The Schwartz space and the Fourier transform

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

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1 Schwartz functions

Let $\mathscr{S}(\mathbb{R}^n)$ be the collection of Schwartz functions $\mathbb{R}^n \to \mathbb{C}$. For $p \geq 0$ and $\phi \in \mathscr{S}$, write

$$\|\phi\|_p^2 = \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} (1+|x|^2)^p |(D^\nu \phi)(x)|^2 dx.$$

With the metric

$$d(\phi, \psi) = \sum_{p \geq 0} 2^{-p} \frac{\|\phi - \psi\|_p}{1 + \|\phi - \psi\|_p},$$

$\mathscr{S}$ is a Fréchet space.

For a multi-index α and for $\phi \in \mathscr{S}$, $x \mapsto x^\alpha \phi(x)$ belongs to $\mathscr{S}$ and we define $X^\alpha : \mathscr{S} \to \mathscr{S}$ by $(X^\alpha \phi)(x) = x^\alpha \phi(x)$. $D^\alpha \phi \in \mathscr{S}$ and

$$\|D^\alpha \phi\|_p^2 = \sum_{|\nu| \leq p} \int_{\mathbb{R}} (1+|x|^2)^p |(D^{\nu+\alpha}\phi)(x)|^2 dx \leq \|\phi\|_{p+|\alpha|}^2.$$

Because $|\{\mu : |\mu| = k\}| = \binom{n+k-1}{k}$,[1]

$$|\{\mu : \mu \leq \nu\}| \leq |\{\mu : |\mu| \leq |\nu|\}| \leq \binom{n+|\nu|}{|\nu|}.$$

The product rule states

$$D^\nu(fg) = \sum_{\mu \leq \nu} \binom{\nu}{\mu} (D^\mu f)(D^{\nu-\mu} g),$$

[1] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, p. 71, Identity 143 and p. 74, Identity 149.

and with the Cauchy-Schwarz inequality we obtain for $|\nu| \le p$,

$$
\begin{aligned}
|D^\nu (X^\alpha \phi)|^2 &= \left| \sum_{\mu \le \nu} \binom{\nu}{\mu} (D^\mu \phi)(D^{\nu-\mu} X^\alpha) \right|^2 \\
&\le \binom{n+p}{p} \sum_{|\mu| \le p} \binom{\nu}{\mu}^2 |D^\mu \phi|^2 |D^{\nu-\mu} X^\alpha|^2,
\end{aligned}
$$

and with this

$$
\begin{aligned}
\|X^\alpha \phi\|_p^2 &= \sum_{|\nu| \le p} \int_{\mathbb{R}^n} (1+|x|^2)^p |(D^\nu (X^\alpha \phi))(x)|^2 dx \\
&\le \sum_{|\nu| \le p} \int_{\mathbb{R}^n} (1+|x|^2)^p \binom{n+p}{p} \sum_{|\mu| \le p} \binom{\nu}{\mu}^2 |D^\mu \phi|^2 |D^{\nu-\mu} X^\alpha|^2 dx \\
&\le C_p \|\phi\|_{p+|\alpha|}^2 .
\end{aligned}
$$

For $g, \phi \in \mathscr{S}$ we have $g\phi \in \mathscr{S}$, and using the product rule we get

$$
\|g\phi\|_p^2 \le C_{p,g} \|\phi\|_p^2 .
$$

Therefore,

$$
\phi \mapsto D^\alpha \phi, \qquad \phi \mapsto X^\alpha \phi, \qquad \phi \mapsto g\phi
$$

are continuous linear maps $\mathscr{S} \to \mathscr{S}$.

2 Tempered distributions

For $u : \mathscr{S} \to \mathbb{C}$, we write

$$
\langle \phi, u \rangle = u(\phi).
$$

$\mathscr{S}'$ denotes the dual space of $\mathscr{S}$, and the elements of $\mathscr{S}'$ are called **tempered distributions**. We assign $\mathscr{S}'$ the weak-* topology, the coarsest topology on $\mathscr{S}'$ such that for each $\phi \in \mathscr{S}$ the map $u \mapsto \langle \phi, u \rangle$ is continuous $\mathscr{S}' \to \mathbb{C}$.

For $\psi \in \mathscr{S}$, we define $\Lambda_\psi : \mathscr{S} \to \mathbb{C}$ by

$$
\langle \phi, \Lambda_\psi \rangle = \int_{\mathbb{R}^n} \phi(x) \psi(x) dx, \qquad \phi \in \mathscr{S},
$$

and by the Cauchy-Schwarz inequality,

$$
|\langle \phi, \Lambda_\psi \rangle| \le \left(\int_{\mathbb{R}^n} |\phi(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |\psi(x)|^2 dx \right)^{1/2} = \|\psi\|_0 \|\phi\|_0 ,
$$

whence $\Lambda_\psi \in \mathscr{S}'$. It is apparent that $\psi \mapsto \Lambda_\psi$ is linear. Suppose that $\psi_i \to \psi$ in $\mathscr{S}$, and let $\phi \in \mathscr{S}$. Then

$$
|\langle \phi, \Lambda_{\psi_i} \rangle - \langle \phi, \Lambda_\psi \rangle| = |\langle \phi, \Lambda_{\psi_i - \psi} \rangle| \le \|\psi_i - \psi\|_0 \|\phi\|_0 \to 0,
$$

which shows that $\psi \mapsto \Lambda_\psi$ is continuous. If $\Lambda_\psi = 0$, then in particular $\Lambda_\psi \overline{\psi} = 0$, i.e. $\int_{\mathbb{R}^n} |\psi(x)|^2 dx = 0$, which implies that $\psi(x) = 0$ for almost all x and because ψ is continuous, $\psi = 0$. Therefore, $\psi \mapsto \Lambda_\psi$ is a continuous linear injection $\mathscr{S} \to \mathscr{S}'$. It can be proved that $\Lambda(\mathscr{S})$ is dense in $\mathscr{S}'$.[2]

For a multi-index α and $u \in \mathscr{S}'$, we define $D^\alpha u : \mathscr{S} \to \mathbb{C}$ by

$$\langle \phi, D^\alpha u \rangle = (-1)^{|\alpha|} \langle D^\alpha \phi, u \rangle, \qquad \phi \in \mathscr{S}.$$

For $\phi_i \to \phi$ in $\mathscr{S}$, because $D^\alpha : \mathscr{S} \to \mathscr{S}$ and $u : \mathscr{S} \to \mathbb{C}$ are continuous,

$$\langle \phi_i, D^\alpha u \rangle = (-1)^{|\alpha|} \langle D^\alpha \phi_i, u \rangle \to (-1)^{|\alpha|} \langle D^\alpha \phi, u \rangle = \langle \phi, D^\alpha u \rangle,$$

and therefore $D^\alpha u \in \mathscr{S}'$.

We define $X^\alpha u : \mathscr{S} \to \mathbb{C}$ by

$$\langle \phi, X^\alpha u \rangle = \langle X^\alpha \phi, u \rangle, \qquad \phi \in \mathscr{S}.$$

For $\phi_i \to \phi$ in $\mathscr{S}$,

$$\langle \phi_i, X^\alpha u \rangle = \langle X^\alpha \phi_i, u \rangle \to \langle X^\alpha \phi, u \rangle = \langle \phi, X^\alpha u \rangle,$$

and therefore $X^\alpha u \in \mathscr{S}'$.

For $g \in \mathscr{S}$, we define $gu : \mathscr{S} \to \mathbb{C}$ by

$$\langle \phi, gu \rangle = \langle g\phi, u \rangle, \qquad \phi \in \mathscr{S}.$$

For $\phi_i \to \phi$ in $\mathscr{S}$,

$$\langle \phi_i, gu \rangle = \langle g\phi_i, u \rangle \to \langle g\phi, u \rangle = \langle \phi, gu \rangle,$$

and therefore $gu \in \mathscr{S}'$.

For $\psi \in \mathscr{S}$, integrating by parts yields

$$\begin{aligned} \langle \phi, D^\alpha \Lambda_\psi \rangle &= (-1)^{|\alpha|} \langle D^\alpha \phi, \Lambda_\psi \rangle \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (D^\alpha \phi)(x) \psi(x) dx \\ &= \int_{\mathbb{R}^n} \phi(x) (D^\alpha \psi)(x) dx \\ &= \langle \phi, \Lambda_{D^\alpha \psi} \rangle, \end{aligned}$$

which implies that $D^\alpha \Lambda_\psi = \Lambda_{D^\alpha \psi}$.

$$\langle \phi, X^\alpha \Lambda_\psi \rangle = \langle X^\alpha \phi, \Lambda_\psi \rangle = \int_{\mathbb{R}^n} x^\alpha \phi(x) \psi(x) dx = \langle \phi, \Lambda_{X^\alpha \psi} \rangle,$$

which implies that $X^\alpha \Lambda_\psi = \Lambda_{X^\alpha \psi}$.

[2] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged edition, p. 144, Corollary 1 to Theorem V.14.

$$\langle \phi, g\Lambda_\psi \rangle = \langle g\phi, \Lambda_\psi \rangle = \int_{\mathbb{R}^n} g(x)\phi(x)\psi(x)dx = \langle \phi, \Lambda_{g\psi} \rangle ,$$

which implies that $g\Lambda_\psi = \Lambda_{g\psi}$.

Because $\phi \mapsto D^\alpha \phi$, $\phi \mapsto X^\alpha \phi$, and $\phi \mapsto g\phi$ are continuous linear maps $\mathscr{S} \to \mathscr{S}$ and because $\Lambda : \mathscr{S} \to \mathscr{S}'$ is a continuous linear map with dense image, using the above it is proved that

$$u \mapsto D^\alpha u, \qquad u \mapsto X^\alpha u, \qquad u \mapsto gu$$

are continuous linear maps $\mathscr{S}' \to \mathscr{S}'$.[3]

3 The Fourier transform

For Borel measurable functions $f, g : \mathbb{R}^n \to \mathbb{C}$, for those x for which the integral exists we write

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy, \qquad x \in \mathbb{R}^n,$$

and for those Borel measurable $f, g : \mathbb{R}^n \to \mathbb{C}$ for which the integral exists we write

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

For $\xi \in \mathbb{R}^n$ we define

$$e_\xi(x) = e^{2\pi i \xi \cdot x}, \qquad x \in \mathbb{R}^n,$$

and for $\phi \in \mathscr{S}$ we calculate, integrating by parts,

$$(D^\alpha \phi) * e_\xi = (2\pi i \xi)^\alpha \phi * e_\xi.$$

We define $\mathscr{F}\phi : \mathbb{R}^n \to \mathbb{C}$ by

$$(\mathscr{F}\phi)(\xi) = \langle \phi, e_\xi \rangle_{L^2} = \int_{\mathbb{R}^n} \phi(x)\overline{e_\xi(x)}dx = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi}\phi(x)dx, \qquad \xi \in \mathbb{R}^n,$$

which we can write as

$$(\phi * e_\xi)(0) = \int_{\mathbb{R}^n} \phi(y)e_\xi(-y)dy = \int_{\mathbb{R}^n} \phi(y)\overline{e_\xi(y)}dy = (\mathscr{F}\phi)(\xi).$$

By Fubini's theorem,

$$\begin{aligned}
\mathscr{F}(\phi * \psi)(\xi) &= \int_{\mathbb{R}^n} \psi(y) \left(\int_{\mathbb{R}^n} \phi(x-y)\overline{e_\xi(x)}dx \right) dy \\
&= \int_{\mathbb{R}^n} \psi(y) \left(\int_{\mathbb{R}^n} \phi(x)\overline{e_\xi(x+y)}dx \right) dy,
\end{aligned}$$

[3] Richard Melrose, *Introduction to Microlocal Analysis*, `http://math.mit.edu/~rbm/iml/Chapter1.pdf`, p. 17.

whence

$$\mathscr{F}(\phi * \psi) = (\mathscr{F}\phi)(\mathscr{F}\psi).$$

We calculate

$$\mathscr{F}(D^\alpha \phi)(\xi) = ((D^\alpha \phi) * e_\xi)(0) = ((2\pi i \xi)^\alpha \phi * e_\xi)(0) = (2\pi i \xi)^\alpha (\mathscr{F}\phi)(\xi),$$

whence

$$\mathscr{F}(D^\alpha \phi) = (2\pi i)^{|\alpha|} X^\alpha \mathscr{F}\phi.$$

It follows from the dominated convergence theorem

$$\begin{aligned}
(D^\alpha \mathscr{F}\phi)(\xi) &= \int_{\mathbb{R}^n} (-2\pi i x)^\alpha e^{-2\pi i x\cdot\xi} \phi(x) dx \\
&= (-2\pi i)^{|\alpha|} \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} x^\alpha \phi(x) dx \\
&= (-2\pi i)^{|\alpha|} \mathscr{F}(X^\alpha \phi)(\xi).
\end{aligned}$$

Therefore

$$\mathscr{F} D^\alpha = (2\pi i)^{|\alpha|} X^\alpha \mathscr{F}, \qquad D^\alpha \mathscr{F} = (-2\pi i)^{|\alpha|} \mathscr{F} X^\alpha. \tag{1}$$

Using the multinomial theorem,

$$\begin{aligned}
(1+|\xi|^2)^p |(D^\nu \mathscr{F}\phi)(\xi)|^2 &= \sum_{k=0}^p \binom{p}{k} |\xi|^{2k} |(D^\nu \mathscr{F}\phi)(\xi)|^2 \\
&= \sum_{k=0}^p \binom{p}{k} \sum_{|\alpha|=k} \binom{k}{\alpha} \xi^{2\alpha} |(D^\nu \mathscr{F}\phi)(\xi)|^2 \\
&= \sum_{k=0}^p \binom{p}{k} \sum_{|\alpha|=k} \binom{k}{\alpha} |(\xi^\alpha D^\nu \mathscr{F}\phi)(\xi)|^2.
\end{aligned}$$

Applying (1),

$$|(\xi^\alpha D^\nu \mathscr{F}\phi)(\xi)| = (2\pi)^{|\nu|} (2\pi)^{-|\alpha|} |(\mathscr{F} D^\alpha X^\nu \phi)(\xi)|.$$

Then

$$\begin{aligned}
\|\mathscr{F}\phi\|_p^2 &= \sum_{|\nu|\leq p} \int_{\mathbb{R}^n} (1+|\xi|^2)^p |(D^\nu \mathscr{F}\phi)(\xi)|^2 d\xi \\
&= \sum_{|\nu|\leq p} \int_{\mathbb{R}^n} \sum_{k=0}^p \binom{p}{k} \sum_{|\alpha|=k} \binom{k}{\alpha} |(\xi^\alpha D^\nu \mathscr{F}\phi)(\xi)|^2 d\xi \\
&= \sum_{|\nu|\leq p} (2\pi)^{2|\nu|} \sum_{k=0}^p \binom{p}{k} (2\pi)^{-2k} \sum_{|\alpha|=k} \binom{k}{\alpha} \int_{\mathbb{R}^n} |(\mathscr{F} D^\alpha X^\nu \phi)(\xi)|^2 d\xi.
\end{aligned}$$

Applying the Plancherel theorem, the product rule, and the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_{\mathbb{R}^n} |(\mathscr{F} D^\alpha X^\nu \phi)(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |(D^\alpha X^\nu \phi)(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} \left| \sum_{\beta \le \alpha} (D^\beta X^\nu)(D^{\alpha-\beta}\phi) \right|^2 d\xi \\ &\le \int_{\mathbb{R}^n} \sum_{\beta \le \alpha} |(D^\beta X^\nu)(\xi)|^2 \cdot \sum_{\beta \le \alpha} |(D^{\alpha-\beta}\phi)(\xi)|^2.\end{aligned}$$

This yields

$$\|\mathscr{F}\phi\|_p \le C_p \|\phi\|_p ,$$

whence $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is continuous.

For $p > n/2$, using the Cauchy-Schwarz inequality and spherical coordinates[4] we calculate

$$\begin{aligned}|(\mathscr{F}\phi)(\xi)| &\le \int_{\mathbb{R}^n} (1+|x|^2)^{-p/2}(1+|x|^2)^{p/2}|\phi(x)|dx \\ &\le \left(\int_{\mathbb{R}^n} (1+|x|^2)^{-p} dx \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|x|^2)^p |\phi(x)|^2 dx \right)^{1/2} \\ &= \left(\int_0^\infty \int_{S^{n-1}} (1+r^2)^{-p} d\sigma r^{n-1} dr \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|x|^2)^p |\phi(x)|^2 dx \right)^{1/2} \\ &= \left(\frac{\pi^{n/2}\Gamma\left(p-\frac{n}{2}\right)}{\Gamma(p)} \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|x|^2)^p |\phi(x)|^2 dx \right)^{1/2} \\ &\le \left(\frac{\pi^{n/2}\Gamma\left(p-\frac{n}{2}\right)}{\Gamma(p)} \right)^{1/2} \|\phi\|_p .\end{aligned}$$

[4] `http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf`