2.1 The Restricted Three-Body Problem

This project is related to the IA Dynamics and Relativity lecture course, but is self-contained.

1 Introduction

Determining the motion of a number of gravitating bodies is a classical problem. It can be solved analytically for two bodies, but not for three or more. Various simplifications have historically been considered, one of which is the 'restricted three-body problem' in which the third body is taken to be much smaller in mass than the other two and therefore has negligible influence on their motion. The problem is then to solve for the motion of the third body in the known gravitational field of the first two bodies. This project investigates the 'planar circular restricted three-body problem', a particular case where the first two bodies move in (stable) circular orbits around their joint centre-of-mass (taken as origin) and the motion of the third body is confined to the plane of the circles (taken as the x-y-plane).

It is convenient to transform to a rotating frame of reference in which the first two bodies appear stationary. Scalings may be chosen so that the angular velocity of this frame is 1 and the distance between the two bodies is 1. The only parameter then appearing is the quantity $\mu \in (0, 0.5]$ defined such that the two masses are in the ratio $\mu : 1-\mu$ and are situated respectively at the points $(\mu - 1, 0)$ and $(\mu, 0)$, which will be referred to as P_l and P_h. The equation of motion for the third body, whose position at time t is $(x(t), y(t))$ may then be written as:

$$\ddot{x} - 2\dot{y} = -\frac{\partial \Omega}{\partial x}\,, \tag{1a}$$

$$\ddot{y} + 2\dot{x} = -\frac{\partial \Omega}{\partial y}\,, \tag{1b}$$

where

$$\Omega = -\tfrac{1}{2}\left(x^2 + y^2\right) - \frac{\mu}{\sqrt{(x + 1 - \mu)^2 + y^2}} - \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2}}\,, \tag{2}$$

i.e. Ω is the potential of the centrifugal and gravitational forces.

Despite the substantial restriction to the full three-body problem which this represents, it is not possible to solve the system (1a), (1b) and (2) analytically.

Question 1 Show from (1a) and (1b) that the quantity

$$J = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \Omega(x, y) \tag{3}$$

is constant following the motion. Deduce that trajectories must be confined to the region

$$\Omega(x, y) \leqslant \Omega(x_0, y_0) + \tfrac{1}{2}\left(u_0^2 + v_0^2\right)\,, \tag{4}$$

where x_0, y_0, u_0 and v_0 are the initial values of x, y, $\dot{x}$ and $\dot{y}$, respectively.

Programming Task Write a program to solve the system (1a), (1b) and (2) numerically, given suitable initial conditions on x, y, $\dot{x}$ and $\dot{y}$. You may use a black-box ODE solver such as the MATLAB function `ode45` which automatically adapts the time-step according to specified absolute and relative error tolerances (these may need to be adjusted). A fixed-step solver (as in the Ordinary Differential Equations core project) may require very small time-steps.

Whenever you write a computer program to find a numerical solution, it is necessary to check that the program is generating accurate results. Standard checks include (i) testing the program against known analytic solutions (if there are any), and (ii) varying the time-step or error tolerances. For this problem, the fact that J is constant provides not only a useful constraint on the behaviour of solutions, but also another possible check on numerical accuracy.

2 Space travel

Assume that the third body is a spacecraft, with the first two bodies being co-orbiting planets of equal mass, i.e. $\mu = 0.5$ (the so-called 'Copenhagen problem').

Question 2 Consider motion sufficiently close to P_h that its gravitational attraction dominates both that of P_l and the centrifugal force, and (2) may be approximated by

$$\Omega = -\frac{0.5}{\sqrt{(x-\mu)^2 + y^2}} \,. \tag{5}$$

Show that in polar co-ordinates with

$$x(t) - \mu = r(t)\cos\theta(t)\,, \quad y(t) = r(t)\sin\theta(t) \tag{6}$$

the approximate system (1a)–(1b) with (5) is equivalent to

$$\dot\theta = -1 + kr^{-2}\,, \quad \ddot r = -V'(r) \tag{7}$$

where k is an arbitrary constant and $V'(r)$ is to be found, and (for particular initial conditions) has analytic solutions with the spacecraft in a circular orbit of radius a about P_h where a can take any value [though of course (5) is a good approximation to (2) only if a is small]. How is the constant k related to a?

Modify your program to solve (1a)–(1b) with Ω specified by (5) instead of (2). Demonstrate, for one value of a, that the modified program can *accurately* reproduce the analytic circular-orbit solution$\underline{s}$.

Question 3 Return to the original system (1a), (1b) and (2) with $\mu = 0.5$, and take initial conditions $x = 0.32$, $y = 0$, $\dot x = 0$, $\dot y = v_0$ with v_0 = -1.0, -1.5, -1.73, -1.78, -1.853, -1.858, -2.3 and -2.31 in turn. For each case, use your program to integrate from $t = 0$ to $t = 30$ and

(i) display the trajectory in the (x, y) plane, and the forbidden region $\Omega(x, y) > J$ (shaded, say, using the MATLAB function `contourf`), on the same plot,

(ii) state the values of x and y at $t = 30$, giving a reasoned assessment of their accuracy.

Comment on the trajectories, and how these and the allowed region change as $-v_0$ increases. Is the allowed region a useful guide to the size of the trajectory? What value of v_0 would be most suitable to travel from the neighbourhood of P_h to the neighbourhood of P_l?

[It *may* be instructive to try other values of v_0 and/or integrate further in time.]

3 Lagrange points and asteroids

In this part of the project do not restrict attention to the case $\mu = 0.5$.

Question 4 By examining contour plots of Ω show that the system (1a)-(2) generally has five equilibrium points: three 'collinear Lagrange points' on the x-axis and two 'equilateral Lagrange points' at the third vertex of an equilateral triangle whose other two vertices are at P_l and P_h. Display contour plots for three values of $\mu \in (0, 0.5]$ with the equilibrium points marked. (You may wish to use a black-box root-finder such as the MATLAB function `fzero` to locate the collinear points accurately.)

Investigate numerically the *linear* stability of the collinear Lagrange points, i.e. stability to very small, formally *infinitesimal* perturbations, by starting trajectories a small distance away from the equilibrium point and integrating forward in time. Display plots of a few representative trajectories in the (x, y) plane, together with corresponding plots of x and y against t, to illustrate your results. Explain why you are satisfied that these computations, which start with small but necessarily finite perturbations, have captured the behaviour for infinitesimal perturbations.

What do you conclude about the linear stability of the collinear Lagrange points? Does it depend on μ? Confirm your numerical findings analytically by performing a linearised stability analysis about such points. (You may be able to deduce the necessary information about the second derivatives of Ω by considering the shapes of the contours, rather than by detailed calculation.)

Question 5 Continue with a numerical investigation of the linear stability of the equilateral Lagrange points for parameter values $\mu = 0.01$, 0.025, 0.05, 0.1 and 0.5. Illustrate the results in your write-up with at least one trajectory picture, and corresponding plots of x and y against t, for each.

How do the stability properties change with μ? By further numerical experimentation find, to within $\pm 1\%$, the critical value μ_c dividing values of μ for which the point is linearly stable from those for which it is unstable, and present numerical results in support of your conclusion. Confirm it by performing a linearised stability analysis (which this time certainly does require calculation of the second derivatives of Ω). For the stable cases, what does this analysis indicate about the form of the motion?

Question 6 The (Jupiter) Trojans are asteroids observed near the Sun-Jupiter equilateral Lagrange points, for which $\mu = 9.54 \times 10^{-4}$. Is the persistence of the Trojans near these points consistent with your findings above? The Earth-Moon system has $\mu = 0.012141$ but no analogue of the Trojans is observed: can you suggest why?

Reference

Szebehely, V., *Theory of Orbits: The Restricted Problem of Three Bodies*, Academic Press Inc. (1967).

