# THE EULER-MACLAURIN SUMMATION FORMULA

#### JORDAN BELL

### 1. Proof

### 2. References

Whiteside [63, pp. 44, 257] Newton and Collins [8, pp. 186, 199]

Domingues [12, p. 44]

Todhunter [55, p. 192]

Estrada and Kanwal [17, p. 36]

Bourbaki [8, Chapter VI]

[18, pp. 45, 160, 337, 475, 531]

[53, pp. XL-XLIX]

Stirling [58, p. 274]

Euler correspondence R. 1998, 236; p. 53, 113, 137, 433

Institutiones calculi differentialis, E212

 $E19,\ E20,\ E25,\ E43,\ E47,\ E55,\ E125,\ E130,\ E247,\ E352,\ E368,\ E393,\ E432,\ E642,\ E746$ 

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