j \in \mathbb{Z}$. If $x + p^n\mathbb{Z} = y + p^n\mathbb{Z}$, then $x - y \in p^n\mathbb{Z}$, so $x - y = p^n k$ for some $k \in \mathbb{Z}$. Then

$$\Phi_n(a)(x+p^n\mathbb{Z}) = e^{2\pi i a x} = e^{2\pi i \frac{j}{p^n}(p^n k + y)} = e^{2\pi i k + 2\pi i \frac{j}{p^n} y} = e^{2\pi i a y} = \Phi_n(a)(y+p^n\mathbb{Z}),$$

showing that Φ_n is well-defined. Furthermore, one checks that indeed $\Phi_n(a) \in (\mathbb{R}/p^n\mathbb{Z})^*$ for each $a \in \frac{1}{p^n}\mathbb{Z}$.

It is apparent that $\Phi_n(a + b) = \Phi_n(a) \cdot \Phi_n(b)$. Φ is continuous because $\frac{1}{p^n}\mathbb{Z}$ is discrete. If $\Phi_n(a) = \Phi_n(b)$, this means that for all $x + p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z}$, $e^{2\pi i a x} = e^{2\pi i b x}$, equivalently, that $(a - b)x \in \mathbb{Z}$ for all $x \in \mathbb{R}$, whence $a = b$. Thus Φ_n is injective.

Let $\gamma \in (\mathbb{R}/p^n\mathbb{Z})^*$. Define $\Gamma : \mathbb{R} \to S^1$ by $\Gamma = \gamma \circ \pi_n$, so that $\Gamma \in \mathbb{R}^*$. We take as given that because $\Gamma \in \mathbb{R}^*$, there is some $y \in \mathbb{R}$ such that $\Gamma(x) = e^{2\pi i y x}$ for all $x \in \mathbb{R}$. In particular, for $x = p^n$, on the one hand

$$\Gamma(p^n) = \gamma(\pi_n(p^n)) = \gamma(0 + p^n\mathbb{Z}) = 1,$$

and on the other hand

$$\Gamma(p^n) = e^{2\pi i y p^n},$$

so $yp^n \in \mathbb{Z}$, i.e. $y \in \frac{1}{p^n}\mathbb{Z}$, and it follows that $\gamma = \Phi_n(y)$. Therefore Φ_n is surjective.

The **open mapping theorem for topological groups** states that if G, H are locally compact groups, $f : G \to H$ is a surjective morphism, and G is σ-compact, then f is open. $\mathbb{Z}$ is discrete and countable, hence is σ-compact, so Φ_n is open. Therefore Φ_n is an isomorphism of topological groups. $\square$

Because the morphisms $\phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$ are surjective, the morphisms $\phi_{n,m}^* : (\mathbb{R}/p^m\mathbb{Z})^* \to (\mathbb{R}/p^n\mathbb{Z})^*$ are injective. For $m \leq n$, define $\iota_{m,n} : \frac{1}{p^m}\mathbb{Z} \to \frac{1}{p^n}\mathbb{Z}$ by $\iota\left(\frac{j}{p^m}\right) = \frac{j}{p^m} = \frac{p^{n-m}j}{p^n} \in \frac{1}{p^n}\mathbb{Z}$; this is an injective morphism. One checks that the following diagram commutes.

$$\begin{tikzcd}
(\mathbb{R}/p^m\mathbb{Z})^* \arrow[r, "\phi_{n,m}^*"] & (\mathbb{R}/p^n\mathbb{Z})^* \\
\frac{1}{p^m}\mathbb{Z} \arrow[u, "\Phi_m"] \arrow[r, "\iota_{m,n}"] & \frac{1}{p^n}\mathbb{Z} \arrow[u, "\Phi_n"]
\end{tikzcd}$$

The discrete groups $\frac{1}{p^m}\mathbb{Z}$ and the morphisms $\iota_{m,n}$ are a direct system. The **localization of $\mathbb{Z}$ away from** p is the abelian group

$$\mathbb{Z}[1/p] = \left\{ \frac{j}{p^m} : j \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}.$$

We assign $\mathbb{Z}[1/p]$ the discrete topology. One proves that $\mathbb{Z}[1/p]$ with the maps $\iota_m : \frac{1}{p^m}\mathbb{Z} \to \mathbb{Z}[1/p]$ defined by

$$\iota_m\left(\frac{j}{p^m}\right) = \frac{j}{p^m}$$

is the direct limit of this direct system.[3] The direct system $\iota_{m,n} : \frac{1}{p^m}\mathbb{Z} \to \frac{1}{p^n}\mathbb{Z}$ is dual to the inverse system $\phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$. It follows that the Pontryagin dual of the limit of either system is isomorphic as a topological group to the limit of the other system. That is,

$$\mathbb{T}_p^* \cong \mathbb{Z}[1/p], \qquad (\mathbb{Z}[1/p])^* \cong \mathbb{T}_p,$$

as topological groups.

3 p-adic integers

For $n \geq m$, let $\psi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ be the projection map. With the discrete topology, $\mathbb{Z}/p^n\mathbb{Z}$ is a compact abelian group, as it is finite. Then $\psi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ is an inverse system, and its inverse limit is a compact abelian group denoted $\mathbb{Z}_p$, called the **p-adic integers**, with morphisms $\psi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$. Because the morphisms $\psi_{n,m}$ are surjective, the morphisms ψ_n are surjective.

Let $\lambda_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{R}/p^n\mathbb{Z}$ be the inclusion map. Then the morphisms $\Lambda_n = \lambda_n \circ \psi_n : \mathbb{Z}_p \to \mathbb{R}/p^n\mathbb{Z}$ are compatible with the inverse system $\phi_{n,m}$:

[3] A direct limit of discrete abelian groups is the direct limit of abelian groups. On direct limits of abelian groups, cf. Luis Ribes and Pavel Zalesskii, *Profinite Groups*, p. 15, Proposition 1.2.1.

$\mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$, so there is a unique morphism $\Lambda : \mathbb{Z}_p \to \mathbb{T}_p$ such that $\phi_n \circ \Lambda = \Lambda_n$ for all $n \in \mathbb{Z}_{\geq 0}$. Suppose that $x, y \in \mathbb{Z}_p$ are distinct and that $\Lambda(x) = \Lambda(y)$. It is a fact that there is some n such that $\psi_n(x) \neq \psi_n(y)$. Because λ_n is injective, this implies that $\Lambda_n(x) \neq \Lambda_n(y)$, and this contradicts that $\Lambda(x) = \Lambda(y)$. Therefore $\Lambda : \mathbb{Z}_p \to \mathbb{T}_p$ is injective.

It can be proved that $\ker \phi_0 = \Lambda(\mathbb{Z}_p)$, which implies that

$$0 \to \mathbb{Z}_p \to \mathbb{T}_p \to \mathbb{R}/\mathbb{Z} \to 0$$

is a short exact sequence of topological groups.[4]

It can be proved that for each $m \in \mathbb{Z}_{>0}$ such that $\gcd(m, p) = 1$, the p-adic solenoid $\mathbb{T}_p$ has a unique cyclic subgroup of order m, and on the other hand that there is no element in $\mathbb{T}_p$ whose order is a power of p, namely, $\mathbb{T}_p$ has no p-torsion.[5]

4 Further reading

Garrett has written several notes on the p-adic solenoid.[6] The p-adic solenoid occurs in several places in the books of Hofmann and Morris.[7] For properties of the p-adic solenoid involving homological algebra, see the below references.[8]

[4]Alain M. Robert, *A Course in p-adic Analysis*, Chapter 1, Appendix, p. 55.

[5]Alain M. Robert, *A Course in p-adic Analysis*, Chapter 1, Appendix, pp. 55–56.

[6]Paul Garrett, *Solenoids*, `http://www.math.umn.edu/~garrett/m/mfms/notes/02_solenoids.pdf`; Paul Garrett, *Bigger diagrams for solenoids, more automorphisms, colimits*, `http://www.math.umn.edu/~garrett/m/mfms/notes/03_more_autos.pdf`; Paul Garrett, *The ur-solenoid and the adeles*, `http://www.math.umn.edu/~garrett/m/mfms/notes/04_ur_solenoid.pdf`

[7]Karl H. Hofmann and Sidney A. Morris, *The Structure of Compact Groups*, 2nd revised and augmented edition; Karl H. Hofmann and Sidney A. Morris, *The Lie Theory of Connected Pro-Lie Groups*; see also H. Salzmann, T. Grundhöfer, H. Hähl, and R. Löwen, *The Classical Fields: Structural Features of the Real and Rational Numbers*, p. 99.

[8]For $\mathrm{Ext}(\mathbb{Z}, \mathbb{T}_p)$ see Jean Dieudonné, *A History of Algebraic and Differential Topology, 1900 – 1960*, p. 94; see also J. M. Cordier and T. Porter, *Shape Theory: Categorical Methods of Approximation*, p. 83; and `http://mathoverflow.net/questions/4478/torsion-in-homology-or-fundamental-group-of-subsets-of-euclidean-3-space`