L^p norms of a sine sum

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The following is an estimate of the L^p norms of a sum with nonnegative

terms [1, p. 77, no. 38]. Let $\Gamma_n(t) = \sum_{k=1}^n \frac{|\sin kt|}{k}$. As $|\sin kt| \le 1$, we have $\Gamma_n(t) \le \sum_{k=1}^n \frac{1}{k}$, but we can give a sharper upper bound for $\Gamma_n(t)$ using the following two results. First, if $B_n(t) = \sum_{k=1}^n \frac{\cos kt}{k}$, then $B_n(t) \ge -1$ for all t [1, p. 75, no. 28]. Second [1, p. 76, no. 34], for all t,

$$|\sin t| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{i=1}^{\infty} \frac{\cos 2jt}{4j^2 - 1}.$$

Therefore

$$\Gamma_n(t) = \sum_{k=1}^n \frac{1}{k} \left(\frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{\cos 2jkt}{4j^2 - 1} \right)$$

$$= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{1}{4j^2 - 1} \sum_{k=1}^n \frac{\cos 2jkt}{k}$$

$$= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{B_n(2jt)}{4j^2 - 1}$$

$$\leq \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} + \frac{4}{\pi} \sum_{j=1}^\infty \frac{1}{4j^2 - 1}$$

$$= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} + \frac{2}{\pi},$$

using

$$\sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} = \sum_{j=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2j - 1} - \frac{1}{2} \cdot \frac{1}{2j + 1} = \frac{1}{2}.$$

Thus $\|\Gamma_n\|_p \le \|\Gamma_n\|_{\infty} \le \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} + \frac{2}{\pi}$. On the other hand,

$$\|\Gamma_n\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \Gamma_n(t)dt$$

$$= \frac{1}{2\pi} \sum_{k=1}^n \frac{1}{k} \int_0^{2\pi} |\sin kt| dt$$

$$= \frac{1}{2\pi} \sum_{k=1}^n \frac{1}{k^2} \int_0^{2\pi k} |\sin t| dt$$

$$= \frac{1}{2\pi} \sum_{k=1}^n \frac{1}{k} \int_0^{2\pi} |\sin t| dt$$

$$= \frac{1}{2\pi} \sum_{k=1}^n \frac{1}{k} \cdot 4$$

$$= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k}.$$

Therefore $\|\Gamma_n\|_p \ge \|\Gamma_n\|_1 = \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k}$. Hence for $1 \le p \le \infty$,

$$\frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k} \le \|\Gamma_n\|_p \le \frac{2}{\pi} + \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k}.$$

So,

$$\|\Gamma_n\|_p = \frac{2}{\pi} \log n + O(1).$$

References

[1] George Pólya and Gábor Szegő. Problems and theorems in analysis, volume II, volume 216 of Die Grundlehren der mathematischen Wissenschaften. Springer, 1976. Translated from the German by C. E. Billigheimer.