The inverse function theorem for Banach spaces

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

February 27, 2017

1 $\mathscr{L}(E;F)$ and $GL(E;F)$

Let E and F be Banach spaces. It is a fact that a linear map $f : E \to F$ is continuous if and only if[1]

$$\|f\| = \sup_{\|x\| \leq 1} \|f(x)\| < \infty.$$

Denote by

$$\mathscr{L}(E;F)$$

the set of continuous linear maps $E \to F$. It is a fact that $\mathscr{L}(E;F)$ is a Banach space.[2]

Let E, F, G be Banach spaces and let $f \in \mathscr{L}(E;F), g \in \mathscr{L}(F;G)$. One checks that $g \circ f \in \mathscr{L}(E;G)$ and

$$\|g \circ f\| \leq \|g\| \, \|f\| .$$

Let id_E be the identity map $\mathrm{id}_E x = x$. We write

$$I = I_E = \mathrm{id}_E.$$

For $f, g \in \mathscr{L}(E;E)$, write

$$gf = g \circ f.$$

$\mathscr{L}(E;E)$ is a Banach algebra.

Let $GL(E,F)$ be the set of those $f \in \mathscr{L}(E;F)$ for which there is some $g \in \mathscr{L}(F;E)$ such that $g \circ f = \mathrm{id}_E$ and $f \circ g = \mathrm{id}_F$. By the open mapping theorem, for $f \in \mathscr{L}(E;F)$, $f \in GL(E;F)$ if and only if f is a bijection.

We now define the **exponential map** on $\mathscr{L}(E;E)$.[3]

[1] Henri Cartan, *Differential Calculus*, p. 13, Theorem 1.4.1.
[2] Henri Cartan, *Differential Calculus*, p. 14, Theorem 1.4.2.
[3] Henri Cartan, *Differential Calculus*, p. 19, Theorem 1.7.1.

Lemma 1. *If $f \in \mathscr{L}(E;E)$, then $\sum_{k=0}^{n} \frac{1}{k!} f^k$ is a Cauchy sequence in $\mathscr{L}(E;E)$. Define*

$$\exp f = \sum_{k=0}^{\infty} \frac{1}{k!} f^k.$$

$$\exp 0_E = \mathrm{id}_E.$$

If $fg = gf$ then

$$\exp(f+g) = (\exp f)(\exp g).$$

In particular, $\exp f \in GL(E;E)$.

Proof.

$$\left\| \sum_{k=0}^{n} \frac{1}{k!} f^k - \sum_{k=m}^{n} \frac{1}{k!} f^k \right\| = \left\| \sum_{k=m+1}^{n} \frac{1}{k!} f^k \right\| \leq \sum_{k=m+1}^{n} \frac{1}{k!} \|f\|^k.$$

Because $e^{\|f\|} < \infty$, $\sum_{k=m+1}^{n} \frac{1}{k!} \|f\|^k \to 0$ as $m \to \infty$. Thus $\sum_{k=0}^{n} \frac{1}{k!} f^k$ is a Cauchy sequence in $\mathscr{L}(E;E)$. Then define $\exp f = \sum_{k=0}^{\infty} \frac{1}{k!} f^k \in \mathscr{L}(E;E)$.

If $fg = gf$ then applying the binomial theorem,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!} (f+g)^k &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} f^j g^{k-j} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{1}{j!} f^j \frac{1}{(k-j)!} g^{k-j} \\
&= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{1}{j!} f^j \frac{1}{(k-j)!} g^{k-j} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{j!} f^j \frac{1}{k!} g^k \\
&= \left(\sum_{j=0}^{\infty} \frac{1}{j!} f^j \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} g^k \right),
\end{aligned}$$

i.e.

$$\exp(f+g) = (\exp f)(\exp g).$$

Finally, $f(-f) = (-f)f$ so $\exp(f-f) = (\exp f)(\exp(-f))$. But $\exp(f-f) = \exp 0_E = \mathrm{id}_E$, so $\exp f \in GL(E;E)$, and $(\exp f)^{-1} = \exp(-f)$. $\square$

Lemma 2. *If $f \in \mathscr{L}(E;E)$ and $\|f\| < 1$ then $I - f \in GL(E;E)$, and*

$$(I-f)^{-1} = \sum_{k=0}^{\infty} f^k.$$

Proof. Let $g_n = \sum_{k=0}^n f^k$. For $n > m$, $g_n - g_m = \sum_{k=m+1}^n f^k$ and hence

$$\|g_n - g_m\| \le \sum_{k=m+1}^n \left\|f^k\right\| \le \sum_{k=m+1}^n \|f\|^k .$$

Because $\|f\| < 1$, the above inequality shows that g_n is a Cauchy sequence in $\mathscr{L}(E;E)$, hence there is some $g \in \mathscr{L}(E;E)$ such that $g_n \to g$. On the one hand,

$$\begin{aligned} g_n(I-f) &= \sum_{k=0}^n f^k(I-f) \\ &= \sum_{k=0}^n f^k - \sum_{k=0}^n f^{k+1} \\ &= f^0 - f^{n+1} \\ &= I - f^{n+1}, \end{aligned}$$

and because $\|f\| < 1$ this shows that $g_n(I-f) \to I$ as $n \to \infty$. On the other hand, because $g_n \to g$ we get $g_n(I-f) \to g(I-f)$. Therefore $g(I-f) = I$, which shows that $I - f \in GL(E;E)$ and

$$(I-f)^{-1} = g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} \sum_{k=0}^n f^k = \sum_{k=0}^\infty f^k.$$

□

We now prove that $GL(E;F)$ is open in $\mathscr{L}(E;F)$ and that $u \mapsto u^{-1}$ is continuous $GL(E;F) \to \mathscr{L}(F;E)$.[4]

Lemma 3. *$GL(E,F)$ is open in $\mathscr{L}(E,F)$.*

If $GL(E;F) \neq \emptyset$ then $\phi : GL(E;F) \to \mathscr{L}(F;E)$ defined by $\phi(u) = u^{-1}$ is continuous.

Proof. If $GL(E;F)$ is empty then it is open. Otherwise, take $u_0 \in GL(E;F)$. For $u \in \mathscr{L}(E;F)$, $u \in GL(E;F)$ if and only if $u_0^{-1} \circ u \in GL(E;E)$. Define $I_E - v = u_0^{-1}u$, i.e.

$$v = I_E - u_0^{-1}u = u_0^{-1}u_0 - u_0^{-1}u = u_0^{-1}(u_0 - u).$$

By Lemma 2, if $\|v\| < 1$ then $I_E - v \in GL(E;E)$. That is, if $\left\|u_0^{-1}(u_0-u)\right\| < 1$ then $u_0^{-1} \circ u \in GL(E;E)$ and then $u \in GL(E;F)$. But $\left\|u_0^{-1}(u_0-u)\right\| \le \left\|u_0^{-1}\right\| \|u_0 - u\|$, so if $\|u - u_0\| < \left\|u_0^{-1}\right\|^{-1}$ then $u \in GL(E;F)$. This shows that $GL(E;F)$ is open in $\mathscr{L}(E;F)$.

Let $u_0 \in GL(E;F)$. For $\|u - u_0\| < \left\|u_0^{-1}\right\|^{-1}$, let $v = I_E - u_0^{-1}u = u_0^{-1}(u_0 - u)$. Then $\|v\| \le \left\|u_0^{-1}\right\| \|u - u_0\| < 1$, so by Lemma 2 we have $I_E - v \in GL(E;E)$.

[4] Henri Cartan, *Differential Calculus*, p. 20, Theorem 1.7.3.

That is, $u_0^{-1}u \in GL(E;E)$. Now, $u_0^{-1}u = I_E - v$, so $u_0^{-1} = (I_E - v)u^{-1}$ and then $u^{-1} = (I_E - v)^{-1}u_0^{-1}$. Hence

$$\begin{aligned}\phi(u) - \phi(u_0) &= u^{-1} - u_0^{-1} \\ &= (I_E - v)^{-1}u_0^{-1} - u_0^{-1} \\ &= [(I_E - v)^{-1} - I_E]u_0^{-1} \\ &= \left[\sum_{k=1}^{\infty} v^k\right] u_0^{-1},\end{aligned}$$

so

$$\begin{aligned}\|\phi(u) - \phi(u_0)\| &\leq \left[\sum_{k=1}^{\infty} \|v\|^k\right] \left\|u_0^{-1}\right\| \\ &= \left\|u_0^{-1}\right\| \frac{\|v\|}{1 - \|v\|} \\ &= \left\|u_0^{-1}\right\| \frac{\left\|u_0^{-1}(u_0 - u)\right\|}{1 - \left\|u_0^{-1}(u_0 - u)\right\|} \\ &\leq \left\|u_0^{-1}\right\|^2 \frac{\|u - u_0\|}{1 - \left\|u_0^{-1}(u - u_0)\right\|} \\ &\leq \left\|u_0^{-1}\right\|^2 \frac{\|u - u_0\|}{1 - \left\|u_0^{-1}\right\| \|u - u_0\|}.\end{aligned}$$

This shows that ϕ is continuous at u_0. □

Let $E_1, \ldots, E_n$ and F be Banach spaces. Let

$$\mathscr{L}(E_1, \ldots, E_n; F)$$

be the set of continuous multilinear maps $E_1 \times \cdots \times E_n \to F$. It is a fact that a multilinear map $f : E_1 \times \cdots \times E_n \to F$ is continuous if and only if[5]

$$\|f\| = \sup_{\|x_1\| \leq 1, \ldots, \|x_n\| \leq 1} \|f(x_1, \ldots, x_n)\| < \infty.$$

This is a norm with which $\mathscr{L}(E_1, \ldots, E_n; F)$ is a Banach space.[6]

2 Differentiable functions

Let U be an open set in E and let $f_1, f_2 : U \to F$ be functions. We say that f_1 and f_2 are **tangential at** a if

$$m(r) = \sup_{\|x-a\| \leq r} \|f_1(x) - f_2(x)\| = o(r).$$

[5] Henri Cartan, *Differential Calculus*, p. 22, Theorem 1.8.1.
[6] Henri Cartan, *Differential Calculus*, p. 23, Exercise 2.

For $a \in U$, we say that f is **differentiable at** a if there is some $L_a \in \mathscr{L}(E;F)$ such that[7]

$$f(x) - f(a) - L_a(x-a) = o(\|x-a\|), \qquad x \to a.$$

Write

$$f'(a) = df(a) = L_a.$$

We say that f is differentiable if f is differentiable at each point in U. We say that $f : U \to F$ is C^1 if f is differentiable and $f' : U \to \mathscr{L}(E;F)$ is continuous.

We now state the **chain rule**.[8]

Theorem 4 (Chain rule)**.** *Let E, F, G be Banach spaces, let U be open in E, let V be open in F, and let $f : U \to F, g : V \to G$ be continuous. Suppose that $a \in U$, $f(a) \in V$, f is differentiable at $a \in U$, and g is differentiable at $f(a) \in V$. Then $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a).$$

[7] Henri Cartan, *Differential Calculus*, pp. 24–26.
[8] Henri Cartan, *Differential Calculus*, p. 27, Theorem 2.2.1.