# Wiener measure and Donsker's theorem

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## 1 Relatively compact sets of Borel probability measures on *C[0,1]*

Let $E = C[0,1]$, let $\mathscr{B}_E$ be the Borel $\sigma$-algebra of $E$, and let $\mathscr{P}_E$ be the collection of Borel probability measures on $E$. We assign $\mathscr{P}$ the **narrow topology**, the coarsest topology on $\mathscr{P}_E$ such that for each $F \in C_b(E)$ the map $\mu \mapsto \int_E F d\mu$ is continuous.

For $f \in E$ and $\delta > 0$ we define

$$\omega_f(\delta) = \sup_{s,t\in[0,1], |s-t|\leq\delta} |f(s) - f(t)|.$$

For $f \in E$, $\omega_f(\delta) \downarrow 0$ as $\delta \downarrow 0$, and for $\delta > 0$, $f \mapsto \omega_f(\delta)$ is continuous. We shall use the following characterization of a relatively compact subset $A$ of $E$, which is proved using the Arzelà-Ascoli theorem.

**Lemma 1.** *Let $A$ be a subset of $E$. $\overline{A}$ is compact if and only if*

$$\sup_{f\in A} |f(0)| < \infty$$

*and*

$$\sup_{f\in A} \omega_f(\delta) \downarrow 0, \qquad \delta \downarrow 0.$$

We shall use **Prokhorov's theorem**:[1] for $X$ a Polish space and for $\Gamma \subset \mathscr{P}_X$, $\overline{\Gamma}$ is compact if and only if for each $\epsilon > 0$ there is a compact subset $K_\epsilon$ of $X$ such that $\mu(K_\epsilon) \geq 1 - \epsilon$ for all $\mu \in \Gamma$. Namely, a subset of $\mathscr{P}_X$ is relatively compact if and only if it is **tight**. We use Prokhorov's theorem to prove a characterization of relatively compact subsets of $\mathscr{P}_E$, which we then use to prove the characterization in Theorem 3.[2]

[1] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 47, Chapter II, Theorem 6.7.

[2] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 213, Chapter VII, Lemma 2.2.

**Lemma 2.** *Let $\Gamma$ be a subset of $\mathscr{P}_E$. $\overline{\Gamma}$ is compact if and only if for each $\epsilon > 0$ there is some $M_\epsilon < \infty$ and a function $\delta \mapsto \omega_\epsilon(\delta)$ satisfying $\omega_\epsilon(\delta) \downarrow 0$ as $\delta \downarrow 0$ and such that for all $\mu \in \Gamma$,*

$$\mu(A_\epsilon) \geq 1 - \frac{\epsilon}{2}, \qquad \mu(B_\epsilon) \geq 1 - \frac{\epsilon}{2},$$

*where*

$$A_\epsilon = \{f \in E : |f(0)| \leq M_\epsilon\}, \qquad B_\epsilon = \{f \in E : \omega_f(\delta) \leq \omega_\epsilon(\delta) \text{ for all } \delta > 0\}.$$

*Proof.* Suppose that $\Gamma$ satisfies the above conditions. Because $f \mapsto |f(0)|$ is continuous, $A_\epsilon$ is closed. For $\delta > 0$, suppose that $f_n$ is a sequence in $B_\epsilon$ tending to some $f \in E$. Because $g \mapsto \omega_g(\delta)$ is continuous, $\omega_{f_n}(\delta) \to \omega_f(\delta)$, and because $\omega_{f_n}(\delta) \leq \omega_\epsilon(\delta)$ for each $n$, we get $\omega_f(\delta) \leq \omega_\epsilon(\delta)$ and hence $f \in B_\epsilon$, showing that $B_\epsilon$ is closed. Therefore $K_\epsilon = A_\epsilon \cap B_\epsilon$ is closed, i.e. $K_\epsilon = \overline{K_\epsilon}$. The set $K_\epsilon$ satisfies

$$\sup_{f \in K_\epsilon} |f(0)| \leq M_\epsilon$$

and

$$\limsup_{\delta \downarrow 0} \sup_{f \in K_\epsilon} \omega_f(\delta) \leq \limsup_{\delta \downarrow 0} \omega_\epsilon(\delta) = 0,$$

thus by Lemma 1, $K_\epsilon$ is compact. For $\mu \in \Gamma$,

$$\mu(K_\epsilon) \geq 1 - \frac{\epsilon}{2},$$

and because $K_\epsilon$ is compact, this means that $\Gamma$ is tight, so by Prokhorov's theorem, $\Gamma$ is relatively compact.

Now suppose that $\Gamma$ is relatively compact and let $\epsilon > 0$. By Prokhorov's theorem, there is a compact set $K_\epsilon$ in $E$ such that $\mu(K_\epsilon) \geq 1 - \frac{\epsilon}{2}$ for all $\mu \in \Gamma$. Define

$$M_\epsilon = \sup_{f \in K_\epsilon} |f(0)|, \qquad \omega_\epsilon(\delta) = \sup_{f \in K_\epsilon} \omega_f(\delta), \qquad \delta > 0.$$

Because $K_\epsilon$ is compact, by Lemma 1 we get that $M_\epsilon < \infty$ and $\omega_\epsilon(\delta) \downarrow 0$ as $\delta \downarrow 0$. For $\mu \in \Gamma$,

$$\mu(A_\epsilon) \geq \mu(K_\epsilon) \geq 1 - \frac{\epsilon}{2}, \qquad \mu(B_\epsilon) \geq \mu(K_\epsilon) \geq 1 - \frac{\epsilon}{2},$$

showing that $\Gamma$ satisfies the conditions of the theorem. □

We now prove the characterization of relatively compact subsets of $\mathscr{P}_E$ that we shall use in our proof of Donsker's theorem.[3]

**Theorem 3** (Relatively compact sets in $\mathscr{P}$)**.** *Let $\Gamma$ be a subset of $\mathscr{P}_E$. $\overline{\Gamma}$ is compact if and only if the following conditions are satisfied:*

[3] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 214, Chapter VII, Theorem 2.2.

1. *For each $\epsilon > 0$ there is some $M_\epsilon < \infty$ such that*

$$\mu(f : |f(0)| \leq M_\epsilon) \geq 1 - \frac{\epsilon}{2}, \qquad \mu \in \Gamma.$$

2. *For each $\epsilon > 0$ and $\delta > 0$ there is some $\eta = \eta(\epsilon, \delta) > 0$ such that*

$$\mu(f : \omega_f(\eta) \leq \delta) \geq 1 - \frac{\epsilon}{2}, \qquad \mu \in \Gamma.$$

*Proof.* Suppose that $\overline{\Gamma}$ is compact and let $\epsilon > 0$. By Lemma 2, there is some $M_\epsilon < \infty$ and a function $\eta \mapsto \omega_\epsilon(\eta)$ satisfying $\omega_\epsilon(\eta) \downarrow 0$ as $\eta \downarrow 0$ and

$$\mu(A_\epsilon) \geq 1 - \frac{\epsilon}{2}, \qquad \mu(B_\epsilon) \geq 1 - \frac{\epsilon}{2}, \qquad \mu \in \Gamma.$$

For $\delta > 0$, there is some $\eta = \eta(\epsilon, \delta)$ with $\omega_\epsilon(\eta) \leq \delta$. Then for $\mu \in \Gamma$,

$$\mu(f : \omega_f(\eta) \leq \delta) \geq \mu(f : \omega_f(\eta) \leq \omega_\epsilon(\eta)) \geq \mu(B_\epsilon) \geq 1 - \frac{\epsilon}{2}.$$

Now suppose that the conditions of the theorem hold. For each $\epsilon > 0$ and $n \geq 1$ there is some $\eta_{\epsilon,n} > 0$ such that

$$\mu(F_{\epsilon,n}) \geq 1 - \frac{\epsilon}{2^{n+1}}, \qquad \mu \in \Gamma,$$

where

$$F_{\epsilon,n} = \left\{ f : \omega_f(\eta_{\epsilon,n}) \leq \frac{1}{n} \right\}.$$

Let

$$K_\epsilon = \{f : |f(0)| \leq M_\epsilon\} \cap \bigcap_{n=1}^{\infty} F_{\epsilon,n},$$

for which

$$\mu(K_\epsilon) \geq \mu(f : |f(0)| \leq M_\epsilon) \geq 1 - \frac{\epsilon}{2}, \qquad \mu \in \Gamma.$$

For $f \in K_\epsilon$, then for each $n \geq 1$ we have $f \in F_{\epsilon,n}$, which means that $\omega_f(\eta_{\epsilon,n}) \leq \frac{1}{n}$, and therefore

$$\sup_{f \in K_\epsilon} \omega_f(\eta_{\epsilon,n}) \leq \frac{1}{n}.$$

Thus for $n \geq 1$, if $0 < \eta \leq \eta_{\epsilon,n}$ then

$$\sup_{f \in K_\epsilon} \omega_f(\eta) \leq \frac{1}{n},$$

which shows $\sup_{f \in K_\epsilon} \omega_f(\eta) \downarrow 0$ as $\eta \downarrow 0$. Then because

$$\sup_{f \in K_\epsilon} |f(0)| \leq M_\epsilon,$$

applying Lemma 1 we get that $\overline{K_\epsilon}$ is compact. The map $f \mapsto \omega_f(\eta_{\epsilon,n})$ is continuous, so the set $F_{\epsilon,n}$ is closed, and therefore the set $K_\epsilon$ is closed. Because $K_\epsilon$ is compact and $\mu(K_\epsilon) \geq 1 - \frac{\epsilon}{2}$ for all $\mu \in \Gamma$, it follows from by Prokhorov's theorem that $\Gamma$ is relatively compact. $\square$

## 2 Wiener measure

For $t_1, \dots, t_d \in [0,1]$, $t_1 < \cdots < t_d$, define $\pi_{t_1,\dots,t_d} : E \to \mathbb{R}^d$ by

$$\pi_{t_1,\dots,t_d}(f) = (f(t_1), \dots, f(t_d)), \qquad f \in E,$$

which is continuous. We state the following results, which we will use later.[4]

**Theorem 4** (The Borel $\sigma$-algebra of $E$). *$\mathscr{B}_E$ is equal to the $\sigma$-algebra generated by $\{\pi_t : t \in [0,1]\}$.*

*Two elements $\mu$ and $\nu$ of $\mathscr{P}_E$ are equal if and only if for any $d$ and any $t_1 < \cdots < t_d$, the pushforward measures*

$$\mu_{t_1,\dots,t_d} = (\pi_{t_1,\dots,t_d})_* \mu, \qquad \nu_{t_1,\dots,t_d} = (\pi_{t_1,\dots,t_d})_* \nu$$

*are equal.*

Let $(\xi_t)_{t\in[0,1]}$ be a stochastic process with state space $\mathbb{R}$ and sample space $(\Omega, \mathscr{F}, P)$. For $t_1 < \cdots < t_d$, let $\xi_{t_1,\dots,t_d} = \xi_{t_1} \otimes \cdots \otimes \xi_{t_d}$ and let $P_{t_1,\dots,t_d} = (\xi_{t_1,\dots,t_d})_* P$: for $B \in \mathscr{B}_{\mathbb{R}}^d$,

$$P_{t_1,\dots,t_d}(B) = ((\xi_{t_1,\dots,t_d})_* P)(B) = P(\xi_{t_1,\dots,t_d}^{-1}(B)) = P((\xi_{t_1}, \dots, \xi_{t_d}) \in B).$$

$P_{t_1,\dots,t_d}$ is a Borel probability measure on $\mathbb{R}^d$ and is called a **finite-dimensional distribution of the stochastic process**.

The **Kolmogorov continuity theorem**[5] tells us that if there are $\alpha, \beta, K > 0$ such that for all $s, t \in [0,1]$,

$$E|\xi_t - \xi_s|^\alpha \le K|t-s|^{1+\beta},$$

then there is a unique $\mu \in \mathscr{P}_E$ such that for all $k$ and for all $t_1 < \cdots < t_d$,

$$\mu_{t_1,\dots,t_d} = P_{t_1,\dots,t_d}.$$

We now define and prove the existence of **Wiener measure**.[6]

**Theorem 5** (Wiener measure). *There is a unique Borel probability measure $W$ on $E$ satisfying:*

1. *$W(f \in E : f(0) = 0) = 1$.*
2. *For $0 \le t_0 < t_1 < \cdots < t_d \le 1$ the random variables*

$$\pi_{t_1} - \pi_{t_0}, \quad \pi_{t_2} - \pi_{t_1}, \quad \pi_{t_3} - \pi_{t_2}, \quad \pi_{t_d} - \pi_{t_{d-1}}$$

*are independent $(E, \mathscr{B}_E, W) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}})$.*

[4] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 212, Chapter VII, Theorem 2.1.

[5] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 216, Chapter VII, Theorem 3.1

[6] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 218, Chapter VII, Theorem 3.2.

3. *If* $0 \le s < t \le 1$*, the random variable* $\pi_t - \pi_s : (E, \mathscr{B}_E, W) \to (\mathbb{R}, \mathscr{B}_\mathbb{R})$ *is normal with mean* $0$ *and variance* $t - s$*.*

*Proof.* There is a stochastic process $(\xi_t)_{t\in[0,1]}$ with state space $\mathbb{R}$ and some sample space $(\Omega, \mathscr{F}, P)$, such that (i) $P(\xi_0 = 0) = 1$, (ii) $(\xi_t)_{t\in[0,1]}$ has independent increments, and (iii) for $s < t$, $\xi_t - \xi_s$ is a normal random variable with mean 0 and variance $t - s$. (Namely, **Brownian motion with starting point** 0.) Because $\xi_t - \xi_s$ has mean 0 and variance $t-s$, we calculate (cf. Isserlis's theorem)

$$E|\xi_t - \xi_s|^4 = 3|t - s|^2.$$

Thus using the Kolmogorov continuity theorem with $\alpha = 4$, $\beta = 1$, $K = 3$, there is a unique $W \in \mathscr{P}_E$ such that for all $t_1 < \cdots < t_d$,

$$W_{t_1,\ldots,t_d} = P_{t_1,\ldots,t_d},$$

i.e. for $B \in \mathscr{B}_\mathbb{R}^d$,

$$W(\pi_{t_1} \otimes \cdots \otimes \pi_{t_d} \in B) = P(\xi_{t_1} \otimes \cdots \otimes \xi_{t_d} \in B).$$

For $t_1 < \cdots < t_d$ and $B \in \mathscr{B}_\mathbb{R}^d$, with $T : \mathbb{R}^d \to \mathbb{R}^d$ defined by $T(x_1, \ldots, x_d) = (x_1, x_2 - x_1, \ldots, x_d - x_{d-1})$,

$$\begin{aligned}
&W(\pi_{t_1} \otimes (\pi_{t_2} - \pi_{t_1}) \otimes \cdots \otimes (\pi_{t_d} - \pi_{t_{d-1}}) \in B)\\
=&W(T \circ (\pi_{t_1} \otimes \pi_{t_2} \otimes \cdots \otimes \pi_{t_d}) \in B)\\
=&W(\pi_{t_1} \otimes \pi_{t_2} \otimes \cdots \otimes \pi_{t_d} \in T^{-1}(B))\\
=&P(\xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_d} \in T^{-1}(B))\\
=&P(T \circ (\xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_d}) \in B)\\
=&P(\xi_{t_1} \otimes (\xi_{t_2} - \xi_{t_1}) \otimes \cdots \otimes (\xi_{t_d} - \xi_{t_{d-1}}) \in B).
\end{aligned}$$

Hence, because $\xi_{t_1}, \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_d} - \xi_{t_{d-1}}$ are independent,

$$\begin{aligned}
&(\pi_{t_1} \otimes (\pi_{t_2} - \pi_{t_1}) \otimes \cdots \otimes (\pi_{t_d} - \pi_{t_{d-1}}))_* W\\
=&(\xi_{t_1} \otimes (\xi_{t_2} - \xi_{t_1}) \otimes \cdots \otimes (\xi_{t_d} - \xi_{t_{d-1}}))_* P\\
=&(\xi_{t_1})_* P \otimes (\xi_{t_2} - \xi_{t_1})_* P \otimes \cdots \otimes (\xi_{t_d} - \xi_{t_{d-1}})_* P\\
=&(\pi_{t_1})_* W \otimes (\pi_{t_2} - \pi_{t_1})_* W \otimes \cdots \otimes (\pi_{t_d} - \pi_{t_{d-1}})_* W,
\end{aligned}$$

which means that the random variables $\pi_{t_1}, \pi_{t_2} - \pi_{t_1}, \ldots, \pi_{t_d} - \pi_{t_{d-1}}$ are independent.

If $s < t$ and $B_1, B_2 \in \mathscr{B}_\mathbb{R}$, and for $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, y - x)$,

$$\begin{aligned}
W((\pi_s, \pi_t - \pi_s) \in (B_1, B_2)) &= W(T \circ (\pi_s, \pi_t) \in (B_1, B_2))\\
&= P((\xi_s, \xi_t) \in T^{-1}(B_1, B_2))\\
&= P((\xi_s, \xi_t - \xi_s) \in (B_1, B_2)),
\end{aligned}$$

which implies that $(\pi_t - \pi_s)_* W = (\xi_t - \xi_s)_* P$, and because $\xi_t - \xi_s$ is a normal random variable with mean 0 and variance $t - s$, so is $\pi_t - \pi_s$.

Finally,

$$W(f : f(0) = 0) = W(\pi_0 = 0) = P(\xi_0 = 0) = 1.$$

□

$(E, \mathscr{B}_E, W)$ is a probability space, and the stochastic process $(\pi_t)_{t\in[0,1]}$ is a Brownian motion.

# 3 Interpolation and continuous stochastic processes

Let $(\xi_t)_{t\in[0,1]}$ be a **continuous stochastic process** with state space $\mathbb{R}$ and sample space $(\Omega, \mathscr{F}, P)$. To say that the stochastic process is continuous means that for each $\omega \in \Omega$ the map $t \mapsto \xi_t(\omega)$ is continuous $[0,1] \to \mathbb{R}$. Define $\xi : \Omega \to E$ by

$$\xi(\omega) = (t \mapsto \xi_t(\omega)), \qquad \omega \in \Omega.$$

For $t \in [0,1]$ and $B$ a Borel set in $\mathbb{R}$,

$$\xi^{-1}\pi_t^{-1}B = \{\omega \in \Omega : \xi_t(\omega) \in B\} = \xi_t^{-1}B,$$

and because $\xi_t : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B}_\mathbb{R})$ is measurable this belongs to $\mathscr{F}$. But by Theorem 4, $\mathscr{B}_E$ is generated by the collection $\{\pi_t^{-1}B : t \in [0,1], B \in \mathscr{B}_\mathbb{R}\}$. Now, for $f : X \to Y$ and for a nonempty collection $\mathscr{F}$ of subsets of $Y$,[7]

$$\sigma(f^{-1}(\mathscr{F})) = f^{-1}(\sigma(\mathscr{F})).$$

Therefore $\xi^{-1}(\mathscr{B}_E) \subset \mathscr{F}$, which means that $\xi : (\Omega, \mathscr{F}) \to (E, \mathscr{B}_E)$ is measurable. This means that a continuous stochastic proess with index set $[0,1]$ induces a random variable with state space $E$. Then the pushforward measure of $P$ by $\xi$ is a Borel probability measure on $E$. We shall end up constructing a sequence of pushforward measures from a sequence of continuous stochastic processes, that converge in $\mathscr{P}_E$ to Wiener measure $W$.

Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables on a sample space $(\Omega, \mathscr{F}, P)$ with $E(X_n) = 0$ and $V(X_n) = 1$, and let $S_0 = 0$ and

$$S_k = \sum_{i=1}^{k} X_i.$$

Then $E(S_k) = 0$ and $V(S_k) = k$. For $t \geq 0$ let

$$Y_t = S_{[t]} + (t - [t])X_{[t]+1}.$$

[7] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 140, Lemma 4.23.

Thus, for $k \geq 0$ and $k \leq t \leq k+1$,

$$\begin{aligned} Y_t &= S_k + (t-k)X_{k+1} \\ &= S_k + (t-k)(S_{k+1} - S_k) \\ &= (1-t+k)S_k + (t-k)S_{k+1}. \end{aligned}$$

For each $\omega \in \Omega$, the map $t \mapsto Y_t(\omega)$ is piecewise linear, equal to $S_k(\omega)$ when $t = k$, and in particular it is continuous. For $n \geq 1$, define

$$X_t^{(n)} = n^{-1/2} Y_{nt} = n^{-1/2} S_{[nt]} + n^{-1/2}(nt - [nt])X_{[nt]+1}, \qquad t \in [0,1]. \quad (1)$$

For $0 \leq k \leq n$,

$$X_{k/n}^{(n)} = n^{-1/2} S_k.$$

For each $n \geq 1$, $(X_t^{(n)})_{t \in [0,1]}$ is a continuous stochastic process on the sample space $(\Omega, \mathscr{F}, P)$, and we denote by $P_n \in \mathscr{P}_E$ the pushforward measure of $P$ by $X^{(n)}$.

# 4 Donsker's theorem

**Lemma 6.** *If $Z_n$ and $U_n$ are random variables with state space $\mathbb{R}^d$ such that $Z_n \to Z$ in distribution and $U_n \to 0$ in distribution, then $Z_n + U_n \to 0$ in distribution.*

*If $Z_n$ are random variables with state space $\mathbb{R}$ that converge in distribution to some random variable $Z$ and $c_n$ are real numbers that converge to some real number $c$, then $c_n Z_n \to cZ$ in distribution.*

For $\sigma \geq 0$, let $\nu_{\sigma^2}$ be the Gaussian measure on $\mathbb{R}$ with mean 0 and variance $\sigma^2$. The **characteristic function** of $\nu_{\sigma^2}$ is, for $\sigma > 0$,

$$\widetilde{\nu}_{\sigma^2}(\xi) = \int_{\mathbb{R}} e^{i\xi x} d\nu_{\sigma^2}(x) = \int_{\mathbb{R}} e^{i\xi x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = e^{-\frac{1}{2}\sigma^2\xi^2},$$

and $\widetilde{\nu}_0(\xi) = 1$. One checks that $c_*\nu_1 = \nu_{c^2}$ for $c \geq 0$.

In following theorem and in what follows, $X^{(n)}$ is the piecewise linear stochastic process defined in (1). We prove that a sequence of finite-dimensional distributions converge to a Gaussian measure.[8]

**Theorem 7.** *For $0 \leq t_0 < t_1 < t_1 < \cdots < t_d \leq 1$, the random vectors*

$$(X_{t_1}^{(n)} - X_{t_0}^{(n)}, \ldots, X_{t_d}^{(n)} - X_{t_{d-1}}^{(n)}), \qquad (\Omega, \mathscr{F}, P) \to (\mathbb{R}^d, \mathscr{B}_{\mathbb{R}}^d),$$

*converge in distribution to $\nu_{t_1 - t_0} \otimes \cdots \otimes \nu_{t_d - t_{d-1}}$ as $n \to \infty$.*

[8] Bert Fristedt and Lawrence Gray, *A Modern Approach to Probability Theory*, p. 368, §19.1, Lemma 1.

*Proof.* For $0 < j \le d$ and $n \ge 1$ let

$$r_{j,n} = \frac{[nt_j]}{n}, \qquad U_{j,n} = X^{(n)}_{t_j} - X^{(n)}_{r_{j,n}},$$

and for $0 \le j < d$ and $n \ge 1$ let

$$s_{j,n} = \frac{\lceil nt_j \rceil}{n}, \qquad V_{j,n} = X^{(n)}_{s_{j,n}} - X^{(n)}_{t_j},$$

with which

$$\begin{aligned}(X^{(n)}_{t_1} - X^{(n)}_{t_0}, \dots, X^{(n)}_{t_d} - X^{(n)}_{t_{d-1}}) = (X^{(n)}_{r_{1,n}} - X^{(n)}_{s_{0,n}}, \dots, X^{(n)}_{r_{d,n}} - X^{(n)}_{s_{d-1,n}})\\ + (U_{1,n}, \dots, U_{d,n}) + (V_{0,n}, \dots, V_{d-1,n}).\end{aligned}$$

Because $E(X^{(n)}_t) = 0$,

$$E(U_{j,n}) = 0, \qquad E(V_{j,n}) = 0.$$

Furthermore,

$$\begin{aligned}
&V(U_{j,n})\\
=&V(X^{(n)}_{t_j} - X^{(n)}_{r_{j,n}})\\
=&n^{-1}V(S_{[nt_j]} + (nt_j - [nt_j])X_{[nt_j]+1} - S_{[nr_{j,n}]} - (nr_{j,n} - [nr_{j,n}])X_{[nr_{j,n}]+1})\\
=&n^{-1}V(S_{[nt_j]} + (nt_j - [nt_j])X_{[nt_j]+1} - S_{[nt_j]} - ([nt_j] - [nt_j])X_{[nr_{j,n}]+1})\\
=&n^{-1}(nt_j - [nt_j])^2 V(X_{[nt_j]+1})\\
=&n^{-1}(nt_j - [nt_j])^2,
\end{aligned}$$

and because $0 \le nt_j - [nt_j] < 1$ this tends to 0 as $n \to \infty$. Likewise, $V(V_{j,n}) \to 0$ as $n \to \infty$.

For $1 \le j \le d$,

$$\begin{aligned}
X^{(n)}_{r_{j,n}} - X^{(n)}_{s_{j-1,n}} &= n^{-1/2}S_{[nr_{j,n}]} + n^{-1/2}(nr_{j,n} - [nr_{j,n}])X_{[nr_{j,n}]+1}\\
&\quad - n^{-1/2}S_{[ns_{j-1,n}]} - n^{-1/2}(ns_{j-1,n} - [ns_{j-1,n}])X_{[ns_{j-1,n}]+1}\\
&= n^{-1/2}S_{[nt_j]} - n^{-1/2}S_{\lceil nt_{j-1}\rceil}\\
&= n^{-1/2}\frac{([nt_j] - \lceil nt_{j-1}\rceil - 1)^{1/2}}{([nt_j] - \lceil nt_{j-1}\rceil - 1)^{1/2}} \sum_{i=\lceil nt_{j-1}\rceil+1}^{[nt_j]} X_i.
\end{aligned}$$

By the central limit theorem,

$$([nt_j] - \lceil nt_{j-1}\rceil - 1)^{1/2} \sum_{i=\lceil nt_{j-1}\rceil+1}^{[nt_j]} X_i \to \nu_1$$

in distribution as $n \to \infty$. But

$$n^{-1/2}([nt_j] - \lceil nt_{j-1}\rceil - 1)^{1/2} \to (t_j - t_{j-1})^{1/2}$$

as $n \to \infty$, and $(t_j - t_{j-1})_*^{1/2}\nu_1 = \nu_{t_j - t_{j-1}}$, so by Lemma 6,

$$X^{(n)}_{r_{j,n}} - X^{(n)}_{s_{j-1,n}} \to \nu_{t_j - t_{j-1}}$$

in distribution as $n \to \infty$.

For sufficiently large $n$, depending on $t_0, \ldots, t_d$,

$$t_0 \le s_{0,n} < r_{1,n} \le t_1 \le s_{1,n} < r_{2,n} \le \cdots \le t_{d-1} \le s_{d-1,n} < r_{d,n} \le t_d.$$

Check that $(U_{1,n}, \ldots, U_{d,n}) \to 0$ in probability and that $(V_{0,n}, \ldots, V_{d-1,n}) \to 0$ in probability, and hence these random vectors converge to 0 in distribution as $n \to \infty$. The random variables $X^{(n)}_{r_{1,n}} - X^{(n)}_{s_{0,n}}, \ldots, X^{(n)}_{r_{d,n}} - X^{(n)}_{s_{d-1,n}}$ are independent, and therefore their joint distribution is equal to the product of their distributions. Now, if $\mu_n = \mu_n^1 \otimes \cdots \otimes \mu_n^d$ and $\mu_n^j \to \mu^j$ as $n \to \infty$, $1 \le j \le d$, then for $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\widetilde{\mu}_n(\xi) &= \widetilde{\mu}_n^1(\xi_1) \cdots \widetilde{\mu}_n^d(\xi_d) \\
&\to \widetilde{\mu}^1(\xi_1) \cdots \widetilde{\mu}^d(\xi_d) \\
&= (\mu^1 \otimes \cdots \otimes \mu^d)\tilde{}(\xi)
\end{aligned}$$

as $n \to \infty$, and therefore by **Lévy's continuity theorem**, $\mu_n \to \mu^1 \otimes \cdots \otimes \mu^d$ as $n \to \infty$. This means that the joint distribution of $X^{(n)}_{r_{1,n}} - X^{(n)}_{s_{0,n}}, \ldots, X^{(n)}_{r_{d,n}} - X^{(n)}_{s_{d-1,n}}$ converges to

$$\nu_{t_1 - t_0} \otimes \cdots \otimes \nu_{t_d - t_{d-1}}$$

as $n \to \infty$. Because $(U_{1,n}, \ldots, U_{d,n}) \to 0$ in distribution as $n \to \infty$ and $(V_{0,n}, \ldots, V_{d-1,n}) \to 0$ in distribution as $n \to \infty$, applying Lemma 6 we get that

$$(X^{(n)}_{t_1} - X^{(n)}_{t_0}, \ldots, X^{(n)}_{t_d} - X^{(n)}_{t_{d-1}}) \to \nu_{t_1 - t_0} \otimes \cdots \otimes \nu_{t_d - t_{d-1}}$$

in distribution as $n \to \infty$, completing the proof. □

Let $t_0 = 0$ and let $0 < t_1 < \cdots < t_d \le 1$. As $X^{(n)}_0 = 0$, the above lemma tells us that

$$(X^{(n)}_{t_1}, X^{(n)}_{t_2} - X^{(n)}_{t_1}, \ldots, X^{(n)}_{t_d} - X^{(n)}_{t_{d-1}}) \to \nu_{t_1} \otimes \nu_{t_2 - t_1} \otimes \cdots \otimes \nu_{t_d - t_{d-1}}$$

in distribution as $n \to \infty$. Define $g : \mathbb{R}^d \to \mathbb{R}^d$ by

$$g(x_1, x_2, \ldots, x_d) = (x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_d).$$

The function $g$ is continuous and satisfies

$$g \circ (X^{(n)}_{t_1} - X^{(n)}_{t_0}, \ldots, X^{(n)}_{t_d} - X^{(n)}_{t_{d-1}}) = (X^{(n)}_{t_1}, X^{(n)}_{t_2}, \ldots, X^{(n)}_{t_d}).$$

Then by the **continuous mapping theorem**,

$$(X^{(n)}_{t_1}, X^{(n)}_{t_2}, \ldots, X^{(n)}_{t_d}) \to g_*(\nu_{t_1} \otimes \nu_{t_2 - t_1} \otimes \cdots \otimes \nu_{t_d - t_{d-1}}) \tag{2}$$

in distribution as $n \to \infty$.[9]

We prove a result that we use to prove the next lemma, and that lemma is used in the proof of Donsker's theorem.[10]

**Lemma 8.** *For $\epsilon > 0$,*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \frac{1}{\delta} P\left( \max_{1 \le j \le [n\delta]+1} |S_j| > \epsilon n^{1/2} \right) = 0.$$

*Proof.* For each $\delta > 0$, by the central limit theorem,

$$([n\delta] + 1)^{-1/2} S_{[n\delta]+1} \to Z$$

in distribution as $n \to \infty$, where $Z_* P = \nu_1$. Because $\frac{([n\delta]+1)^{1/2}}{(n\delta)^{1/2}} \to 1$ as $n \to \infty$, by Lemma 6 we then get that

$$(n\delta)^{-1/2} S_{[n\delta]+1} \to Z$$

in distribution as $n \to \infty$. Now let $\lambda > 0$, and there is a sequence $\phi_k$ in $C_b(\mathbb{R})$ such that $\phi_k \downarrow 1_{(-\infty,-\lambda] \cup [\lambda,\infty)} = \chi_\lambda$ pointwise as $k \to \infty$. For each $k$, writing $X = S_{[n\delta]+1}$, using the change of variables formula,

$$\begin{aligned} P(|X| \ge \lambda(n\delta)^{1/2}) &= \int_\Omega \chi_{\lambda(n\delta)^{1/2}}(X(\omega)) dP(\omega) \\ &= \int_\Omega \chi_\lambda((n\delta)^{-1/2} X(\omega)) dP(\omega) \\ &\le \int_\Omega \phi_k((n\delta)^{-1/2} X(\omega)) dP(\omega) \\ &= E(\phi_k((n\delta)^{-1/2} X)). \end{aligned}$$

Therefore, by the continuous mapping theorem,

$$\begin{aligned} \limsup_{n \to \infty} P(|S_{[n\delta]+1}| \ge \lambda(n\delta)^{1/2}) &\le \lim_{n \to \infty} E(\phi_k((n\delta)^{-1/2} S_{[n\delta]+1})) \\ &= E(\phi_k \circ Z). \end{aligned}$$

Because $\phi_k \downarrow \chi_\lambda$ pointwise as $k \to \infty$, using the monotone convergence theorem and then using Chebyshev's inequality,

$$E(\phi_k \circ Z) \to E(\chi_\lambda \circ Z) = P(|Z| \ge \lambda) \le \lambda^{-3} E|Z|^3.$$

We have established that for each $\lambda > 0$,

$$\limsup_{n \to \infty} P(|S_{[n\delta]+1}| \ge \lambda(n\delta)^{1/2}) \le \lambda^{-3} E|Z|^3. \tag{3}$$

[9] Allan Gut, *Probability: A Graduate Course*, second ed., p. 245, Chapter 5, Theorem 10.4.

[10] Ioannis Karatzas and Steven E. Shreve, *Brownian Motion and Stochastic Calculus*, second ed., p. 68, Lemma 4.18.

Define

$$\tau = \min\{j \geq 1 : |S_j| > n^{1/2}\epsilon\}.$$

For $0 < \delta < \epsilon^2/2$, it is a fact that

$$\begin{aligned}
&P\left(\max_{0\leq j\leq [n\delta]+1} |S_j| > n^{1/2}\epsilon\right) \\
\leq &P(|S_{[n\delta]+1}| \geq n^{1/2}(\epsilon - (2\delta)^{1/2})) \\
&+ \sum_{j=1}^{[n\delta]} P(|S_{[n\delta]+1}| < n^{1/2}(\epsilon - (2\delta)^{1/2})|\tau = j)P(\tau = j).
\end{aligned}$$

If $\tau(\omega) = j$ and $|S_{[n\delta]+1}(\omega)| < n^{1/2}(\epsilon - (2\delta)^{1/2})$ then

$$|S_j(\omega) - S_{[n\delta]+1}(\omega)| \geq |S_j(\omega)| - |S_{[n\delta]+1}(\omega)| > n^{1/2}\epsilon - n^{1/2}(\epsilon - (2\delta)^{1/2}) = (2n\delta)^{1/2}.$$

But by Chebyshev's inequality and the fact that the random variables $X_1, X_2, \ldots$ are independent with mean 0 and variance 1,

$$P(|S_j - S_{[n\delta]+1}| > (2n\delta)^{1/2}) \leq \frac{1}{2n\delta}E((S_j - S_{[n\delta]+1})^2) = \frac{1}{2n\delta}([n\delta] - j) \leq \frac{1}{2},$$

so

$$P(|S_{[n\delta]+1}(\omega)| < n^{1/2}(\epsilon - (2\delta)^{1/2})|\tau = j) \leq \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
&P\left(\max_{0\leq j\leq [n\delta]+1} |S_j| > n^{1/2}\epsilon\right) \\
\leq &P(|S_{[n\delta]+1}| \geq n^{1/2}(\epsilon - (2\delta)^{1/2})) + \sum_{j=1}^{[n\delta]} \frac{1}{2} \cdot P(\tau = j) \\
= &P(|S_{[n\delta]+1}| \geq n^{1/2}(\epsilon - (2\delta)^{1/2})) + \frac{1}{2}P(\tau \leq [n\delta]) \\
= &P(|S_{[n\delta]+1}| \geq n^{1/2}(\epsilon - (2\delta)^{1/2})) + \frac{1}{2}P\left(\max_{0\leq j\leq [n\delta]+1} |S_j| > n^{1/2}\epsilon\right),
\end{aligned}$$

so

$$P\left(\max_{0\leq j\leq [n\delta]+1} |S_j| > n^{1/2}\epsilon\right) \leq 2P(|S_{[n\delta]+1}| \geq n^{1/2}(\epsilon - (2\delta)^{1/2})).$$

Now using (3) with $\lambda = (\epsilon - (2\delta)^{1/2})\delta^{-1/2}$,

$$\limsup_{n\to\infty} P(|S_{[n\delta]+1}| \geq (\epsilon - (2\delta)^{1/2})\delta^{-1/2}(n\delta)^{1/2}) \leq (\epsilon - (2\delta)^{1/2})^{-3}\delta^{3/2}E|Z|^3,$$

hence

$$\limsup_{n\to\infty} P\left(\max_{0\leq j\leq [n\delta]+1} |S_j| > n^{1/2}\epsilon\right) \leq 2(\epsilon - (2\delta)^{1/2})^{-3}\delta^{3/2}E|Z|^3.$$

Dividing both sides by $\delta$ and then taking $\delta \downarrow 0$ we obtain the claim. $\square$

We prove one more result that we use to prove Donsker's theorem.[11]

**Lemma 9.** *For $T > 0$ and $\epsilon > 0$,*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P\left( \max_{0 \leq k \leq [nT]+1} \max_{1 \leq j \leq [n\delta]+1} |S_{j+k} - S_k| > n^{1/2}\epsilon \right) = 0.$$

*Proof.* For $0 < \delta \leq T$, let $m = \lceil T/\delta \rceil$, so $T/m < \delta \leq T/(m-1)$. Then

$$\lim_{n \to \infty} \frac{[nT]+1}{[n\delta]+1} = \frac{T}{\delta} < m,$$

so for all $n \geq n_\delta$ it is the case that $[nT] + 1 < ([n\delta] + 1)m$. Suppose that $\omega \in \Omega$ is such that there are $1 \leq j \leq [n\delta] + 1$ and $0 \leq k \leq [nT] + 1$ satisfying

$$|S_{j+k}(\omega) - S_k(\omega)| > n^{1/2}\epsilon,$$

and then let $p = [k/([n\delta] + 1)]$, which satisfies $0 \leq p \leq m - 1$ and

$$([n\delta] + 1)p \leq k < ([n\delta] + 1)(p + 1).$$

Because $1 \leq j \leq [n\delta] + 1$, either

$$([n\delta] + 1)p < k + j \leq ([n\delta] + 1)(p + 1)$$

or

$$([n\delta] + 1)(p + 1) < k + j < ([n\delta] + 1)(p + 2).$$

We separate the first case into the cases

$$|S_k(\omega) - S_{([n\delta]+1)p}(\omega)| > \frac{1}{2}n^{1/2}\epsilon$$

and

$$|S_{j+k}(\omega) - S_{([n\delta]+1)p}(\omega)| > \frac{1}{2}n^{1/2}\epsilon,$$

and we separate the second case into the cases

$$|S_k - S_{([n\delta]+1)p}(\omega)| > \frac{1}{3}n^{1/2}\epsilon,$$

and

$$|S_{([n\delta]+1)p}(\omega) - S_{([n\delta]+1)(p+1)}(\omega)| > \frac{1}{3}n^{1/2}\epsilon,$$

and

$$|S_{([n\delta]+1)(p+1)}(\omega) - S_{([n+\delta]+1)(p+2)}(\omega)| > \frac{1}{3}n^{1/2}\epsilon.$$

[11] Ioannis Karatzas and Steven E. Shreve, *Brownian Motion and Stochastic Calculus*, second ed., p. 69, Lemma 4.19.

It follows that[12]

$$\begin{aligned}&\left\{\max_{1\le j\le[n\delta]+1}\max_{0\le k\le[nT]+1}|S_{j+k}-S_k|>n^{1/2}\epsilon\right\}\\ \subset&\bigcup_{p=0}^{m-1}\left\{\max_{1\le j\le[n\delta]+1}|S_{j+([n\delta]+1)p}-S_{([n\delta]+1)p}|>\frac{1}{3}n^{1/2}\epsilon\right\}.\end{aligned}$$

For $0\le p\le m-1$,

$$\begin{aligned}&P\left(\max_{1\le j\le[n\delta]+1}|S_{j+([n\delta]+1)p}-S_{([n\delta]+1)p}|>\frac{1}{3}n^{1/2}\epsilon\right)\\ \le&P\left(\max_{1\le j\le[n\delta]+1}|S_j|>\frac{1}{3}n^{1/2}\epsilon\right),\end{aligned}$$

so

$$\begin{aligned}&P\left\{\max_{1\le j\le[n\delta]+1}\max_{0\le k\le[nT]+1}|S_{j+k}-S_k|>n^{1/2}\epsilon\right\}\\ \le&\sum_{p=0}^{m-1}P\left(\max_{1\le j\le[n\delta]+1}|S_j|>\frac{1}{3}n^{1/2}\epsilon\right)\\ =&mP\left(\max_{1\le j\le[n\delta]+1}|S_j|>\frac{1}{3}n^{1/2}\epsilon\right).\end{aligned}$$

Lemma 8 tells us

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{1}{\delta}P\left(\max_{1\le j\le[n\delta]+1}|S_j|>\frac{1}{3}n^{1/2}\epsilon\right)=0,$$

and because $m\le\frac{T}{\delta}+1=\frac{T+\delta}{\delta}$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}P\left\{\max_{1\le j\le[n\delta]+1}\max_{0\le k\le[nT]+1}|S_{j+k}-S_k|>n^{1/2}\epsilon\right\}=0,$$

proving the claim. □

In the following, $P_n\in\mathscr{P}_E$ denotes the pushforward measure of $P$ by $X^{(n)}$, for $X^{(n)}$ defined in (1). We now prove **Donsker's theorem**.[13]

**Theorem 10** (Donsker's theorem)**.** *$P_n\to W$.*

*Proof.* We shall use Theorem 3 to prove that $\Gamma=\{P_n:n\ge 1\}$ is relatively compact in $\mathscr{P}_E$. For $n\ge 1$,

$$P_n(f\in E:|f(0)|=0)=P(\omega\in\Omega:|X_0^{(n)}(\omega)|=0)=1,$$

[12]This should be worked out more carefully. In Karatzas and Shreve, there is $m+1$ where I have $m$.

[13]Ioannis Karatzas and Steven E. Shreve, *Brownian Motion and Stochastic Calculus*, second ed., p. 70, Theorem 4.20.

thus the first condition of Theorem 3 is satisfied with $M_\epsilon = 0$. For the second condition of Theorem 3 to be satisfied it suffices that for each $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P\left( \sup_{0 \le s,t \le 1, |s-t| \le \delta} |X^{(n)}(s) - X^{(n)}(t)| > \epsilon \right) = 0.$$

Now,

$$P\left( \sup_{0 \le s,t \le 1, |s-t| \le \delta} |X_s^{(n)} - X_t^{(n)}| > \epsilon \right) = P\left( \sup_{0 \le s,t \le n, |s-t| \le n\delta} |Y_s - Y_t| > n^{1/2}\epsilon \right).$$

Also,

$$\begin{aligned} \sup_{0 \le s,t \le n, |s-t| \le n\delta} |Y_s - Y_t| \le & \sup_{0 \le s,t \le n, |s-t| \le n\delta} |Y - s - Y_t| \\ \le & \max_{1 \le j \le [n\delta]+1} \max_{0 \le k \le n+1} |S_{j+k} - S_k|, \end{aligned}$$

so applying Lemma 9,

$$\begin{aligned} & \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P\left( \sup_{0 \le s,t \le 1, |s-t| \le \delta} |X_s^{(n)} - X_t^{(n)}| > \epsilon \right) \\ \le & \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P\left( \max_{1 \le j \le [n\delta]+1} \max_{0 \le k \le n+1} |S_{j+k} - S_k| > n^{1/2}\epsilon \right) \\ \to & 0, \end{aligned}$$

from which we get that $\Gamma$ is tight in $\mathscr{P}_E$. $\square$