Vitali coverings on the real line

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For $x \in \mathbb{R}$ and $r > 0$ write

$$B(x,r) = \{y \in \mathbb{R} : |y - x| < r\}.$$

Let λ be Lebesgue measure on the Borel σ-algebra of $\mathbb{R}$ and let λ^* be Lebesgue outer measure on $\mathbb{R}$.

A **Vitali covering** of a set $E \subset \mathbb{R}$ is a collection $\mathcal{V}$ of closed intervals such that for $\epsilon > 0$ and for $x \in E$ there is some $I \in \mathcal{V}$ with $x \in I$ and $0 < \lambda(I) < \epsilon$.

The following is the **Vitali covering theorem**.[1]

Theorem 1 (Vitali covering theorem)**.** *Let U be an open set in $\mathbb{R}$ with $\lambda(U) < \infty$, let $E \subset U$, and let $\mathcal{V}$ be a Vitali covering of E each interval of which is contained in U. Then for any $\epsilon > 0$, there are disjoint $I_1, \ldots, I_n \in \mathcal{V}$ such that*

$$\lambda^* \left(E \setminus \bigcup_{j=1}^{n} I_j \right) < \epsilon.$$

Proof. Suppose that $I_1, \ldots, I_n \in \mathcal{V}$ are pairwise disjoint. If $E \subset \bigcup_{j=1}^{n} I_j$ then $I_1, \ldots, I_n$ satisfy the claim, and otherwise, let

$$U_n = U \setminus \bigcup_{j=1}^{n} I_j,$$

and there exists some $x \in E \cap U_n$. As $x \in U_n$ and U_n is open, there is some $\eta > 0$ such that $B(x, \eta) \subset U_n$ and then as $\mathcal{V}$ is a Vitali covering of E there is some $I \in \mathcal{V}$ with $x \in I \subset B(x, \eta) \subset U_n$. Thus $\delta_n > 0$ for

$$\delta_n = \sup \{\lambda(I) : I \in \mathcal{V}, I \subset U_n\},$$

and there is some $I_{n+1} \in \mathcal{V}$ with $I_{n+1} \subset U_n$ and $\lambda(I_{n+1}) > \frac{\delta_n}{2}$.

[1] Klaus Bichteler, *Integration – A Functional Approach*, p. 161, Lemma 10.5; John J. Benedetto and Wojciech Czaja, *Integration and Modern Analysis*, p. 179, Theorem 4.3.1; Russell A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, p. 52, Lemma 4.6.

For $j \geq 1$ write $I_j = [x_j - r_j, x_j + r_j]$ and let $J_j = [x_j - 5r_j, x_j + 5r_j]$, namely J_j is concentric with I_j and $\lambda(J_j) = 5\lambda(I_j)$. Then, as the intervals $I_1, I_2, \dots$ are pairwise disjoint Borel sets each contained in U,

$$\sum_{j=1}^{\infty} \lambda(J_j) = 5 \sum_{j=1}^{\infty} \lambda(I_j) = 5\lambda\left(\bigcup_{j=1}^{\infty} I_j\right) \leq 5\lambda(U) < \infty$$

and it follows from $\sum_{j=1}^{\infty} \lambda(J_j) < \infty$ that $\sum_{j=M}^{\infty} \lambda(J_j) \to 0$ as $M \to \infty$, which with

$$\lambda\left(\bigcup_{j=M}^{\infty} J_j\right) \leq \sum_{j=M}^{\infty} \lambda(J_j)$$

yields $\lambda\left(\bigcup_{j=M}^{\infty} J_j\right) \to 0$ as $M \to \infty$.

Let $M \geq 1$. If $x \in E \setminus \bigcup_{j=1}^{\infty} I_j$ then $x \in E \setminus \bigcup_{j=1}^{M} I_j$ and so $x \in U_M$, and as U_M is open there is some $\eta > 0$ with $B(x, \eta) \subset U_M$. But $x \in E$ and $\mathcal{V}$ is a Vitali covering of E, so there is some $I \in \mathcal{V}$ with $x \in I$ and $I \subset B(x, \eta) \subset U_M$. Now, $\lambda(I_{j+1}) > \frac{\delta_j}{2}$ and $\sum_{j=1}^{\infty} \lambda(I_j) < \infty$ together imply $\delta_n \to 0$ as $n \to \infty$, so there is some n for which $\delta_n < \lambda(I)$. By the definition of δ_n as a supremum, this means that $I \not\subset U_n$ and so it makes sense to define N to be a minimal positive integer such that $I \not\subset U_N$. $M < N$: if $M \geq N$ then $I \subset U_M \subset U_N$, contradicting $I \not\subset U_N$. (We shall merely use that $M \leq N$.) The fact that $I \not\subset U_N$ and $I \subset U_{N-1}$ means that $I \cap I_N \neq \emptyset$ and also, by the definition of δ_{N-1}, $\lambda(I) \leq \delta_{N-1} < 2\lambda(I_N)$. Write $I = [y - r, y + r]$. $I \cap I_N \neq \emptyset$ tells us $y - r \leq x_N + r_N$ and $y + r \geq x_N - r_N$, and $\lambda(I) < 2\lambda(I_N)$ tells us $2r < 4r_N$, hence

$$y + r \leq x_N + r_N + 2r \leq x_N + 5r_N, \qquad y - r \geq x_N - r_N - 2r \geq x_N - 5r_N,$$

showing that

$$x \in I = [y - r, y + r] \subset J_N \subset \bigcup_{j=M}^{\infty} J_j.$$

This is true for each $x \in E \setminus \bigcup_{j=1}^{\infty} I_j$, which means that

$$E \setminus \bigcup_{j=1}^{\infty} I_j \subset \bigcup_{j=M}^{\infty} J_j.$$

Because $\lambda(\bigcup_{j=M}^{\infty} J_j) \to 0$ as $M \to \infty$, this yields

$$\lambda^*\left(E \setminus \bigcup_{j=1}^{\infty} I_j\right) = 0.$$

But $E \setminus \bigcup_{j=1}^{n} I_j$ is an increasing sequence of sets tending to $E \setminus \bigcup_{j=1}^{\infty} I_j$, therefore

$$\lambda^*\left(E \setminus \bigcup_{j=1}^{n} I_j\right) \to \lambda^*\left(E \setminus \bigcup_{j=1}^{\infty} I_j\right) = 0, \qquad n \to \infty,$$

so there is some n such that $\lambda^* \left(E \setminus \bigcup_{j=1}^{n} I_j \right) < \epsilon$ and then $I_1, \ldots, I_n$ satisfy the claim. $\square$