Haar wavelets and multiresolution analysis

Jordan Bell

April 3, 2014

1 Introduction

Let

$$\psi(x) = \begin{cases} 0 & x < 0, \\ 1 & 0 \le x < \frac{1}{2}, \\ -1 & \frac{1}{2} \le x < 1, \\ 0 & x \ge 1. \end{cases}$$

For $n, k \in \mathbb{Z}$, we define

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k), \qquad x \in \mathbb{R}.$$

$L^2(\mathbb{R})$ is a complex Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

We will prove that ψ satisfies the following definition of an *orthonormal wavelet*.[1]

Definition 1 (Orthonormal wavelet)**.** If $\Psi \in L^2(\mathbb{R})$, $\Psi_{n,k}(x) = 2^{n/2}\Psi(2^n x - k)$, and the set $\{\Psi_{n,k} : n, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then Ψ is called an *orthonormal wavelet*.

Lemma 2. $\{\psi_{n,k} : n, k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$.

Proof. If $n, n', k, k' \in \mathbb{Z}$, then

$$\begin{aligned} \int_{\mathbb{R}} \psi_{n,k}(x)\overline{\psi_{n',k'}(x)}dx &= \int_{\mathbb{R}} 2^{n/2}\psi(2^n x - k)2^{n'/2}\psi(2^{n'}x - k')dx \\ &= \int_{\mathbb{R}} 2^{(n'-n)/2}\psi(x-k)\psi(2^{n'-n}x - k')dx \\ &= 2^{(n'-n)/2}\delta_{k,k'}\int_0^1 \psi(x)\psi(2^{(n'-n)/2}x)dx \\ &= \delta_{k,k'} \cdot \delta_{n,n'}, \end{aligned}$$

hence $\{\psi_{n,k} : n, k \in \mathbb{Z}\}$ is an orthonormal set. $\square$

[1] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 303, Definition 6.4.1.

Bessel's inequality states that if $\mathscr{E}$ is an orthonormal set in a Hilbert space H, then for any $f \in H$ we have $\sum_{e\in\mathscr{E}} |\langle f, e\rangle|^2 \leq \|f\|_2^2$, from which it follows that $\sum_{e\in\mathscr{E}} \langle f, e\rangle e \in H$. To say that a subset $\mathscr{E}$ of a Hilbert space H is an orthonormal basis is equivalent to saying that $\mathscr{E}$ is an orthonormal set and that

$$\mathrm{id}_H = \sum_{e\in\mathscr{E}} e \otimes e$$

in the strong operator topology. In other words, for $\mathscr{E}$ to be an orthonormal basis of H means that $\mathscr{E}$ is an orthonormal set and that for every $f \in H$ we have

$$f = \sum_{e\in\mathscr{E}} \langle f, e\rangle e.$$

From Lemma 2 and Bessel's inequality, we know that for each $f \in L^2(\mathbb{R})$,

$$\sum_{n,k\in\mathbb{Z}} |\langle f, \psi_{n,k}\rangle|^2 \leq \|f\|_2^2, \qquad \sum_{n,k\in\mathbb{Z}} \langle f, \psi_{n,k}\rangle \psi_{n,k} \in L^2(\mathbb{R}).$$

We have not yet proved that f is equal to the series $\sum_{n,k\in\mathbb{Z}} \langle f, \psi_{n,k}\rangle \psi_{n,k}$, and this will not be accomplished until later in this note.

2 Coarser σ-algebras

For $n, k \in \mathbb{Z}$, let

$$I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right),$$

and let $\mathscr{F}_n$ be the σ-algebra generated by $\{I_{k,n} : k \in \mathbb{Z}\}$. $\mathbb{R} = \bigcup_{k\in\mathbb{Z}} I_{n,k}$, and if $k \neq k'$ then $I_{n,k} \cap I_{n,k'} = \emptyset$. If $n < n'$ then

$$\mathscr{F}_n \subset \mathscr{F}_{n'} \subset \mathscr{F},$$

where $\mathscr{F}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. An element of $L^2(\mathbb{R}, \mathscr{F}_n)$ is an element of $L^2(\mathbb{R}, \mathscr{F})$ that is constant on each set $I_{n,k}$, $k \in \mathbb{Z}$. In other words, an element of $L^2(\mathbb{R}, \mathscr{F}_n)$ is a function $f : \mathbb{R} \to \mathbb{C}$ such that if $k \in \mathbb{Z}$ then the image $f(I_{n,k})$ is a single element of $\mathbb{R}$ and such that

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}} \int_{I_{n,k}} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}} \frac{1}{2^n} \cdot |f(I_{n,k})|^2 < \infty.$$

If $n < n'$, then

$$L^2(\mathbb{R}, \mathscr{F}_n) \subset L^2(\mathbb{R}, \mathscr{F}_{n'}) \subset L^2(\mathbb{R}, \mathscr{F}).$$

3 Integral kernels

We define

$$\phi(x) = \begin{cases} 0 & x < 0, \\ 1 & 0 \le x < 1, \\ 0 & x \ge 1. \end{cases}$$

For $n \in \mathbb{Z}$ we define

$$K_n(x, y) = 2^n \sum_{k \in \mathbb{Z}} \phi(2^n x - k)\phi(2^n y - k), \qquad x, y \in \mathbb{R}.$$

We have

$$K_n(x, y) \in \{0, 2^n\}.$$

$K_n(x, y) = 2^n$ if and only if there is some $k \in \mathbb{Z}$ such that $2^n x - k, 2^n y - k \in [0, 1)$, equivalently there is some $k \in \mathbb{Z}$ with $2^n x, 2^n y \in [k, k + 1)$, which is equivalent to there being some $k \in \mathbb{Z}$ such that

$$x, y \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) = I_{n,k}.$$

We define

$$P_n f(x) = \int_{\mathbb{R}} K_n(x, y) f(y) dy.$$

If $x \in \mathbb{R}$ then there is a unique $k_x \in \mathbb{Z}$ with $x \in I_{n,k_x}$, and

$$P_n f(x) = 2^n \int_{I_{n,k_x}} f(y) dy. \tag{1}$$

It is straightforward to check that $L^2(\mathbb{R}, \mathscr{F}_n)$ is a closed subspace of $L^2(\mathbb{R}, \mathscr{F})$, and in the following theorem we prove that P_n is the orthogonal projection onto $L^2(\mathbb{R}, \mathscr{F}_n)$.

Lemma 3. If $n \in \mathbb{Z}$, then P_n is the orthogonal projection of $L^2(\mathbb{R}, \mathscr{F})$ onto $L^2(\mathbb{R}, \mathscr{F}_n)$.

Proof. For each $k \in \mathbb{Z}$, the function $P_n f$ is constant on the interval $I_{n,k}$, and using (1) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|P_n f\|_2^2 &= \sum_{k\in\mathbb{Z}} \int_{I_{n,k}} |P_n f(x)|^2 dx \\
&= \sum_{k\in\mathbb{Z}} \int_{I_{n,k}} \left| 2^n \int_{I_{n,k}} f(y)dy \right|^2 dx \\
&= 2^n \sum_{k\in\mathbb{Z}} \left| \int_{I_{n,k}} f(y)dy \right|^2 \\
&\le 2^n \sum_{k\in\mathbb{Z}} \left(\int_{I_{n,k}} |f(y)|^2 dy \right) \left(\int_{I_{n,k}} dy \right) \\
&= \sum_{k\in\mathbb{Z}} \int_{I_{n,k}} |f(y)|^2 dy \\
&= \int_{\mathbb{R}} |f(y)|^2 dy.
\end{aligned}$$

Therefore, $P_n : L^2(\mathbb{R}, \mathscr{F}) \to L^2(\mathbb{R}, \mathscr{F}_n)$. Moreover, the left-hand side of the above inequality is equal to $\|P_n f\|_2^2$ and the right-hand side is equal to $\|f\|_2^2$, hence we have $\|P_n f\|_2 \le \|f\|_2$, giving $\|P_n\| \le 1$.

If $f \in L^2(\mathbb{R}, \mathscr{F}_n)$, then

$$\begin{aligned}
P_n f(x) &= \int_{\mathbb{R}} K_n(x,y) f(y) dy \\
&= 2^n \int_{I_{n,k_x}} f(y)dy \\
&= f(I_{n,k_x}) \\
&= f(x),
\end{aligned}$$

hence if $f \in L^2(\mathbb{R}, \mathscr{F}_n)$ then $P_n f = f$. □

For $n \in \mathbb{Z}$, we define

$$L_n = K_{n+1} - K_n,$$

and the following lemma gives a different expression for L_n.[2]

Lemma 4. If $n \in \mathbb{Z}$, then

$$L_n(x,y) = \sum_{k\in\mathbb{Z}} \psi_{n,k}(x)\psi_{n,k}(y), \qquad x, y \in \mathbb{R}.$$

[2]Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 293, §6.3.2.

Proof. $\psi(2^n x - k) = 1$ means that $0 \leq 2^n x - k < \frac{1}{2}$, which is equivalent to $\frac{k}{2^n} \leq x < \frac{k+\frac{1}{2}}{2^n}$, which is equivalent to $\frac{2k}{2^{n+1}} \leq x < \frac{2k+1}{2^{n+1}}$, which is equivalent to $x \in I_{n+1,2k}$. $\psi(2^n x - k) = -1$ means that $\frac{1}{2} \leq 2^n x - k < 1$, which is equivalent to $\frac{k+\frac{1}{2}}{2^n} \leq x < \frac{k+1}{2^n}$, and this is equivalent to $x \in I_{n+1,2k+1}$. $\psi(2^n x - k) = 0$ if and only if $x \notin I_{n+1,2k} \cup I_{n+1,2k+1}$. Therefore,

$$\psi_{n,k}(x)\psi_{n,k}(y) = \begin{cases} 2^n & (x,y) \in I_{n+1,2k} \times I_{n+1,2k} \cup I_{n+1,2k+1} \times I_{n+1,2k+1}, \\ -2^n & (x,y) \in I_{n+1,2k} \times I_{n+1,2k+1} \cup I_{n+1,2k+1} \times I_{n+1,2k}, \\ 0 & \text{otherwise.} \end{cases}$$

If there is no $k \in \mathbb{Z}$ such that $(x,y) \in I_{n,k} \times I_{n,k}$, then $L_n(x,y) = 0$. Otherwise, suppose that $k \in \mathbb{Z}$ and that $(x,y) \in I_{n,k} \times I_{n,k}$. We have

$$I_{n,k} = I_{n+1,2k} \cup I_{n+1,2k+1}.$$

If $(x,y) \in I_{n+1,2k} \times I_{n+1,2k}$, then

$$L_n(x,y) = K_{n+1}(x,y) - K_n(x,y) = 2^{n+1} - 2^n = 2^n;$$

if $(x,y) \in I_{n+1,2k+1} \times I_{n+1,2k+1}$, then

$$L_n(x,y) = K_{n+1}(x,y) - K_n(x,y) = 2^{n+1} - 2^n = 2^n;$$

if $(x,y) \in I_{n+1,2k} \times I_{n+1,2k+1}$, then

$$L_n(x,y) = K_{n+1}(x,y) - K_n(x,y) = 0 - 2^n = -2^n;$$

and if $(x,y) \in I_{n+1,2k+1} \times I_{n+1,2k}$, then

$$L_n(x,y) = K_{n+1}(x,y) - K_n(x,y) = 0 - 2^n = -2^n.$$

It follows that

$$L_n(x,y) = \sum_{k \in \mathbb{Z}} \psi_{n,k}(x)\psi_{n,k}(y).$$

□

4 Continuous functions

Let $C_0(\mathbb{R})$ denote those continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that if $\epsilon > 0$ then there is some compact subset K of $\mathbb{R}$ such that $x \notin K$ implies that $|f(x)| < \epsilon$. We say that an element of $C_0(\mathbb{R})$ is a continuous function that *vanishes at infinity*. Let $C_c(\mathbb{R})$ denote the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\text{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$

is a compact set.

In the following lemma, we prove that the larger the intervals over which we average a continuous function vanishing at infinity, the smaller the supremum of the averaged function.[3]

[3] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 295, Lemma 6.3.2.

Lemma 5. If $f \in C_0(\mathbb{R})$, then $\|P_n f\|_\infty \to 0$ as $n \to -\infty$.

Proof. If $g \in C_c(\mathbb{R})$ and $x \in \mathbb{R}$, then

$$\begin{aligned} |P_n g(x)| &= \left| \int_{\mathbb{R}} K_n(x,y) g(y) dy \right| \\ &= \left| \int_{\mathrm{supp}(g)} K_n(x,y) g(y) dy \right| \\ &\le \int_{\mathrm{supp}(g)} K_n(x,y) |g(y)| dy \\ &\le \int_{\mathrm{supp}(g)} 2^n |g(y)| dy \\ &\le 2^n \cdot \mu(\mathrm{supp}(g)) \cdot \|g\|_\infty , \end{aligned}$$

hence

$$\|P_n g\|_\infty \le 2^n \cdot \mu(\mathrm{supp}(g)) \cdot \|g\|_\infty . \tag{2}$$

If $f \in C_0(\mathbb{R})$ and $\epsilon > 0$ then there is some $g \in C_c(\mathbb{R})$ with $\|f - g\|_\infty < \epsilon$. Hence,

$$\|P_n f\|_\infty \le \|P_n(f-g)\|_\infty + \|P_n g\|_\infty .$$

If $x \in \mathbb{R}$, then

$$|P_n(f-g)(x)| = 2^n \left| \int_{I_{n,k_x}} (f-g)(y) dy \right| \le 2^n \int_{I_{n,k_x}} |(f-g)(y)| dy \le \|f-g\|_\infty ,$$

hence $\|P_n(f-g)\|_\infty \le \|f-g\|_\infty$. Using this and (2) we obtain

$$\|P_n f\|_\infty \le \|f-g\|_\infty + 2^n \cdot \mu(\mathrm{supp}(g)) \cdot \|g\|_\infty < \epsilon + 2^n \cdot \mu(\mathrm{supp}(g)) \cdot \|g\|_\infty .$$

Hence,

$$\limsup_{n\to-\infty} \|P_n f\|_\infty \le \limsup_{n\to-\infty} \left(\epsilon + 2^n \cdot \mu(\mathrm{supp}(g)) \cdot \|g\|_\infty \right) = \epsilon.$$

This is true for every $\epsilon > 0$, so

$$\lim_{n\to-\infty} \|P_n f\|_\infty = 0.$$

□

Lemma 6. If $f \in L^2(\mathbb{R})$, then $\|P_n f\|_2 \to 0$ as $n \to -\infty$.

Proof. If $\epsilon > 0$ then there is some $g \in C_c(\mathbb{R})$ such that $\|f - g\|_2 < \epsilon$. Say $\mathrm{supp}(g) \subseteq [-K, K]$. If $2^m > K$, then we have by (1) and because $\mathrm{supp}(g) \subseteq$

$I_{-m,-1} \cup I_{-m,0}$,

$$
\begin{aligned}
\|P_{-m}g\|_2^2 &= \int_{\mathbb{R}} \left| 2^{-m} \int_{I_{-m,k_x}} g(y)dy \right|^2 dx \\
&= 2^m \left| 2^{-m} \int_{I_{-m,-1}} g(y)dy \right|^2 + 2^m \left| 2^{-m} \int_{I_{-m,0}} g(y)dy \right|^2 \\
&= 2^{-m} \left| \int_{-K}^{0} g(y)dy \right|^2 + 2^{-m} \left| \int_{0}^{K} g(y)dy \right|^2 \\
&\le 2^{-m}\mu([-K,0]) \|g\|_2^2 + 2^{-m}\mu([0,K]) \|g\|_2^2 \\
&= 2K \cdot 2^{-m} \|g\|_2^2 .
\end{aligned}
$$

Therefore, when $2^m > K$ we have $\|P_{-m}g\|_2 \le 2^{-\frac{m}{2}}\sqrt{2K}\,\|g\|_2$, and so, as the operator norm of P_{-m} on $L^2(\mathbb{R})$ is 1,

$$
\begin{aligned}
\|P_{-m}f\|_2 &\le \|P_{-m}(f-g)\|_2 + \|P_{-m}g\|_2 \\
&\le \|f-g\|_2 + \|P_{-m}g\|_2 \\
&< \epsilon + 2^{-\frac{m}{2}}\sqrt{2K}\,\|g\|_2 .
\end{aligned}
$$

Thus, if $\epsilon > 0$ then

$$
\limsup_{m\to\infty} \|P_{-m}f\|_2 \le \epsilon.
$$

This is true for all $\epsilon > 0$, so we obtain

$$
\lim_{m\to\infty} \|P_{-m}f\|_2 = 0.
$$

□

The following lemma shows that if $f \in C_c(\mathbb{R})$, then $P_n f$ converges to f in the L^2 norm and in the L^∞ norm as $n \to \infty$.[4]

Lemma 7. If $f \in C_c(\mathbb{R})$, then $P_n f \to f$ in the L^2 norm and in the L^∞ norm as $n \to \infty$.

Proof. Suppose that $\operatorname{supp}(f) \subseteq [-2^M, 2^M]$ for $M \ge 0$. f is uniformly continuous on the compact set $[-2^M, 2^M]$, thus, if $\epsilon > 0$ then there is some $\delta > 0$ such that $x, y \in [-2^M, 2^M]$ and $|x-y| < \delta$ imply that $|f(x) - f(y)| < \frac{\epsilon}{2^M}$. Let $2^{-n} \le \delta$.

[4] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 296, Lemma 6.3.3.

For each $x \in \mathbb{R}$, there is some $k_x \in \mathbb{Z}$ such that $x \in I_{n,k_x}$ and we have

$$\begin{aligned}
|P_n f(x) - f(x)| &= \left| 2^n \int_{I_{n,k_x}} f(y)dy - f(x) \right| \\
&= 2^n \left| \int_{I_{n,k_x}} f(y) - f(x)dy \right| \\
&\leq 2^n \int_{I_{n,k_x}} |f(y) - f(x)|dy \\
&< 2^n \int_{I_{n,k_x}} \frac{\epsilon}{2^M} dy \\
&= \frac{\epsilon}{2^M}.
\end{aligned}$$

This tells us that if $2^{-n} \leq \delta$ then $\|P_n f - f\|_\infty \leq \frac{\epsilon}{2^M}$. Therefore, if $\epsilon > 0$ then for sufficiently large n we have $\|P_n f - f\|_\infty \leq \frac{\epsilon}{2^M}$, showing that

$$\lim_{n\to\infty} \|P_n f - f\|_\infty = 0.$$

Furthermore, if $n \geq 0$ then

$$\|P_n f - f\|_2^2 = \int_{\mathbb{R}} |P_n f(x) - f(x)|^2 dx = \int_{-2^M}^{2^M} |P_n f(x) - f(x)|^2 dx \leq 2 \cdot 2^M \cdot \|P_n f - f\|_\infty^2 ,$$

and because $\|P_n f - f\|_\infty \to 0$ as $n \to \infty$ we get $\|P_n f - f\|_2 \to 0$ as $n \to \infty$. $\square$

From Lemma 4, we get

$$\begin{aligned}
(P_{n+1} - P_n) f(x) &= \int_{\mathbb{R}} K_{n+1}(x, y) f(y)dy - \int_{\mathbb{R}} K_n(x, y) f(y)dy \\
&= \int_{\mathbb{R}} L_n(x, y) f(y)dy \\
&= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \psi_{n,k}(x)\psi_{n,k}(y) f(y)dy \\
&= \sum_{k\in\mathbb{Z}} \langle f, \psi_{n,k} \rangle \psi_{n,k}(x),
\end{aligned}$$

thus

$$P_{n+1} - P_n = \sum_{k\in\mathbb{Z}} \psi_{n,k} \otimes \psi_{n,k} \tag{3}$$

in the strong operator topology. Using (3), we obtain for $n \geq 0$ that

$$\begin{aligned}
P_{n+1} &= P_0 + \sum_{j=0}^{n} P_{j+1} - P_j \\
&= P_0 + \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} \psi_{j,k} \otimes \psi_{j,k}
\end{aligned}$$

in the strong operator topology. For $n < 0$,

$$\begin{aligned} P_n &= P_0 - \sum_{j=-n}^{-1} P_{j+1} - P_j \\ &= P_0 - \sum_{j=-n}^{-1} \sum_{k\in\mathbb{Z}} \psi_{j,k} \otimes \psi_{j,k} \end{aligned}$$

in the strong operator topology.

We have already shown in Lemma 2 that $\{\psi_{n,k} : n, k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$, and we now prove that it is an orthonormal basis for $L^2(\mathbb{R})$.

Theorem 8. In the strong operator topology,

$$\mathrm{id}_{L^2(\mathbb{R})} = \sum_{n,k\in\mathbb{Z}} \psi_{n,k} \otimes \psi_{n,k}.$$

Proof. Let $f \in L^2(\mathbb{R})$ and suppose $\epsilon > 0$. By Lemma 6, there is some M such that $m \geq M$ implies that $\|P_{-m}f\|_2 < \frac{\epsilon}{2}$. There is some $g \in C_c(\mathbb{R})$ satisfying $\|f - g\|_2 < \frac{\epsilon}{6}$, and by Lemma 7 there is some N such that $n \geq N$ implies that $\|P_n g - g\|_2 < \frac{\epsilon}{6}$. Hence, if $n \geq N$ then

$$\begin{aligned} \|P_n f - f\|_2 &\leq \|P_n f - P_n g\|_2 + \|P_n g - g\|_2 + \|g - f\|_2 \\ &\leq 2\|f - g\|_2 + \|P_n g - g\|_2 \\ &< \frac{2\epsilon}{6} + \frac{\epsilon}{6} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Therefore, if $m \geq M$ and $n \geq N$, then

$$\left\|(P_n - P_{-m} - \mathrm{id}_{L^2(\mathbb{R})})f\right\|_2 \leq \|P_n f - f\|_2 + \|P_{-m}f\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

For $m, n > 0$, we have

$$\begin{aligned} P_{n+1} - P_{-m} &= \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} \psi_{j,k} \otimes \psi_{j,k} + \sum_{j=-m}^{-1} \sum_{k\in\mathbb{Z}} \psi_{j,k} \otimes \psi_{j,k} \\ &= \sum_{j=-m}^{n} \sum_{k\in\mathbb{Z}} \psi_{j,k} \otimes \psi_{j,k} \end{aligned}$$

in the strong operator topology. □

5 Other function spaces

Let $C_b(\mathbb{R})$ denote those continuous functions $\mathbb{R} \to \mathbb{C}$ that are bounded. We have

$$C_c(\mathbb{R}) \subset C_0(\mathbb{R}) \subset C_b(\mathbb{R}) \subset C(\mathbb{R}).$$

Lemma 9. If $n \in \mathbb{Z}$ and $f \in C_b(\mathbb{R})$, then $\|P_n f\|_\infty \le \|f\|_\infty$.

Proof. If $x \in \mathbb{R}$, then there is a unique $k_x \in \mathbb{Z}$ with $x \in I_{n,k_x}$, and

$$|P_n f(x)| = \left| 2^n \int_{I_{n,k_x}} f(y)dy \right| \le 2^n \int_{I_{n,k_x}} |f(y)|dy \le \|f\|_\infty .$$

$\square$

Theorem 10. If $f \in C_0(\mathbb{R})$, then the series $\sum_{n,k\in\mathbb{Z}} \langle f, \psi_{n,k}\rangle \psi_{n,k}$ converges to f uniformly on $\mathbb{R}$.

Proof. If $\epsilon > 0$ then there is some $g \in C_c(\mathbb{R})$ with $\|f - g\|_\infty < \frac{\epsilon}{6}$. By Lemma 5, there is some M such that $m \ge M$ implies that $\|P_{-m} g\|_\infty < \frac{\epsilon}{3}$, hence

$$\begin{aligned} \|P_{-m} f\|_\infty &\le \|P_{-m} f - P_{-m} g\|_\infty + \|P_{-m} g\|_\infty \\ &\le \|f - g\|_\infty + \|P_{-m} g\|_\infty \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{3} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

By Lemma 7, there is some N such that $n \ge N$ implies that $\|P_n g - g\|_\infty < \frac{\epsilon}{6}$, hence

$$\begin{aligned} \|P_n f - f\|_\infty &\le \|P_n f - P_n g\|_\infty + \|P_n g - g\|_\infty + \|g - f\|_\infty \\ &\le 2\|f - g\|_\infty + \|P_n g - g\|_\infty \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Therefore, if $n \ge N$ and $m \ge M$, then

$$\|P_n f - P_{-m} f - f\|_\infty \le \|P_n f - f\|_\infty + \|P_{-m} f\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

The following theorem states that P_n is an operator on $L^p(\mathbb{R})$ with operator norm ≤ 1.[5] In particular, it asserts that if $f \in L^p(\mathbb{R})$ then the averaged function $P_n f$ is also an element of $L^p(\mathbb{R})$.

Theorem 11. If $1 \le p < \infty$, $n \in \mathbb{Z}$, and $f \in L^p(\mathbb{R})$, then $\|P_n f\|_p \le \|f\|_p$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$, so $q = \frac{p}{p-1}$. (If $p = 1$ then $q = \infty$.) If $x \in \mathbb{R}$, then there

[5] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 297, Lemma 6.3.9.

is a unique $k_x \in \mathbb{Z}$ with $x \in I_{n,k_x}$, and using Hölder's inequality we get

$$
\begin{aligned}
|P_n f(x)| &= \left| 2^n \int_{I_{n,k_x}} f(y)dy \right| \\
&\le 2^n \left(\int_{I_{n,k_x}} |f(y)|^p dy \right)^{1/p} (\mu(I_{n,k_x}))^{1/q} \\
&= 2^n \left(\int_{I_{n,k_x}} |f(y)|^p dy \right)^{1/p} 2^{-n/q}.
\end{aligned}
$$

Therefore, if $k \in \mathbb{Z}$ then

$$
\begin{aligned}
\int_{I_{n,k}} |P_n f(x)|^p dx &\le \int_{I_{n,k}} 2^{np} 2^{-np/q} \int_{I_{n,k_x}} |f(y)|^p dy dx \\
&= \int_{I_{n,k}} 2^{np} 2^{-np/q} \int_{I_{n,k}} |f(y)|^p dy dx \\
&= 2^{-n} 2^{np} 2^{-np/q} \int_{I_{n,k}} |f(y)|^p dy \\
&= \int_{I_{n,k}} |f(y)|^p dy.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\|P_n f\|_p^p &= \sum_{k \in \mathbb{Z}} \int_{I_{n,k}} |P_n f(x)|^p dx \\
&\le \sum_{k \in \mathbb{Z}} \int_{I_{n,k}} |f(y)|^p dy \\
&= \int_{\mathbb{R}} |f(y)|^p dy \\
&= \|f\|_p^p,
\end{aligned}
$$

giving $\|P_n f\|_p \le \|f\|_p$. □

6 Multiresolution analysis

For $a \in \mathbb{R}$, we define $m_a : \mathbb{R} \to \mathbb{R}$ by $m_a(x) = ax$, and we define $\tau_a : \mathbb{R} \to \mathbb{R}$ by $\tau_a(x) = x - a$.

Definition 12 (Multiresolution analysis)**.** A *multiresolution analysis of* $L^2(\mathbb{R})$ is a set $\{V_n : n \in \mathbb{Z}\}$ of closed subspaces of the Hilbert space $L^2(\mathbb{R})$ and a function $\Phi \in L^2(\mathbb{R})$ satisfying

1. If $n \in \mathbb{Z}$, then $f \in V_n$ if and only if $f \circ m_2 \in V_{n+1}$.

2. $V_n \subseteq V_{n+1}$.
3. $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.
4. $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$.
5. $\{\Phi \circ \tau_k : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

It is straightforward to prove the following theorem using what we have established so far.

Theorem 13. The closed subspaces $\{L^2(\mathbb{R}, \mathscr{F}_n) : n \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ and the function $\phi = \chi_{[0,1)}$ is a multiresolution analysis of $L^2(\mathbb{R})$.

The following lemma shows that if P_n is the projection onto V_n, where V_n is a closed subspace of a multiresolution analysis of $L^2(\mathbb{R})$, then $P_n \to 0$ in the strong operator topology as $n \to -\infty$.[6]

Lemma 14. If $\{V_n : n \in \mathbb{Z}\}$ and $\Phi \in L^2(\mathbb{R})$ is a multiresolution analysis of $L^2(\mathbb{R})$, $P_n : L^2(\mathbb{R}) \to V_n$ is the orthogonal projection onto V_n, and $f \in L^2(\mathbb{R})$, then

$$\lim_{n\to-\infty} P_n f = 0.$$

Proof. Define $\Phi_{n,k}(x) = 2^{n/2}\Phi(2^n x - k)$. The set $\{\Phi_{0,k} : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, and one checks that the set $\{\Phi_{n,k} : k \in \mathbb{Z}\}$ is an orthonormal basis for V_n. Therefore

$$P_n = \sum_{k\in\mathbb{Z}} \Phi_{n,k} \otimes \Phi_{n,k}$$

in the strong operator topology.

For $R > 0$, let $f_R = f\chi_{[-R,R]}$. If $2^n R < \frac{1}{2}$, then, using the Cauchy-Schwarz

[6] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 313, Lemma 6.4.28.

inequality,

$$\begin{aligned}
\|P_n f_R\|_2^2 &= \sum_{k\in\mathbb{Z}} |\left\langle P_n f_R, \Phi_{n,k}\right\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}} |\left\langle f_R, \Phi_{n,k}\right\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}} |\left\langle f_R, \chi_{[-R,R]}\Phi_{n,k}\right\rangle|^2 \\
&\le \sum_{k\in\mathbb{Z}} \left(\int_{-R}^{R} |f_R(x)|^2 dx\right)\left(\int_{-R}^{R} |\Phi_{n,k}(x)|^2 dx\right) \\
&= \|f_R\|_2^2 \sum_{k\in\mathbb{Z}} \int_{-R}^{R} |\Phi_{n,k}(x)|^2 dx \\
&= \|f_R\|_2^2 \sum_{k\in\mathbb{Z}} 2^n \int_{-R}^{R} |\Phi(2^n x - k)|^2 dx \\
&= \|f_R\|_2^2 \sum_{k\in\mathbb{Z}} \int_{-2^n R-k}^{2^n R-k} |\Phi(x)|^2 dx \\
&= \|f_R\|_2^2 \int_{U_n} |\Phi(x)|^2 dx,
\end{aligned}$$

where

$$U_n = \bigcup_{k\in\mathbb{Z}} (-k - 2^n R, -k + 2^n R);$$

the intervals are disjoint because $2^n R < \frac{1}{2}$. Define $F_n(x) = |\Phi(x)|^2 \chi_{U_n}(x)$. For all $x \in \mathbb{R}$ we have $|F_n(x)| \le |\Phi(x)|^2$, and if $x \in \mathbb{R}$ then

$$\lim_{n\to-\infty} F_n(x) \to |\Phi(x)|^2 \chi_{\mathbb{Z}}(x),$$

where $\mathbb{Z} = \bigcap_{n\in\mathbb{Z}} U_n$. Thus by the dominated convergence theorem we get

$$\lim_{n\to-\infty} \int_{\mathbb{R}} F_n(x) dx = \int_{\mathbb{R}} |\Phi(x)|^2 \chi_{\mathbb{Z}}(x) dx = 0,$$

because $\mu(\mathbb{Z}) = 0$. Therefore,

$$\lim_{n\to-\infty} \|P_n f_R\|_2 = 0.$$

If $\epsilon > 0$ then there is some R such that $\|f - f_R\|_2 < \epsilon$. We have, because P_n is an orthogonal projection,

$$\begin{aligned}
\limsup_{n\to-\infty} \|P_n f\|_2 &\le \limsup_{n\to-\infty} \|P_n f - P_n f_R\|_2 + \limsup_{n\to-\infty} \|P_n f_R\|_2 \\
&= \limsup_{n\to-\infty} \|P_n f - P_n f_R\|_2 \\
&\le \limsup_{n\to-\infty} \|f - f_R\|_2 \\
&< \epsilon.
\end{aligned}$$

This is true for all $\epsilon > 0$, so we obtain

$$\lim_{n \to -\infty} \|P_n f\|_2 = 0.$$

$\square$

If $S_\alpha, \alpha \in I$, are subsets of a Hilbert space H, we denote by $\bigvee_{\alpha \in I} S_\alpha$ the closure of the span of $\bigcup_{\alpha \in I} S_\alpha$. If S is a subset of H, let $S^\perp$ be the set of all $x \in H$ such that $y \in S$ implies that $\langle x, y \rangle = 0$. If $S_n, n \in \mathbb{Z}$, are mutually orthogonal closed subspaces of a Hilbert space, we write

$$\bigoplus_{n \in \mathbb{Z}} S_n = \bigvee_{n \in \mathbb{Z}} S_n.$$

The following theorem shows a consequence of Definition 12.

Theorem 15. If $\{V_n : n \in \mathbb{Z}\}$ are the closed subspaces of a multiresolution analysis of $L^2(\mathbb{R})$ and $W_n = V_{n+1} \cap V_n^\perp$, then

$$L^2(\mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} W_n.$$

Proof. Because $W_n = V_{n+1} \cap V_n^\perp$ is the intersection of two closed subspaces, it is itself a closed subspace. Suppose that $n < n'$, $f \in W_n, g \in W_{n'}$. $n + 1 \leq n'$, and hence $V_{n+1} \subseteq V_{n'}$. Therefore

$$W_{n'} = V_{n'+1} \cap V_{n'}^\perp \subset V_{n'}^\perp \subseteq V_{n+1}^\perp.$$

But $f \in W_n \subset V_{n+1}$ and $g \in W_{n'} \subset V_{n+1}^\perp$, so $\langle f, g \rangle = 0$. Therefore $W_n \perp W_{n'}$.

If $f \in V_n$ and $f \neq 0$, then there is a minimal N such that $f \in V_N$; this minimal N exists because $V_n \subseteq V_{n+1}$ and $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$. We have

$$V_N = V_{N-1} \oplus W_{N-1},$$

hence $f = f_{N-1} + g_{N-1}$, with $f_{N-1} \in V_{N-1}$ and $g_{N-1} \in W_{N-1}$. Likewise,

$$V_{N-1} = V_{N-2} \oplus W_{N-2},$$

hence $f_{N-1} = f_{N-2} + g_{N-2}$, with $f_{N-2} \in V_{N-2}$ and $g_{N-2} \in W_{N-2}$. In this way, for any $M \geq 0$ we obtain

$$f = f_{N-M} + \sum_{m=1}^{M} g_{N-m},$$

where $f_{N-M} \in V_{N-M}$ and $g_{N-m} \in W_{N-m}$. Check that f_{N-M} is the orthogonal projection of f onto V_{N-M}. It thus follows from Lemma 14 that $f_{N-M} \to 0$ as $M \to \infty$. Thus, for any $\epsilon > 0$ there is some M with $\|f_{N-M}\|_2 < \epsilon$ and

$f \in f_{N-M} + \bigoplus_{m=1}^{M} W_{N-m}$. Therefore, if $f \in \bigcup_{n\in\mathbb{Z}} V_n$ then there is some $g \in \bigoplus_{n\in\mathbb{Z}} W_n$ satisfying $\|f-g\|_2 < \infty$. Thus

$$\overline{\bigcup_{n\in\mathbb{Z}} V_n} \subseteq \bigoplus_{n\in\mathbb{Z}} W_n,$$

and so

$$L^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n.$$

□

7 The unit interval

$L^2([0,1))$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

If $n \geq 0$, then $I_{n,0} = \left[0, \frac{1}{2^n}\right)$ and $I_{n,2^n-1} = \left[1 - \frac{1}{2^n}, 1\right)$, and we have

$$[0,1) = \bigcup_{k=0}^{2^n-1} I_{n,k}.$$

Let $n \geq 0$, let $\mathscr{G}_n$ be the σ-algebra generated by $\{I_{n,k} : 0 \leq k \leq 2^n - 1\}$, and let $\mathscr{G}$ be the σ-algebra of Lebesgue measurable subsets of $[0,1)$. If $n < n'$, then

$$\mathscr{G}_n \subset \mathscr{G}_{n'} \subset \mathscr{G}.$$

An element of $L^2([0,1), \mathscr{G}_n)$ is an element of $L^2([0,1), \mathscr{G})$ that is constant on each set $I_{n,k}, 0 \leq k \leq 2^n - 1$. Equivalently, an element of $L^2([0,1), \mathscr{G}_n)$ is a function $f : [0,1) \to \mathbb{C}$ that is constant on each set $I_{n,k}, 0 \leq k \leq 2^n - 1$; because $[0,1)$ is a union of finitely many $I_{n,k}$, any such function will be an element of $L^2([0,1), \mathscr{G})$. It is apparent that

$$L^2([0,1), \mathscr{G}_n) \subset L^2([0,1), \mathscr{G}_{n'}) \subset L^2([0,1), \mathscr{G}).$$

We check that $L^2([0,1), \mathscr{G}_n)$ is a complex vector space of dimension 2^n.

$I_{n,k} = I_{n+1,2k} \cup I_{n+1,2k+1}$. If $x \in I_{n+1,2k}$, then $\frac{2k}{2^{n+1}} \leq x < \frac{2k+1}{2^{n+1}}$, so $\frac{k}{2^n} \leq x < \frac{k}{2^n} + \frac{1}{2^{n+1}}$, hence $0 \leq 2^n x - k < \frac{1}{2}$. If $x \in I_{n+1,2k+1}$, then $\frac{2k+1}{2^{n+1}} \leq x < \frac{2k+2}{2^{n+1}}$, hence $\frac{k}{2^n} + \frac{1}{2^{n+1}} \leq x < \frac{k+1}{2^n}$, and so $\frac{1}{2} \leq 2^n x - k < 1$. Thus, if $x \in I_{n+1,2k}$ then

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k) = 2^{n/2}$$

and if $x \in I_{n+1,2k+1}$ then

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k) = -2^{n/2}.$$

Otherwise $x \notin I_{n,k}$, for which $\psi_{n,k}(x) = 0$. It follows that $\psi_{n,k} \in L^2([0,1), \mathscr{G}_{n+1})$.

Theorem 16. If
$$\mathscr{B}_0 = \{\chi_{[0,1)}\}$$
and, for $n \geq 0$,
$$\mathscr{B}_{n+1} = \{\psi_{n,k} : 0 \leq k \leq 2^n - 1\},$$
then
$$\bigcup_{n=0}^{N} \mathscr{B}_n$$
is an orthonormal basis of $L^2([0,1), \mathscr{G}_N)$.

Proof. It follows from Lemma 2 that $\bigcup_{n=1}^{N} \mathscr{B}_n$ is orthonormal in $L^2([0,1))$, as it is a subset of an orthonormal set. If $0 \leq n \leq N$ then $\mathscr{B}_n \subset L^2([0,1), \mathscr{G}_N)$, hence $\bigcup_{n=1}^{N} \mathscr{B}_n$ is orthonormal in $L^2([0,1), \mathscr{G}_N)$. If $0 < n \leq N$ and $0 \leq k \leq 2^{n-1} - 1$, then $\psi_{n-1,k} \in \mathscr{B}_n$ and
$$\begin{aligned}
\left\langle \psi_{n-1,k}, \chi_{[0,1)} \right\rangle &= \int_0^1 \psi_{n-1,k}(x) \overline{\chi_{[0,1)}(x)} dx \\
&= \int_0^1 \psi_{n-1,k}(x) dx \\
&= \int_{I_{n,2k}} \psi_{n-1,k}(x) dx + \int_{I_{n,2k+1}} \psi_{n-1,k}(x) dx \\
&= \int_{I_{n,2k}} 2^{(n-1)/2} dx + \int_{I_{n,2k+1}} -2^{(n-1)/2} dx \\
&= 0.
\end{aligned}$$
Therefore, $\bigcup_{n=0}^{N} \mathscr{B}_n$ is orthonormal in $L^2([0,1), \mathscr{G}_N)$.

$|\mathscr{B}_0| = 1$, and if $n \geq 1$ then $|\mathscr{B}_n| = 2^{n-1}$. Therefore the number of elements of $\bigcup_{n=0}^{N} \mathscr{B}_n$ is
$$1 + \sum_{n=1}^{N} 2^{n-1} = 1 + \sum_{n=0}^{N-1} 2^n = 2^N.$$
As $\dim L^2([0,1), \mathscr{G}_N) = 2^N$, the orthonormal set $\bigcup_{n=0}^{N} \mathscr{B}_n$ is an orthonormal basis for $L^2([0,1), \mathscr{G}_N)$. □

By Theorem 16, if $N \geq 0$ then $\bigcup_{n=0}^{N} \mathscr{B}_n$ is an orthonormal set in $L^2([0,1))$. Hence
$$\mathscr{B} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$$
is an orthonormal set in $L^2([0,1))$: if $f, g \in \mathscr{B}$ then there is some N with $f, g \in \bigcup_{n=0}^{N} \mathscr{B}_n$, which is an orthonormal set. The following theorem shows that $\mathscr{B}$ is an orthonormal basis for the Hilbert space $L^2([0,1))$.[7]

[7] John K. Hunter and Bruno Nachtergaele, *Applied Analysis*, p. 177, Lemma 7.13.

Theorem 17. $\mathscr{B}$ is an orthonormal basis for $L^2([0,1))$.

Proof. If $f \in L^2([0,1))$ and $\epsilon > 0$ then there is some $g \in C([0,1])$ with $\|f-g\|_2 < \frac{\epsilon}{2}$. g is uniformly continuous on the compact set $[0,1]$, so there is some $\delta > 0$ such that $|x-y| < \delta$ implies that $|g(x)-g(y)| < \frac{\epsilon}{2}$. Let $2^{-n} \leq \delta$, and define $h : [0,1) \to \mathbb{C}$ by

$$h(x) = \sum_{k=0}^{2^n-1} g\left(\frac{k}{2^n}\right) \chi_{I_{n,k}}(x).$$

If $x \in [0,1)$ then there is a unique $k_x, 0 \leq k_x \leq 2^n - 1$, with $x \in I_{n,k_x}$, and for this k_x we have $\left|x - \frac{k_x}{2^n}\right| < 2^{-n} \leq \delta$, and hence

$$|g(x) - h(x)| = \left| g(x) - g\left(\frac{k_x}{2^n}\right)\right| < \frac{\epsilon}{2}.$$

Therefore $\|g-h\|_\infty \leq \frac{\epsilon}{2}$.

We have $h \in L^2([0,1), \mathscr{G}_n)$, and

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2 < \frac{\epsilon}{2} + \|g-h\|_\infty \leq \epsilon.$$

We have shown that if $f \in L^2([0,1))$ and $\epsilon > 0$ then there is some n and some $h \in L^2([0,1), \mathscr{G}_n)$ with $\|f-h\|_2 < \epsilon$. This tells us that $\bigcup_{n=0}^\infty L^2([0,1), \mathscr{G}_n)$ is a dense subset of $L^2([0,1))$. Since $\mathscr{B}$ is orthonormal and $\operatorname{span} \mathscr{B} = \bigcup_{n=0}^\infty L^2([0,1), \mathscr{G}_n)$, $\mathscr{B}$ is an orthonormal basis for $L^2([0,1))$. $\square$

8 References

Useful references on wavelets and multiresolution analysis are Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*; P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*; Yves Meyer, *Wavelets and Operators*; Eugenio Hernández and Guido Weiss, *A First Course on Wavelets*.