The Bernstein and Nikolsky inequalities for trigonometric polynomials

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January 28, 2015

1 Introduction

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For a function $f : \mathbb{T} \to \mathbb{C}$ and $\tau \in \mathbb{T}$, we define $f_\tau : \mathbb{T} \to \mathbb{C}$ by $f_\tau(t) = f(t - \tau)$. For measurable $f : \mathbb{T} \to \mathbb{C}$ and $0 < r < \infty$, write

$$\|f\|_r = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^r dt \right)^{1/r}.$$

For $f, g \in L^1(\mathbb{T})$, write

$$(f * g)(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) g(x - t) dt, \qquad x \in \mathbb{T},$$

and for $f \in L^1(\mathbb{T})$, write

$$\hat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt} dt, \qquad k \in \mathbb{Z}.$$

This note works out proofs of some inequalities involving the support of $\hat{f}$ for $f \in L^1(\mathbb{T})$.

Let $\mathscr{T}_n$ be the set of trigonometric polynomials of degree $\leq n$. We define the **Dirichlet kernel** $D_n : \mathbb{T} \to \mathbb{C}$ by

$$D_n(t) = \sum_{|j| \leq n} e^{ijt}, \qquad t \in \mathbb{T}.$$

It is straightforward to check that if $T \in \mathscr{T}_n$ then

$$D_n * T = T.$$

2 Bernstein's inequality for trigonometric polynomials

DeVore and Lorentz attribute the following inequality to Szegö.[1]

[1] Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, p. 97, Theorem 1.1.

Theorem 1. *If $T \in \mathscr{T}_n$ and T is real valued, then for all $x \in \mathbb{T}$,*

$$T'(x)^2 + n^2 T(x)^2 \leq n^2 \|T\|_\infty^2 .$$

Proof. If $T = 0$ the result is immediate. Otherwise, take $x \in \mathbb{T}$, and for real $c > 1$ define

$$P_c(t) = \frac{T(t+x)\operatorname{sgn} T'(x)}{c\,\|T\|_\infty}, \qquad t \in \mathbb{T}.$$

$P_c \in \mathscr{T}_n$, and satisfies

$$P_c'(0) = \frac{T'(x)\operatorname{sgn} T'(x)}{c\,\|T\|_\infty} \geq 0$$

and $\|P_c\|_\infty \leq \frac{1}{c} < 1$. Since $\|P_c\|_\infty < 1$, in particular $|P_c(0)| < 1$ and so there is some α, $|\alpha| < \frac{\pi}{2n}$, such that $\sin n\alpha = P_c(0)$. We define $S \in \mathscr{T}_n$ by

$$S(t) = \sin n(t+\alpha) - P_c(t), \qquad t \in \mathbb{T},$$

which satisfies $S(0) = \sin n\alpha - P_c(0) = 0$. For $k = -n, \dots, n$, let $t_k = -\alpha + \frac{(2k-1)\pi}{2n}$, for which we have

$$\sin n(t_k + \alpha) = \sin \frac{(2k-1)\pi}{2} = (-1)^{k+1}.$$

Because $\|P_c\|_\infty < 1$,

$$\operatorname{sgn} S(t_k) = (-1)^{k+1},$$

so by the intermediate value theorem, for each $k = -n, \dots, n-1$ there is some $c_k \in (t_k, t_{k+1})$ such that $S(c_k) = 0$. Because

$$t_n - t_{-n} = \frac{(2n-1)\pi}{2n} - \frac{(-2n-1)\pi}{2n} = 2\pi,$$

it follows that if $j \neq k$ then c_j and c_k are distinct in $\mathbb{T}$. It is a fact that a trigonometric polynomial of degree n has $\leq 2n$ distinct roots in $\mathbb{T}$, so if $t \in (t_k, t_{k+1})$ and $S(t) = 0$, then $t = c_k$. It is the case that $t_1 = -\alpha + \frac{\pi}{2n} > 0$ and $t_0 = -\alpha - \frac{\pi}{2} < 0$, so $0 \in (t_0, t_1)$. But $S(0) = 0$, so $c_0 = 0$. Using $S(t_1) = 1 > 0$ and the fact that S has no zeros in $(0, t_1)$ we get a contradiction from $S'(0) < 0$, so $S'(0) \geq 0$. This gives

$$0 \leq P_c'(0) = n\cos n\alpha - S'(0) \leq n\cos n\alpha = n\sqrt{1 - \sin^2 n\alpha} = n\sqrt{1 - P_c(0)^2}.$$

Thus

$$P_c'(0) \leq n\sqrt{1 - P_c(0)^2},$$

or

$$n^2 P_c(0) + P_c'(0)^2 \leq n^2.$$

Because

$$P_c(0)^2 = \frac{T(x)^2}{c^2 \|T\|_\infty^2}, \qquad P_c'(0)^2 = \frac{T'(x)^2}{c^2 \|T\|_\infty^2}$$

we get

$$n^2 T(x)^2 + T'(x)^2 \le c^2 n^2 \|T\|_\infty^2 .$$

Because this is true for all $c > 1$,

$$n^2 T(x)^2 + T'(x)^2 \le n^2 \|T\|_\infty^2 ,$$

completing the proof. □

Using the above we now prove Bernstein's inequality.[2]

Theorem 2 (Bernstein's inequality)**.** *If $T \in \mathscr{T}_n$, then*

$$\|T'\|_\infty \le n \|T\|_\infty .$$

Proof. There is some $x_0 \in \mathbb{T}$ such that $|T'(x_0)| = \|T'\|_\infty$. Let $\alpha \in \mathbb{R}$ be such that $e^{i\alpha} T'(x_0) = \|T'\|_\infty$. Define $S(x) = \mathrm{Re}\,(e^{i\alpha} T(x))$ for $x \in \mathbb{T}$, which satisfies $S'(x) = \mathrm{Re}\,(e^{i\alpha} T'(x))$ and in particular

$$S'(x_0) = \mathrm{Re}\,(e^{i\alpha} T'(x_0)) = e^{i\alpha} T'(x_0) = \|T'\|_\infty .$$

Because $S \in \mathscr{T}_n$ and S is real valued, Theorem 1 yields

$$S'(x_0)^2 + n^2 S(x_0)^2 \le n^2 \|S\|_\infty^2 .$$

A fortiori,

$$S'(x_0)^2 \le n^2 \|S\|_\infty^2 ,$$

giving, because $S'(x_0) = \|T'\|_\infty$ and $\|S\|_\infty \le \|T\|_\infty$,

$$\|T'\|_\infty^2 \le n^2 \|T\|_\infty^2 ,$$

proving the claim. □

The following is a version of Bernstein's inequality.[3]

Theorem 3. *If $T \in \mathscr{T}_n$ and $A \subset \mathbb{T}$ is a Borel set, there is some $x_0 \in \mathbb{T}$ such that*

$$\int_A |T'(t)| dt \le n \int_{A - x_0} |T(t)| dt.$$

[2] Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, p. 98.

[3] Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, p. 101, Theorem 2.4.

Proof. Let $A \subset \mathbb{T}$ be a Borel set with indicator function χ_A. Define $Q : \mathbb{T} \to \mathbb{C}$ by

$$Q(x) = \int_{\mathbb{T}} \chi_A(t) T(t+x) \operatorname{sgn} T'(t) dt, \qquad x \in \mathbb{T},$$

which we can write as

$$\begin{aligned} Q(x) &= \int_{\mathbb{T}} \chi_A(t) \sum_j \widehat{T}(j) e^{ij(t+x)} \operatorname{sgn} T'(t) dt \\ &= \sum_j \widehat{T}(j) \left(\int_{\mathbb{T}} \chi_A(t) e^{ijt} \operatorname{sgn} T'(t) dt \right) e^{ijx}, \end{aligned}$$

showing that $Q \in \mathscr{T}_n$. Also,

$$Q'(x) = \int_{\mathbb{T}} \chi_A(t) T'(t+x) \operatorname{sgn} T'(t) dt, \qquad x \in \mathbb{T}.$$

Let $x_0 \in \mathbb{T}$ with $|Q(x_0)| = \|Q\|_\infty$. Applying Theorem 2 we get

$$\|Q'\|_\infty \leq n \|Q\|_\infty .$$

Using

$$Q'(0) = \int_{\mathbb{T}} \chi_A(t) T'(t) \operatorname{sgn} T'(t) dt = \int_{\mathbb{T}} \chi_A(t) |T'(t)| dt,$$

this gives

$$\begin{aligned} \int_{\mathbb{T}} \chi_A(t) |T'(t)| dt &\leq n \|Q\|_\infty \\ &= n |Q(t_0)| \\ &= n \left| \int_{\mathbb{T}} \chi_A(t) T(t+x_0) \operatorname{sgn} T'(t) dt \right| \\ &\leq n \int_{\mathbb{T}} \chi_A(t) |T(t+x_0)| dt \\ &= n \int_{\mathbb{T}} \chi_{A-x_0}(t) |T(t)| dt. \end{aligned}$$

□

Applying the above with $A = \mathbb{T}$ gives the following version of Bernstein's inequality, for the L^1 norm.

Theorem 4 (L^1 Bernstein's inequality)**.** *If $T \in \mathscr{T}_n$, then*

$$\|T'\|_1 \leq n \|T\|_1 .$$

3 Nikolsky's inequality for trigonometric polynomials

DeVore and Lorentz attribute the following inequality to Sergey Nikolsky.[4]

Theorem 5 (Nikolsky's inequality)**.** *If $T \in \mathscr{T}_n$ and $0 < q \le p \le \infty$, then for $r \ge \frac{q}{2}$ an integer,*

$$\|T\|_p \le (2nr+1)^{\frac{1}{q}-\frac{1}{p}} \|T\|_q .$$

Proof. Let $m = nr$. Then $T^r \in \mathscr{T}_m$, so $T^r * D_m = T^r$, and using this and the Cauchy-Schwarz inequality we have, for $x \in \mathbb{T}$,

$$\begin{aligned}
|T(x)^r| &= \left| \frac{1}{2\pi} \int_{\mathbb{T}} T(t)^r D_m(x-t) \right| \\
&\le \frac{1}{2\pi} \int_{\mathbb{T}} |T(t)|^r |D_m(x-t)| dt \\
&\le \|T\|_\infty^{r-\frac{q}{2}} \cdot \frac{1}{2\pi} \int_{\mathbb{T}} |T(t)|^{\frac{q}{2}} |D_m(x-t)| dt \\
&\le \|T\|_\infty^{r-\frac{q}{2}} \left\| |T|^{q/2} \right\|_2 \|D_m\|_2 \\
&= \|T\|_\infty^{r-\frac{q}{2}} \|T\|_q^{\frac{q}{2}} \left\| \widehat{D_m} \right\|_{\ell^2(\mathbb{Z})} \\
&= \sqrt{2m+1} \|T\|_\infty^{r-\frac{q}{2}} \|T\|_q^{\frac{q}{2}} .
\end{aligned}$$

Hence

$$\|T\|_\infty^r \le \sqrt{2m+1} \|T\|_\infty^{r-\frac{q}{2}} \|T\|_q^{\frac{q}{2}} ,$$

thus

$$\|T\|_\infty \le (2m+1)^{\frac{1}{q}} \|T\|_q .$$

Then, using $\|T\|_p \le \|T\|_\infty^{1-\frac{q}{p}} \|T\|_q^{\frac{q}{p}}$, we have

$$\|T\|_p \le (2m+1)^{\frac{1}{q}-\frac{1}{p}} \|T\|_q^{1-\frac{q}{p}} \|T\|_q^{\frac{q}{p}} = (2m+1)^{\frac{1}{q}-\frac{1}{p}} \|T\|_q .$$

□

4 The complementary Bernstein inequality

We define a **homogeneous Banach space** to be a linear subspace B of $L^1(\mathbb{T})$ with a norm $\|f\|_{L^1(\mathbb{T})} \le \|f\|_B$ with which B is a Banach space, such that if $f \in B$ and $\tau \in \mathbb{T}$ then $f_\tau \in B$ and $\|f_\tau\|_B = \|f\|_B$, and such that if $f \in B$ then $f_\tau \to f$ in B as $\tau \to 0$.

[4] Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, p. 102, Theorem 2.6.

Fejér's kernel is, for $n \geq 0$,

$$K_n(t) = \sum_{|j| \leq n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \sum_{j \in \mathbb{Z}} \chi_n(j) \left(1 - \frac{|j|}{n+1}\right) e^{ijt} \qquad t \in \mathbb{T}.$$

One calculates that, for $t \not\in 4\pi\mathbb{Z}$,

$$K_n(t) = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2} t}{\sin \frac{1}{2} t} \right)^2 .$$

Bernstein's inequality is a statement about functions whose Fourier transform is supported only on low frequencies. The following is a statement about functions whose Fourier transform is supported only on high frequencies.[5] In particular, for $1 \leq p < \infty$, $L^p(\mathbb{T})$ is a homogeneous Banach space, and so is $C(\mathbb{T})$ with the supremum norm.

Theorem 6. *Let B be a homogeneous Banach space and let m be a positive integer. Define C_m as $C_m = m + 1$ if m is even and $C_m = 12m$ if m is odd. If*

$$f(t) = \sum_{|j| \geq n} a_j e^{ijt}, \qquad t \in \mathbb{T},$$

is m times differentiable and $f^{(m)} \in B$, then $f \in B$ and

$$\|f\|_B \leq C_m n^{-m} \left\| f^{(m)} \right\|_B .$$

Proof. Suppose that m is even. It is a fact that if $a_j, j \in \mathbb{Z}$, is an even sequence of nonnegative real numbers such that $a_j \to 0$ as $|j| \to \infty$ and such that for each $j > 0$,

$$a_{j-1} + a_{j+1} - 2a_j \geq 0,$$

then there is a nonnegative function $f \in L^1(\mathbb{T})$ such that $\hat{f}(j) = a_j$ for all $j \in \mathbb{Z}$.[6] Define

$$a_j = \begin{cases} j^{-m} & |j| \geq n \\ n^{-m} + (n - |j|)(n^{-m} - (n+1)^{-m}) & |j| \leq n - 1. \end{cases}$$

It is apparent that a_j is even and tends to 0 as $|j| \to \infty$. For $1 \leq j \leq n - 2$,

$$a_{j-1} + a_{j+1} - 2a_j = 0.$$

For $j = n - 1$,

$$\begin{aligned} a_{j-1} + a_{j+1} - 2a_j &= n^{-m} + (n - (n-2))(n^{-m} - (n+1)^{-m}) + n^{-m} \\ &\quad - 2\left(n^{-m} + (n - (n-1))(n^{-m} - (n+1)^{-m})\right) \\ &= 0. \end{aligned}$$

[5] Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 55, Theorem 8.4.

[6] Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 24, Theorem 4.1.

The function $j \mapsto j^{-m}$ is convex on $\{n, n+1, \ldots\}$, as $m \geq 1$, so for $j \geq n$ we have $a_{j-1} + a_{j+1} - 2a_j \geq 0$. Therefore, there is some nonnegative $\phi_{m,n} \in L^1(\mathbb{T})$ such that

$$\widehat{\phi_{m,n}}(j) = a_j, \qquad j \in \mathbb{Z}.$$

Because $\phi_{m,n}$ is nonnegative, and using $n^{-m} - (n+1)^{-m} < \frac{m}{n} n^{-m}$,

$$\|\phi_{m,n}\|_1 = \widehat{\phi_{m,n}}(0) = n^{-m} + n(n^{-m} - (n+1)^{-m}) < (m+1)n^{-m}.$$

Define $d\mu_{m,n}(t) = \frac{1}{2\pi}\phi_{m,n}(t)dt$. For $|j| \geq n$,

$$\begin{aligned}
\widehat{f^{(m)} * \mu_{m,n}}(j) &= \widehat{f^{(m)}}(j)\widehat{\mu_{m,n}}(j) \\
&= (ij)^m \hat{f}(j)\widehat{\phi_{m,n}}(j) \\
&= (ij)^m \hat{f}(j) \cdot |j|^{-m} \\
&= i^m \hat{f}(j).
\end{aligned}$$

For $|j| < n$, since $\hat{f}(j) = 0$ we have

$$\widehat{f^{(m)} * \mu_{m,n}}(j) = (ij)^m \hat{f}(j)\widehat{\phi_{m,n}}(j) = 0 = i^m \hat{f}(j),$$

so for all $j \in \mathbb{Z}$,

$$\widehat{f^{(m)} * \mu_{m,n}}(j) = i^m \hat{f}(j).$$

This implies that $f^{(m)} * \mu_{m,n} = i^m f$, which in particular tells us that $f \in B$. Then,

$$\begin{aligned}
\|f\|_B &= \|i^m f\|_B \\
&= \left\| f^{(m)} * \mu_{m,n} \right\|_B \\
&\leq \left\| f^{(m)} \right\|_B \|\mu_{m,n}\|_{M(\mathbb{T})} \\
&= \|\phi_{m,n}\|_1 \left\| f^{(m)} \right\|_B \\
&\leq (m+1)n^{-m} \left\| f^{(m)} \right\|_B.
\end{aligned}$$

This shows what we want in the case that m is even, with $C_m = m+1$.

Suppose that m is odd. For l a positive integer, define $\psi_l : \mathbb{T} \to \mathbb{C}$ by

$$\psi_l(t) = \left(e^{2lit} + \frac{1}{2} e^{3lit} \right) K_{l-1}(t), \qquad t \in \mathbb{T}.$$

There is a unique l_n such that $n \in \{2l_n, 2l_n + 1\}$. For $k \geq 0$ an integer, define $\Psi_{n,k} : \mathbb{T} \to \mathbb{C}$ by

$$\Psi_{n,k}(t) = \psi_{l_n 2^k}(t), \qquad t \in \mathbb{T}.$$

$\Psi_{n,k}$ satisfies

$$\|\Psi_{n,k}\|_1 \leq \frac{3}{2} \|K_{k-1}\|_1 = \frac{3}{2} \cdot \frac{1}{2\pi} \int_{\mathbb{T}} |K_{k-1}(t)| dt = \frac{3}{2} \cdot \frac{1}{2\pi} \int_{\mathbb{T}} K_{k-1}(t) dt = \frac{3}{2}.$$

On the one hand, for $j \leq 0$, from the definition of ψ_l we have $\widehat{\Psi_{n,k}}(j) = 0$, hence $\sum_{k=0}^{\infty} \widehat{\Psi_{n,k}}(j) = 0$. On the other hand, for $j \geq n$ we assert that

$$\sum_{k=0}^{\infty} \widehat{\Psi_{n,k}}(j) = 1.$$

We define $\Phi_n : \mathbb{T} \to \mathbb{C}$ by

$$\Phi_n(t) = \sum_{k=0}^{\infty} (\Psi_{n,k} * \phi_{1,n2^k})(t), \qquad t \in \mathbb{T}.$$

We calculate the Fourier coefficients of Φ_n. For $j \geq n$,

$$\widehat{\Phi_n}(j) = \sum_{k=0}^{\infty} \widehat{\Phi_{n,k}}(j) \widehat{\phi_{1,n2^k}}(j) = \frac{1}{j} \sum_{k=0}^{\infty} \widehat{\Phi_{n,k}}(j) = \frac{1}{j}.$$

As well,

$$\|\Phi_n\|_1 \leq \sum_{k=0}^{\infty} \left\|\Psi_{n,k} * \phi_{1,n2^k}\right\|_1 \leq \sum_{k=0}^{\infty} \|\Psi_{n,k}\|_1 \left\|\phi_{1,n2^k}\right\|_1 \leq \frac{3}{2} \sum_{k=0}^{\infty} 2(n2^k)^{-1} = \frac{6}{n}$$

We now define

$$d\mu_{1,n}(t) = \frac{1}{2\pi}(\Phi_n(t) - \Phi_n(-t))dt,$$

which satisfies for $|j| \geq n$,

$$\widehat{\mu_{1,n}}(j) = \widehat{\Phi_n}(j) - \widehat{\Phi_n}(-j) = \frac{1}{j}$$

and hence

$$\widehat{f' * \mu_{1,n}}(j) = \widehat{f'}(j)\widehat{\mu_{1,n}}(j) = ij\hat{f}(j) \cdot \frac{1}{j} = i\hat{f}(j).$$

Because $\hat{f}(j) = 0$ for $|j| < n$, $\widehat{f' * \mu_{1,n}}(j) = 0$ for $|j| < n$, it follows that for any $j \in \mathbb{Z}$,

$$\widehat{f' * \mu_{1,n}}(j) = i\hat{f}(j),$$

and therefore,

$$f' * \mu_{1,n} = if.$$

Then

$$\|f\|_B = \|if\|_B = \|f' * \mu_{1,n}\|_B \leq \|\mu_{1,n}\|_{M(\mathbb{T})} \|f'\|_B \leq 2\|\Phi_n\|_1 \|f'\|_B \leq \frac{12}{n} \|f'\|_B.$$

That is, with $C_1 = 12$ we have

$$\|f\|_B \leq 12n^{-1} \|f'\|_B.$$

For $m = 2\nu + 1$, we define

$$\mu_{m,n} = \mu_{1,n} * \mu_{2\nu,n},$$

for which we have, for $|j| \geq n$,

$$\widehat{f^{(m)} * \mu_{m,n}}(j) = (ij)^m \hat{f}(j)\widehat{\mu_{1,n}}(j)\widehat{\mu_{2\nu,n}}(j) = (ij)^m \hat{f}(j) \cdot \frac{1}{j} \cdot j^{-2\nu} = i^m \hat{f}(j).$$

It follows that

$$f^{(m)} * \mu_{m,n} = i^m f,$$

whence

$$\begin{aligned}
\|f\|_B &= \|i^m f\|_B \\
&= \left\| f^{(m)} * \mu_{m,n} \right\|_B \\
&\leq \|\mu_{m,n}\|_{M(\mathbb{T})} \left\| f^{(m)} \right\|_B \\
&\leq \|\mu_{1,n}\|_{M(\mathbb{T})} \|\mu_{2\nu,n}\|_{M(\mathbb{T})} \left\| f^{(m)} \right\|_B \\
&\leq \frac{12}{n} \cdot (2\nu + 1) n^{-2\nu} \left\| f^{(m)} \right\|_B \\
&= 12 m n^{-m} \left\| f^{(m)} \right\|_B .
\end{aligned}$$

That is, with $C_m = 12m$, we have

$$\|f\|_B \leq C_m n^{-m} \left\| f^{(m)} \right\|_B ,$$

completing the proof. $\square$