

The Banach algebra of functions of bounded variation and the pointwise Helly selection theorem

Jordan Bell

January 22, 2015

1 $BV[a, b]$

Let $a < b$. For $f : [a, b] \rightarrow \mathbb{R}$, we define¹

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|,$$

and if $\|f\|_\infty < \infty$ we say that f is **bounded**. We define $B[a, b]$ to be the set of bounded functions $[a, b] \rightarrow \mathbb{R}$, which with the norm $\|\cdot\|_\infty$ is a Banach algebra.

A **partition** of $[a, b]$ is a set $P = \{t_0, \dots, t_n\}$ such that $a = t_0 < \dots < t_n = b$. For example, $P = \{a, b\}$ is a partition of $[a, b]$. If Q is a partition of $[a, b]$ and $P \subset Q$, we say that Q is a **refinement** of P . For $f : [a, b] \rightarrow \mathbb{R}$, we define

$$V(f, P) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})|.$$

It is straightforward to show using the triangle inequality that if Q is a refinement of P then

$$V(f, P) \leq V(f, Q).$$

In particular, any partition P is a refinement of $\{a, b\}$, so

$$|f(b) - f(a)| \leq V(f, P).$$

The **total variation** of $f : [a, b] \rightarrow \mathbb{R}$ is

$$V_a^b f = \sup\{V(f, P) : P \text{ is a partition of } [a, b]\},$$

and if $V_a^b f < \infty$ we say that f is of **bounded variation**. We denote by $BV[a, b]$ the set of functions $[a, b] \rightarrow \mathbb{R}$ of bounded variation. For a function $f \in BV[a, b]$, we define $v : [a, b] \rightarrow \mathbb{R}$ by $v(x) = V_a^x f$ for $x \in [a, b]$, called the **variation of f** .

¹In this note we speak about functions that take values in \mathbb{R} , because this makes it simpler to talk about monotone functions. Once the machinery is established we can then apply it to the real and imaginary parts of a function that takes values in \mathbb{C} .

If $f : [a, b] \rightarrow \mathbb{R}$ is monotone, it is straightforward to check that $V_a^b f = |f(b) - f(a)|$, hence that f is of bounded variation.

We first show that $BV[a, b] \subset B[a, b]$.

Lemma 1. *If $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation, then*

$$\|f\|_\infty \leq |f(a)| + V_a^b f.$$

Proof. Let $x \in [a, b]$. If $x = a$ the result is immediate. If $x = b$, then

$$|f(b)| \leq |f(a)| + |f(b) - f(a)| \leq |f(a)| + V_a^b f.$$

Otherwise, $P = \{a, x, b\}$ is a partition of $[a, b]$ and

$$|f(x) - f(a)| \leq V(f, P) \leq V_a^b f.$$

□

The total variation of functions has several properties. The following lemma and that fact that functions of bounded variation are bounded imply that $BV[a, b]$ is an algebra.²

Lemma 2. *If $f, g \in BV[a, b]$ and $c \in \mathbb{R}$, then the following statements are true.*

1. $V_a^b f = 0$ if and only if f is constant.
2. $V_a^b(cf) = |c|V_a^b(f)$.
3. $V_a^b(f + g) \leq V_a^b f + V_a^b g$.
4. $V_a^b(fg) \leq \|f\|_\infty V_a^b g + \|g\|_\infty V_a^b f$.
5. $V_a^b |f| \leq V_a^b f$.
6. $V_a^b f = V_a^x f + V_x^b f$ for $a \leq x \leq b$.

Lemma 3. *If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on (a, b) and $\|f'\|_\infty < \infty$, then*

$$V_a^b f \leq \|f'\|_\infty (b - a).$$

Proof. Suppose that $P = \{a = t_0 < \dots < t_n = b\}$ is a partition of $[a, b]$. By the mean value theorem, for each $j = 1, \dots, n$ there is some $x_j \in (t_{j-1}, t_j)$ at which

$$f'(x_j) = \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}}.$$

²N. L. Carothers, *Real Analysis*, p. 204, Lemma 13.3.

Then

$$\begin{aligned}
V(f, P) &= \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\
&= \sum_{j=1}^n (t_j - t_{j-1}) |f'(x_j)| \\
&\leq \|f'\|_{\infty} \sum_{j=1}^n (t_j - t_{j-1}) \\
&= \|f'\|_{\infty} (b - a).
\end{aligned}$$

□

Lemma 4. *If $f \in C^1[a, b]$, then*

$$V_a^b f \leq \int_a^b |f'(t)| dt.$$

Proof. Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a, b]$. Then, by the fundamental theorem of calculus,

$$\begin{aligned}
V(f, P) &= \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\
&\leq \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} f'(t) dt \right| \\
&\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |f'(t)| dt \\
&= \int_a^b |f'(t)| dt.
\end{aligned}$$

Therefore

$$V_a^b f = \sup_P V(f, P) \leq \int_a^b |f'(t)| dt.$$

□

Lemma 5. *If $f : [a, b] \rightarrow \mathbb{R}$ is a polynomial, then*

$$V_a^b f = \int_a^b |f'(t)| dt.$$

Proof. Because f is a polynomial, f' is also, so f' is piecewise monotone, say $f' = c_j|f'|$ on (t_{j-1}, t_j) for $j = 1, \dots, n$, for some $c_j \in \{+1, -1\}$ and $a = t_0 < \dots < t_n = b$. Then

$$\int_{t_{j-1}}^{t_j} |f'(t)| dt = c_j \int_{t_{j-1}}^{t_j} f'(t) dt = c_j(f(t_j) - f(t_{j-1})),$$

giving, because $t_0 < \dots < t_n$ is a partition of $[a, b]$,

$$\begin{aligned} \int_a^b |f'(t)| dt &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |f'(t)| dt \\ &= \sum_{j=1}^n c_j(f(t_j) - f(t_{j-1})) \\ &\leq \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\ &\leq V_a^b f. \end{aligned}$$

□

Lemma 6. *If f_m is a sequence of functions $[a, b] \rightarrow \mathbb{R}$ that converges pointwise to some $f : [a, b] \rightarrow \mathbb{R}$ and P is some partition of $[a, b]$, then*

$$V(f_m, P) \rightarrow V(f, P).$$

If f_m is a sequence in $BV[a, b]$ that converges pointwise to some $f : [a, b] \rightarrow \mathbb{R}$, then

$$V_a^b f \leq \liminf_{m \rightarrow \infty} V_a^b f_m.$$

Proof. Say $P = \{t_0, \dots, t_n\}$. Then, because taking the limit of convergent sequences is linear,

$$\begin{aligned} \lim_{m \rightarrow \infty} V(f_m, P) &= \lim_{m \rightarrow \infty} \sum_{j=1}^n |f_m(t_j) - f_m(t_{j-1})| \\ &= \sum_{j=1}^n \lim_{m \rightarrow \infty} |f_m(t_j) - f_m(t_{j-1})| \\ &= \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\ &= V(f, P). \end{aligned}$$

Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a, b]$. Then

$$\begin{aligned} V(f, P) &= \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\ &= \sum_{j=1}^n \lim_{m \rightarrow \infty} |f_m(t_j) - f_m(t_{j-1})| \\ &= \lim_{m \rightarrow \infty} V(f_m, P) \\ &\leq \liminf_{m \rightarrow \infty} V_a^b f_m. \end{aligned}$$

This is true for any partition P of $[a, b]$, which yields

$$V_a^b f \leq \liminf_{m \rightarrow \infty} V_a^b f_m.$$

□

We now prove that $BV[a, b]$ is a Banach space.³

Theorem 7. *With the norm*

$$\|f\|_{BV} = |f(a)| + V_a^b f.$$

$BV[a, b]$ is a Banach space.

Proof. Using Lemma 2, it is straightforward to check that $BV[a, b]$ is a normed linear space. Suppose that f_m is a Cauchy sequence in $BV[a, b]$. By Lemma 1 it follows that f_m is a Cauchy sequence in $B[a, b]$, and thus converges in $B[a, b]$ to some $f \in B[a, b]$.

Let P be a partition of $[a, b]$ and let $\epsilon > 0$. Because f_n is a Cauchy sequence in $BV[a, b]$, there is some N such that if $n, m \geq N$ then $\|f_m - f_n\|_{BV} < \epsilon$. For $n \geq N$, Lemma 6 yields

$$\begin{aligned} \|f - f_n\|_{BV} &\leq |f(a) - f_n(a)| + V(f - f_n, P) \\ &= \lim_{m \rightarrow \infty} (|f_m(a) - f_n(a)| + V(f_m - f_n, P)) \\ &\leq \sup_{m \geq N} (|f_m(a) - f_n(a)| + V(f_m - f_n, P)) \\ &= \sup_{m \geq N} \|f_m - f_n\|_{BV} \\ &\leq \epsilon. \end{aligned}$$

Because $f - f_N \in BV[a, b]$ and $f_N \in BV[a, b]$ and $BV[a, b]$ is an algebra, $f = (f - f_N) + f_N \in BV[a, b]$. That is, the Cauchy sequence f_n converges in $BV[a, b]$ to $f \in BV[a, b]$, showing that $BV[a, b]$ is a complete metric space and thus a Banach space. □

³N. L. Carothers, *Real Analysis*, p. 206, Theorem 13.4.

The following theorem shows that a function of bounded variation can be written as the difference of nondecreasing functions.⁴

Theorem 8. *Let $f \in BV[a, b]$ and let v be the variation of f . Then $v - f$ and v are nondecreasing.*

Proof. If $x, y \in [a, b]$, $x < y$, then, using Lemma 2,

$$\begin{aligned} v(y) - v(x) &= V_a^y f - V_a^x f \\ &= V_x^y f \\ &\geq |f(y) - f(x)| \\ &\geq f(y) - f(x). \end{aligned}$$

That is, $v(y) - f(y) \geq v(x) - f(x)$, showing that $v - f$ is nondecreasing, and because f is nondecreasing we have $f(y) - f(x) \geq 0$ and so $v(y) - v(x) \geq 0$. \square

The following theorem tells us that a function of bounded variation is right or left continuous at a point if and only if its variation is respectively right or left continuous at the point.⁵

Theorem 9. *Let $f \in BV[a, b]$ and let v be the variation of f . For $x \in [a, b]$, f is right (respectively left) continuous at x if and only if v is right (respectively left) continuous at x .*

Proof. Assume that v is right continuous at x . If $\epsilon > 0$, there is some $\delta > 0$ such that $x \leq y < x + \delta$ implies that $v(y) - v(x) = |v(y) - v(x)| < \epsilon$. If $x \leq y < x + \delta$, then

$$|f(y) - f(x)| \leq v(y) - v(x) < \epsilon,$$

showing that f is right continuous at x .

Assume that f is right continuous at x , with $a \leq x < b$. Let $\epsilon > 0$. There is some $\delta > 0$ such that $x \leq y < x + \delta$ implies that $|f(y) - f(x)| < \frac{\epsilon}{2}$. Because $V_x^b f$ is a supremum over partitions of $[x, b]$, there is some partition $P = \{t_0, t_1, \dots, t_n\}$ of $[x, b]$ such that $V_x^b f - \frac{\epsilon}{2} \leq V(f, P)$. Let $x \leq y < \min\{\delta, t_1 - x\}$. Then $Q = \{t_0, y, t_1, \dots, t_n\}$ is a refinement of P , so

$$\begin{aligned} V_x^b f - \frac{\epsilon}{2} &\leq V(f, P) \\ &\leq V(f, Q) \\ &= |f(y) - f(t_0)| + V(f, \{y, t_1, \dots, t_n\}) \\ &< \frac{\epsilon}{2} + V_y^b f. \end{aligned}$$

Hence

$$\epsilon > V_x^b f - V_y^b f = V_x^y f = v(y) - v(x) = |v(y) - v(x)|,$$

showing that v is right continuous at x . \square

⁴N. L. Carothers, *Real Analysis*, p. 207, Theorem 13.5.

⁵N. L. Carothers, *Real Analysis*, p. 207, Theorem 13.9.

For $f \in BV[a, b]$ and for v the variation of f , we define the **positive variation of f** as

$$p(x) = \frac{v(x) + f(x) - f(a)}{2}, \quad x \in [a, b],$$

and the **negative variation of f** as

$$n(x) = \frac{v(x) - f(x) + f(a)}{2}, \quad x \in [a, b].$$

We can write the variation as $v = p + n$. We now establish properties of the positive and negative variations.⁶

Theorem 10. *Let $f \in BV[a, b]$, let v be its variation, let p be its positive variation, and let n be its negative variation. Then $0 \leq p \leq v$ and $0 \leq n \leq v$, and p and n are nondecreasing.*

Proof. For $x \in [a, b]$, $v(x) = V_a^x f \geq |f(x) - f(a)|$. Because $v(x) \geq -(f(x) - f(a))$, we have $p(x) \geq 0$, and because $v(x) \geq f(x) - f(a)$ we have $n(x) \geq 0$. And then $v = p + n$ implies that $p \leq v$ and $n \leq v$.

For $x < y$,

$$\begin{aligned} p(y) - p(x) &= \frac{v(y) + f(y) - v(x) - f(x)}{2} \\ &= \frac{1}{2} (V_x^y f + (f(y) - f(x))) \\ &\geq \frac{1}{2} (|f(y) - f(x)| + (f(y) - f(x))) \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} n(y) - n(x) &= \frac{v(y) - f(y) - v(x) + f(x)}{2} \\ &= \frac{1}{2} (V_x^y f - (f(y) - f(x))) \\ &\geq \frac{1}{2} (|f(y) - f(x)| - (f(y) - f(x))) \\ &\geq 0. \end{aligned}$$

□

We now prove that $BV[a, b]$ is a Banach algebra.⁷

Theorem 11. *$BV[a, b]$ is a Banach algebra.*

⁶N. L. Carothers, *Real Analysis*, p. 209, Proposition 13.11.

⁷N. L. Carothers, *Real Analysis*, p. 209, Proposition 13.12.

Proof. For $f_1, f_2 \in BV[a, b]$, let $v_1, v_2, p_1, p_2, n_1, n_2$ be their variations, positive variations, and negative variations, respectively. Then

$$\begin{aligned} f_1 f_2 &= (f_1(a) + p_1 - n_1)(f_2(a) + p_2 - n_2) \\ &= f_1(a)f_2(a) + p_1 p_2 + n_1 n_2 - n_1 p_2 - n_2 p_1 \\ &\quad + f_1(a)p_2 + f_2(a)p_1 - f_1(a)n_2 - f_2(a)n_1. \end{aligned}$$

Using this and the fact that if f is nondecreasing then $V_a^b f = f(b) - f(a)$,

$$\begin{aligned} \|f_1 f_2\|_{BV} &= |f_1(a)||f_2(a)| + V_a^b(f_1 f_2) \\ &\leq |f_1(a)||f_2(a)| + V_a^b(p_1 p_2) + V_a^b(n_1 n_2) + V_a^b(n_1 p_2) + V_a^b(n_2 p_1) \\ &\quad + |f_1(a)|V_a^b p_2 + |f_2(a)|V_a^b p_1 + |f_1(a)|V_a^b n_2 + |f_2(a)|V_a^b n_1 \\ &= |f_1(a)||f_2(a)| + p_1(b)p_2(b) + n_1(b)n_2(b) + n_1(b)p_2(b) + n_2(b)p_1(b) \\ &\quad + |f_1(a)|p_2(b) + |f_2(a)|p_1(b) + |f_1(a)|n_2(b) + |f_2(a)|n_1(b) \\ &= (|f_1(a)| + p_1(b) + n_1(b))(|f_2(a)| + p_2(b) + n_2(b)) \\ &= (|f_1(a) + v_1(b)|)(|f_2(a) + v_2(b)|) \\ &= \|f_1\|_{BV} \|f_2\|_{BV}, \end{aligned}$$

which shows that $BV[a, b]$ is a normed algebra. And $BV[a, b]$ is a Banach space, so $BV[a, b]$ is a Banach algebra. \square

Theorem 12. If $f \in C^1[a, b]$, then

$$V_a^b f = \int_a^b |f'(t)| dt.$$

Let $(f')^+$ and $(f')^-$ be the positive and negative parts of f' and let p and n be the positive and negative variations of f . Then, for $x \in [a, b]$,

$$p(x) = \int_a^x (f')^+(t) dt, \quad n(x) = \int_a^x (f')^-(t) dt.$$

Proof. Lemma 4 states that $V_a^b \leq \int_a^b |f'(t)| dt$. Because f' is continuous it is Riemann integrable, hence for any $\epsilon > 0$ there is some partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ such that if $x_j \in [t_{j-1}, t_j]$ for $j = 1, \dots, n$ then

$$\left| \int_a^b |f'(t)| dt - \sum_{j=1}^n |f'(x_j)|(t_j - t_{j-1}) \right| < \epsilon.$$

By the mean value theorem, for each $j = 1, \dots, n$ there is some $x_j \in (t_{j-1}, t_j)$ such that $f'(x_j) = \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}}$. Then

$$V(f, P) = \sum_{j=1}^n |f(t_j) - f(t_{j-1})| = \sum_{j=1}^n |f'(x_j)|(t_j - t_{j-1}),$$

so

$$\left| \int_a^b |f'(t)| dt - V(f, P) \right| < \epsilon,$$

and thus

$$\int_a^b |f'(t)| dt < V(f, P) + \epsilon \leq V_a^b f + \epsilon.$$

This is true for all $\epsilon > 0$, therefore

$$\int_a^b |f'(t)| dt \leq V_a^b f,$$

which is what we wanted to show.

Write

$$g(t) = (f')^+(t) = \max\{f'(t), 0\}, \quad h(t) = (f')^-(t) = -\min\{f'(t), 0\}.$$

These satisfy $g + h = |f'|$ and $g - h = f'$. Using the fundamental theorem of calculus,

$$\begin{aligned} p(x) &= \frac{1}{2} (v(x) + f(x) - f(a)) \\ &= \frac{1}{2} \left(V_a^x f + \int_a^x f'(t) dt \right) \\ &= \frac{1}{2} \left(\int_a^b |f'(t)| dt + \int_a^b f'(t) dt \right) \\ &= \int_a^b g(t) dt \end{aligned}$$

and

$$\begin{aligned} n(x) &= \frac{1}{2} (v(x) - f(x) + f(a)) \\ &= \frac{1}{2} \left(V_a^x f - \int_a^x f'(t) dt \right) \\ &= \frac{1}{2} \left(\int_a^x |f'(t)| dt - \int_a^x f'(t) dt \right) \\ &= \int_a^x h(t) dt. \end{aligned}$$

□

2 Helly's selection theorem

We will use the following lemmas in the proof of the Helly selection theorem.⁸

⁸N. L. Carothers, *Real Analysis*, p. 210, Theorem 13.13; p. 211, Lemma 13.14; p. 211, Lemma 13.15.

Lemma 13. Suppose that X is a set, that $f_n : X \rightarrow \mathbb{R}$ is a sequence of functions, and that there is some K such that $\|f_n\|_\infty \leq K$ for all n . If D is a countable subset of X , then there is a subsequence of f_n that converges pointwise on D to some $\phi : D \rightarrow \mathbb{R}$, which satisfies $\|\phi\|_\infty \leq K$.

Proof. Say $D = \{x_k : k \geq 1\}$. Write $f_n^0 = f_n$. The sequence of real numbers $f_n^0(x_1)$ satisfies $f_n^0(x_1) \in [-K, K]$ for all n , and since the set $[-K, K]$ is compact there is a subsequence $f_n^1(x_1)$ of $f_n^0(x_1)$ that converges, say to $\phi(x_1) \in [-K, K]$. Suppose that $f_n^m(x_m)$ is a subsequence of $f_n^{m-1}(x_m)$ that converges to $\phi(x_m) \in [-K, K]$. Then the sequence of real numbers $f_n^m(x_{m+1})$ satisfies $f_n^m(x_{m+1}) \in [-K, K]$ for all n , and so there is a subsequence $f_n^{m+1}(x_{m+1})$ of $f_n^m(x_{m+1})$ that converges, say to $\phi(x_{m+1}) \in [-K, K]$. Let $k \geq 1$. Then one checks that $f_n^n(x_k) \rightarrow \phi(x_k)$ as $n \rightarrow \infty$, namely, f_n^n is a subsequence of f_n that converges pointwise on D to ϕ , and for each k we have $\phi(x_k) \in [-K, K]$. \square

Lemma 14. Let $D \subset [a, b]$ with $a \in D$ and $b = \sup D$. If $\phi : D \rightarrow \mathbb{R}$ is nondecreasing, then $\Phi : [a, b] \rightarrow \mathbb{R}$ defined by

$$\Phi(x) = \sup\{\phi(t) : t \in [a, x] \cap D\}$$

is nondecreasing and the restriction of Φ to D is equal to ϕ .

Lemma 15. If $f_n : [a, b] \rightarrow \mathbb{R}$ is a sequence of nondecreasing functions and there is some K such that $\|f_n\|_\infty \leq K$ for all n , then there is a nondecreasing function $f : [a, b] \rightarrow \mathbb{R}$, satisfying $\|f\|_\infty \leq K$, and a subsequence of f_n that converges pointwise to f .

Proof. Let $D = (\mathbb{Q} \cap [a, b]) \cup \{a\}$. By Lemma 13, there is a function $\phi : D \rightarrow \mathbb{R}$ and a subsequence f_{a_n} of f_n that converges pointwise on D to ϕ , and $\|\phi\|_\infty \leq K$. Because each f_n is nondecreasing, if $x, y \in D$ and $x < y$ then

$$\phi(x) = \lim_{n \rightarrow \infty} f_{a_n}(x) \leq \lim_{n \rightarrow \infty} f_{a_n}(y) = \phi(y),$$

namely, ϕ is nondecreasing. D is a dense subset of $[a, b]$ and $a \in D$, so applying Lemma 14, there is a nondecreasing function $\Phi : [a, b] \rightarrow \mathbb{R}$ such that for $x \in D$,

$$\Phi(x) = \phi(x) = \lim_{n \rightarrow \infty} f_{a_n}(x).$$

Suppose that Φ is continuous at $x \in [a, b]$ and let $\epsilon > 0$. Using the fact that Φ is continuous at x , there are $p, q \in \mathbb{Q} \cap [a, b]$ such that $p < x < q$ and $\Phi(q) - \Phi(p) = |\Phi(q) - \Phi(p)| < \frac{\epsilon}{2}$. Because $p, q \in D$, $f_{a_n}(p) \rightarrow \Phi(p)$ and $f_{a_n}(q) \rightarrow \Phi(q)$, so there is some N such that $n \geq N$ implies that both $|f_{a_n}(p) - \Phi(p)| < \frac{\epsilon}{2}$ and $|f_{a_n}(q) - \Phi(q)| < \frac{\epsilon}{2}$. Then for $n \geq N$, because each function f_{a_n} is nondecreasing,

$$\begin{aligned} f_{a_n}(x) &\geq f_{a_n}(p) \\ &\geq \Phi(p) - \frac{\epsilon}{2} \\ &\geq \Phi(q) - \epsilon \\ &\geq \Phi(x) - \epsilon. \end{aligned}$$

Likewise, for $n \geq N$,

$$\begin{aligned} f_{a_n}(x) &\leq f_{a_n}(q) \\ &\leq \Phi(q) + \frac{\epsilon}{2} \\ &< \Phi(p) + \epsilon \\ &\leq \Phi(x) + \epsilon. \end{aligned}$$

This shows that if Φ is continuous at $x \in [a, b]$ then $f_{a_n}(x) \rightarrow \Phi(x)$.

Let $D(\Phi)$ be the collection of those $x \in [a, b]$ such that Φ is not continuous at x . Because Φ is monotone, $D(\Phi)$ is countable. So we have established that if $x \in [a, b] \setminus D(\Phi)$ then $f_{a_n}(x) \rightarrow \Phi(x)$. Because $f_{a_n} : [a, b] \rightarrow \mathbb{R}$ satisfies $\|f_{a_n}\|_\infty \leq K$ and $D(\Phi)$ is countable, Lemma 13 tells us that there is a function $F : D \rightarrow \mathbb{R}$ and a subsequence f_{b_n} of f_{a_n} such that f_{b_n} converges pointwise on D to F , and $\|F\|_\infty \leq K$. We define $f : [a, b] \rightarrow \mathbb{R}$ by $f(x) = \Phi(x)$ for $x \notin D(\Phi)$ and $f(x) = F(x)$ for $x \in D(\Phi)$. $\|f\|_\infty \leq K$. For $x \notin D(\Phi)$, $f_{a_n}(x)$ converges to $\Phi(x) = f(x)$, and $f_{b_n}(x)$ is a subsequence of $f_{a_n}(x)$ so $f_{b_n}(x)$ converges to $f(x)$. For $x \in D(\Phi)$, $f_{b_n}(x)$ converges to $F(x) = f(x)$. Therefore, for any $x \in [a, b]$ we have that $f_{b_n}(x) \rightarrow f(x)$, namely, f_{b_n} converges pointwise to f . Because each function f_{b_n} is nondecreasing, it follows that f is nondecreasing. \square

Finally we prove the **pointwise Helly selection theorem**.⁹

Theorem 16. *Let f_n be a sequence in $BV[a, b]$ and suppose there is some K with $\|f_n\|_{BV} \leq K$ for all n . There is some subsequence of f_n that converges pointwise to some $f \in BV[a, b]$, satisfying $\|f\|_{BV} \leq K$.*

Proof. Let v_n be the variation of f_n . This satisfies, for any n ,

$$\|v_n\|_\infty = V_a^b f_n \leq K$$

and

$$\|v_n - f_n\|_\infty \leq \|v_n\|_\infty + \|f_n\|_\infty \leq K + \|f_n\|_{BV} \leq 2K.$$

Theorem 8 tells us that $v_n - f_n$ and v_n are nondecreasing, so we can apply Lemma 15 to get that there is a nondecreasing function $g : [a, b] \rightarrow \mathbb{R}$ and a subsequence $v_{a_n} - f_{a_n}$ of $v_n - f_n$ that converges pointwise to g . Then we use Lemma 15 again to get that there is a nondecreasing function $h : [a, b] \rightarrow \mathbb{R}$ and a subsequence v_{b_n} of v_{a_n} that converges pointwise to h . Because g and h are pointwise limits of nondecreasing functions, they are each nondecreasing and so belong to $BV[a, b]$. We define $f = h - g \in BV[a, b]$. For $x \in [a, b]$,

$$\begin{aligned} \lim_{n \rightarrow \infty} f_{b_n}(x) &= \lim_{n \rightarrow \infty} v_{b_n}(x) - \lim_{n \rightarrow \infty} (v_{b_n}(x) - f_{b_n}(x)) \\ &= h(x) - g(x) \\ &= f(x), \end{aligned}$$

⁹N. L. Carothers, *Real Analysis*, p. 212, Theorem 13.16.

namely the subsequence f_{b_n} of f_n converges pointwise to f . By Lemma 6, because f_{b_n} is a sequence in $BV[a, b]$ that converges pointwise to f we have

$$\begin{aligned}
\|f\|_{BV} &= |f(a)| + V_a^b f \\
&\leq |f(a)| + \liminf_{n \rightarrow \infty} V_a^b f_{b_n} \\
&= \liminf_{n \rightarrow \infty} (|f_{b_n}(a)| + V_a^b f_{b_n}) \\
&= \liminf_{n \rightarrow \infty} \|f_{b_n}\|_{BV} \\
&\leq K,
\end{aligned}$$

completing the proof. □