

Gaussian integrals

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August 30, 2015

1 One dimensional Gaussian integrals

For $p \in \mathbb{C}$, let¹

$$h(p) = \int_{\mathbb{R}} e^{-x^2/2} e^{-ipx} dx.$$

Then we check that

$$h'(p) = -i \int_{\mathbb{R}} x e^{-x^2/2} e^{-ipx} dx = i \int_{\mathbb{R}} \frac{d}{dx} (e^{-x^2/2}) e^{-ipx} dx.$$

Integrating by parts yields

$$h'(p) = -p \int_{\mathbb{R}} e^{-x^2/2} e^{-ipx} dx = -ph(p).$$

Since $h'(p) = -ph(p)$,²

$$h(p) = h(0) e^{-p^2/2}.$$

Now, using Fubini's theorem and then polar coordinates,

$$\begin{aligned} h(0)^2 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-x^2/2} dx \right) e^{-y^2/2} dy \\ &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy \\ &= \int_0^\infty \left(\int_{S^1} e^{-r^2/2} e^{-r^2/2} d\sigma(\theta) \right) r dr \\ &= 2\pi \int_0^\infty r e^{-r^2/2} dr \\ &= 2\pi, \end{aligned}$$

¹Eberhard Zeidler, *Quantum Field Theory I: Basics in Mathematics and Physics*, p. 493, Problem 7.1.

²cf. Einar Hille, *Ordinary Differential Equations in the Complex Domain*.

so

$$h(p) = (2\pi)^{1/2} e^{-p^2/2}.$$

For $a > 0$ and $p \in \mathbb{C}$, doing the change of variable $y = a^{1/2}x$,

$$\begin{aligned} (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{-ipx} dx &= (2\pi)^{-1/2} a^{-1/2} \int_{\mathbb{R}} e^{-y^2/2} e^{-ipa^{-1/2}y} dy \\ &= (2\pi)^{-1/2} a^{-1/2} h(pa^{-1/2}) \\ &= a^{-1/2} e^{-a^{-1}p^2/2}. \end{aligned}$$

For $t > 0$ and $m \in \mathbb{R}$, doing the change of variable $y = x - m$, and using the above with $a = t^{-2}$ and $p = 0$,

$$\begin{aligned} (2\pi t^2)^{-1/2} \int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2t^2}\right) dx &= (2\pi)^{-1/2} t^{-1} \int_{\mathbb{R}} e^{-ay^2/2} dx \\ &= t^{-1} \cdot a^{-1/2} \\ &= 1. \end{aligned}$$

Theorem 1. For $a > 0$ and $p \in \mathbb{C}$,

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{-ipx} dx = a^{-1/2} e^{-a^{-1}p^2/2}.$$

For $t > 0$ and $m \in \mathbb{R}$,

$$(2\pi t^2)^{-1/2} \int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2t^2}\right) dx = 1.$$

For $t > 0$ and $x \in \mathbb{R}$, let

$$p_t(x) = (2\pi t^2)^{-1/2} \exp\left(-\frac{x^2}{2t^2}\right).$$

For $\phi \in \mathcal{S}(\mathbb{R})$, doing the change of variable $x = ty$,

$$\int_{\mathbb{R}} \phi(x) p_t(x) dx = (2\pi)^{-1/2} \int_{\mathbb{R}} \phi(ty) e^{-y^2/2} dy = \int_{\mathbb{R}} \phi(tx) p_1(x) dx.$$

Then as $t \downarrow 0$, using the dominated convergence theorem,

$$\int_{\mathbb{R}} \phi(x) p_t(x) dx \rightarrow \int_{\mathbb{R}} \phi(0) p_1(x) dx = \phi(0).$$

For $\phi \in L^1(\mathbb{R}^N)$, let

$$\widehat{\phi}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\langle \xi, x \rangle} \phi(x) dx, \quad \xi \in \mathbb{R}^N.$$

By Theorem 1, with $a = t^{-2}$,

$$\begin{aligned}\widehat{p}_t(\xi) &= (2\pi t^2)^{-1/2} \cdot (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{-i\xi x} dx \\ &= (2\pi t^2)^{-1/2} a^{-1/2} e^{-a^{-1}\xi^2/2} \\ &= (2\pi)^{-1/2} e^{-t^2\xi^2/2}.\end{aligned}$$

2 Moments

For $a > 0$, define for $Z \in \mathbb{C}$,

$$Z(J) = a^{1/2}(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{iJx} dx.$$

By Theorem 1,

$$Z(J) = e^{-a^{-1}J^2/2}. \quad (1)$$

By the dominated convergence theorem,

$$Z^{(n)}(J) = a^{1/2}(2\pi)^{-1/2} i^n \int_{\mathbb{R}} x^n e^{-ax^2/2} e^{iJx} dx,$$

and so

$$a^{1/2}(2\pi)^{-1/2} \int_{\mathbb{R}} x^n e^{-ax^2/2} dx = i^{-n} \frac{dZ}{dJ}(0).$$

From (1) we calculate

$$Z'(J) = -a^{-1}JZ(J), \quad Z''(J) = -a^{-1}Z(J) + a^{-2}J^2Z(J),$$

so $Z''(0) = -a^{-1}Z(0) = -a^{-1}$, and thus for $t > 0$ and $a = t^{-1}$,

$$(2\pi t)^{-1/2} \int_{\mathbb{R}} x^2 e^{-t^{-2}x^2/2} dx = t,$$

i.e.

$$\int_{\mathbb{R}} x^2 p_t(x) dx = t.$$

3 N -dimensional Gaussian integrals

Let $S(x) = \frac{\langle x, x \rangle}{2}$ for $x \in \mathbb{R}^N$. For $\chi \in \mathcal{D}(\mathbb{R}^N)$ and $t > 0$, **Laplace's method**³ tells us that

$$\int_{\mathbb{R}^N} e^{-tS(x)} \chi(x) dx = (2\pi t^{-1})^{N/2} (\det \text{Hess } S(0))^{-1/2} e^{-tS(0)} \chi(0) (1 + O(t^{-1}))$$

³cf. <http://individual.utoronto.ca/jordanbell/notes/stationaryphase.pdf>, p. 5, §5.

as $t \rightarrow \infty$. Here, $\text{Hess } S(x) = I$ for all x and $S(0) = 0$, so

$$\int_{\mathbb{R}^N} e^{-t \frac{\langle x, x \rangle}{2}} \chi(x) dx = (2\pi t^{-1})^{N/2} \chi(0) (1 + O(t^{-1}))$$

as $t \rightarrow \infty$.

For A an $N \times N$ matrix, we write $A > 0$ if A is symmetric and has positive eigenvalues. It is proved that⁴

$$\begin{aligned} & \int_{\mathbb{R}^N} \exp \left(-\frac{1}{2} \langle Ax, x \rangle - i \langle \xi, x \rangle \right) dx \\ &= (\det A)^{-1/2} (2\pi)^{N/2} \exp \left(-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle \right) \end{aligned}$$

for all $\xi \in \mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} \exp \left(-\frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle \right) dx = (\det A)^{-1/2} (2\pi)^{N/2} \exp \left(\frac{1}{2} \langle A^{-1} b, b \rangle \right).$$

for all $b \in \mathbb{R}^N$. Let

$$Z_A = \int_{\mathbb{R}^N} e^{-\frac{1}{2} \langle Ax, x \rangle} dx = (\det A)^{-1/2} (2\pi)^{N/2}.$$

Let λ_N be Lebesgue measure on \mathbb{R}^N and let μ_A be the following Borel probability measure on \mathbb{R}^N :⁵

$$d\mu_A(x) = \frac{1}{Z_A} e^{-\frac{1}{2} \langle Ax, x \rangle} d\lambda_N(x) = (\det A)^{1/2} (2\pi)^{-N/2} e^{-\frac{1}{2} \langle Ax, x \rangle} d\lambda_N(x).$$

For $\xi \in \mathbb{R}^N$,

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-i \langle \xi, x \rangle} d\mu_A(x) &= (\det A)^{1/2} (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{1}{2} \langle Ax, x \rangle} e^{-i \langle \xi, x \rangle} d\lambda_N(x) \\ &= (\det A)^{1/2} (2\pi)^{-N/2} \cdot (\det A)^{-1/2} (2\pi)^{N/2} e^{-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle} \\ &= e^{-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle}, \end{aligned}$$

and for $b \in \mathbb{R}^N$,

$$\begin{aligned} \int_{\mathbb{R}^N} e^{\langle b, x \rangle} d\mu_A(x) &= (\det A)^{1/2} (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\langle b, x \rangle} e^{-\frac{1}{2} \langle Ax, x \rangle} d\lambda_N(x) \\ &= (\det A)^{1/2} (2\pi)^{-N/2} \cdot (\det A)^{-1/2} (2\pi)^{N/2} e^{\frac{1}{2} \langle A^{-1} b, b \rangle} \\ &= e^{\frac{1}{2} \langle A^{-1} b, b \rangle}. \end{aligned}$$

⁴<http://individual.utoronto.ca/jordanbell/notes/stationaryphase.pdf>, Theorem 2 and Theorem 3.

⁵See <http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf>, p. 6, §4.

Theorem 2. For $\xi \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} e^{-i\langle \xi, x \rangle} d\mu_A(x) = e^{-\frac{1}{2}\langle A^{-1}\xi, \xi \rangle},$$

and for $b \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} e^{\langle b, x \rangle} d\mu_A(x) = e^{\frac{1}{2}\langle A^{-1}b, b \rangle}.$$

Let⁶

$$L = L^A = \sum_{j,k=1}^N A_{j,k}^{-1} \partial_j \partial_k.$$

We work out the semigroup whose infinitesimal generator is $L/2$.⁷

Theorem 3. For $f \in C^1(\mathbb{R}^N)$ that is μ_A -integrable and for $t > 0$,

$$(e^{tL/2}f)(x) = \int_{\mathbb{R}^N} f(x - t^{1/2}y) d\mu_A(y), \quad x \in \mathbb{R}^N.$$

Proof. For $\xi \in \mathbb{R}^N$ define $f(x) = e^{\langle \xi, x \rangle} = e^{\xi_1 x_1 + \dots + \xi_N x_N}$. On the one hand,

$$Lf = \sum_{j,k=1}^n A_{j,k}^{-1} \xi_j \xi_k f = \langle A^{-1}\xi, \xi \rangle f.$$

Then

$$\exp(tL/2)f = \exp\left(\frac{1}{2}t\langle A^{-1}\xi, \xi \rangle\right)f.$$

On the other hand, for $x \in \mathbb{R}^N$, applying Theorem 2,

$$\begin{aligned} \int_{\mathbb{R}^N} f(x - t^{1/2}y) d\mu_A(y) &= \int_{\mathbb{R}^N} e^{\langle \xi, x - t^{1/2}y \rangle} d\mu_A(y) \\ &= e^{\langle \lambda, x \rangle} \int_{\mathbb{R}^N} e^{\langle -t^{1/2}\xi, y \rangle} d\mu_A(y) \\ &= e^{\langle \lambda, x \rangle} e^{\frac{1}{2}\langle A^{-1}(-t^{1/2}\xi), (-t^{1/2}\xi) \rangle} \\ &= e^{\frac{1}{2}t\langle A^{-1}\xi, \xi \rangle} f. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} f(x - t^{1/2}y) d\mu_A(y) = e^{tL/2}f.$$

□

⁶See <http://www.math.ucsd.edu/~bdriver/247A-Winter2012/>

⁷cf. the Ornstein-Uhlenbeck semigroup, <http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf>, §6.

4 Concentration of measure

Let γ_N be the Borel probability measure on \mathbb{R}^N defined by

$$d\gamma_N(x) = (2\pi)^{-N/2} e^{-\frac{1}{2}\langle x, x \rangle} d\lambda_N(x).$$

We estimate the mass γ_N assigns to a spherical shell about the sphere of radius $N^{1/2}$.⁸

Theorem 4. For $\delta \geq 0$,

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq \left(\frac{N}{N + \delta}\right)^{-N/2} e^{-\delta/2},$$

and for $0 < \delta \leq N$,

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \leq N - \delta\} \leq \left(\frac{N}{N - \delta}\right)^{-N/2} e^{\delta/2}.$$

Proof. For $0 < \lambda < 1$, if $\|x\|^2 \geq N + \delta$ then $\lambda \|x\|^2 / 2 \geq \lambda(N + \delta)/2$ and then $e^{\lambda \|x\|^2 / 2} \geq e^{\lambda(N + \delta)/2}$. Hence

$$\begin{aligned} \gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} &= e^{-\lambda(N + \delta)/2} \int_{\|x\|^2 \geq N + \delta} e^{\lambda(N + \delta)/2} d\gamma_N(x) \\ &\leq e^{-\lambda(N + \delta)/2} \int_{\|x\|^2 \geq N + \delta} e^{\lambda \|x\|^2 / 2} d\gamma_N(x) \\ &\leq e^{-\lambda(N + \delta)/2} \int_{\mathbb{R}^N} e^{\lambda \|x\|^2 / 2} d\gamma_N(x) \\ &= e^{-\lambda(N + \delta)/2} \cdot (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\lambda \|x\|^2 / 2} e^{-\frac{1}{2}\|x\|^2} d\lambda_N(x) \\ &= e^{-\lambda(N + \delta)/2} \cdot \prod_{k=1}^N (2\pi)^{-1/2} \int_{\mathbb{R}} e^{(\lambda - 1)u^2 / 2} du. \end{aligned}$$

For $a = -\lambda + 1 > 0$, we have by Theorem 1

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-au^2 / 2} du = a^{-1/2},$$

so

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq e^{-\lambda(N + \delta)/2} a^{-N/2} = e^{-\lambda(N + \delta)/2} (1 - \lambda)^{-N/2}.$$

For $\lambda = \frac{\delta}{N + \delta}$ this is

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq e^{-\delta/2} \left(\frac{N}{N + \delta}\right)^{-N/2}.$$

□

⁸Alexander Barvinok, *Measure Concentration*, <http://www.math.lsa.umich.edu/~barvinok/total710.pdf>, p. 5, Proposition 2.2.

Let $\Sigma_N = \{x \in \mathbb{R}^N : \|x\| = N^{1/2}\}$, and let μ_N be the unique $SO(N)$ -invariant Borel probability measure on S^{N-1} (any Borel probability measure on a metric space is regular so we need not explicitly demand this to ensure uniqueness).⁹ Let $\pi_N : \Sigma_N \rightarrow \mathbb{R}$ be the projection

$$\pi_N(x) = \pi_N(x_1, \dots, x_N) = x_1,$$

and let $\nu_N = (\pi_N)_* \mu_N$, the pushforward measure which is itself a Borel probability measure on \mathbb{R} . The following theorem states that the measures ν_N converges strongly to the standard Gaussian measure γ_1 .¹⁰

Theorem 5. *For A a Borel set in \mathbb{R} ,*

$$\nu_N(A) \rightarrow \gamma_1(A)$$

as $N \rightarrow \infty$.

5 Zeta functions

Let $A > 0$, with eigenvalues $\lambda_1, \dots, \lambda_N$, counted according to multiplicity. For $s \in \mathbb{C}$, define¹¹

$$\zeta_A(s) = \sum_{k=1}^N \lambda_k^{-s} = \sum_{k=1}^N e^{-s \log \lambda_k}.$$

The derivative of ζ_A is

$$\zeta'_A(s) = \sum_{k=1}^N -\log \lambda_k \cdot \lambda_k^{-s},$$

so

$$\zeta'_A(0) = -\sum_{k=1}^N \log \lambda_k,$$

hence

$$e^{-\zeta'_A(0)} = \prod_{k=1}^N \lambda_k = \det A.$$

Theorem 6. *For $\xi \in \mathbb{R}^N$,*

$$(2\pi)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \langle Ax, x \rangle - i \langle \xi, x \rangle\right) dx = e^{\zeta'_A(0)/2} \exp\left(-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle\right).$$

⁹<http://individual.utoronto.ca/jordanbell/notes/harmonicpolynomials.pdf>, §2.

¹⁰Alexander Barvinok, *Measure Concentration*, <http://www.math.lsa.umich.edu/~barvinok/total710.pdf>, p. 54, Theorem 13.2.

¹¹Eberhard Zeidler, *Quantum Field Theory I: Basics in Mathematics and Physics*, p. 434, §7.23.3.

Let $\lambda_k > 0$, $k \geq 1$, and let $Ae_k = \lambda_k e_k$, and if it makes sense let¹²

$$\det A = \prod_{k=1}^{\infty} \lambda_k.$$

For those complex s for which the expression makes sense, let¹³

$$\zeta_A(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} = \sum_{k=1}^{\infty} e^{-s \log \lambda_k}.$$

Then, if the above makes sense in a neighborhood of $s = 0$,

$$\zeta'_A(0) = - \sum_{k=1}^{\infty} \log \lambda_k,$$

so

$$e^{-\zeta'_A(0)} = \det A.$$

We calculate, doing the change of variables $t = \lambda_k u$,

$$\begin{aligned} \Gamma(s) \zeta_A(s) &= \int_0^{\infty} t^{s-1} e^{-t} dt \cdot \sum_{k=1}^{\infty} \lambda_k^{-s} \\ &= \sum_{k=1}^{\infty} \lambda_k^{-s} \int_0^{\infty} t^{s-1} e^{-t} dt \\ &= \sum_{k=1}^{\infty} \lambda_k^{-s} \int_0^{\infty} (\lambda_k u)^{s-1} e^{-\lambda_k u} \lambda_k du \\ &= \sum_{k=1}^{\infty} \int_0^{\infty} u^{s-1} e^{-\lambda_k u} du. \end{aligned}$$

Thus

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{k=1}^{\infty} e^{-\lambda_k u} du.$$

For $\gamma > 0$, the eigenvalues of γA are $\gamma \lambda_k$, and doing the change of variables $v = \gamma u$,

$$\begin{aligned} \zeta_{\gamma A}(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{k=1}^{\infty} e^{-\gamma \lambda_k u} du \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \gamma^{-s} v^{s-1} \sum_{k=1}^{\infty} e^{-\lambda_k v} dv \\ &= \gamma^{-s} \zeta_A(s). \end{aligned}$$

¹²cf. <http://individual.utoronto.ca/jordanbell/notes/fredholm.pdf>

¹³cf. <http://individual.utoronto.ca/jordanbell/notes/functionaldeterminant.pdf>

Taking the derivative,

$$\zeta'_{\gamma A}(s) = -\log \gamma \cdot \gamma^{-s} \cdot \zeta_A(s) + \gamma^{-s} \gamma'_A(s),$$

and then

$$\zeta'_{\gamma A}(0) = -\log \gamma \cdot \zeta_A(0) + \zeta'_A(0).$$

Then

$$\det(\gamma A) = e^{-\zeta'_{\gamma A}(0)} = e^{\log \gamma \cdot \zeta_A(0) - \zeta'_A(0)} = \gamma^{\zeta_A(0)} \det A.$$