Hardy spaces

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

April 19, 2016

1 Hardy spaces

Let $D_r = \{z : |z| < r\}$. For a continuous function $f : D_1 \to \mathbb{C}$, let

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \qquad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{0 \leq \theta \leq 2\pi} |f(r^{i\theta})|.$$

Let H^p be the collection of analytic functions $f : D_1 \to \mathbb{C}$ such that $\|f\|_{H^p} < \infty$, where

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f).$$

Let h^p be the collection of harmonic functions $u : D_1 \to \mathbb{R}$ such that $\|u\|_{H^p} < \infty$.

Lemma 1. *For $0 < p \leq \infty$, H^p and h^p are linear spaces. If $p < q$ then $H^q \subset H^p$ and $h^q \subset h^p$. For an analytic function $f : D_1 \to \mathbb{C}$, $f \in H^p$ if and only if $\operatorname{Re} f, \operatorname{Im} f \in h^p$.*

Proof. For $a \geq 0$, $b \geq 0$,

$$(a + b)^p \leq \begin{cases} a^p + b^p & 0 < p < 1 \\ 2^{p-1}(a^p + b^p) & p > 1. \end{cases}$$

$$\begin{aligned} \|f + g\|_{H^p}^p &= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |(f + g)(re^{i\theta})|^p d\theta \\ &\leq 2^p(\|f\|_{H^p}^p + \|g\|_{H^p}^p) \\ &\leq 2^p \cdot 2(\|f\|_{H^p} + \|g\|_{H^p})^p, \end{aligned}$$

hence $\|f + g\|_{H^p} \leq 2^{1+1/p}(\|f\|_{H^p} + \|g\|_{H^p})$. It follows that H^p and h^p are linear spaces. $\square$

2 Subharmonic functions

Let D be a domain, namely, a nonempty connected open set in $\mathbb{C}$. A function $g : D \to \mathbb{R}$ is called **subharmonic** if it is continuous, and for any domain B with $\overline{B} \subset D$ and continuous function $U : \overline{B} \to \mathbb{R}$ such that $U|B$ is harmonic and such that $g(z) \leq U(z)$ for all $z \in \partial B$, it follows that $g(z) \leq U(z)$ for all $z \in B$.[1]

Theorem 2. *Let $g : D \to \mathbb{R}$ be continuous. g is subharmonic if and only if for any $a \in D$ there is some $r_a > 0$ such that $D_{r_a}(a) \subset D$ and for each $0 < r < r_a$,*

$$g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + re^{i\theta}) d\theta.$$

Lemma 3. *If $f : D \to \mathbb{C}$ is analytic and $0 < p < \infty$ then $g(z) = |f(z)|^p$ is subharmonic.*

Theorem 4. *Let $g : D_1 \to \mathbb{R}$ be subharmonic and define*

$$m(r) = \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) d\theta, \qquad 0 \leq r < 1.$$

m is increasing and $r \mapsto m(e^r)$ is convex.

[2]

Theorem 5 (Fejér-Riesz inequality)**.** *If $f \in H^p$, then*

$$\int_{-1}^{1} |f(x)|^p dx \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta.$$

[3]

Theorem 6. *If $1 < p < \infty$ there is some A_p such that if $u \in h^p$ and v is the harmonic conjugate of u, $v(0) = 0$, then*

$$M_p(r, v) \leq A_p M_p(r, u), \qquad 0 \leq r < 1.$$

[1] Peter L. Duren, *Theory of H^p Spaces*, p. 7, Theorem 1.4.
[2] Peter L. Duren, *Theory of H^p Spaces*, p. 46, Theorem 3.13.
[3] Peter L. Duren, *Theory of H^p Spaces*, p. 54, Theorem 4.1.