

Adaptive dating and fast proposals: revisiting the phylogenetic relaxed clock model

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S1 Appendix: Rate quantiles

1 Piecewise linear approximation

In this article we introduced a linear piecewise approximation of the i-CDF (inverse cumulative distribution function) to improve the computational performance of the *quant* parameterisation. Let $\hat{F}^{-1}(\mathcal{R}_i)$ be the piecewise approximation of the i-CDF $F^{-1}(\mathcal{R}_i)$. The approximation consists of n pieces (where $n = 100$ is fixed). Due to the nonlinear nature of small and large quantiles in a Log-normal distribution, the first and last pieces are not linear approximations but rather equal to the underlying distribution itself.

$$\hat{F}^{-1}(q) = \begin{cases} F^{-1}(q) & \text{if } q \leq \frac{1}{n} \text{ or } q \geq \frac{n-1}{n} \\ F^{-1}(\lfloor v \rfloor) + \left(F^{-1}(\lfloor v \rfloor + 1) - F^{-1}(\lfloor v \rfloor) \right) (v - \lfloor v \rfloor) & \text{otherwise.} \end{cases} \quad (1)$$

where $v = q(n - 1)$ indexes quantile q into piece number $\lfloor v \rfloor$. Values from the underlying function F^{-1} are cached, enabling rapid computation.

2 Tree operators for rate quantiles

Zhang and Drummond 2020 introduced several tree operators for the *real* parameterisation – including `ConstantDistance`, `SimpleDistance`, and `SmallPulley` [1]. In this appendix, these three operators are extended to the *quant* parameterisation. Following the notation presented in the main article, let t_i be the time of node i , let $0 < q_i < 1$ be the rate quantile of node i , and let $r_i = \hat{F}^{-1}(q_i)$ be the real rate of node i where \hat{F}^{-1} is the linear approximation of the i-CDF.

Constant Distance

Let \mathcal{X} be a uniformly-at-random sampled internal node on tree \mathcal{T} . Let \mathcal{L} and \mathcal{R} be the left and right child of \mathcal{X} , respectively, and let \mathcal{P} be the parent of \mathcal{X} . Under the *quant* parameterisation, the **ConstantDistance** operator works as follows:

Step 1. Propose a new height for $t_{\mathcal{X}}$:

$$t_{\mathcal{X}'} \leftarrow t_{\mathcal{X}} + s\Sigma \quad (2)$$

where Σ is drawn from a proposal transition distribution (Uniform or Bactrian), and s is a tunable step size. Ensure that $\max\{t_{\mathcal{L}}, t_{\mathcal{R}}\} < t_{\mathcal{X}'} < t_{\mathcal{P}}$, and if the constraint is broken then reject the proposal.

Step 2. Recalculate $q_{\mathcal{X}}$ as:

$$\begin{aligned} q_{\mathcal{X}'} &\leftarrow \hat{F}(r_{\mathcal{X}'}) \\ &\leftarrow \hat{F}\left(\frac{t_{\mathcal{P}} - t_{\mathcal{X}}}{t_{\mathcal{P}} - t_{\mathcal{X}'}} r_{\mathcal{X}}\right) \\ &\leftarrow \hat{F}\left(\frac{t_{\mathcal{P}} - t_{\mathcal{X}}}{t_{\mathcal{P}} - t_{\mathcal{X}'}} F^{\hat{-1}}(q_{\mathcal{X}})\right). \end{aligned} \quad (3)$$

This ensures that the genetic distance between \mathcal{X} and \mathcal{P} remains constant after the operation by enforcing the constraint:

$$r_{\mathcal{X}}(t_{\mathcal{P}} - t_{\mathcal{X}}) = r_{\mathcal{X}'}(t_{\mathcal{P}} - t_{\mathcal{X}'}). \quad (4)$$

Step 3. Similarly, propose new rate quantiles for the two children $\mathcal{C} \in \{\mathcal{L}, \mathcal{R}\}$:

$$\begin{aligned} q_{\mathcal{C}'} &\leftarrow \hat{F}(r_{\mathcal{C}'}) \\ &\leftarrow \hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{C}}}{t_{\mathcal{X}'} - t_{\mathcal{C}}} \times r_{\mathcal{C}}\right) \\ &\leftarrow \hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{C}}}{t_{\mathcal{X}'} - t_{\mathcal{C}}} \times F^{\hat{-1}}(q_{\mathcal{C}})\right). \end{aligned} \quad (5)$$

Ensure that $0 < q_i' < 1$ for all proposed nodes $i \in \{\mathcal{X}, \mathcal{L}, \mathcal{R}\}$, and if the constraint is broken then reject the proposal. This constraint can only be broken from numerical issues.

Step 4. Finally, in order to calculate the Metropolis-Hastings-Green ratio, return the determinant of the Jacobian matrix:

$$\begin{aligned}
J &= \begin{bmatrix} \frac{\partial t_{\mathcal{X}'}}{\partial t_{\mathcal{X}}} & \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{X}}} & \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{L}}} & \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{R}}} \\ \frac{\partial q_{\mathcal{X}'}}{\partial t_{\mathcal{X}}} & \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{X}}} & \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{R}}} \\ \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{L}'}}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{R}'}}{\partial q_{\mathcal{L}}} \\ \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{R}}} & \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{R}}} & \frac{\partial q_{\mathcal{L}'}}{\partial q_{\mathcal{R}}} & \frac{\partial q_{\mathcal{R}'}}{\partial q_{\mathcal{R}}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial t_{\mathcal{X}'}}{\partial t_{\mathcal{X}}} & 0 & 0 & 0 \\ \frac{\partial q_{\mathcal{X}'}}{\partial t_{\mathcal{X}}} & \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{X}}} & 0 & 0 \\ \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{L}}} & 0 & \frac{\partial q_{\mathcal{L}'}}{\partial q_{\mathcal{L}}} & 0 \\ \frac{\partial t_{\mathcal{X}'}}{\partial q_{\mathcal{R}}} & 0 & 0 & \frac{\partial q_{\mathcal{R}'}}{\partial q_{\mathcal{R}}} \end{bmatrix}.
\end{aligned} \tag{6}$$

As J is triangular, its determinant $|J|$ is equal to the product of diagonal elements:

$$\begin{aligned}
\ln |J| &= \ln \left\{ \frac{\partial t_{\mathcal{X}'}}{\partial t_{\mathcal{X}}} \times \frac{\partial q_{\mathcal{X}'}}{\partial q_{\mathcal{X}}} \times \frac{\partial q_{\mathcal{L}'}}{\partial q_{\mathcal{L}}} \times \frac{\partial q_{\mathcal{R}'}}{\partial q_{\mathcal{R}}} \right\} \\
&= \ln 1 + \ln D\hat{F}\left(\frac{t_{\mathcal{P}} - t_{\mathcal{X}}}{t_{\mathcal{P}} - t_{\mathcal{X}'}} \times F^{\hat{-1}}(q_{\mathcal{X}})\right) + \ln \frac{\partial}{\partial q_{\mathcal{X}}} \frac{t_{\mathcal{P}} - t_{\mathcal{X}}}{t_{\mathcal{P}} - t_{\mathcal{X}'}} F^{\hat{-1}}(q_{\mathcal{X}}) \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}'} - t_{\mathcal{L}}} \times F^{\hat{-1}}(q_{\mathcal{L}})\right) + \ln \frac{\partial}{\partial q_{\mathcal{L}}} \frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}'} - t_{\mathcal{L}}} F^{\hat{-1}}(q_{\mathcal{L}}) \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}'} - t_{\mathcal{R}}} \times F^{\hat{-1}}(q_{\mathcal{R}})\right) + \ln \frac{\partial}{\partial q_{\mathcal{R}}} \frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}'} - t_{\mathcal{R}}} F^{\hat{-1}}(q_{\mathcal{R}}) \\
&= \ln D\hat{F}\left(\frac{t_{\mathcal{P}} - t_{\mathcal{X}}}{t_{\mathcal{P}} - t_{\mathcal{X}'}} \times F^{\hat{-1}}(q_{\mathcal{X}})\right) + \ln DF^{\hat{-1}}(q_{\mathcal{X}}) + \ln \frac{t_{\mathcal{P}} - t_{\mathcal{X}}}{t_{\mathcal{P}} - t_{\mathcal{X}'}} \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}'} - t_{\mathcal{L}}} \times F^{\hat{-1}}(q_{\mathcal{L}})\right) + \ln DF^{\hat{-1}}(q_{\mathcal{L}}) + \ln \frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}'} - t_{\mathcal{L}}} \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}'} - t_{\mathcal{R}}} \times F^{\hat{-1}}(q_{\mathcal{R}})\right) + \ln DF^{\hat{-1}}(q_{\mathcal{R}}) + \ln \frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}'} - t_{\mathcal{R}}}.
\end{aligned} \tag{7}$$

The derivatives $D\hat{F}$ and $DF^{\hat{-1}}$ are computed using numerical approximations for the first and last pieces, or as the gradient of the linear approximation for internal pieces. As its final step, the operator returns $\ln |J|$.

Simple Distance

While `ConstantDistance` proposes internal node heights, `SimpleDistance` operates on the root. Let \mathcal{X} be the root node and let \mathcal{L} and \mathcal{R} be its two children.

Step 1. Propose a new height for $t_{\mathcal{X}}$:

$$t_{\mathcal{X}'} \leftarrow t_{\mathcal{X}} + s\Sigma. \tag{8}$$

Ensure that $\max\{t_{\mathcal{L}}, t_{\mathcal{R}}\} < t_{\mathcal{X}'}$, and if the constraint is broken then reject the proposal.

Step 2. Propose new rate quantiles for the two children $\mathcal{C} \in \{\mathcal{L}, \mathcal{R}\}$:

$$\begin{aligned}
q_{\mathcal{C}}' &\leftarrow \hat{F}(r_{\mathcal{C}}') \\
&\leftarrow \hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{C}}}{t_{\mathcal{X}}' - t_{\mathcal{C}}} \times r_{\mathcal{C}}\right) \\
&\leftarrow \hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{C}}}{t_{\mathcal{X}}' - t_{\mathcal{C}}} \times F^{\hat{-1}}(q_{\mathcal{C}})\right).
\end{aligned} \tag{9}$$

These proposals ensure that the genetic distance between \mathcal{X} and its children \mathcal{C} remain constant after the operation by enforcing the constraint:

$$r_{\mathcal{C}}(t_{\mathcal{X}} - t_{\mathcal{C}}) = r_{\mathcal{C}}'(t_{\mathcal{X}}' - t_{\mathcal{C}}). \tag{10}$$

Ensure that $0 < q_{\mathcal{C}}' < 1$, and if the constraint is broken then reject the proposal.

Step 3. Finally, in order to calculate the Metropolis-Hastings-Green ratio, return the determinant of the Jacobian matrix:

$$\begin{aligned}
J &= \begin{bmatrix} \frac{\partial t_{\mathcal{X}}'}{\partial t_{\mathcal{X}}} & \frac{\partial t_{\mathcal{X}}'}{\partial q_{\mathcal{L}}} & \frac{\partial t_{\mathcal{X}}'}{\partial q_{\mathcal{R}}} \\ \frac{\partial q_{\mathcal{L}}'}{\partial t_{\mathcal{X}}} & \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{R}}} \\ \frac{\partial q_{\mathcal{R}}'}{\partial t_{\mathcal{X}}} & \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial t_{\mathcal{X}}'}{\partial t_{\mathcal{X}}} & 0 & 0 \\ \frac{\partial q_{\mathcal{L}}'}{\partial t_{\mathcal{X}}} & \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} & 0 \\ \frac{\partial q_{\mathcal{R}}'}{\partial t_{\mathcal{X}}} & 0 & \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \end{bmatrix}.
\end{aligned} \tag{11}$$

As J is triangular, its determinant $|J|$ is equal to the product of diagonal elements:

$$\begin{aligned}
\ln |J| &= \ln \left\{ \frac{\partial t_{\mathcal{X}}'}{\partial t_{\mathcal{X}}} \times \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} \times \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \right\} \\
&= \ln \frac{\partial t_{\mathcal{X}}'}{\partial t_{\mathcal{X}}} + \ln \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} + \ln \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \\
&= \ln 1 \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}}' - t_{\mathcal{L}}} \times F^{\hat{-1}}(q_{\mathcal{L}})\right) + \ln \frac{\partial}{\partial q_{\mathcal{L}}} \frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}}' - t_{\mathcal{L}}} F^{\hat{-1}}(q_{\mathcal{L}}) \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}}' - t_{\mathcal{R}}} \times F^{\hat{-1}}(q_{\mathcal{R}})\right) + \ln \frac{\partial}{\partial q_{\mathcal{R}}} \frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}}' - t_{\mathcal{R}}} F^{\hat{-1}}(q_{\mathcal{R}}) \\
&= \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}}' - t_{\mathcal{L}}} \times F^{\hat{-1}}(q_{\mathcal{L}})\right) + \ln DF^{\hat{-1}}(q_{\mathcal{L}}) + \ln \frac{t_{\mathcal{X}} - t_{\mathcal{L}}}{t_{\mathcal{X}}' - t_{\mathcal{L}}} \\
&\quad + \ln D\hat{F}\left(\frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}}' - t_{\mathcal{R}}} \times F^{\hat{-1}}(q_{\mathcal{R}})\right) + \ln DF^{\hat{-1}}(q_{\mathcal{R}}) + \ln \frac{t_{\mathcal{X}} - t_{\mathcal{R}}}{t_{\mathcal{X}}' - t_{\mathcal{R}}}.
\end{aligned} \tag{12}$$

As its final step, the operator returns $\ln |J|$.

Small Pulley

Just like the previous operator, **SmallPulley** operates on the root. Let \mathcal{X} be the root node and let \mathcal{L} and \mathcal{R} be its two children. However, unlike **SimpleDistance**, this operator alters the two genetic distances $d_{\mathcal{L}} = r_{\mathcal{L}}(t_{\mathcal{X}} - t_{\mathcal{L}}) = F^{\hat{-1}}(q_{\mathcal{L}})(t_{\mathcal{X}} - t_{\mathcal{L}})$ and $d_{\mathcal{R}} = r_{\mathcal{R}}(t_{\mathcal{X}} - t_{\mathcal{R}}) = F^{\hat{-1}}(q_{\mathcal{R}})(t_{\mathcal{X}} - t_{\mathcal{R}})$, while conserving their sum $d_{\mathcal{L}} + d_{\mathcal{R}}$.

Step 1. Propose new genetic distances for $d_{\mathcal{L}}$ and $d_{\mathcal{R}}$:

$$d_{\mathcal{L}}' \leftarrow d_{\mathcal{L}} + s\Sigma \quad (13)$$

$$d_{\mathcal{R}}' \leftarrow d_{\mathcal{R}} - s\Sigma \quad (14)$$

Ensure that $0 < d_{\mathcal{L}}' < d_{\mathcal{L}} + d_{\mathcal{R}}$, and if the constraint is broken then reject the proposal.

Step 2. Propose new rate quantiles for the two children \mathcal{L} and \mathcal{R} :

$$\begin{aligned} q_{\mathcal{L}}' &\leftarrow \hat{F}\left(\frac{d_{\mathcal{L}}'}{t_{\mathcal{X}} - t_{\mathcal{L}}}\right) \\ &\leftarrow \hat{F}\left(\frac{F^{\hat{-1}}(q_{\mathcal{L}})(t_{\mathcal{X}} - t_{\mathcal{L}}) + \Sigma}{t_{\mathcal{X}} - t_{\mathcal{L}}}\right) \end{aligned} \quad (15)$$

$$\begin{aligned} q_{\mathcal{R}}' &\leftarrow \hat{F}\left(\frac{d_{\mathcal{R}}'}{t_{\mathcal{X}} - t_{\mathcal{R}}}\right) \\ &\leftarrow \hat{F}\left(\frac{F^{\hat{-1}}(q_{\mathcal{R}})(t_{\mathcal{X}} - t_{\mathcal{R}}) - \Sigma}{t_{\mathcal{X}} - t_{\mathcal{R}}}\right). \end{aligned} \quad (16)$$

Step 3. Return the determinant of the Jacobian matrix:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{R}}} \\ \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{L}}} & \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} & 0 \\ 0 & \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \end{bmatrix}. \end{aligned} \quad (17)$$

As J is triangular/diagonal, its determinant $|J|$ is equal to the product of diagonal elements:

$$\begin{aligned}
\ln |J| &= \ln \left\{ \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} \times \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \right\} \\
&= \ln \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} + \ln \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \\
&= \ln D\hat{F}\left(\frac{F^{\hat{-1}}(q_{\mathcal{L}})(t_{\mathcal{X}} - t_{\mathcal{L}}) + \Sigma}{t_{\mathcal{X}} - t_{\mathcal{L}}}\right) + \ln \frac{\partial q_{\mathcal{L}}'}{\partial q_{\mathcal{L}}} \frac{F^{\hat{-1}}(q_{\mathcal{L}})(t_{\mathcal{X}} - t_{\mathcal{L}}) + \Sigma}{t_{\mathcal{X}} - t_{\mathcal{L}}} \\
&\quad + \ln D\hat{F}\left(\frac{F^{\hat{-1}}(q_{\mathcal{R}})(t_{\mathcal{X}} - t_{\mathcal{R}}) - \Sigma}{t_{\mathcal{X}} - t_{\mathcal{R}}}\right) + \ln \frac{\partial q_{\mathcal{R}}'}{\partial q_{\mathcal{R}}} \frac{F^{\hat{-1}}(q_{\mathcal{R}})(t_{\mathcal{X}} - t_{\mathcal{R}}) - \Sigma}{t_{\mathcal{X}} - t_{\mathcal{R}}} \\
&= \ln D\hat{F}\left(\frac{F^{\hat{-1}}(q_{\mathcal{L}})(t_{\mathcal{X}} - t_{\mathcal{L}}) + \Sigma}{t_{\mathcal{X}} - t_{\mathcal{L}}}\right) + \ln DF^{\hat{-1}}(q_{\mathcal{L}}) \\
&\quad + \ln D\hat{F}\left(\frac{F^{\hat{-1}}(q_{\mathcal{R}})(t_{\mathcal{X}} - t_{\mathcal{R}}) - \Sigma}{t_{\mathcal{X}} - t_{\mathcal{R}}}\right) + \ln DF^{\hat{-1}}(q_{\mathcal{R}}). \tag{18}
\end{aligned}$$

Thus, as its final step, the operator returns $\ln |J|$.

3 CisScale operator

CisScale was originally introduced by Zhang and Drummond 2020 for the *real* parameterisation (therein named `ucldstdevScaleOperator`). Under the *quant* configuration, the **CisScale** operator works as follows.

Step 1. Propose a new value for the relaxed clock standard deviation σ

$$\sigma' \leftarrow \sigma \times e^{s\Sigma}. \quad (19)$$

Step 2. Recalculate all branch substitution rate quantiles q such that their rates r remain constant

$$\text{let } r = \hat{F}^{-1}(q|\sigma) \quad (20)$$

$$\text{let } r' = r \quad (21)$$

$$q' \leftarrow \hat{F}(r'|\sigma') = \hat{F}(\hat{F}^{-1}(q|\sigma)|\sigma'). \quad (22)$$

Step 3. Return the log Hastings-Green ratio of this proposal. If Σ was drawn from a symmetric proposal kernel (such as the Bactrian distribution), this is equal to:

$$|J| = \log(e^{s\Sigma}) + \log\left(\frac{\delta}{\delta q} \hat{F}(\hat{F}^{-1}(q|\sigma)|\sigma')\right) \quad (23)$$

$$= s\Sigma + \log D\hat{F}(\hat{F}^{-1}(q|\sigma)|\sigma') + \log D\hat{F}^{-1}(q|\sigma), \quad (24)$$

where derivatives $D\hat{F}$ and $D\hat{F}^{-1}$ can be approximated using either the piecewise linear model or standard numerical libraries.

4 Narrow exchange rates

The `NarrowExchangeRate` operator is also compatible with rate quantiles. This operator behaves the same as presented in the main article however the Hastings-Green ratio requires further augmentation due to changes in dimension throughout the proposal.

Step 1. Apply `NarrowExchange` to the current tree topology as described in the main article. This will return a Hastings ratio H due to the asymmetry of this proposal.

Step 2. Compute the relevant branch rates r_i for $r \in \{A, B, C, D\}$ of the current state from their respective quantile parameters.

$$r_i = \hat{F}^{-1}(q_i). \quad (25)$$

Step 3. Propose new rates and node heights and compute the Hastings-Green ratio of the real-space component of the proposal (e.g. Algorithms 1-2 of the main article).

$$(r'_A, r'_B, r'_C, r'_D, t'_D, |J_r|) \leftarrow \text{PROPOSAL}(r_A, r_B, r_C, r_D, t_D). \quad (26)$$

Step 4. Transform the rates back into quantiles.

$$q'_i = \hat{F}(r'_i). \quad (27)$$

Step 5. Compute the log Hastings-Green ratio of the interconversion between rates and quantiles.

$$\log |J_q| = \log \hat{F}(q) + \log \hat{F}^{-1}(r'). \quad (28)$$

Step 6. Return the total log Hastings-Green ratio of this proposal: $\log H + \log |J_r| + \log |J_q|$.

References

- [1] Zhang R, Drummond A. Improving the performance of Bayesian phylogenetic inference under relaxed clock models. BMC Evolutionary Biology. 2020;20:1–28.