Total variation, absolute continuity, and the Borel σ-algebra of $C(I)$

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1 Total variation

Let $a < b$. A **partition** of $[a, b]$ is a sequence $t_0, t_1, \ldots, t_n$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

The **total variation** of a function $f : [a, b] \to \mathbb{C}$ is

$$\mathrm{Var}_f[a, b] = \sup \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : t_0, t_1, \ldots, t_n \text{ is a partition of } [a, b] \right\}.$$

If $\mathrm{Var}_f[a, b] < \infty$ then we say that f has **bounded variation**.

Lemma 1. *If $a \le c < e < d \le b$, then*

$$\mathrm{Var}_f[c, d] = \mathrm{Var}_f[c, e] + \mathrm{Var}_f[e, d].$$

The following theorem establishes properties of functions of bounded variation.[1]

Theorem 2. *Suppose that $f : [a, b] \to \mathbb{R}$ is of bounded variation and define*

$$F(x) = \mathrm{Var}_f[a, x], \qquad x \in [a, b].$$

Then:

1. $|f(y) - f(x)| \le F(y) - F(x)$ *for all* $a \le x < y \le b$.
2. F *is a nondecreasing function.*
3. $F - f$ *and* $F + f$ *are nondecreasing functions.*
4. *For* $x_0 \in [a, b]$, f *is continuous at* x_0 *if and only if* F *is continuous at* x_0.

[1] Charalambos D. Aliprantis and Owen Burkinshaw, *Principles of Real Analysis*, third ed., p. 377, Theorem 39.10.

Proof. If $t_0, \ldots, t_n$ is a partition of $[a, x]$ then $t_0, \ldots, t_n, y$ is a partition of $[a, y]$, so

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| + |f(y) - f(x)| \leq F(y).$$

Since this is true for any partition $t_0, \ldots, t_n$ of $[a, x]$,

$$F(x) + |f(y) - f(x)| \leq F(y).$$

This shows in particular that $F(x) \leq F(y)$, and thus that F is nondecreasing.

For $a \leq x < y \leq b$,

$$f(y) - f(x) \leq |f(y) - f(x)| \leq F(y) - F(x),$$

thus

$$F(x) - f(x) \leq F(y) - f(y),$$

showing that $x \mapsto F(x) - f(x)$ is nondecreasing. Likewise,

$$f(x) - f(y) \leq |f(y) - f(x)| \leq F(y) - F(x),$$

thus

$$f(x) + F(x) \leq f(y) + F(y),$$

showing that $x \mapsto F(x) + f(x)$ is nondecreasing.

Suppose that F is continuous at x_0 and let $\epsilon > 0$. There is some $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|F(x) - F(x_0)| < \epsilon$. If $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq |F(x) - F(x_0)| < \epsilon,$$

showing that f is continuous at x_0.

Suppose that f is continuous at x_0 and let $\epsilon > 0$. Then there is some $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$, and such that $x_0 - \delta > a$. Let $x_0 - \delta < s < x_0$, and let $t_0, \ldots, t_n$ be a partition of $[s, b]$ such that

$$\mathrm{Var}_f[s, b] < \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| + \epsilon$$

and such that none of $t_0, \ldots, t_n$ is equal to x_0. Say that $t_k < x_0 < t_{k+1}$. Then

$$t_0, \ldots, t_k, x_0, t_{k+1}, \ldots, t_n$$

is a partition of $[s,b]$. For $t_k < x < x_0$ we have $|x - x_0| < \delta$ and therefore

$$\begin{aligned}
\mathrm{Var}_f[s,x] + \mathrm{Var}_f[x,b] &= \mathrm{Var}_f[s,b] \\
&< \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| + \epsilon \\
&\leq \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| + |f(x) - f(t_k)| \\
&+ |f(x_0) - f(x)| \\
&+ |f(t_{k+1}) - f(x_0)| + \sum_{i=k+2}^{n} |f(t_i) - f(t_{i-1})| + \epsilon \\
&\leq \mathrm{Var}_f[s,x] + |f(x) - f(x_0)| + \mathrm{Var}_f[x_0,b] + \epsilon \\
&< \mathrm{Var}_f[s,x] + \mathrm{Var}_f[x_0,b] + 2\epsilon,
\end{aligned}$$

giving

$$\mathrm{Var}_f[x,b] - \mathrm{Var}_f[x_0,b] < 2\epsilon.$$

As $\mathrm{Var}_f[a,b] = \mathrm{Var}_f[a,x] + \mathrm{Var}_f[x,b]$ and also $\mathrm{Var}_f[a,b] = \mathrm{Var}_f[a,x_0] + \mathrm{Var}_f[x_0,b]$, we have $F(x) + \mathrm{Var}_f[x,b] = F(x_0) + \mathrm{Var}_f[x_0,b]$, and therefore

$$F(x_0) - F(x) < 2\epsilon.$$

Thus, if $t_k < x < x_0$ then $|F(x_0) - F(x)| < 2\epsilon$, showing that F is left-continuous at x_0. It is straightforward to show in the same way that F is right-continuous at x_0, and thus continuous at x_0. □

If $f : [a,b] \to \mathbb{R}$ is of bounded variation, then Theorem 2 tells us that F and $F + f$ are nondecreasing functions. A monotone function is differentiable almost everywhere,[2] and it follows that $f = (F + f) - F$ is differentiable almost everywhere.

2 Absolute continuity

Let $a < b$ and let $I = [a,b]$. A function $f : I \to \mathbb{C}$ is said to be **absolutely continuous** if for any $\epsilon > 0$ there is some $\delta > 0$ such that for any n and any collection of pairwise disjoint intervals $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ satisfying

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta,$$

we have

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \epsilon.$$

It is immediate that if f is absolutely continuous then f is uniformly continuous.

[2] Charalambos D. Aliprantis and Owen Burkinshaw, *Principles of Real Analysis*, third ed., p. 375, Theorem 39.9.

Lemma 3. *If $f : [a, b] \to \mathbb{C}$ is absolutely continuous then f has bounded variation.*

Proof. Because f is absolutely continuous, there is some $\delta > 0$ such that if $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are pairwise disjoint and

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta,$$

then

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < 1.$$

Let N be an integer that is $> \frac{b-a}{\delta}$ and let $a = x_0 < \cdots < x_N = b$ such that $x_i - x_{i-1} < \frac{b-a}{N}$ for each $i = 1, \ldots, N$. Then

$$\mathrm{Var}_f[a, b] = \sum_{i=1}^{N} \mathrm{Var}_f[x_{i-1}, x_i] \leq N,$$

showing that f has bounded variation. □

Let λ be Lebesgue measure on $\mathbb{R}$ and let $\mathfrak{M}$ be the collection of Lebesgue measurable subsets of $\mathbb{R}$.

The following theorem establishes connections between absolute continuity of a function and Lebesgue measure.[3] In the following theorem, we extend $f : [a, b] \to \mathbb{R}$ to $\mathbb{R} \to \mathbb{R}$ by defining $f(x) = f(b)$ for $x > b$ and $f(x) = f(a)$ for $x < a$. In particular, for any $x > b$, $f'(x)$ exists and is equal to 0, and for any $x < a$, $f'(x)$ exists and is equal to 0.

Theorem 4. *Suppose that $I = [a, b]$ and that $f : I \to \mathbb{R}$ is continuous and nondecreasing. Then the following statements are equivalent.*

1. *f is absolutely continuous.*
2. *If $E \subset I$ and $\lambda(E) = 0$ then $\lambda(f(E)) = 0$. (In words: f has the **Luzin property**.)*
3. *f is differentiable λ-almost everywhere on I, $f' \in L^1(\lambda)$, and*

$$f(x) - f(a) = \int_a^x f'(t) d\lambda(t), \qquad a \leq x \leq b.$$

Proof. Assume that f is absolutely continuous and let $E \subset I$ with $\lambda(E) = 0$. Let $E_0 = E \setminus \{a, b\}$; to prove that $\lambda(f(E)) = 0$ it suffices to prove that $\lambda(f(E_0)) = 0$.

[3] Walter Rudin, *Real and Complex Analysis*, third ed., p. 146, Theorem 7.18.

Let $\epsilon > 0$. As f is absolutely continuous, there is some $\delta > 0$ such that for any n and any collection of pairwise disjoint intervals $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$ satisfying

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta,$$

we have

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \epsilon.$$

There is an open set V such that $E_0 \subset V \subset I$ and such that $\lambda(V) < \delta$. (Lebesgue measure is outer regular.) There are countably many pairwise disjoint intervals (α_i, β_i) such that $V = \bigcup_i (\alpha_i, \beta_i)$. Then

$$\sum_{i} (\beta_i - \alpha_i) = \lambda(V) < \delta,$$

so for any n,

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta,$$

and because f is absolutely continuous it follows that

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \epsilon.$$

This is true for all n, so

$$\sum_{i} |f(\beta_i) - f(\alpha_i)| \leq \epsilon.$$

Because f is continuous and nondecreasing, $f(\alpha_i, \beta_i) = (f(\alpha_i), f(\beta_i))$ for each i. Therefore

$$f(V) = f\left(\bigcup_i (\alpha_i, \beta_i) \right) = \bigcup_i f(\alpha_i, \beta_i) = \bigcup_i (f(\alpha_i), f(\beta_i)),$$

which gives

$$\lambda(f(V)) = \sum_i (f(\beta_i) - f(\alpha_i)) = \sum_i |f(\beta_i) - f(\alpha_i)| \leq \epsilon.$$

This is true for all $\epsilon > 0$, so $\lambda(f(V)) = 0$. Because $f(E_0) \subset f(V)$, it follows that $f(E_0) \in \mathfrak{M}$ (Lebesgue measure is complete) and that $\lambda(f(E_0)) = 0$.

Assume that for all $E \subset I$ with $\lambda(E) = 0$, $\lambda(f(E)) = 0$. Define $g : I \to \mathbb{R}$ by

$$g(x) = x + f(x), \qquad x \in I.$$

Because f is continuous and nondecreasing, g is continuous and strictly increasing. Thus if $(\alpha,\beta)\subset I$ then $g(\alpha,\beta)=(g(\alpha),g(\beta))$ and so

$$\lambda(g(\alpha,\beta))=g(\beta)-g(\alpha)=\beta+f(\beta)-(\alpha+f(\alpha))=\beta-\alpha+f(\beta)-f(\alpha),$$

showing that if $J\subset I$ is an interval then $\lambda(g(J))=\lambda(J)+\lambda(f(J))$. Suppose that $E\subset I$ and $\lambda(E)=0$, and let $\epsilon>0$. There are countably many pairwise disjoint intervals (α_i,β_i) such that $E\subset\bigcup_i(\alpha_i,\beta_i)$ and $\sum_i(\beta_i-\alpha_i)<\epsilon$, and because $\lambda(f(E))=0$, there are countably many pairwise disjoint intervals (γ_i,δ_i) such that $f(E)\subset\bigcup_i(\gamma_i,\delta_i)$ and $\sum_i(\delta_i-\gamma_i)<\epsilon$. Let

$$N=f^{-1}\left(\bigcup_i(\gamma_i,\delta_i)\right)\cap\bigcup_i(\alpha_i,\beta_i)=\bigcup_{i,j}(f^{-1}(\gamma_i,\delta_i)\cap(\alpha_i,\beta_i))\in\mathfrak{M}.$$

We check that

$$\lambda(g(N))=\lambda(N)+\lambda(f(N)),$$

and because

$$\lambda(N)+\lambda(f(N))\leq\sum_i(\beta_i-\alpha_i)+\sum_i(\delta_i-\gamma_i)<2\epsilon$$

we have

$$\lambda(g(N))<2\epsilon.$$

Finally, $E\subset N$ so $g(E)\subset g(N)$. Therefore, for every $\epsilon>0$ there is some $N\in\mathfrak{M}$ with $g(E)\subset g(N)$ and $\lambda(g(N))<\epsilon$, from which it follows that $\lambda(g(E))=0$.

Suppose that $E\subset I$ belongs to $\mathfrak{M}$. Because $E\in\mathfrak{M}$, there are $E_0,E_1\in\mathfrak{M}$ such that $E=E_0\cup E_1$, $\lambda(E_0)=0$, and E_1 is a countable union of closed sets (namely, an F_σ-set). On the one hand, as $E_1\subset I$, E_1 is a countable union of compact sets, and because g is continuous, $g(E_1)$ is a countable union of compact sets, and in particular belongs to $\mathfrak{M}$. On the other hand, because $\lambda(E_0)=0$, $g(E_0)\in\mathfrak{M}$. Therefore $g(E)=g(E_0)\cup g(E_1)\in\mathfrak{M}$. Define $\mu:\mathfrak{M}\to[0,\infty)$ by

$$\mu(E)=\lambda(g(E\cap I)),\qquad E\in\mathfrak{M}.$$

If E_i are countably many pairwise disjoint elements of $\mathfrak{M}$, then $g(E_i\cap I)$ are pairwise disjoint elements of $\mathfrak{M}$, hence

$$\begin{aligned}\mu\left(\bigcup_i E_i\right)&=\lambda\left(g\left(\left(\bigcup_i E_i\right)\cap I\right)\right)\\&=\lambda\left(\bigcup_i g(E_i\cap I)\right)\\&=\sum_i\lambda(g(E_i\cap I))\\&=\sum_i\mu(E_i),\end{aligned}$$

showing that μ is a measure. If $\lambda(E) = 0$, then $\lambda(E \cap I) = 0$ so $\lambda(g(E \cap I)) = 0$, i.e. $\mu(E) = 0$. This shows that μ is absolutely continuous with respect to λ. Therefore by the Radon-Nikodym theorem[4] there is a unique $h \in L^1(\lambda)$ such that

$$\mu(E) = \int_E h d\lambda, \qquad E \in \mathfrak{M}.$$

$h(x) \geq 0$ for λ-almost all $x \in \mathbb{R}$.

Suppose that $x \in \mathbb{R}$ and let $E = [a, x]$. Then $g(E) = [g(a), g(x)]$, and

$$\mu(E) = \int_E h(t) d\lambda(t) = \int_a^x h(t) d\lambda(t).$$

On the other hand,

$$\mu(E) = \lambda(g(E)) = \lambda([g(a), g(x)]) = g(x) - g(a) = x + f(x) - (a + f(a)).$$

Hence

$$f(x) - f(a) = \int_a^x h(t) d\lambda(t) - (x - a),$$

i.e.,

$$f(x) - f(a) = \int_a^x (h(t) - 1) d\lambda(t).$$

By the Lebesgue differentiation theorem,[5] $f'(x) = h(x) - 1$ for λ-almost all $x \in \mathbb{R}$, and it follows that $f' \in L^1(\lambda)$ and

$$f(x) - f(a) = \int_a^x f'(t) d\lambda(t), \qquad x \in I.$$

Assume that f is differentiable λ-almost everywhere in I, $f' \in L^1(\lambda)$, and

$$f(x) - f(a) = \int_a^x f'(t) d\lambda(t), \qquad x \in I.$$

Let $\epsilon > 0$ and let $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ be pairwise disjoint intervals satisfying

$$\sum_{i=1}^n (\beta_i - \alpha_i) < \delta.$$

Because f is nondecreasing, for λ-almost all $x \in I$, $f'(x) \geq 0$, and hence the measure μ defined by $d\mu = f' d\lambda$ is absolutely continuous with respect to λ. It follows[6] that there is some $\delta > 0$ such that for $E \in \mathfrak{M}$, $\lambda(E) < \delta$ implies that $\mu(E) < \epsilon$. This gives us

$$\mu\left(\bigcup_{i=1}^n (\alpha_i, \beta_i)\right) < \epsilon,$$

[4] Walter Rudin, *Real and Complex Analysis*, third ed., p. 121, Theorem 6.10.
[5] Walter Rudin, *Real and Complex Analysis*, third ed., p. 141, Theorem 7.11.
[6] Walter Rudin, *Real and Complex Analysis*, third ed., p. 124, Theorem 6.11.

and as

$$\mu(\alpha_i, \beta_i) = \int_{\alpha_i}^{\beta_i} f'(t) d\lambda(t) = f(\beta_i) - f(\alpha_i),$$

we get

$$\sum_{i=1}^{n} f(\beta_i) - f(\alpha_i) < \epsilon.$$

This shows that f is absolutely continuous, completing the proof. $\square$

The following lemma establishes properties of the total variation of absolutely continuous functions.[7]

Lemma 5. *Suppose that $I = [a, b]$ and that $f : I \to \mathbb{R}$ is absolutely continuous. Then the function $F : I \to \mathbb{R}$ defined by*

$$F(x) = \mathrm{Var}_f[a, x], \qquad x \in I$$

is absolutely continuous.

Proof. Let $\epsilon > 0$. Because f is absolutely continuous, there is some $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_m, b_m)$ are disjoint intervals with $\sum_{k=1}^{m}(b_k - a_k) < \delta$, then

$$\sum_{k=1}^{m} |f(b_k) - f(a_k)| < \epsilon.$$

Suppose that $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are disjoint intervals with $\sum_{i=1}^{n}(\beta_i - \alpha_i) < \delta$. If $\alpha_i = t_{i,0} < \cdots < t_{i,m_i} = \beta_i$ for $i = 1, \ldots, n$, then $(t_{i,j-1}, t_{i,j})$, $1 \leq i \leq n$, $1 \leq j \leq m_i$, are disjoint intervals whose total length is $< \delta$, hence

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} |f(t_{i,j}) - f(t_{i,j-1})| < \epsilon.$$

It follows that

$$\sum_{i=1}^{n} |F(\beta_i) - F(\alpha_i)| = \sum_{i=1}^{n} \mathrm{Var}_f[\alpha_i, \beta_i] \leq \epsilon,$$

which shows that F is absolutely continuous. $\square$

We now prove the **fundamental theorem of calculus** for absolutely continuous functions.[8]

Theorem 6. *Suppose that $I = [a, b]$ and that $f : I \to \mathbb{R}$ is absolutely continuous. Then f is differentiable at almost all x in I, $f' \in L^1(\lambda)$, and*

$$f(x) - f(a) = \int_a^x f'(t) d\lambda(t), \qquad x \in I.$$

[7] Walter Rudin, *Real and Complex Analysis*, third ed., p. 147, Theorem 7.19.

[8] Walter Rudin, *Real and Complex Analysis*, third ed., p. 148, Theorem 7.20.

Proof. Define $F : I \to \mathbb{R}$ by

$$F(x) = \mathrm{Var}_f[a, x], \qquad x \in I.$$

By Lemma 3, f has bounded variation, and then using Theorem 2, $F - f$ and $F + f$ are nondecreasing. Furthermore, by Lemma 5, F is absolutely continuous, so $F - f$ and $F + f$ are absolutely continuous. Let

$$f_1 = \frac{F + f}{2}, \qquad f_2 = \frac{F - f}{2},$$

which are thus nondecreasing and absolutely continuous. Applying Theorem 4, we get that f_1, f_2 are differentiable at almost all $x \in I$, $f_1', f_2' \in L^1(\lambda)$, and

$$f_1(x) - f_1(a) = \int_a^x f_1'(t) d\lambda(t), \qquad a \leq x \leq b$$

and

$$f_2(x) - f_2(a) = \int_a^x f_2'(t) d\lambda(t), \qquad a \leq x \leq b.$$

Because $f = f_1 - f_2$, f is differentiable at almost all $x \in I$, $f' = f_1' - f_2' \in L^1(\lambda)$, and

$$f(x) - f(a) = \int_a^x f'(t) d\lambda(t), \qquad a \leq x \leq b,$$

proving the claim. □

3 Borel sets

Let $I = [a, b]$. Denote by $C(I)$ the set of continuous functions $I \to \mathbb{C}$, which with the norm

$$\|f\|_{C(I)} = \sup_{x \in I} |f(x)|, \qquad f \in C(I),$$

is a Banach space. Denote by $AC(I)$ the set of absolutely continuous functions $I \to \mathbb{C}$. Let $\mathscr{B}_{C(I)}$ be the Borel σ-algebra of $C(I)$. We have $AC(I) \subset C(I)$, and in the following theorem we prove that $AC(I)$ is a Borel set in $C(I)$.

Theorem 7. $AC(I) \in \mathscr{B}_{C(I)}$.

Proof. If X, Y are Polish spaces, $f : X \to Y$ is continuous, $A \in \mathscr{B}_X$, and $f|A$ is injective, then $f(A) \in \mathscr{B}_Y$.[9] Let $X = \mathbb{C} \times L^1(I)$, which is a Banach space with the norm

$$\|(A, g)\|_X = |A| + \int_a^b |g| d\lambda, \qquad (A, g) \in X.$$

[9] Alexander Kechris, *Classical Descriptive Set Theory*, p. 89, Theorem 15.1.

Furthermore, $\mathbb{C}$ and $L^1(I)$ are separable and thus so is X, so X is indeed a Polish space. The Banach space $C(I)$ is separable and thus is a Polish space. Define $\Phi : X \to C(I)$ by

$$\Phi(A, g)(x) = A + \int_a^x g(t) d\lambda(t), \qquad (A, g) \in X, \qquad x \in I.$$

For $(A_1, g_1), (A_2, g_2) \in X$,

$$\begin{aligned} \|\Phi(A_1, g_1) - \Phi(A_2, g_2)\|_{C(I)} &= \left\|(A_1 - A_2) + \int_a^x (g_1(t) - g_2(t)) d\lambda(t)\right\|_{C(I)} \\ &= |A_1 - A_2| + \sup_{x \in I} \left|\int_a^x (g_1(t) - g_2(t)) d\lambda(t)\right| \\ &\leq |A_1 - A_2| + \int_a^b |g_1(t) - g_2(t)| d\lambda(t) \\ &= \|(A_1, g_1) - (A_2, g_2)\|_X, \end{aligned}$$

which shows that $\Phi : X \to C(I)$ is continuous.

Let $(A, g) \in X$ and $\epsilon > 0$. Because $g \in L^1(I)$, there is some $\delta > 0$ such that if $\lambda(E) < \delta$ then $\int_E |g| d\lambda < \epsilon$.[10] If $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are disjoint intervals whose total length is $< \delta$, then, with $E = \bigcup_{i=1}^n (\alpha_i, \beta_i)$,

$$\begin{aligned} \sum_{i=1}^n |\Phi(A, g)(\beta_i) - \Phi(A, g)(\alpha_i)| &= \sum_{i=1}^n \left|\int_{\alpha_i}^{\beta_i} g(t) d\lambda(t)\right| \\ &\leq \sum_{i=1}^n \int_{\alpha_i}^{\beta_i} |g(t)| d\lambda(t) \\ &= \int_E |g| d\lambda \\ &< \epsilon, \end{aligned}$$

showing that $\Phi(A, g)$ is absolutely continuous. On the other hand, let $f \in AC(I)$. From Theorem 6, f is differentiable at almost all $x \in I$, $f' \in L^1(I)$, and

$$f(x) - f(a) = \int_a^x f'(t) d\lambda(t), \qquad x \in I.$$

Then $(f(a), f') \in X$, and the above gives us, for all $x \in I$,

$$\Phi(f(a), f')(x) = f(a) + \int_a^x f'(t) d\lambda(t) = f(x),$$

thus $\Phi(f(a), f') = f$. Therefore

$$\Phi(X) = AC(I).$$

[10] Walter Rudin, *Real and Complex Analysis*, third ed., p. 32, exercise 1.12.

If $\Phi(A_1, g_1) = \Phi(A_2, g_2)$, then $\Phi(A_1, g_1)(a) = \Phi(A_2, g_2)(a)$ gives $A_1 = A_2$. Using this, and defining $G : I \to \mathbb{C}$ by $G = \int_a^x (g_1(t) - g_2(t)) d\lambda(t)$, we have $G(x) = 0$ for all $x \in I$. Then $G'(x) = 0$ for all $x \in I$, and by the Lebesgue differentiation theorem[11] we have $G'(x) = g_1(x) - g_2(x)$ for almost all $x \in I$. That is, $g_1(x) = g_2(x)$ for almost all $x \in I$, and thus in $L^1(I)$ we have $g_1 = g_2$. Therefore $\Phi : X \to C(I)$ is injective.

Therefore $\Phi(X) \in \mathscr{B}_{C(I)}$. $\square$

[11] Walter Rudin, *Real and Complex Analysis*, third ed., p. 141, Theorem 7.11.