Fatou's theorem, Bergman spaces, and Hardy spaces on the circle

Jordan Bell

April 3, 2014

1 Introduction

In this note I am writing out proofs of some facts about Fourier series, Bergman spaces, and Hardy spaces. §§1–3 follow the presentation in Stein and Shakarchi's *Real Analysis* and *Fourier Analysis*. The questions in Halmos's *Hilbert Space Problem Book* that deal with Hardy spaces are: §§24–35, 67, 116–117, 124–125, 127, 193–199, and I present solutions to some of these in §§4–5, on Bergman spaces, and §6, on Hardy spaces.

2 Poisson kernel

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let

$$\|f\|_{L^p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p dt\right)^{1/p}.$$

If $f \in L^1(\mathbb{T})$, let

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt.$$

Define

$$P_r(t) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{int} = \frac{1-r^2}{1-2r\cos t + r^2}, \qquad 0 \le r < 1, t \in \mathbb{T}.$$

One checks that P_r is an *approximation to the identity*, which implies that for $f \in L^1(\mathbb{T})$, for almost all $\theta \in \mathbb{T}$ we have $(f * P_r)(\theta) \to f(\theta)$ as $r \to 1^-$.[1]

For $f \in L^1(\mathbb{T})$, for any θ we have

$$\left\| f(t) \sum_{|n|\le N} r^{|n|}e^{in(\theta-t)} \right\|_{L^1} \le \frac{1+r}{1-r}\|f\|_{L^1},$$

[1] If f is continuous, then $f * P_r$ converges to f uniformly on $\mathbb{T}$ as $r \to 1^-$. This is proved for example in Lang's *Complex Analysis*, fourth ed., chapter VIII, §5.

hence by the dominated convergence theorem we have

$$\begin{aligned}\lim_{N\to\infty}\sum_{|n|\le N} r^{|n|}\int_0^{2\pi} f(t)e^{in(\theta-t)}dt &= \lim_{N\to\infty}\int_0^{2\pi} f(t)\sum_{|n|\le N} r^{|n|}e^{in(\theta-t)}dt\\ &= \int_0^{2\pi} f(t)\sum_{n\in\mathbb{Z}} r^{|n|}e^{in(\theta-t)}dt,\end{aligned}$$

and so

$$\begin{aligned}(f*P_r)(\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)P_r(\theta-t)dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\sum_{n\in\mathbb{Z}} r^{|n|}e^{in(\theta-t)}dt\\ &= \sum_{n\in\mathbb{Z}} r^{|n|}\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{in(\theta-t)}dt\\ &= \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}\hat{f}(n).\end{aligned}$$

3 Harmonic functions

For $f \in L^1(\mathbb{T})$, define u_f on $|z| < 1$ by

$$u_f(re^{i\theta}) = (f*P_r)(\theta).$$

In polar coordinates, the Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Then

$$\begin{aligned}(\Delta u_f)(re^{i\theta}) &= \Delta\left(\sum_{n<0} r^{-n}e^{in\theta}\hat{f}(n) + \sum_{n\ge 0} r^{n}e^{in\theta}\hat{f}(n)\right)\\ &= \sum_{n<0}\hat{f}(n)\Delta\left(r^{-n}e^{in\theta}\right) + \sum_{n\ge 0}\hat{f}(n)\Delta\left(r^{n}e^{in\theta}\right)\\ &= \sum_{n<0}\hat{f}(n)\cdot 0 + \sum_{n\ge 0}\hat{f}(n)\cdot 0\\ &= 0.\end{aligned}$$

Hence u_f is harmonic on the open unit disc.

4 Fatou's theorem

Let $D = \{z : |z| < 1\}$. If $F : D \to \mathbb{C}$ is holomorphic, let it have the power series

$$F(z) = \sum_{n \geq 0} a_n z^n, \qquad a_n \in \mathbb{C}.$$

By the Cauchy integral formula, for $n \geq 0$ and for any $0 < r < 1$, $\gamma_r(\theta) = re^{i\theta}$, we have

$$\begin{aligned} F^{(n)}(0) &= \frac{n!}{2\pi i} \int_{\gamma_r} \frac{F(\zeta)}{\zeta^{n+1}} d\zeta \\ &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{F(re^{i\theta})}{(re^{i\theta})^{n+1}} rie^{i\theta} d\theta \\ &= \frac{n!}{2\pi r^n} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

Hence, for $n \geq 0$ and for $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta = a_n r^n.$$

On the other hand, for $n < 0$, we have

$$\frac{n!}{2\pi r^n} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{F(\zeta)}{\zeta^{n+1}} d\zeta,$$

and because $\frac{F(\zeta)}{z^{n+1}}$ is holomorphic on D for $n < 0$, by the residue theorem the right-hand side of the above equation is equal to 0. Hence, for $n < 0$ and for $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta = 0.$$

Let $F : D \to \mathbb{C}$ be holomorphic, and suppose there is some M such that $|F(z)| \leq M$ for all $z \in D$. For $0 < r < 1$, define $f_r : \mathbb{T} \to \mathbb{C}$ by $f_r(\theta) = F(re^{i\theta})$. From our above work, we have

$$\widehat{f_r}(n) = \begin{cases} a_n r^n & n \geq 0, \\ 0 & n < 0. \end{cases}$$

For $0 < r < 1$, note that $\|f_r\|_{L^2} \leq \|f_r\|_{L^\infty} \leq M$, so, by Parseval's identity,

$$\sum_{n \in \mathbb{Z}} |\widehat{f_r}(n)|^2 \leq M^2.$$

On the other hand,

$$\sum_{n \in \mathbb{Z}} |\widehat{f_r}(n)|^2 = \sum_{n \geq 0} |a_n|^2 r^{2n}.$$

It follows that

$$\sum_{n\geq 0} |a_n|^2 \leq M^2.$$

Define $f \in L^2(\mathbb{T})$ by

$$\hat{f}(n) = \begin{cases} a_n & n \geq 0, \\ 0 & n < 0; \end{cases}$$

this defines an element of $L^2(\mathbb{T})$ if and only if $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 < \infty$, and indeed

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \leq M^2.$$

As $f \in L^2(\mathbb{T})$, $f \in L^1(\mathbb{T})$. Then by our work in §2, for almost all $\theta \in \mathbb{T}$ we have

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \hat{f}(n) = f(\theta),$$

which means here that for almost all $\theta \in \mathbb{T}$,

$$\lim_{r\to 1^-} \sum_{n\geq 0} a_n r^n e^{in\theta} = f(\theta).$$

Thus, for almost all $\theta \in \mathbb{T}$,

$$\lim_{r\to 1^-} F(re^{i\theta}) = f(\theta).$$

In words, we have proved that if F is a bounded holomorphic function on the unit disc, then it has radial limits at almost every angle. This is *Fatou's theorem*.

5 Bergman spaces

This section somewhat follows Problem 24 of Halmos. Let μ be Lebesgue measure on D. $d\mu(z) = dx \wedge dy = \frac{dz\wedge d\overline{z}}{-2i}$.

If U is a nonempty bounded open subset of $\mathbb{C}$ and $1 \leq p < \infty$, let $A^p(U)$ denote the set of functions $f : U \to \mathbb{C}$ that are holomorphic and that satisfy

$$\|f\|_{A^p(U)} = \left(\int_U |f(z)|^p d\mu(z) \right)^{1/p} < \infty,$$

and let $A^\infty(U)$ denote the set of functions $f : U \to \mathbb{C}$ that are holomorphic and that satisfy

$$\|f\|_{A^\infty(U)} = \sup_{z\in U} |f(z)| < \infty.$$

It is apparent that $A^p(U)$ is a vector space over $\mathbb{C}$. By Minkowski's inequality, $\|\cdot\|_{A^p(U)}$ is a norm, and thus $A^p(U)$ is a normed space. If $p \leq q$ then by Jensen's inequality we have

$$\|f\|_{A^p(U)} \leq \mu(D)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{A^q(U)},$$

and so

$$A^q(U) \subseteq A^p(U).$$

$A^p(U)$ is called a *Bergman space*. It is not apparent that it is a complete metric space. We show this using the following lemmas. We use the following lemma to prove the lemma after it, and use that lemma to prove the theorem.

Lemma 1. *If $z_0 \in \mathbb{C}$, $R > 0$, and $f \in A^1(D(z_0, R))$, then*

$$f(z_0) = \frac{1}{\pi R^2} \int_{D(z_0,R)} f(z) d\mu(z).$$

Proof. Put $F_n(z) = \sum_{k=0}^n a_k(z - z_0)^k$, with $a_k = \frac{f^{(k)}(z_0)}{k!}$. For $0 < r < R$, define

$$\|g\|_r = \sup_{|z-z_0|\le r} |g(z)|.$$

We have $\|F_n - f\|_r \to 0$ as $n \to \infty$. Then,

$$\begin{aligned}
\left|\int_{D(z_0,r)} f(z)d\mu(z) - \int_{D(z_0,r)} F_n(z)d\mu(z)\right| &= \left|\int_{D(z_0,r)} f(z) - F_n(z)d\mu(z)\right| \\
&\le \int_{D(z_0,r)} |f(z) - F_n(z)|d\mu(z) \\
&\le \int_{D(z_0,r)} \|f - F_n\|_r \, d\mu(z) \\
&= \|f - F_n\|_r \cdot \pi r^2,
\end{aligned}$$

which tends to 0 as $n \to \infty$. Thus

$$\begin{aligned}
\int_{D(z_0,r)} f(z)d\mu(z) &= \lim_{n\to\infty} \int_{D(z_0,r)} F_n(z)d\mu(z) \\
&= \lim_{n\to\infty} \int_{D(z_0,r)} \sum_{k=0}^n a_k(z - z_0)^k d\mu(z) \\
&= \lim_{n\to\infty} \sum_{k=0}^n a_k \int_{D(z_0,r)} (z - z_0)^k d\mu(z) \\
&= \lim_{n\to\infty} \sum_{k=0}^n a_k \int_{D(0,r)} z^k d\mu(z).
\end{aligned}$$

For $k \ge 1$, using polar coordinates we have

$$\begin{aligned}
\int_{D(0,r)} z^k d\mu(z) &= \int_0^r \int_0^{2\pi} (\rho e^{i\theta})^k \rho d\theta d\rho \\
&= \int_0^r \int_0^{2\pi} \rho^{k+1} e^{ik\theta} d\theta d\rho \\
&= \int_0^r \rho^{k+1} \cdot \frac{0}{k} d\rho \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}\int_{D(z_0,r)} f(z)d\mu(z) &= \lim_{n\to\infty} a_0 \cdot \pi r^2 \\ &= a_0 \cdot \pi r^2.\end{aligned}$$

That is, for each $0 < r < R$ we have

$$f(z_0) = \frac{1}{\pi r^2} \int_{D(z_0,r)} f(z)d\mu(z). \tag{1}$$

Because $f \in L^1(D(z_0, R))$,

$$\lim_{r\to R} \int_{D(z_0,r)} f(z)d\mu(z) = \int_{D(z_0,R)} f(z)d\mu(z).$$

Thus, taking the limit as $r \to R$ of (1), we obtain

$$f(z_0) = \frac{1}{\pi R^2} \int_{D(z_0,R)} f(z)d\mu(z).$$

□

If $z_0 \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, denote

$$d(z_0, S) = \inf_{z\in S} |z_0 - z|,$$

and for $z_0 \in U$, let

$$r(z_0) = d(z_0, \partial U).$$

This is the radius of the largest open disc centered at z_0 that is contained in U (it is equal to the union of all open discs centered at z_0 that are contained in U, and thus makes sense). As U is open, $r(z_0) > 0$, and as U is bounded, $r(z_0) < \infty$.

Lemma 2. *If $1 \le p \le \infty$, $z_0 \in U$, and $f \in A^p(U)$, then*

$$|f(z_0)| \le \left(\frac{1}{\pi r(z_0)^2}\right)^{1/p} \|f\|_{A^p(U)}.$$

Proof. As $f \in A^p(U)$ we have $f \in A^p(D(z_0, r(z_0))) \subseteq A^1(D(z_0, r(z_0)))$. Using

Lemma 1 and Hölder's inequality, we get, with $\frac{1}{p}+\frac{1}{q}=1$ (q is infinite if $p=1$),

$$
\begin{aligned}
|f(z_0)| &= \left|\frac{1}{\pi r(z_0)^2}\int_{D(z_0,r(z_0))} f(z)d\mu(z)\right| \\
&\leq \frac{1}{\pi r(z_0)^2}\int_{D(z_0,r(z_0))} |f(z)|d\mu(z) \\
&\leq \frac{1}{\pi r(z_0)^2}\mu(D(z_0,r(z_0)))^{1/q}\,\|f\|_{A^p(D(z_0,r(z_0)))} \\
&= \frac{1}{\pi r(z_0)^2}(\pi r(z_0)^2)^{1/q}\,\|f\|_{A^p(D(z_0,r(z_0)))} \\
&\leq \frac{1}{\pi r(z_0)^2}(\pi r(z_0)^2)^{1/q}\,\|f\|_{A^p(U)} \\
&= \frac{1}{\pi r(z_0)^2}(\pi r(z_0)^2)^{1-\frac{1}{p}}\,\|f\|_{A^p(U)} \\
&= \left(\frac{1}{\pi r(z_0)^2}\right)^{1/p}\|f\|_{A^p(U)}.
\end{aligned}
$$

□

Now we prove that $A^p(U)$ is a complete metric space, showing that it is a Banach space.

Theorem 3. *If $1\leq p\leq\infty$, then $A^p(U)$ is a Banach space.*

Proof. Suppose that $f_n\in A^p(U)$ is a Cauchy sequence. We have to show that there is some $f\in A^p(U)$ such that $f_n\to f$ in $A^p(U)$. The space $H(U)$ of holomorphic functions on U is a Fréchet space: there is an increasing sequence of compact sets $K_i\subset U$ whose union is U, and the p_{K_i} seminorms on $H(U)$ are the supremum of a function on K_i. (See Henri Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, §V.1.3.) For each of these compact sets K_i, let r_i be the distance between K_i and ∂U, which are both compact sets. If $z_0\in K_i$ then $r(z_0)\geq r_i$. Thus if $z_0\in K_i$ and $g\in A^p(U)$, using Lemma 2 we get

$$
|g(z_0)|\leq\left(\frac{1}{\pi r(z_0)^2}\right)^{1/p}\|g\|_{A^p(U)}\leq\left(\frac{1}{\pi r_i^2}\right)^{1/p}\|g\|_{A^p(U)}.
$$

From this and the fact that $\|f_n-f_m\|_{A^p(U)}\to 0$ as $m,n\to\infty$, we get that

$$
p_{K_i}(f_n-f_m)\to 0,\qquad m,n\to\infty.
$$

That is, f_n is a Cauchy sequence in each of the seminorms p_{K_i}, and as $H(U)$ is a Fréchet space it follows that there is some $f\in H(U)$ such that $f_n\to f$ in $H(U)$. In particular, for all $z_0\in U$ we have $f_n(z_0)\to f(z_0)$ as $n\to\infty$ (because each z_0 is included in one of the compact sets K_i, on which the f_n converge uniformly to f and hence pointwise to f).

On the other hand, $L^p(U)$ is a Banach space, and hence there is some $g \in L^p(U)$ such that $\|f_n - g\|_{L^p(U)} \to 0$ as $n \to \infty$. This implies that there is some subsequence $f_{a(n)}$ such that for almost all $z_0 \in U$, $f_{a(n)}(z_0) \to g(z_0)$. Thus, for almost all $z_0 \in U$ we have $f(z_0) = g(z_0)$. Therefore, in $L^p(U)$ we have $f = g$ and so

$$\|f_n - f\|_{A^p(U)} = \|f_n - f\|_{L^p(U)} \to 0, \qquad n \to \infty.$$

□

6 Inner products

In this section we follow Problem 25 of Halmos. In this section we restrict our attention to the Bergman space $A^2(D)$, where D is the open unit disc, on which we define the inner product

$$\langle f, g \rangle = \int_D f g^* d\mu = \int_D f(z)\overline{g(z)} d\mu(z).$$

As $\langle f, f \rangle = \|f\|^2_{A^2(D)}$, it follows that $A^2(D)$ is a Hilbert space with this inner product. If we have a Hilbert space we would like to find an explicit orthonormal basis.

Theorem 4. *If $n \geq 0$ and $z \in D$, define $e_n : D \to \mathbb{C}$ by*

$$e_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot z^n.$$

Then e_n are an orthonormal basis for $A^2(D)$.

Proof. If E is a subset of a Hilbert space and $v \in H$, we write $v \perp E$ if $\langle v, e \rangle = 0$ for all $e \in E$. If E is an orthonormal set in H, E is an orthonormal basis if and only if $v \perp E$ implies that $v = 0$. This is proved in John B. Conway, *A Course in Functional Analysis*, second ed., p. 16, Theorem 4.13. For $n \neq m$,

$$\begin{aligned}
\langle e_n, e_m \rangle &= \int_D \sqrt{\frac{n+1}{\pi}} z^n \sqrt{\frac{m+1}{\pi}} \overline{z}^m d\mu(z) \\
&= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_D z^n \overline{z}^m d\mu(z) \\
&= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_0^1 \int_0^{2\pi} (re^{i\theta})^n (re^{-i\theta})^m r d\theta dr \\
&= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_0^1 \int_0^{2\pi} r^{n+m+1} e^{i\theta(n-m)} d\theta dr \\
&= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_0^1 \int_0^{2\pi} r^{n+m+1} e^{i\theta(n-m)} d\theta dr \\
&= 0,
\end{aligned}$$

while

$$
\begin{aligned}
\langle e_n, e_n \rangle &= \frac{n+1}{\pi} \int_0^1 \int_0^{2\pi} r^{2n+1} d\theta dr \\
&= 2(n+1) \int_0^1 r^{2n+1} dr \\
&= 1.
\end{aligned}
$$

Therefore e_n is an orthonormal set. Hence, to show that it is an orthonormal basis for $A^2(D)$ we have to show that if $\langle f, e_n \rangle = 0$ for all $n \geq 0$ then $f = 0$.

For $0 < r < 1$, let D_r be the open disc centered at 0 of radius r, and let $\|g\|_r = \sup_{|z| \leq r} |g(z)|$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and for each $0 < r < 1$ this power series converges uniformly in D_r. Then

$$
\begin{aligned}
\int_{D_r} f e_m^* d\mu &= \int_{D_r} \sum_{n=0}^{\infty} a_n z^n \overline{z}^m d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \int_{D_r} z^n \overline{z}^m d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \int_0^r \int_0^{2\pi} \rho^{n+m+1} e^{i\theta(n-m)} d\theta d\rho \\
&= \sum_{n=0}^{\infty} a_n \int_0^r \rho^{n+m+1} \cdot 2\pi \cdot \delta_{n,m} d\rho \\
&= 2\pi a_m \int_0^r \rho^{2m+1} d\rho \\
&= 2\pi a_m \frac{r^{2m+2}}{2m+2}
\end{aligned}
$$

One checks that $f e_m^* \in A^1(D)$, and hence

$$
\lim_{r \to 1} \int_{D_r} f e_m^* d\mu(z) = \int_D f e_m^* d\mu(z).
$$

Therefore

$$
\langle f, e_m \rangle = \pi a_m \frac{1}{m+1}.
$$

As $\langle f, e_m \rangle = 0$ for each m, this gives us that $a_m = 0$ for all m and hence $f = 0$. This shows that e_n is an orthonormal basis for $A^2(D)$. □

Steven G. Krantz, *Geometric Function Theory: Explorations in Complex Analysis*, p. 9, §1.2, writes about the Bergman space $A^2(\Omega)$, where Ω is a connected open subset of $\mathbb{C}$, not necessarily bounded.

7 Hardy spaces

In a Hilbert space H, if $S_\alpha, \alpha \in I$ are subsets of H, let $\bigvee_{\alpha\in I} S_\alpha$ denote the closure in H of $\bigcup_{\alpha\in I} S_\alpha$. Thus, to say that a set $\{v_\alpha\}$ is an orthonormal basis for a Hilbert space H is to say that $\{v_\alpha\}$ is orthonormal and that $\bigvee_{\alpha\in I}\{v_\alpha\} = H$.

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and let μ be normalized arc length, so that $\mu(S^1) = 1$. Define $e_n : S^1 \to \mathbb{C}$ by $e_n(z) = z^n$, for $n \in \mathbb{Z}$. It is a fact that $e_n, n \in \mathbb{Z}$ are an orthonormal basis for the Hilbert space $L^2(S^1)$, with inner product

$$\langle f, g\rangle = \int_{S^1} fg^* d\mu.$$

We define the *Hardy space* $H^2(S^1)$ to be $\bigvee_{n\geq 0}\{e_n\}$. As it is a closed subspace of the Hilbert space $L^2(S^1)$, it is itself a Hilbert space. For $f \in L^2(S^1)$, we denote $f^*(z) = \overline{f(z)}$.

The following is Problem 26 of Halmos. Note $f^*(z) = \overline{f(z)}$.

Theorem 5. *If $f \in H^2(S^1)$ and $f^* = f$, then f is constant.*

Proof. If $g_n \in L^2(S^1)$ and $g_n \to g \in L^2(S^1)$, then

$$\|g_n^* - g^*\| = \|g_n - g\| \to 0.$$

Thus $g \mapsto g^*$ is continuous $L^2(S^1) \to L^2(S^1)$.

If $g \in L^2(S^1)$, then, as $e_n, n \in \mathbb{Z}$ is an orthonormal basis for $L^2(S^1)$, we have $g = \lim_{N\to\infty} \sum_{|n|\leq N} \langle g, e_n\rangle e_n$, and so, as $e_n^* = e_{-n}$,

$$g^* = \lim_{N\to\infty} \sum_{|n|\leq N} (\langle g, e_n\rangle e_n)^* = \lim_{N\to\infty} \sum_{|n|\leq N} \overline{\langle g, e_n\rangle} e_{-n} = \lim_{N\to\infty} \sum_{|n|\leq N} \overline{\langle g, e_{-n}\rangle} e_n.$$

Therefore if $n \in \mathbb{Z}$ then

$$\langle g^*, e_n\rangle = \overline{\langle g, e_{-n}\rangle}. \tag{2}$$

For $n > 0$,

$$\langle f, e_n\rangle = \langle f^*, e_n\rangle = \overline{\langle f, e_{-n}\rangle} = 0;$$

the first equality is because $f^* = f$, the second equality is by what we showed for any element of $L^2(S^1)$, and the third equality is because $f \in H^2(S^1)$. It follows that $f \in \operatorname{span}\{e_0\}$, and thus that f is constant. □

If $g \in L^2(S^1)$, define $\operatorname{Re} g \in L^2(S^1)$ by

$$\operatorname{Re} g = \frac{g + g^*}{2}$$

and $\operatorname{Re} g \in L^2(S^1)$ by

$$\operatorname{Im} g = \frac{g - g^*}{2i}.$$

$g = \sum_{n\in\mathbb{Z}} \langle g, e_n\rangle e_n$ and, by (2), $g^* = \sum_{n\in\mathbb{Z}} \langle g^*, e_n\rangle e_n = \sum_{n\in\mathbb{Z}} \overline{\langle g, e_{-n}\rangle} e_n$, so

$$\operatorname{Re} g = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} \langle g, e_n\rangle e_n + \sum_{n\in\mathbb{Z}} \overline{\langle g, e_n\rangle} e_n^*\right) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\langle g, e_n\rangle + \overline{\langle g, e_{-n}\rangle}\right) e_n,$$

and

$$\operatorname{Im} g = \frac{1}{2i}\left(\sum_{n\in\mathbb{Z}} \langle g, e_n\rangle e_n - \sum_{n\in\mathbb{Z}} \overline{\langle g, e_{-n}\rangle} e_n\right) = \frac{1}{2i}\sum_{n\in\mathbb{Z}} \left(\langle g, e_n\rangle - \overline{\langle g, e_{-n}\rangle}\right) e_n. \tag{3}$$

$g = \operatorname{Re} g + i\operatorname{Im} g$, and we have $(\operatorname{Re} g)^* = \operatorname{Re} g$ and $(\operatorname{Im} g)^* = \operatorname{Im} g$; that is, both $\operatorname{Re} g$ and $\operatorname{Im} g$ are real valued, like how the real and imaginary parts of a complex number are both real numbers.

The following is Problem 35 of Halmos. In words, it states that a real valued L^2 function u has a corresponding real valued L^2 function v (made unique by demanding that v have 0 constant term) such that the sum $u + iv$ is an element of the Hardy space H^2. This v is called the *Hilbert transform* of u. This is analogous to how if u is harmonic on an open subset Ω of $\mathbb{R}^2$, then $g(x+iy) = u_x(x, y) - iu_y(x, y)$ satisfies the Cauchy-Riemann equations at every point in Ω and hence is holomorphic on Ω. Since g is holomorphic on Ω, for every $z_0 \in \Omega$ there is some open neighborhood of z on which g has a primitive f (g might not have a primitive defined on Ω, e.g. $g(z) = \frac{1}{z}$ on $\Omega = \mathbb{C}\setminus\{0\}$), and there is a constant c such that $u(x, y) = \operatorname{Re} f(x + iy) + c$ for all (x, y) in this neighborhood. u and $v(x, y) = \operatorname{Im} f(x + iy) + c$ are called *harmonic conjugates*.

Theorem 6. *If $u \in L^2(S^1)$ and $u^* = u$, then there is a unique $v \in L^2(S^1)$ such that $v^* = v$, $\langle v, e_0\rangle = 0$, and $u + iv \in H^2(S^1)$.*

Proof. Define $D : \{u \in L^2(S^1) : u^* = u\} \to H^2(D)$ by

$$\langle Du, e_n\rangle = \begin{cases} \langle u, e_0\rangle & n = 0, \\ \langle u, e_n\rangle + \overline{\langle u, e_{-n}\rangle} & n > 0, \\ 0 & n < 0. \end{cases}$$

As $|a + b|^2 \le 2|a|^2 + 2|b|^2$, and using Parseval's identity,

$$\begin{aligned} \sum_{n\ge 0} |\langle Du, e_n\rangle|^2 &= |\langle u, e_0\rangle|^2 + \sum_{n>0} |\langle u, e_n\rangle + \overline{\langle u, e_{-n}\rangle}|^2 \\ &\le |\langle u, e_0\rangle|^2 + 2\sum_{n>0} |\langle u, e_n\rangle|^2 + |\overline{\langle u, e_{-n}\rangle}|^2 \\ &= |\langle u, e_0\rangle|^2 + 2\sum_{n\ne 0} |\langle u, e_n\rangle|^2 \\ &\le 2\|u\|^2. \end{aligned}$$

This is finite, hence $Du \in H^2(S^1)$.

For any $g \in L^2(S^1)$ and $n \in \mathbb{Z}$, by (2) we have $\langle g^*, e_n \rangle = \overline{\langle g, e_{-n} \rangle}$. As $u^* = u$, if $n \in \mathbb{Z}$ then $\langle u, e_n \rangle = \overline{\langle u, e_{-n} \rangle}$. Using this, we check that $\operatorname{Re} Du = u$.

Put $v = \operatorname{Im} Du$, hence $Du = u + iv$. $\langle u, e_0 \rangle = \overline{\langle u, e_0 \rangle}$ gives $\langle Du, e_0 \rangle = \overline{\langle Du, e_0 \rangle}$, and applying this and (3) we get $\langle v, e_0 \rangle = 0$. Thus v satisfies the conditions $v^* = v$, $\langle v, e_0 \rangle = 0$, and $u + iv \in H^2(S^1)$. We are not obliged to do so, but let's write out the Fourier coefficients of v. If $n \in \mathbb{Z}$ then, using $\langle u, e_n \rangle = \overline{\langle u, e_{-n} \rangle}$,

$$\begin{aligned}
\langle v, e_n \rangle &= \langle \operatorname{Im} Du, e_n \rangle \\
&= \frac{1}{2i} \left(\langle Du, e_n \rangle - \overline{\langle Du, e_{-n} \rangle} \right) \\
&= \begin{cases} 0 & n = 0 \\ \frac{1}{2i} \left(\langle u, e_n \rangle + \overline{\langle u, e_{-n} \rangle} \right) & n > 0 \\ -\frac{1}{2i} \overline{\left(\langle u, e_{-n} \rangle + \overline{\langle u, e_n \rangle} \right)} & n < 0 \end{cases} \\
&= \begin{cases} 0 & n = 0 \\ \frac{1}{2i} \left(\langle u, e_n \rangle + \overline{\langle u, e_{-n} \rangle} \right) & n > 0 \\ -\frac{1}{2i} \left(\langle u, e_n \rangle + \overline{\langle u, e_{-n} \rangle} \right) & n < 0 \end{cases} \\
&= \begin{cases} 0 & n = 0 \\ \frac{1}{i} \langle u, e_n \rangle & n > 0 \\ -\frac{1}{i} \langle u, e_n \rangle & n < 0. \end{cases}
\end{aligned}$$

Thus $\langle v, e_n \rangle = -i\operatorname{sgn}(n) \langle u, e_n \rangle$.

If $f \in H^2(S^1)$, then, as $\langle \operatorname{Re} f, e_n \rangle = \frac{\langle f, e_n \rangle + \overline{\langle f, e_{-n} \rangle}}{2}$,

$$\begin{aligned}
\langle D\operatorname{Re} f, e_n \rangle &= \begin{cases} \frac{\langle f, e_0 \rangle + \overline{\langle f, e_0 \rangle}}{2} & n = 0 \\ \frac{\langle f, e_n \rangle + \overline{\langle f, e_{-n} \rangle}}{2} + \frac{\overline{\langle f, e_{-n} \rangle + \overline{\langle f, e_n \rangle}}}{2} & n > 0 \\ 0 & n < 0. \end{cases} \\
&= \begin{cases} \frac{\langle f, e_0 \rangle + \overline{\langle f, e_0 \rangle}}{2} & n = 0 \\ \langle f, e_n \rangle & n > 0 \\ 0 & n < 0 \end{cases} \\
&= \begin{cases} \frac{\langle f, e_0 \rangle + \overline{\langle f, e_0 \rangle}}{2} & n = 0 \\ \langle f, e_n \rangle & n \neq 0 \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
\langle f - D\operatorname{Re} f, e_n \rangle &= \begin{cases} \frac{\langle f, e_0 \rangle - \overline{\langle f, e_0 \rangle}}{2} & n = 0 \\ 0 & n \neq 0 \end{cases} \\
&= \begin{cases} i \cdot \langle \operatorname{Im} f, e_0 \rangle & n = 0 \\ 0 & n \neq 0 \end{cases}
\end{aligned}$$

□