Regular Expressions

Definitions

Equivalence to Finite Automata

RE's: Introduction

- *Regular expressions* describe languages by an algebra.
- They describe exactly the regular languages.
- If E is a regular expression, then L(E) is the language it defines.
- We'll describe RE's and their languages recursively.

Operations on Languages

- RE's use three operations: union, concatenation, and Kleene star.
- The union of languages is the usual thing, since languages are sets.
- Example: $\{01,111,10\} \cup \{00, 01\} = \{01,111,10,00\}$.

Concatenation

- The *concatenation* of languages L and M is denoted LM.
- It contains every string wx such that w is in L and x is in M.
- Example: {01,111,10}{00, 01} = {0100, 0101, 11100, 11101, 1000, 1001}.

Kleene Star

- If L is a language, then L*, the *Kleene star* or just "star," is the set of strings formed by concatenating zero or more strings from L, in any order.
- $L^* = \{\epsilon\} \cup L \cup LL \cup LLL \cup \ldots$
- Example: $\{0,10\}^* = \{\epsilon, 0, 10, 00, 010, 100, 1010, \ldots\}$

RE's: Definition

- Basis 1: If *a* is any symbol, then **a** is a RE, and L(**a**) = {a}.
  - Note: {a} is the language containing one string, and that string is of length 1.
- Basis 2: ϵ is a RE, and $L(\epsilon) = \{\epsilon\}$.
- Basis 3: $\emptyset$ is a RE, and $L(\emptyset) = \emptyset$.

RE's: Definition – (2)

- Induction 1: If E_1 and E_2 are regular expressions, then E_1+E_2 is a regular expression, and $L(E_1+E_2) = L(E_1) \cup L(E_2)$.
- Induction 2: If E_1 and E_2 are regular expressions, then E_1E_2 is a regular expression, and $L(E_1E_2) = L(E_1)L(E_2)$.
- Induction 3: If E is a RE, then E* is a RE, and $L(E^*) = (L(E))^*$.

Precedence of Operators

- Parentheses may be used wherever needed to influence the grouping of operators.
- Order of precedence is * (highest), then concatenation, then + (lowest).

Examples: RE's

- L(**01**) = {01}.
- L(**01**+**0**) = {01, 0}.
- L(**0**(**1**+**0**)) = {01, 00}.
  - Note order of precedence of operators.
- L(**0***) = {ϵ, 0, 00, 000,… }.
- L((**0**+**10**)*(ϵ+**1**)) = all strings of 0's and 1's without two consecutive 1's.

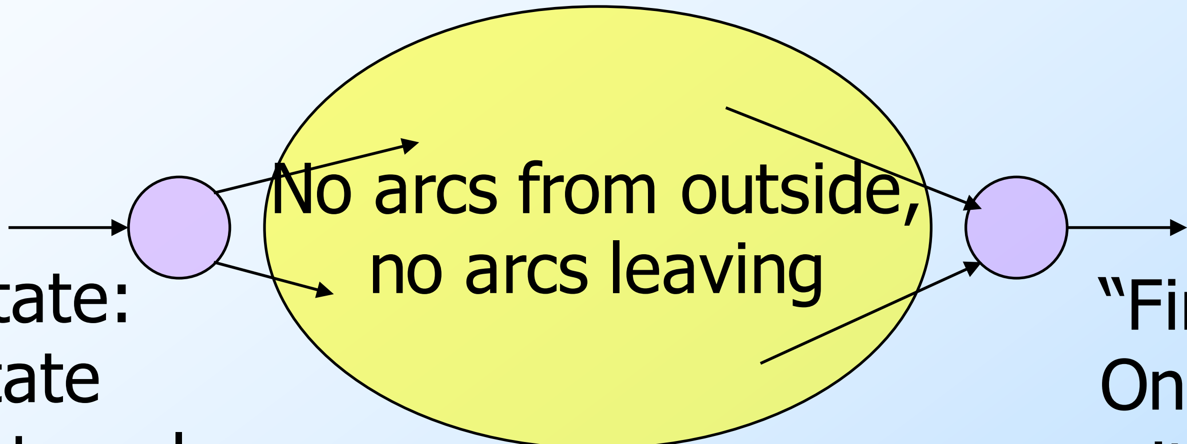
Equivalence of RE's and Finite Automata

- We need to show that for every RE, there is a finite automaton that accepts the same language.
  - Pick the most powerful automaton type: the ϵ-NFA.
- And we need to show that for every finite automaton, there is a RE defining its language.
  - Pick the most restrictive type: the DFA.

Converting a RE to an ϵ-NFA

- Proof is an induction on the number of operators (+, concatenation, *) in the RE.
- We always construct an automaton of a special form (next slide).

Form of ϵ-NFA's Constructed

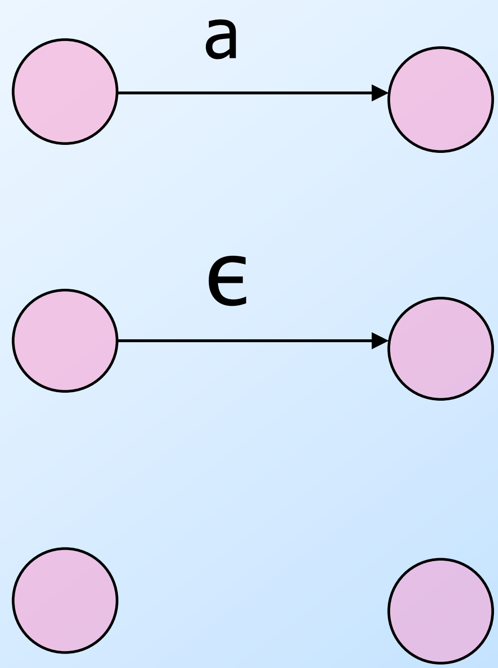


Start state:
Only state
with external
predecessors

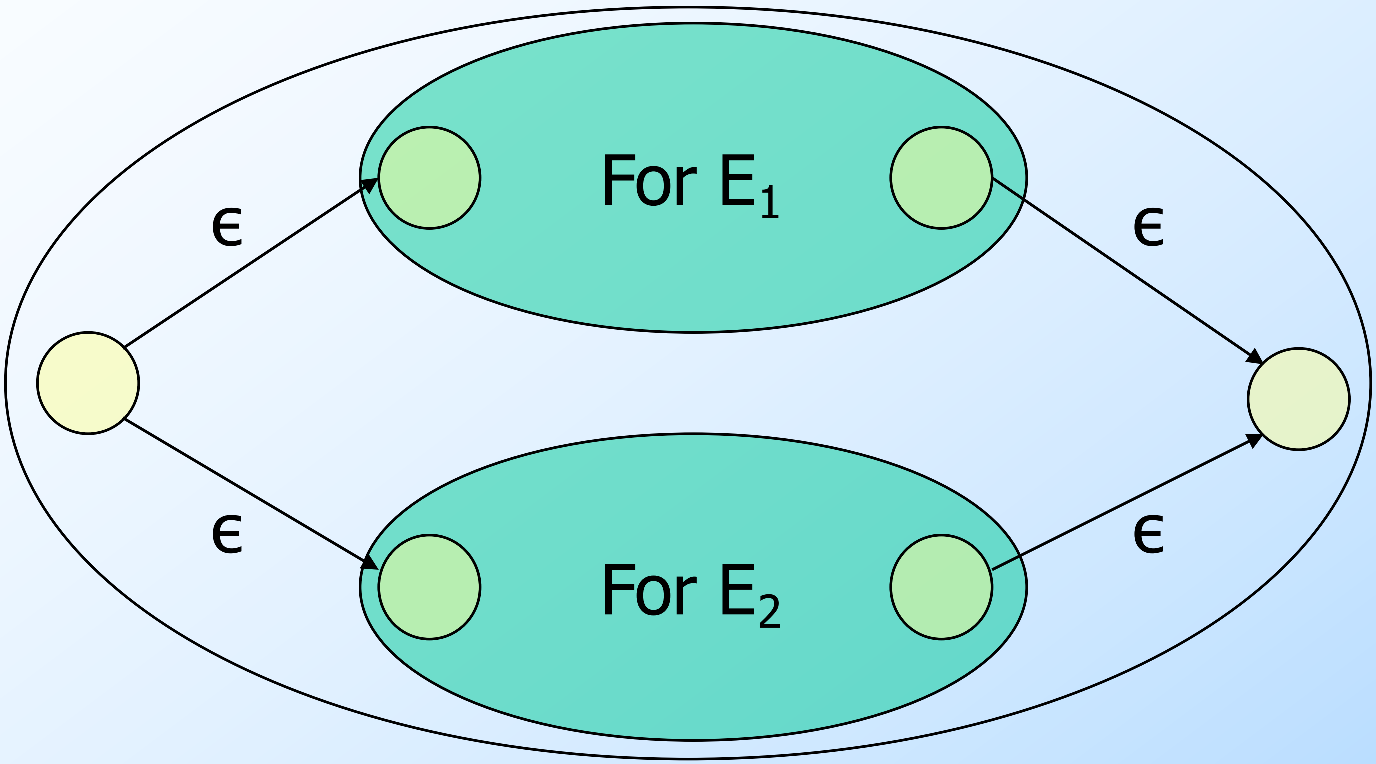
"Final" state:
Only state
with external
successors

RE to ϵ-NFA: Basis

- Symbol **a**:
- ϵ:
- ∅:

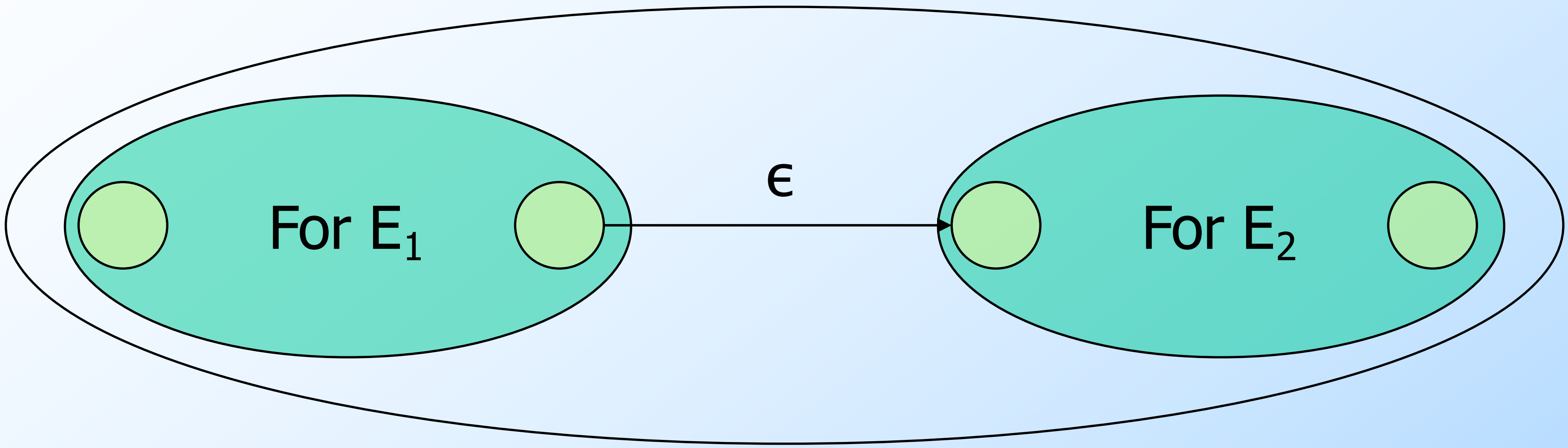


RE to ϵ-NFA: Induction 1 – Union



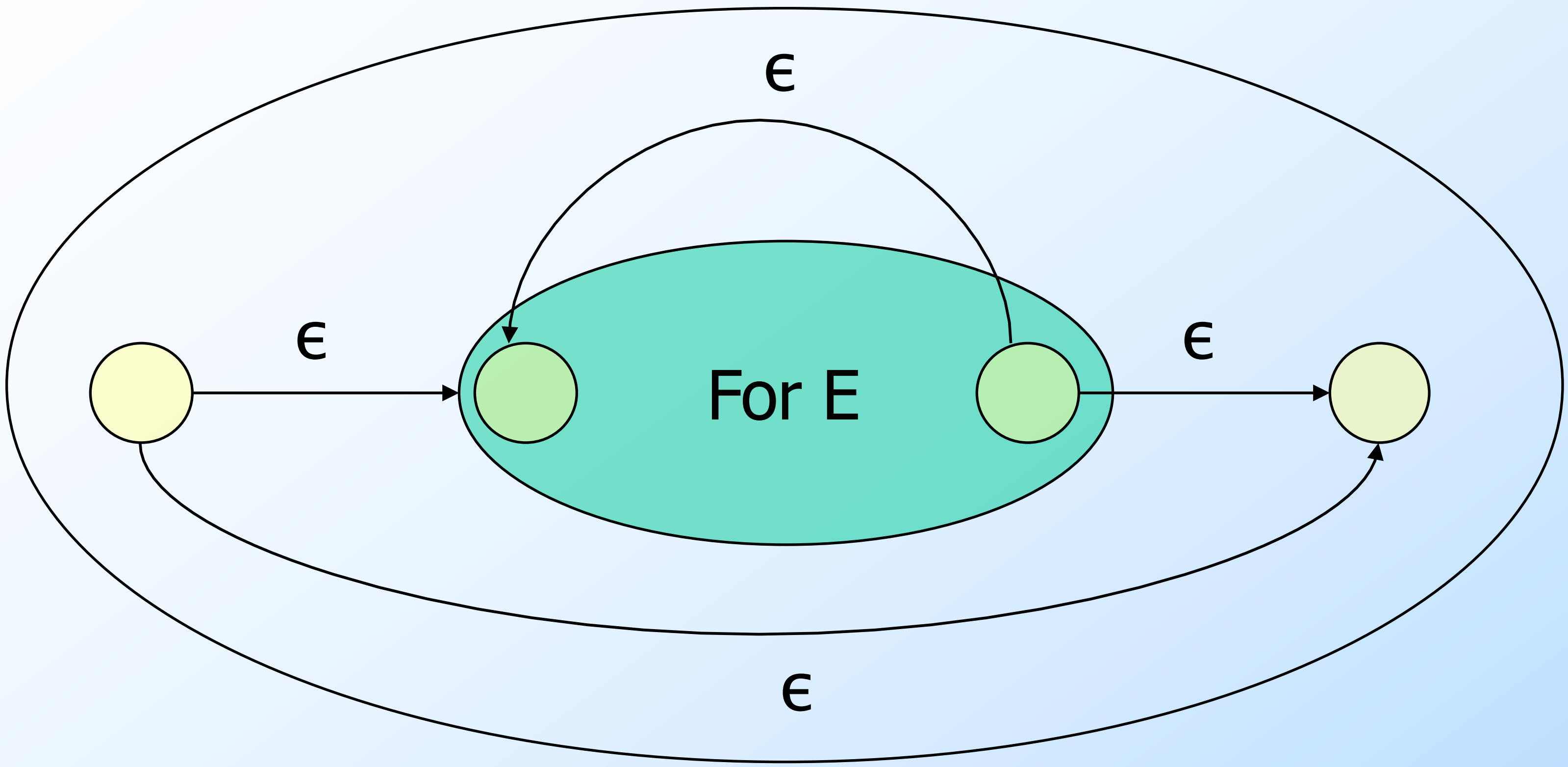
For $E_1 \cup E_2$

RE to ϵ-NFA: Induction 2 – Concatenation



For E_1E_2

RE to ϵ-NFA: Induction 3 – Closure



For E*

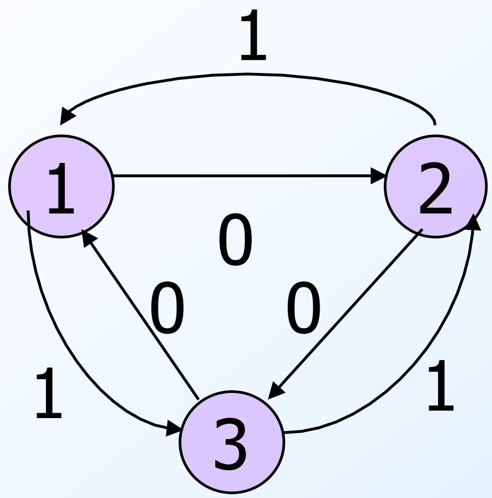
DFA-to-RE

- A strange sort of induction.
- States of the DFA are named 1,2,…,n.
- Induction is on k, the maximum state number we are allowed to traverse along a path.

k-Paths

- A k-path is a path through the graph of the DFA that goes through no state numbered higher than k.
- Endpoints are not restricted; they can be any state.
- n-paths are unrestricted.
- RE is the union of RE's for the n-paths from the start state to each final state.

Example: k-Paths



0-paths from 2 to 3:
RE for labels = **0**.

1-paths from 2 to 3:
RE for labels = **0**+**11**.

2-paths from 2 to 3:
RE for labels =
(**10**)***0**+**1**(**01**)***1**

3-paths from 2 to 3:
RE for labels = ??

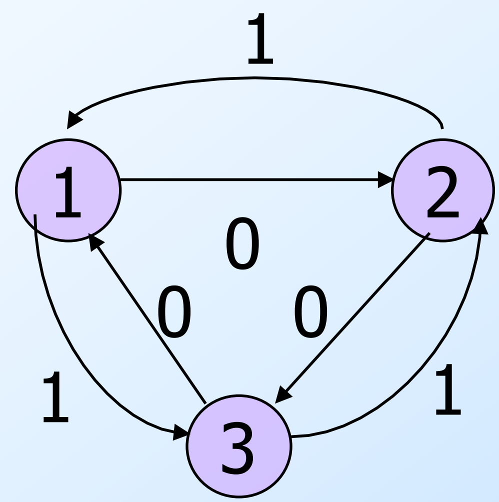
DFA-to-RE

- Basis: k = 0; only arcs or a node by itself.
- Induction: construct RE's for paths allowed to pass through state k from paths allowed only up to k-1.

k-Path Induction

- Let R_{ij}^k be the regular expression for the set of labels of k-paths from state i to state j.
- Basis: k=0. R_{ij}^0 = sum of labels of arc from i to j.
  - $\emptyset$ if no such arc.
  - But add ϵ if i=j.

Example: Basis



- $R_{12}^{0} = \mathbf{0}$.
- $R_{11}^{0} = \varnothing + \epsilon = \epsilon$.

Notice algebraic law:
$\varnothing$ plus anything =
that thing.

k-Path Inductive Case

- A k-path from i to j either:
  1. Never goes through state k, or
  2. Goes through k one or more times.

$$R_{ij}^{k} = R_{ij}^{k-1} + R_{ik}^{k-1}(R_{kk}^{k-1})^{*}\ R_{kj}^{k-1}.$$

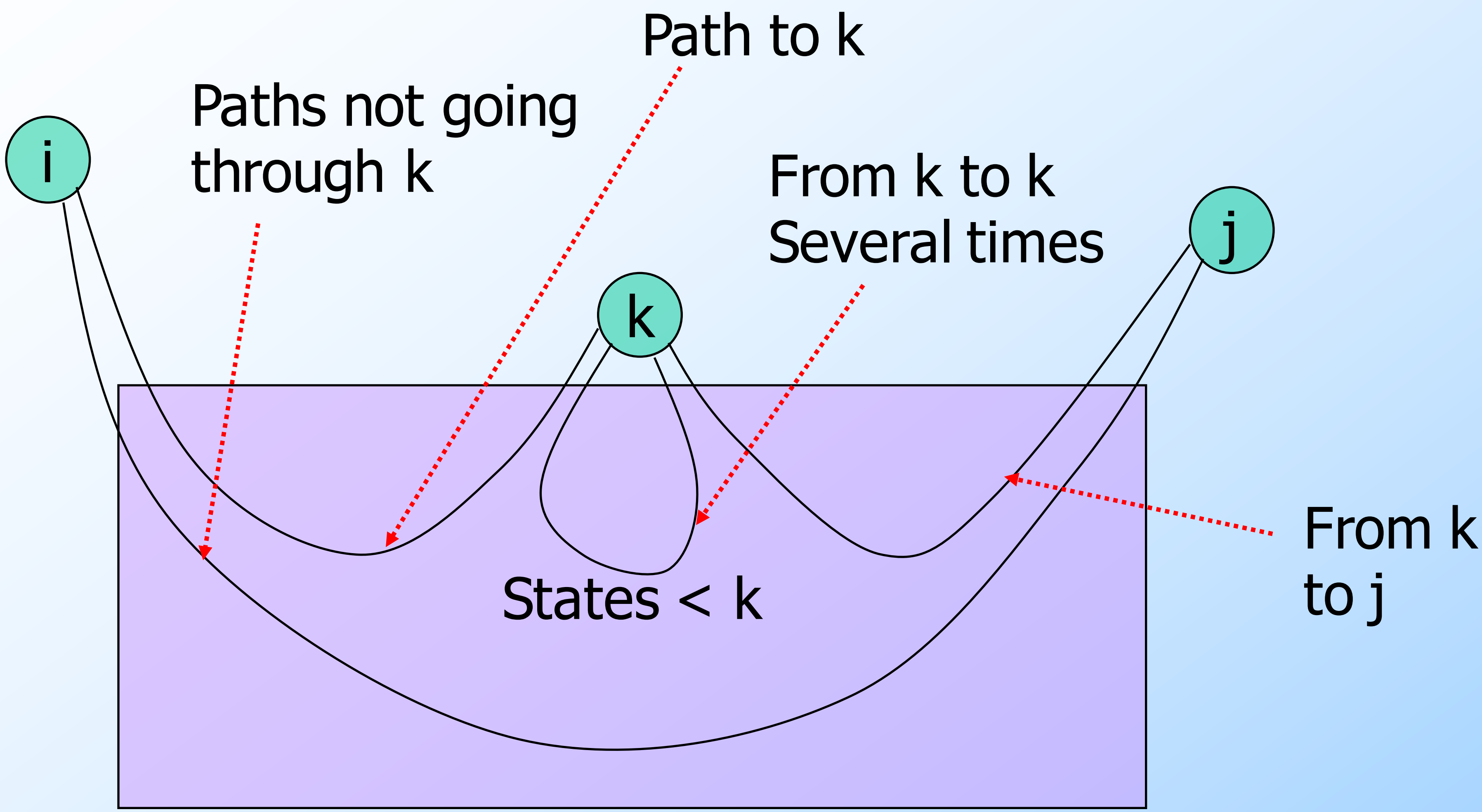
Doesn't go through k

Goes from i to k the first time

Zero or more times from k to k

Then, from k to j

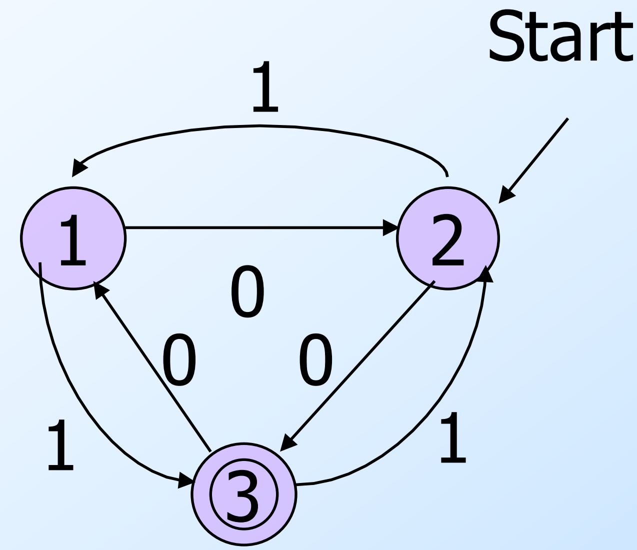
Illustration of Induction



Final Step

- The RE with the same language as the DFA is the sum (union) of R_{ij}^n, where:
  1. n is the number of states; i.e., paths are unconstrained.
  2. i is the start state.
  3. j is one of the final states.

Example



- $R_{23}^3 = R_{23}^2 + R_{23}^2(R_{33}^2)^*R_{33}^2 = R_{23}^2(R_{33}^2)^*$
- $R_{23}^2 = (\mathbf{10})^*\mathbf{0}+\mathbf{1}(\mathbf{01})^*\mathbf{1}$
- $R_{33}^2 = \epsilon + \mathbf{0}(\mathbf{01})^*(\mathbf{1}+\mathbf{00}) + \mathbf{1}(\mathbf{10})^*(\mathbf{0}+\mathbf{11})$
- $R_{23}^3 = [(\mathbf{10})^*\mathbf{0}+\mathbf{1}(\mathbf{01})^*\mathbf{1}]\,[\epsilon + (\mathbf{0}(\mathbf{01})^*(\mathbf{1}+\mathbf{00}) + \mathbf{1}(\mathbf{10})^*(\mathbf{0}+\mathbf{11}))]^*$

Summary

- Each of the three types of automata (DFA, NFA, ϵ-NFA) we discussed, and regular expressions as well, define exactly the same set of languages: the regular languages.

Algebraic Laws for RE's

- Union and concatenation behave sort of like addition and multiplication.
  - + is commutative and associative; concatenation is associative.
  - Concatenation distributes over +.
  - Exception: Concatenation is not commutative.

Identities and Annihilators

- $\emptyset$ is the identity for +.
  - $R + \emptyset = R$.
- ϵ is the identity for concatenation.
  - $\epsilon R = R\epsilon = R$.
- $\emptyset$ is the annihilator for concatenation.
  - $\emptyset R = R\emptyset = \emptyset$.