Model Documentation Risk Management

Jordan Boucher - jpb2232

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1 Executive Summary

The purpose of this document is to review the mathematical intuition behind three common Value at Risk (VaR) and Expected Shortfall (ES) techniques: Monte Carlo, parametric, and historical. We aim to compare and contrast these different methods and provide recommendations on their usage in various circumstances.

The focus of our analysis is on investments in US equities, although the same logic applied in our model can be extended to other asset classes. Our model takes historical data and portfolio construction inputs, generating VaR and ES metrics, along with back test results for the specified portfolio. This approach provides a robust estimate of the future risk associated with the portfolio in the context of the model's theoretical performance in the past.

The model is designed for use on portfolios comprising both long and short positions on equities, as well as vanilla options on equities. It is essential to note that a limitation of parametric VaR is the absence of closed-form formulas for VaR and ES in more complex portfolios involving options or a mix of long and short positions. Consequently, we will only be able to provide Monte Carlo and historical VaR results for such portfolios.

The validation of our model will occur through two distinct approaches. Firstly, we will compare our results directly with models developed by previous risk managers. Given the widespread use of our model, we anticipate locating and contrasting our outcomes with those of these past models. Secondly, we will validate our results through VaR backtests. Fortunately, VaR inherently comes with a certain probability of breach, allowing us to assess the success of our model by comparing expected outcomes with actual results.

2 Introduction

In the upcoming sections, we will delve into the methodologies employed in calculating VaR within the context of our model. Among these methods, the parametric VaR is the most mathematically intensive. This complexity is unsurprising given that parametric VaR necessitates the transformation of a portfolio of assets, subject to geometric Brownian motion, into a singular formula for VaR.

The primary objective of our model is to asses the risk of portfolios of equities, which includes both long and short positions as well as basic vanilla options. In addition to exploring VaR calculations, we will provide a brief overview of the fundamental mechanics underlying options.

The model's development has undergone frequent review in the preceding months, involving several "mini" model validation reports.[1] Each of these assessments introduced new elements, such as alternative VaR calculation methods, additional financial instruments, and diverse portfolio positions. This in-

[1] The homework assignments for this class.

cremental approach allowed for the gradual construction of this comprehensive model.

By providing insights into VaR calculation techniques, we offer a comprehensive understanding of the model's capabilities, applications, and limitations.

3 Product Description

The model was designed for use in portfolios containing US equities, where both long and short positions are possible. Additionally, the model accepts portfolios that use vanilla options. The basic payoff of vanilla call and put options is given below in the figure:

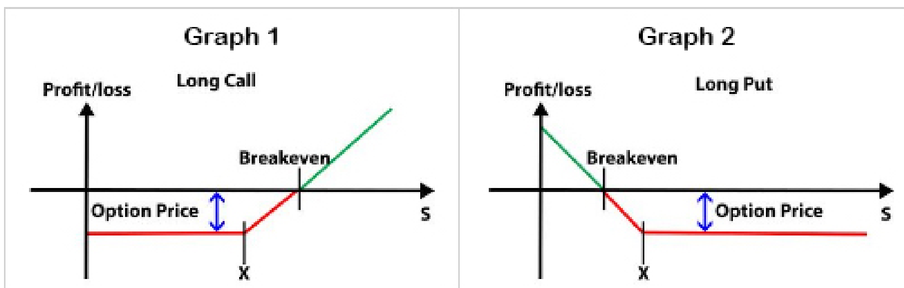

Figure 1: "Hockey stick" payoff function of vanilla calls (left) and puts (right).

Indeed, given some strike price K, the payoff at expiration of a vanilla call is the maximum of the difference between the stock price at expiration S_T and K and 0 and for puts, it is the maximum of the difference between K and S_T and 0. We will use the Black-Scholes model to price these options, which we will discuss in the next section.

4 Model Description

4.1 Model Theory

Stock Price Dynamics

We first begin by stating that we will fit the stocks and the portfolio to geometric Brownian motion. A stochastic process S_t follows geometric Brownian motion if its dynamics are modeled as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

In the case of correlated geometric Brownian motions $\{S_t\}_i$, such as stocks, we say

$$dS_t^i = \mu_i S_t^i dt + \sum_{j=1}^{d} \sigma_{i,j} S_t^i dW_t^j$$

That is to say, the dynamics are defined by the different realizations of $\{W_t\}_j$ through the covariance term $\sigma_{i,j}$. Note that for Monte Carlo VaR (defined below), we model the stocks seperately and combine the correlated paths to model the portfolio. For parametric VaR (defined below) we instead model a single Brownian motion to the entire portfolio. The results from the two methods are very similar. However, it is clear based on the above formulas that, at most, one of these methods is correct. We include both because of the mathematical advantages for the purposes of numerical solutions and for the sake of completeness.

Value at Risk (VaR)

Value at Risk (VaR) is a statistical measure which quantifies the potential loss on a portfolio over a specific time period within a certain level of confidence. In simpler terms, it provides an estimate of the maximum loss that could be incurred given that the loss is within the top returns (defined by the confidence level). Note that VaR is often taken relative to some fixed portfolio value. This is due to the fact that, otherwise, the VaR of a portfolio would increase in dollar terms as the portfolio increases in value simply due to there being more money invested as opposed to there being fundamentally more risk.

VaR is a popular choice among financial institutions for several reasons. For one, it is easy to interpret and gives a notion of risk and probability simultaneously. Simple standard deviation, for example, can be converted into a probability statement if an assumption is made about the underlying distribution of returns, but such a simplification is not necessary for VaR. Another reason to use VaR is for regulatory compliance. Indeed, stress testing and other test of bank strength require VaR calculations, which have made the measure the industry standard.

Historical VaR

Historical VaR–as suggested by the name–is a method for estimating the potential losses of a portfolio by looking at historical price data. Unlike other VaR methods that use statistical models to predict future price movements, historical VaR relies on past historical price changes to estimate the likelihood of losses. The basic idea is to look at the current portfolio and create a time series of the change in the portfolio had it been traded in the past. The key assumption is that the magnitude of the (downside) movements in the past are a strong indicator of future movements.

The calculation involves ordering historical returns from worst to best, selecting the returns that correspond to the chosen confidence level, and using the worst return as the estimate for VaR. As with all types of VaR, the confidence level is an input. Additionally, the window over which the historical returns are considered is also an input. The advantage of using a longer window is that VaR is more stable and less influenced by outliers, and the advantage of using a shorter window is that more recent data has a larger impact. This could be important if, for example, we believe that the dynamics of the portfolio have changed recently due to stress correlation.

Assumptions

1. Stationarity of Returns: Said another way, we assume that the statistical properties of returns do not change over time. While this simplification is common, it may be challenged during periods of economic shifts or crises. It may also be challenged due to changes sector dynamics, regulation change, and the maturity stage of the company.

2. Representative Historical Period: Our assumption is that the past is a reasonable guide to the future. However, this assumption is challenged if the historical period chosen includes unique events or market conditions that are unlikely to be repeated. We can somewhat reduce the impact of this assumption by allowing the lookback period to be a hyperparameter in the model.

3. No Changes in Portfolio Composition: While this assumption is not unique to historical VaR, a constant portfolio composition is important to point out because it overlooks the impact of changes in holdings, by definition.

Parametric VaR

Unlike historical VaR, parametric VaR relies on statistical models and simplifying assumptions to estimate potential losses in a portfolio. Indeed, we use historical data to fit the parameters of some stochastic model and consider dynamics under that model.

Assumptions

1. Normal Distribution of Returns: We can (somewhat) justify normality by the Central Limit Theorem, which states that the sum of many independent and identically distributed random variables approaches a normal distribution. In the context of a portfolio, the theorem suggests that returns tend to be approximately normally distributed if their are enough uncorrelated assets. In reality, we know that this assumption is often violated.

2. Geometric Brownian Motion for Portfolio: This assumption is based on empirical observations that stock prices exhibit random and continuous movements. That being said, it's important to acknowledge that assuming the entire portfolio follows GBM might oversimplify the true dynamics

of the underlying assets. Stock prices and returns are influenced by various factors, and the assumption of a joint GBM process for the entire portfolio may not capture the complexities of the inter-dependencies between individual stocks.

3. Constant Volatility and Expected Returns: This assumption simplifies the modeling process and makes a closed form solution for VaR tenable. However, in reality, these parameters will vary over time due to changing market conditions. In that sense, it is more of a compromise than an assumption.

4. Linear Relationship between Stocks: Said another way, we assume that the correlation between stocks is constant. As with the previous assumption, we have strong empirical evidence against this idea. As previously mentioned, stress correlation does exist.

Monte Carlo VaR

Monte Carlo VaR is a simulation-based approach used to estimate the potential losses of a portfolio by generating a large number of random scenarios for future asset prices. Unlike parametric approaches, Monte Carlo VaR does not rely on closed-form solutions but instead leverages statistical sampling to model the uncertainty in asset returns. Each stock in the portfolio is modeled using correlated geometric Brownian motion, as opposed to the entire portfolio being simulated directly. By running numerous simulations, the method generates a distribution of potential portfolio values, allowing us to estimate the VaR at a desired confidence level.

Monte Carlo VaR offers several advantages. Firstly, it accommodates complex portfolio structures, including a mixture of long and short positions, as well as options. This flexibility makes it well-suited for portfolios with diverse financial instruments. Secondly, Monte Carlo simulations provide a more realistic representation of the uncertainty in financial markets by incorporating a range of potential future scenarios. This makes it particularly useful in capturing non-linearities and tail risk, offering a more comprehensive risk assessment.

However, Monte Carlo VaR also has its drawbacks. The method can be computationally intensive, requiring substantial computational resources to generate a sufficient number of simulations for accurate results. The accuracy of Monte Carlo VaR is highly dependent on the quality of the underlying stochastic models and the correlation structures chosen for the assets. Inaccuracies in these models can lead to biased estimates. Additionally, Monte Carlo VaR may struggle to capture sudden changes in market conditions or extreme events not well-represented in historical data. The approach requires careful validation and calibration to ensure its effectiveness in capturing the portfolio's risk dynamics.

Assumptions

1. Sufficient Number of Simulations: The accuracy of Monte Carlo VaR depends on the number of simulations performed. A larger number of simulations better captures the range of potential portfolio outcomes, based on the law of large numbers. The trade off, naturally is computation time. We assume that have enough simulations to get a reasonable result.

2. Constant Volatility over time and moneyness: The Black-Scholes model assumes that volatility remains constant over the option's life and over various moneyness. This assumption is clearly wrong as we observe both an implied volatility smile as well as term structure skew. We need this assumption, though, to make the calculations simple enough.

3. No-Arbitrage Implication: In a continuous and frictionless market, the assumption is that there are no arbitrage opportunities. In reality, financial markets are not perfectly continuous and frictionless. Transaction costs, bid-ask spreads, and other market frictions can create conditions where arbitrage opportunities may temporarily exist. For the purposes of our analysis, this should not have a huge impact.

Expected Shortfall

Expected Shortfall (ES), is a risk measure that is an extension of VaR. While VaR provides an estimate of the maximum loss at a specified confidence level, ES goes a step further by quantifying the average loss that may occur beyond the VaR threshold. In essence, ES measures the conditional expectation of losses that exceed the VaR. Mathematically, ES represents the average of all losses beyond the VaR level, taking into account the probability of such events. ES provides a more comprehensive view of tail risk, offering insight into the severity of losses during extreme market conditions.

4.2 Mathematical Description

Value at Risk

As heuristically described above, VaR has the following formal mathematical meaning:

$$\begin{aligned} VaR_\alpha(X) &= -inf\{x \in \mathbb{R} : F_x(x) > \alpha\} \\ &= -F_Y^{-1}(1-\alpha), \end{aligned}$$

given that X is a profit and loss distribution and α is the confidence level. We can intuitively derive a formula for historical VaR (HVAR) as well:

$$HVaR_\alpha = -percentile_\alpha\{r_t\}_{t=T-w,\ldots,T},$$

where $\{r_t\}_{t=T-w,\ldots,T}$ is the set of daily returns between the day on which VaR is calculated and the first day of the lookback window. Finding the paremtric

VaR (PVaR) is a bit more challenging. By assumption of geometric Brownian motion, we have that

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}$$

We now need to find an expression for the probability of S_T being below some level and find the level for which the probability is equal to the confidence level:

$$P[S_T < X] = P\left[W_T < \frac{log(X/S_0) - (\mu - \sigma^2/2)T}{\sigma}\right]$$

where $W_T \sim N(0, T)$. From this, we can see that

$$\phi^{-1}(1-p) = \frac{log(X/S_0) - (\mu - \sigma^2/2)T}{\sigma}$$

Solving for X, we get a final answer of

$$VaR(S, T, p) = S_0(1 - e^{\sigma\sqrt{T}\phi^{-1}(1-p) + (\mu + \sigma^2/2)T})$$

Naturally, this will require us to fit μ and σ, which we will do in the next section. Finally, we derive an intuitive definition for Monte Carlo VaR:

$$\begin{aligned} MCVaR_\alpha &= -percentile_\alpha\{ln(S_T^s/S_0)\}_{s=1,\ldots,S} \\ S_T^s &= S_0 e^{(\mu - \sigma^2/2)T + \sigma W_T^s} \\ W_T^s &\sim N(0, T) \end{aligned}$$

That is to say, for each of our S simulations, we take a draw from the $N(0, T)$ distribution and calculate the ending value of the stock based on that draw. We then calculate the log return of this simulation. Finally, considering the entire set of simulated values, we take the negative of the αth percentile.

Expected Shortfall

Now that we have our VaR calculation, we can now focus on expected shortfall, which is defined as follows:

$$ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha VaR_\gamma d\gamma$$

While VaR considers the losses given the αth percentile outcome, ES consider the expected losses given that the losses are, at best, the αth percentile. It is no surprise, then, that ES is at least as large as VaR for the same confidence level.

4.3 Numerical Model Implementation

Now that we have the mathematical foundations of VaR and ES, we can now discuss how these concepts are translated to real data.

Fitting μ, σ, and ρ to GBM

We need to fit parameters to the following model:

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
S_T &= S_0 e^{(\mu - \sigma^2/2)T + \sigma W_T} \\
log(S_{t_2}/S_{t_1}) &= (\mu - \sigma^2/2)T + \sigma(W_{T_2} - W_{T_1})
\end{aligned}
$$

We will use historical mean and standard deviation to accomplish this:

$$
\begin{aligned}
\bar{\mu} &\approx E[log(S_{t_2}/S_{t_1})] \\
&= (\mu - \sigma^2/2)(t_2 - t_1) \\
\bar{\sigma}^2 &\approx \mathrm{Var}[log(S_{t_2}/S_{t_1})] \\
&= \sigma^2(t_2 - t_1)
\end{aligned}
$$

Assuming we use one year of daily data, we get a final of:

$$
\begin{aligned}
\sigma &\approx \bar{\sigma}\sqrt{252} \\
\mu &\approx 252\bar{\mu} + \sigma^2/2 \\
\rho &\approx 252\frac{\bar{\sigma}_{i,j}}{\sigma_i \sigma_j}
\end{aligned}
$$

Windowing and Exponential Weighting

There still remains a question of how to calculate the historical sample means and standard deviaitons. We will discuss two primary ways. The first is windowing. For daily returns $\{a_i\}$:

$$
\begin{aligned}
\bar{\mu}_{wind} &= \frac{1}{N}\sum a_i \\
&= \frac{S_N - S_0}{S_0} \\
\sigma_{wind} &= \sqrt{\frac{1}{N}\sum a_i^2 - \bar{a}^2}
\end{aligned}
$$

That is to say, the mean return is equal to the mean of the daily arithemetic returns and the standard deviation is given by the standard deviation of those same returns. Moving on to exponential weighting, we have:

$$
\begin{aligned}
\bar{\mu}_{expon} &= (1 - \lambda)a_N + \lambda\bar{\mu}_{N-1} \\
\sigma_{expon} &= \sqrt{r_N - m_N^2} \\
r_N &= (1 - \lambda)a_N^2 + \lambda r_{N-1}
\end{aligned}
$$

These functions are recursive in the sense that the previous values directly affect the current value. This makes exponential weighting faster to calculate because the old values are combined with only one new data point (as opposed to recalculating with the entire sequence again).

5 Validation

Validation takes places in the Test Plan and Results documentation.

6 Conclusions

Value at Risk (VaR) models play a crucial role in risk management by quantifying the potential losses in a financial portfolio under various market conditions. Parametric VaR relies on assumptions of normal distribution and geometric Brownian motion, providing a computationally efficient method for estimating risk. Historical VaR utilizes past market data, assuming that historical performance is indicative of future outcomes, offering insights into real-world scenarios. Monte Carlo VaR introduces simulation techniques, accommodating complex instruments and dynamic market conditions through repeated random sampling. These models are essential in assessing and mitigating financial risk, aiding decision-makers in understanding the potential downside of their portfolios. By employing diverse methodologies, risk managers gain a comprehensive view of uncertainty, enabling them to make informed decisions and establish risk tolerance levels to safeguard against adverse market movements. The flexibility of these models allows risk management strategies to be tailored to the specific characteristics of different portfolios and market environments.