Tauber's theorem and Karamata's proof of the Hardy-Littlewood tauberian theorem

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November 11, 2017

The following lemma is attributed to Kronecker by Knopp.[1]

Lemma 1 (Kronecker's lemma). *If $b_n \to 0$ then*

$$\frac{b_0 + b_1 + \cdots + b_n}{n+1} \to 0.$$

Proof. Suppose that $|b_n| \le K$ for all n, and let $\epsilon > 0$. As $b_n \to 0$ there is some n_0 such that $n \ge n_0$ implies that $|b_n| < \epsilon$. If $n \ge \frac{(n_0+1)K}{\epsilon}$, then

$$\begin{aligned}\left|\frac{b_0 + b_1 + \cdots + b_n}{n+1}\right| &\le \left|\frac{b_0 + b_1 + \cdots + b_{n_0}}{n+1}\right| + \left|\frac{b_{n_0} + \cdots + b_n}{n+1}\right| \\ &\le \frac{(n_0+1)K}{n+1} + \frac{(n-n_0)\epsilon}{n+1} \\ &\le \epsilon + \epsilon.\end{aligned}$$

□

We now use the above lemma to prove Tauber's theorem.[2]

Theorem 2 (Tauber's theorem). *If $a_n = o(1/n)$ and $\sum_{n=0}^\infty a_n x^n \to s$ as $x \to 1^-$, then*

$$\sum_{n=0}^\infty a_n = s.$$

Proof. Let $\epsilon > 0$. Because $\sum_{n=0}^\infty a_n x^n \to s$ as $x \to 1^-$, there is some $\delta > 0$ such that $x > 1 - \delta$ implies that

$$\left|\sum_{n=0}^\infty a_n x^n - s\right| < \epsilon.$$

[1] Konrad Knopp, *Theory and Application of Infinite Series*, p. 129, Theorem 3.
[2] cf. E. C. Titchmarsh, *The Theory of Functions*, second ed., p. 10, §1.23.

Next, because $n|a_n| \to 0$, there is some $N > \frac{1}{\delta}$ such that (i) if $n \geq N$ then $n|a_n| < \epsilon$ and by Lemma 1, (ii) $\frac{1}{N+1}\sum_{n=0}^{N} n|a_n| < \epsilon$.

Take $x = 1 - \frac{1}{N}$, so $N = \frac{1}{1-x}$ and $1 - x = \frac{1}{N}$. We have

$$\begin{aligned}
\left|\sum_{n=N+1}^{\infty} a_n x^n\right| &= \left|\sum_{n=N+1}^{\infty} n a_n \cdot \frac{x^n}{n}\right| \\
&< \sum_{n=N+1}^{\infty} \epsilon \cdot \frac{x^n}{N+1} \\
&< \frac{\epsilon}{N+1} \cdot \frac{1}{1-x} \\
&= \epsilon \cdot \frac{N}{N+1} \\
&< \epsilon.
\end{aligned}$$

Also, using

$$1 - x^n = (1-x)(1 + x + \cdots + x^{n-1}) < (1-x)n$$

we have

$$\begin{aligned}
\left|\sum_{n=0}^{N} a_n(1-x^n)\right| &\leq \sum_{n=0}^{N} |a_n|(1-x^n) \\
&< \sum_{n=0}^{N} |a_n|(1-x)n \\
&= \sum_{n=0}^{N} \frac{|a_n|n}{N} \\
&= \frac{N+1}{N} \cdot \frac{1}{N+1} \sum_{n=0}^{N} n|a_n| \\
&< \frac{N+1}{N} \cdot \epsilon \\
&< 2\epsilon.
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{n=0}^{N} a_n - s &= \sum_{n=0}^{N} a_n - \sum_{n=0}^{N} a_n x^n + \sum_{n=0}^{N} a_n x^n - s \\
&= \sum_{n=0}^{N} a_n(1-x^n) + \sum_{n=0}^{N} a_n x^n - s \\
&= \sum_{n=0}^{N} a_n(1-x^n) + \sum_{n=0}^{N} a_n x^n + \sum_{n=N+1}^{\infty} a_n x^n - \sum_{n=N+1}^{\infty} a_n x^n - s \\
&= \sum_{n=0}^{N} a_n(1-x^n) + \sum_{n=0}^{\infty} a_n x^n - s - \sum_{n=N+1}^{\infty} a_n x^n
\end{aligned}$$

and then

$$\begin{aligned}
\left|\sum_{n=0}^{N} a_n - s\right| &\le \left|\sum_{n=0}^{N} a_n(1-x^n)\right| + \left|\sum_{n=0}^{\infty} a_n x^n - s\right| + \left|\sum_{n=N+1}^{\infty} a_n x^n\right| \\
&< 2\epsilon + \epsilon + \epsilon,
\end{aligned}$$

proving the claim. $\square$

Lemma 3. *Let $g : [0,1] \to \mathbb{R}$ and $0 < c < 1$. Suppose that the restrictions of g to $[0, c)$ and $[c, 1]$ are continuous and that*

$$g(c-0) = \lim_{x \to c^-} g(x) \le g(c).$$

For $\epsilon > 0$, there are are polynomials $p(x)$ and $P(x)$ such that

$$p(x) \le g(x) \le P(x), \qquad 0 \le x \le 1$$

and

$$\|g - p\|_1 \le \epsilon, \quad \|g - P\|_1 \le \epsilon.$$

Proof. There is some $\delta > 0$ such that $c - \delta \le x < c$ implies that

$$g(c-0) - \frac{\epsilon}{2} \le g(x) \le g(c-0) + \frac{\epsilon}{2};$$

further, take $\delta < \frac{\epsilon}{g(c)-g(c-0)}$ and $\delta < \frac{1}{2}$.

Take L to be the linear function satisfying

$$L(c-\delta) = g(c-\delta) + \frac{\epsilon}{2}, \qquad L(c) = g(c) + \frac{\epsilon}{2}.$$

For $c-\delta \le x < c$,

$$\begin{aligned}
L(x)-g(x) &= L(x)-g(c-\delta)+g(c-\delta)-g(c-0)+g(c-0)-g(x) \\
&= L(x)-L(c-\delta)+\frac{\epsilon}{2}+g(c-\delta)-g(c-0)+g(c-0)-g(x) \\
&\le L(c)-L(c-\delta)+\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2} \\
&= g(c)-g(c-\delta)+\frac{3\epsilon}{2} \\
&= g(c)-g(c-0)+g(c-0)-g(c-\delta)+\frac{3\epsilon}{2} \\
&< \frac{\epsilon}{\delta}+\frac{\epsilon}{2}+\frac{3\epsilon}{2} \\
&< \frac{2\epsilon}{\delta}.
\end{aligned}$$

Define $\Phi : [0,1] \to \mathbb{R}$ by

$$\Phi(x) = \begin{cases} g(x)+\frac{\epsilon}{2} & 0 \le x < c-\delta \\ \max\{L(x), g(x)+\frac{\epsilon}{2}\} & c-\delta \le x \le c \\ g(x)+\frac{\epsilon}{2} & c < x \le 1. \end{cases}$$

Φ is continuous and $\Phi \ge g+\frac{\epsilon}{2}$. We have

$$\begin{aligned}
\|g-\Phi\|_1 &= \int_0^1 (\Phi(x)-g(x))dx \\
&= \int_0^{c-\delta} \frac{\epsilon}{2}dx + \int_{c-\delta}^c (\Phi(x)-g(x))dx + \int_c^1 \frac{\epsilon}{2}dx \\
&< \frac{\epsilon}{2} + \int_{c-\delta}^c (\Phi(x)-g(x))dx \\
&\le \frac{\epsilon}{2} + \int_{c-\delta}^c \max\left\{L(x)-g(x), \frac{\epsilon}{2}\right\} dx \\
&\le \frac{\epsilon}{2} + \int_{c-\delta}^c \max\left\{\frac{2\epsilon}{\delta}, \frac{\epsilon}{2}\right\} dx \\
&= \frac{\epsilon}{2} + \delta \cdot \frac{2\epsilon}{\delta} \\
&= \frac{5\epsilon}{2}.
\end{aligned}$$

Because Φ is continuous, by the Weierstrass approximation theorem there is a polynomial $P(x)$ such that $\|\Phi - P\|_\infty \le \frac{\epsilon}{2}$. Then,

$$g(x) \le P(x), \qquad 0 \le x \le 1,$$

and

$$\|g-P\|_1 \le \|g-\Phi\|_1 + \|\Phi-P\|_1 < \frac{5\epsilon}{2} + \|\Phi-P\|_\infty \le \frac{5\epsilon}{2}+\frac{\epsilon}{2} = 3\epsilon.$$

On the other hand, take l to be the linear function satisfying

$$l(c-\delta) = g(c-\delta) - \frac{\epsilon}{2}, \qquad l(c) = g(c) - \frac{\epsilon}{2}.$$

One checks that for $c - \delta \le x < c$.

$$g(x) - l(x) < \frac{2\epsilon}{\delta},$$

Define $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} g(x) - \frac{\epsilon}{2} & 0 \le x < c - \delta \\ \min\{l(x), g(x) - \frac{\epsilon}{2}\} & c - \delta \le x \le c \\ g(x) - \frac{\epsilon}{2} & c < x \le 1, \end{cases}$$

which is continuous and satisfies $\phi \le g - \frac{\epsilon}{2}$. One checks that

$$\|g - \phi\|_1 < \frac{5\epsilon}{2}.$$

Because ϕ is continuous, there is a polynomial $p(x)$ such that $\|\phi - p\|_\infty \le \frac{\epsilon}{2}$. Then,

$$p(x) \le g(x), \qquad 0 \le x \le 1,$$

and

$$\|g - p\|_1 \le \|g - \phi\|_1 + \|\phi - p\|_1 < \frac{5\epsilon}{2} + \|\phi - p\|_\infty \le \frac{5\epsilon}{2} + \frac{\epsilon}{2} = 3\epsilon.$$

□

The following is the Hardy-Littlewood tauberian theorem.[3]

Theorem 4 (Hardy-Littlewood tauberian theorem)**.** *If $a_n \ge 0$ for all n and*

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x}, \qquad x \to 1^-,$$

then

$$s_n = \sum_{\nu=0}^{n} a_\nu \sim n.$$

[3]E. C. Titchmarsh, *The Theory of Functions*, second ed., p. 227, §7.53, attributed to Karamata.

Proof. For any $k \geq 0$,

$$\begin{aligned}
(1-x)\sum_{n=0}^{\infty} a_n x^n (x^n)^k &= \frac{1-x}{1-x^{k+1}}(1-x^{k+1})\sum_{n=0}^{\infty} a_n (x^{k+1})^n \\
&= \frac{1}{1+x+\cdots+x^k}(1-x^{k+1})\sum_{n=0}^{\infty} a_n (x^{k+1})^n \\
&\to \frac{1}{k+1}\cdot 1 \\
&= \int_0^1 t^k dt,
\end{aligned}$$

as $x \to 1^-$. Hence if $P(x)$ is a polynomial, then

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n P(x^n) = \int_0^1 P(t)dt. \tag{1}$$

Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & 0 \leq t < e^{-1} \\ t^{-1} & e^{-1} \leq t \leq 1. \end{cases}$$

Let $\epsilon > 0$. By Lemma 3, there are polynomials $p(x), P(x)$ such that

$$p(x) \leq g(x) \leq P(x), \qquad 0 \leq x \leq 1$$

and

$$\|g-p\|_1 \leq \epsilon, \qquad \|P-g\|_1 \leq \epsilon.$$

Because the coefficients a_n are nonnegative, taking upper limits and then using (1) we obtain

$$\begin{aligned}
\limsup_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) &\leq \limsup_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n P(x^n) \\
&= \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n P(x^n) \\
&= \int_0^1 P(t)dt \\
&< \int_0^1 g(t)dt + \epsilon.
\end{aligned}$$

Taking lower limits and then using (1) we obtain

$$\begin{aligned}
\liminf_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) &\geq \liminf_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n p(x^n)\\
&= \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n p(x^n)\\
&= \int_0^1 p(t)dt\\
&> \int_0^1 g(t)dt - \epsilon.
\end{aligned}$$

The above two inequalities do not depend on the polynomials $p(x), P(x)$ but only on ϵ, and taking $\epsilon \to 0$ yields

$$\limsup_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) \leq \int_0^1 g(t)dt$$

and

$$\liminf_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) \geq \int_0^1 g(t)dt.$$

Thus

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) = \int_0^1 g(t)dt = \int_{e^{-1}}^1 t^{-1}dt = 1. \tag{2}$$

For $x = e^{-1/N}$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n g(x^n) &= \sum_{n=0}^{\infty} a_n e^{-n/N} g(e^{-n/N})\\
&= \sum_{n=0}^{N} a_n e^{-n/N} e^{n/N}\\
&= s_N.
\end{aligned}$$

Thus, (2) tells us that

$$\lim_{N\to\infty}(1-e^{-1/N})s_N = 1.$$

That is,

$$s_N \sim \frac{1}{1-e^{-1/N}},$$

and using

$$\frac{1}{1-e^{-1/N}} = N + \frac{1}{2} + O(N^{-1})$$

we get

$$s_N \sim N,$$

completing the proof. $\square$