

Oscillatory integrals

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August 4, 2014

1 Oscillatory integrals

Suppose that $\Phi \in C^\infty(\mathbb{R}^d)$, $\psi \in \mathcal{D}(\mathbb{R}^d)$, and that Φ is real-valued. Define $I : (0, \infty) \rightarrow \mathbb{C}$ by

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x)} \psi(x) dx, \quad \lambda > 0.$$

We call Φ a **phase** and ψ an **amplitude**, and $I(\lambda)$ an **oscillatory integral**.

The following proof follows Stein and Shakarchi.¹

Theorem 1. *If there is some $c > 0$ such that $|(\nabla\Phi)(x)| \geq c$ for all $x \in \text{supp } \psi$, then for each nonnegative integer N there is some $c_N \geq 0$ such that*

$$|I(\lambda)| \leq c_N \lambda^{-N}, \quad \lambda > 0.$$

Proof. There is some $h \in \mathcal{D}(\mathbb{R}^d)$, $h \geq 0$, such that $h(x) = 1$ for $x \in \text{supp } \psi$.² Define $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$a = h \frac{\nabla\Phi}{|\nabla\Phi|^2},$$

whose entries each belong to $\mathcal{D}(\mathbb{R}^d)$, and define $L : C^\infty(\mathbb{R}^d) \rightarrow \mathcal{D}(\mathbb{R}^d)$ by

$$Lf = \frac{1}{i\lambda} \sum_{k=1}^d a_k \partial_k f = \frac{1}{i\lambda} (a \cdot \nabla) f.$$

L satisfies, doing integration by parts and using the fact that a has compact support,

$$\int_{\mathbb{R}^d} (Lf) g dx = \frac{1}{i\lambda} \sum_{k=1}^d \int_{\mathbb{R}^d} a_k (\partial_k f) g dx = \frac{1}{i\lambda} \sum_{k=1}^d - \int_{\mathbb{R}^d} f \partial_k (ga) dx.$$

Thus the **transpose** of L is

$$L^t g = -\frac{1}{i\lambda} \sum_{k=1}^d \partial_k (ga) = -\frac{1}{i\lambda} \nabla \cdot (ga).$$

¹Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 325, Proposition 2.1.

²Walter Rudin, *Functional Analysis*, second ed., p. 162, Theorem 6.20.

Furthermore, in $\text{supp } \psi$,

$$\begin{aligned} L(e^{i\lambda\Phi}) &= e^{i\lambda\Phi} \sum_{k=1}^d a_k (\partial_k \Phi) \\ &= e^{i\lambda\Phi} \sum_{k=1}^d \frac{\partial_k \Phi}{|\nabla \Phi|^2} \partial_k \Phi \\ &= e^{i\lambda\Phi}. \end{aligned}$$

Thus for any positive integer N and for $x \in \text{supp } \psi$, $L(e^{i\lambda\Phi})(x) = e^{i\lambda\Phi(x)}$, hence

$$I(\lambda) = \int_{\mathbb{R}^d} L^N(e^{i\lambda\Phi}) \psi dx = \int_{\mathbb{R}^d} e^{i\lambda\Phi} (L^t)^N \psi dx.$$

But

$$\int_{\mathbb{R}^d} |(L^t)^N \psi| dx = \int_{\mathbb{R}^d} |\lambda^{-N} A_N| dx,$$

where $A_1 = \nabla \cdot (\psi a)$ and $A_n = \nabla \cdot (A_{n-1} a)$. With

$$c_N = \int_{\mathbb{R}^d} |A_N| dx < \infty,$$

we obtain

$$|I(\lambda)| = \left| \int_{\mathbb{R}^d} e^{i\lambda\Phi} (L^t)^N \psi dx \right| \leq \int_{\mathbb{R}^d} |(L^t)^N \psi| dx = c_N \lambda^{-N},$$

completing the proof. \square

The following is an estimate for a one-dimensional oscillatory integral without an amplitude term.³

Lemma 2. *Let $a < b$, and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued, that either $\Phi''(x) \geq 0$ for all $x \in [a, b]$ or $\Phi''(x) \leq 0$ for all $x \in [a, b]$, and that $\Phi'(x) \geq 1$ for all $x \in [a, b]$. Then*

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 3\lambda^{-1}, \quad \lambda > 0.$$

Proof. Write

$$L = \frac{1}{i\lambda\Phi'} \frac{d}{dx},$$

which satisfies

$$\int_a^b (Lf)g dx = \int_a^b \frac{1}{i\lambda\Phi'} f' g dx = \frac{1}{i\lambda\Phi'} f g \Big|_a^b - \int_a^b f \left(\frac{g}{i\lambda\Phi'} \right)' dx.$$

³Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 326, Proposition 2.2.

With $f = e^{i\lambda\Phi}$ and $g = 1$, we have $Lf = e^{i\lambda\Phi}$ and hence

$$\begin{aligned}\int_a^b e^{i\lambda\Phi} dx &= \frac{e^{i\lambda\Phi}}{i\lambda\Phi'} \Big|_a^b - \int_a^b e^{i\lambda\Phi} \left(\frac{1}{i\lambda\Phi'} \right)' dx \\ &= \frac{e^{i\lambda\Phi}}{i\lambda\Phi'} \Big|_a^b + \frac{1}{i\lambda} \int_a^b e^{i\lambda\Phi} (\Phi')^{-2} \Phi'' dx.\end{aligned}$$

For $\lambda > 0$, using that $\Phi'(x) \geq 1$ for all $x \in [a, b]$ the boundary terms have absolute value

$$\left| \frac{e^{i\lambda\Phi(b)}}{i\lambda\Phi'(b)} - \frac{e^{i\lambda\Phi(a)}}{i\lambda\Phi'(a)} \right| \leq \frac{1}{\lambda|\Phi'(b)|} + \frac{1}{\lambda|\Phi'(a)|} \leq \frac{2}{\lambda}.$$

Because $\Phi'' \geq 0$ or $\Phi'' \leq 0$ on $[a, b]$,

$$\begin{aligned}\frac{1}{\lambda} \left| \int_a^b e^{i\lambda\Phi} (\Phi')^{-2} \Phi'' dx \right| &\leq \frac{1}{\lambda} \int_a^b |(\Phi')^{-2} \Phi''| dx \\ &= \frac{1}{\lambda} \left| \int_a^b (\Phi')^{-2} \Phi'' dx \right| \\ &= \frac{1}{\lambda} \left| \frac{1}{\Phi'(a)} - \frac{1}{\Phi'(b)} \right| \\ &\leq \frac{1}{\lambda};\end{aligned}$$

the final inequality uses the fact that the two terms inside the absolute value are both ≥ 1 , and thus the absolute value can be bounded by the larger of them. Putting together the two inequalities,

$$\left| \int_a^b e^{i\lambda\Phi} dx \right| \leq \frac{2}{\lambda} + \frac{3}{\lambda} = 3\lambda^{-1}, \quad \lambda > 0,$$

proving the claim. \square

Lemma 3. *Let $a < b$, and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued, that either $\Phi''(x) \geq 0$ for all $x \in [a, b]$ or $\Phi''(x) \leq 0$ for all $x \in [a, b]$, and that there is some $\mu > 0$ such that $|\Phi'(x)| \geq \mu$ for all $x \in [a, b]$. Then*

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 3\mu^{-1}\lambda^{-1}, \quad \lambda > 0.$$

Proof. Φ' is continuous on $[a, b]$, so, by the intermediate value theorem, either $\Phi'(x) \geq \mu$ for all $x \in [a, b]$ or $\Phi'(x) \leq -\mu$ for all $x \in [a, b]$. Let $\epsilon = 1$ in the first case and $\epsilon = -1$ in the second case, and define $\Phi_0 = \epsilon \frac{\Phi}{\mu}$. Then applying Lemma 2, for $\lambda > 0$ we have, writing $\lambda_0 = \mu\lambda$,

$$\left| \int_a^b e^{i\lambda_0\Phi_0(x)} dx \right| \leq 3\lambda_0^{-1},$$

i.e.

$$\left| \int_a^b e^{i\epsilon\lambda\Phi(x)} dx \right| \leq 3(\mu\lambda)^{-1}.$$

If $\epsilon = 1$ this is the claim. If $\epsilon = -1$, then the above integral is the complex conjugate of the integral in the claim, and these have the same absolute values. \square

Theorem 4. *Let $a < b$, and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued, that either $\Phi''(x) \geq 0$ for all $x \in [a, b]$ or $\Phi''(x) \leq 0$ for all $x \in [a, b]$, and there is some $\mu > 0$ such that $|\Phi'(x)| \geq \mu$ for all $x \in [a, b]$. Suppose also that $\psi \in C^1(\mathbb{R})$. Then with*

$$c_\psi = 3 \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right),$$

we have

$$\left| \int_a^b e^{i\lambda\Phi(x)} \psi(x) dx \right| \leq c_\psi \mu^{-1} \lambda^{-1}.$$

Proof. Define $J : [a, b] \rightarrow \mathbb{C}$ by

$$J(x) = \int_a^x e^{i\lambda\Phi(u)} du,$$

which satisfies $J'(x) = e^{i\lambda\Phi(x)}$. Integrating by parts,

$$\int_a^b e^{i\lambda\Phi(x)} \psi(x) dx = \int_a^b J'(x) \psi(x) dx = J(x) \psi(x) \Big|_a^b - \int_a^b J(x) \psi'(x) dx,$$

and as $J(a) = 0$ this is equal to

$$J(b) \psi(b) - \int_a^b J(x) \psi'(x) dx.$$

Lemma 3 tells us that $|J(x)| \leq 3\mu^{-1}\lambda^{-1}$ for all $x \in [a, b]$, so

$$\left| J(b) \psi(b) - \int_a^b J(x) \psi'(x) dx \right| \leq 3\mu^{-1}\lambda^{-1} |\psi(b)| + 3\mu^{-1}\lambda^{-1} \int_a^b |\psi'(x)| dx,$$

proving the claim. \square

The following is **van der Corput's lemma**.⁴

Lemma 5 (van der Corput's lemma). *Let $a < b$ and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued and satisfies $\Phi''(x) \geq 1$ for all $x \in [a, b]$. Then*

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 8\lambda^{-1/2}, \quad \lambda > 0.$$

⁴Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 328, Proposition 2.3.

Proof. Because Φ' is strictly increasing on $[a, b]$, Φ' has at most one zero in this interval. If $\Phi'(x_0) = 0$, then for $x \geq x_0 + \lambda^{-1/2}$ we have $\Phi'(x) \geq \lambda^{-1/2}$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$,

$$\left| \int_{[x_0 + \lambda^{-1/2}, b]} e^{i\lambda\Phi(x)} dx \right| \leq 3\mu^{-1}\lambda^{-1} = 3\lambda^{-1/2}.$$

For $x \leq x_0 - \lambda^{-1/2}$ we have $\Phi'(x) \leq -\lambda^{-1/2}$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$,

$$\left| \int_{[a, x_0 - \lambda^{-1/2}]} e^{i\lambda\Phi(x)} dx \right| \leq 3\mu^{-1}\lambda^{-1} = 3\lambda^{-1/2}.$$

But

$$\left| \int_{[x_0 - \lambda^{-1/2}, x_0 + \lambda^{-1/2}] \cap [a, b]} e^{i\lambda\Phi(x)} dx \right| \leq \int_{[x_0 - \lambda^{-1/2}, x_0 + \lambda^{-1/2}] \cap [a, b]} dx \leq 2\lambda^{-1/2},$$

and

$$\int_a^b = \int_{[a, x_0 - \lambda^{-1/2}]} + \int_{[x_0 - \lambda^{-1/2}, x_0 + \lambda^{-1/2}] \cap [a, b]} + \int_{[x_0 + \lambda^{-1/2}, b]},$$

so

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 3\lambda^{-1/2} + 2\lambda^{-1/2} + 3\lambda^{-1/2} = 8\lambda^{-1/2}.$$

If there is no $x_0 \in [a, b]$ such that $\Phi'(x_0) = 0$, then either $\Phi' > 0$ on $[a, b]$ or $\Phi' < 0$ on $[a, b]$. In the first case, because Φ' is strictly increasing on $[a, b]$, $\Phi'(x) > \lambda^{-1/2}$ for $x \in [a + \lambda^{-1/2}, b]$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$ gives

$$\begin{aligned} \left| \int_a^b e^{i\lambda\Phi(x)} dx \right| &\leq \left| \int_{[a, a + \lambda^{-1/2}] \cap [a, b]} e^{i\lambda\Phi(x)} dx \right| + \left| \int_{[a + \lambda^{-1/2}, b]} e^{i\lambda\Phi(x)} dx \right| \\ &\leq \lambda^{-1/2} + 3\mu^{-1}\lambda^{-1} \\ &= 4\lambda^{-1/2}. \end{aligned}$$

In the second case, $\Phi'(x) < -\lambda^{-1/2}$ for $x \in [a, b - \lambda^{-1/2}]$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$ also gives

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 4\lambda^{-1/2}.$$

Therefore, if Φ' does not have a zero on $[a, b]$ then

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 4\lambda^{-1/2} < 8\lambda^{-1/2}.$$

□

Lemma 6. *Let $a < b$ and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued and that there is some $\mu > 0$ such that $|\Phi''(x)| \geq \mu$ for all $x \in [a, b]$. Then*

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 8\mu^{-1/2}\lambda^{-1/2}, \quad \lambda > 0.$$

Proof. Φ'' is continuous on $[a, b]$, so by the intermediate value theorem either $\Phi''(x) \geq \mu$ for all $x \in [a, b]$ or $\Phi''(x) \leq -\mu$ for all $x \in [a, b]$. Let $\epsilon = 1$ in the first case and $\epsilon = -1$ in the second case, and define $\Phi_0 = \epsilon \frac{\Phi}{\mu}$. Then $\Phi_0''(x) \geq 1$ for all $x \in [a, b]$, and applying Lemma 5,

$$\left| \int_a^b e^{i\mu\lambda\Phi_0(x)} dx \right| \leq 8(\mu\lambda)^{-1/2}, \quad \lambda > 0,$$

i.e.

$$\left| \int_a^b e^{i\epsilon\lambda\Phi(x)} dx \right| \leq 8(\mu\lambda)^{-1/2}, \quad \lambda > 0.$$

If $\epsilon = 1$ this is the inequality in the claim. If $\epsilon = -1$, then the above integral is the complex conjugate of the integral in the claim, and these have the same absolute values. \square

We use the above to prove the following estimate which involves an amplitude.⁵

Theorem 7. *Let $a < b$ and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued and that there is some $\mu > 0$ such that $|\Phi''(x)| \geq \mu$ for all $x \in [a, b]$. Suppose also that $\psi \in C^1(\mathbb{R})$. Then with*

$$c_\psi = 8 \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right),$$

we have

$$\left| \int_a^b e^{i\lambda\Phi(x)} \psi(x) dx \right| \leq c_\psi \mu^{-1/2} \lambda^{-1/2}, \quad \lambda > 0.$$

Proof. Define $J : [a, b] \rightarrow \mathbb{C}$ by

$$J(x) = \int_a^x e^{i\lambda\Phi(u)} du,$$

which satisfies $J'(x) = e^{i\lambda\Phi(x)}$. Integrating by parts,

$$\int_a^b e^{i\lambda\Phi(x)} \psi(x) dx = \int_a^b J'(x) \psi(x) dx = J(x) \psi(x) \Big|_a^b - \int_a^b J(x) \psi'(x) dx.$$

⁵Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 328, Corollary 2.4.

and as $J(a) = 0$ this is equal to

$$J(b)\psi(b) - \int_a^b J(x)\psi'(x)dx.$$

But for each $x \in [a, b]$ we have by Lemma 6 that $|J(x)| \leq 8\mu^{-1/2}\lambda^{-1/2}$, so

$$\left| J(b)\psi(b) - \int_a^b J(x)\psi'(x)dx \right| \leq 8\mu^{-1/2}\lambda^{-1/2}|\psi(b)| + 8\mu^{-1/2}\lambda^{-1/2} \int_a^b |\psi'(x)|dx,$$

completing the proof. \square

2 Bessel functions

For $n \in \mathbb{Z}$, the n th **Bessel function of the first kind** $J_n : \mathbb{R} \rightarrow \mathbb{R}$ is

$$J_n(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\lambda \sin x} e^{-inx} dx, \quad \lambda \in \mathbb{R}.$$

Let

$$I_1 = \left[0, \frac{\pi}{4}\right], \quad I_2 = \left[\frac{3\pi}{4}, \pi\right], \quad I_3 = \left[\pi, \frac{5\pi}{4}\right], \quad I_4 = \left[\frac{7\pi}{4}, 2\pi\right],$$

on which $|\cos x| \geq \frac{1}{\sqrt{2}}$, and

$$I_5 = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right], \quad I_6 = \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right],$$

on which $|\sin x| \geq \frac{1}{\sqrt{2}}$. Write $\Phi(x) = \sin x$ and $\psi(x) = e^{-inx}$. $\Phi'(x) = \cos(x)$

and $\Phi''(x) = -\sin(x)$, and for I_1, I_2, I_3, I_4 we apply Theorem 4 with $\mu = \frac{1}{\sqrt{2}}$.

For each of I_1, I_2, I_3, I_4 we compute $c_\psi = 3\left(1 + \frac{\pi n}{4}\right)$, which gives us

$$\left| \int_{I_k} e^{i\lambda\Phi(x)} \psi(x) dx \right| \leq c_\psi \mu^{-1} \lambda^{-1} = 3 \left(1 + \frac{\pi n}{4}\right) \cdot \sqrt{2} \cdot \lambda^{-1}.$$

For I_5 and I_6 , we apply Theorem 7 with $\mu = \frac{1}{\sqrt{2}}$. For each of I_5 and I_6 we compute $c_\psi = 8\left(1 + \frac{\pi n}{2}\right)$, which gives us

$$\left| \int_{I_k} e^{i\lambda\Phi(x)} \psi(x) dx \right| \leq c_\psi \mu^{-1/2} \lambda^{-1/2} = 8 \left(1 + \frac{\pi n}{2}\right) \cdot 2^{1/4} \cdot \lambda^{-1/2}.$$

Therefore

$$|J_n(\lambda)| \leq 4 \cdot \frac{1}{2\pi} \cdot 3 \left(1 + \frac{\pi n}{4}\right) \cdot \sqrt{2} \cdot \lambda^{-1} + 2 \cdot \frac{1}{2\pi} \cdot 8 \left(1 + \frac{\pi n}{2}\right) \cdot 2^{1/4} \cdot \lambda^{-1/2},$$

which shows that for each $n \in \mathbb{Z}$,

$$J_n(\lambda) = O_n(\lambda^{-1/2})$$

as $\lambda \rightarrow \infty$.