Mu Alpha Theta Calculus Review

1 Introduction

This is a very brief review of AP Calculus for the purposes of doing well at the Mu Alpha Theta State competition. It is by no means a way to learn calculus, and it does not go over basic facts that the reader is assumed to know. Most of the things covered here will probably be useful at Mu Alpha Theta, so it is a good idea to go over it, especially if you do not remember your calculus from the beginning of the year.

2 Limits

2.1 Undefined Limits

These are generally when you have something in the denominator that equals zero and something in the numerator that is not zero, for example,

$$\lim_{x \to 4} \frac{x^2 + 3x - 6}{x - 4} = \frac{4^2 + 3(4) - 6}{4 - 4} = \frac{22}{0}.$$

The limit is considered undefined or ∞.

2.2 Removable Discontinuities

These are generally when both the numerator and the denominator are equal to zero and we can "plug" the hole in the graph, for example,

$$\lim_{x \to 4} \frac{x^2 + 3x - 28}{x - 4} = \lim_{x \to 4} \frac{(x+7)(x-4)}{x - 4} = \lim_{x \to 4} (x + 7) = 11.$$

Just because the denominator is zero does not mean the limit is undefined!

2.3 L'Hopital's Rule

This states that if $f(x)$ and $g(x)$ are functions such that $\frac{f(c)}{g(c)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

For example, if we wanted to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$, we know that both go to zero, so we can apply L'Hopital's Rule to get

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Be cautious, though; always check that it satisfies one of the indeterminate forms before using L'Hopital's Rule.

2.4 Using the Logarithm

Sometimes, you may need to take the logarithm of a limit and then apply L'Hopital's Rule to evaluate it. For example, if we wanted to evaluate

$$\lim_{x\to 0} x^x$$

plugging it in gives 0^0 which is not defined. So take the logarithm of it to get $\ln(x^x) = x\ln x$ and now take the limit by using L'Hopital cleverly:

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0.$$

But remember at the end to exponentiate; so our original limit would be

$$\lim_{x\to 0} x^x = e^{\lim_{x\to 0} x\ln x} = e^0 = 1.$$

3 Differentiation

3.1 Definition

For a function to be differentiable at a point, it must also be continuous; the reverse statement is not necessarily true, however. If the function is differentiable, the derivative of $f(x)$ is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Beware of problems that look like limits but are really the definition of the derivative; using this knowledge can simplify them greatly. For example,

$$\lim_{h\to 0} \frac{((3+h)^2 + (3+h)) - (3^2+3)}{h}$$

is just the derivative of $f(x) = x^2 + x$ at $x = 3$.

3.2 Basic Things

3.2.1 Critical Points and Points of Inflection

The critical points of a function $f(x)$ are the points at which $f'(x) = 0$. There are several classifications of critical points: local minimum/maximum, absolution minimum/maximum, or just none of them. Know how to find them and determine what they are.
Points of inflection of $f(x)$ are, for the most part, points at which $f''(x) = 0$. Don't forget the definition!

3.2.2 The Chain Rule

The chain rule is very important and often forgotten. It states that

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

Always remember to use this when you have a composition of functions.

3.3 Differentiation Techniques

3.3.1 Logarithmic Differentiation

Like with limits, sometimes we need the logarithm to differentiate. We will use the same example: suppose we want to find $f'(x)$, where $f(x) = x^x$. Well, take the logarithm of both sides to get

$$\ln f(x) = x \ln x.$$

Now take the derivative, being careful with the chain rule. We have

$$\frac{f'(x)}{f(x)} = \ln x + 1 \Rightarrow f'(x) = f(x)(\ln x + 1).$$

But since $f(x) = x^x$, we substitute that and get $f'(x) = x^x(\ln x + 1)$. Logarithmic differentiation is a very powerful tool; keep it handy.

3.3.2 Implicit Differentiation

When the x's and y's are mixed together and you need to find a derivative, implicit differentiation is usually a good way to go. Let's say we want to find $\frac{dy}{dx}$ in the equation

$$x^2 + y^2 - 4x + 6y - 3 = 0.$$

We want to take the derivative with respect to x, but what is the derivative of a function of y with respect to x? Use the chain rule; that tells us that $\frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$. Using this we can take the derivative implicitly:

$$\frac{d}{dx}[x^2 + y^2 - 4x + 6y - 3] = 2x + 2y \cdot \frac{dy}{dx} - 4 + 6 \cdot \frac{dy}{dx} = 0.$$

Solving for the derivative, we get

$$\frac{dy}{dx} = \frac{4 - 2x}{2y + 6}.$$

3.4 Applications

3.4.1 Functions of Motion

If we represent the position of a particle as a function of time, $s(t)$ for example, we can also find the velocity and acceleration of the particle. These are, respectively, $v(t) = s'(t)$ and $a(t) = s''(t)$.

3.4.2 Linear Approximations

One nice thing about derivatives is that they let us approximate functions. For example, if we wanted to approximate $\sqrt[3]{29}$, we could do the following:

- Find a close known value, i.e. $\sqrt[3]{27} = 3$.

- Take the derivative of the function, i.e. $\frac{d}{dx}[\sqrt[3]{x}] = \frac{1}{3x^{\frac{2}{3}}}$.
- Plug in your known value, i.e. $\frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{27}$.
- Use this to linearly approximate the new value, i.e. $\sqrt[3]{29} \approx \sqrt[3]{27} + (29-27) \cdot \frac{1}{27} = \frac{83}{27}$.

4 Integration

4.1 Definition

Like the definition of the derivative, the definition of the integral is via a limit and often integrals are disguised as limits. The integral of a function $f(x)$ on the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}),$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $t_1, t_2, \ldots, t_n$ are in the intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$, respectively.
What this allows us to do is evaluate a limit like

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \right),$$

because if we use $f(x) = \frac{1}{x}$ and $t_1 = x_0 = n, t_2 = x_1 = n+1, \ldots, t_n = x_{n-1} = 2n-1$ and $x_n = 2n$, the above limit is equivalent to

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) = \int_n^{2n} \frac{1}{x} dx = \big[\ln x \big]_n^{2n} = \ln(2n) - \ln(n) = \ln 2.$$

4.2 Integration Techniques

4.2.1 u-substitution

This is the king of all integration techniques and you cannot forget it! When evaluating an integral, we can substitute $u = g(x)$, but we must also substitute accordingly for the dx term. For example, if we wanted to evaluate

$$\int \frac{1}{x \ln x} dx$$

we could set $u = \ln x$, but we would also have to take the derivative of the substitution to get $du = \frac{1}{x} dx$. Now replacing the terms in the integral, we see

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln u + C = \ln \ln x + C.$$

A common u-substitution problem is $\int x\sqrt{x-1} dx$. Make sure you can do this.

4.2.2 Partial Fractions

Suppose we wanted to integrate $\frac{1}{x^2+5x+6}$. We cannot do this regularly, so we try to split it up into parts, hence the name. Since $x^2+5x+6=(x+2)(x+3)$, we assume

$$\frac{1}{x^2+5x+6}=\frac{A}{x+2}+\frac{B}{x+3}.$$

There are a variety of ways to solve this, but we get

$$\frac{1}{x^2+5x+6}=\frac{1}{x+2}-\frac{1}{x+3},$$

both of which we know how to integrate easily, so we are set.

4.2.3 Trigonometric Substitution

If we encounter an integral like

$$\int \frac{1}{\sqrt{x^2+1}}dx,$$

we should look for a trigonometric substitution. The square root of the sum of two squares is a good giveaway. One possibility is $x=\tan\theta$, which gives us $dx=\sec^2\theta d\theta$. Substitute now to get

$$\int \frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}}d\theta=\int\frac{\sec^2\theta}{\sec\theta}d\theta=\int\sec\theta d\theta=\ln|\sec\theta+\tan\theta|+C=\ln|\sqrt{x^2+1}+x|+C.$$

4.2.4 Integration by Parts

If we have an integral of two functions multiplied by each other, there is a good chance we need to use integration by parts. Suppose we have

$$\int x\ln x dx,$$

which cannot be integrated by any of the above methods. Integration by parts says that

$$\int udv=uv-\int vdu.$$

In general, we pick dv that integrates nicely and the rest is u. In this case, we can pick $dv=xdx$ and $u=\ln x$ so $v=\frac{x^2}{2}$ and $du=\frac{1}{x}dx$. We get

$$\int x\ln x dx=\frac{x^2}{2}\ln x-\int\frac{x}{2}dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

Whenever you use integration by parts, make sure you pick good u's and v's. Otherwise, it can lead to a nightmare.

4.3 Applications

4.3.1 Differential Equations

The differential equations you will see will probably be very easy. Examples include solving for y given

$$\frac{dy}{dx} = ky.$$

The basic idea is the separate the variables, i.e. move all the y's to one side and all the x's to the other. We would find something like

$$\frac{dy}{y} = kdx \Rightarrow \int \frac{dy}{y} = \int kdx \Rightarrow \ln y = kx + C \Rightarrow y = C_0 e^{kx}.$$

In some cases, you may have to use a partial fraction decomposition on the y side.

4.3.2 Arc Length

Just memorize the equation here:

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

.

4.3.3 Volume: Disk Method

To find the volume of a rotated graph using the disk method, we need one function: r, the radius of the disks. It may be $r(x)$ or $r(y)$, depending on what you are rotating around. Just remember this: disk method rotating horizontally is x and rotating vertically is y. So, we have one of the following integrals:

$$\pi \int_{x_a}^{x_b} [r(x)]^2 dx \quad \text{or} \quad \pi \int_{y_a}^{y_b} [r(y)]^2 dy.$$

Simply choose r to be the distance that f is away from the axis of revolution; most of the time, r is either f (rotating around x-axis) or f^{-1} (rotating around y-axis).

4.3.4 Volume: Shell Method

If the disk method fails, you can always resort to the shell method for finding volume, but this time we will need two functions: r and h, the radius and height of the cylindrical shells, respectively. They will be functions of the same variable. Also, it is the opposite of the disk method, so horizontally is y and vertically is x. Again, it will be one of the following two integrals:

$$2\pi \int_{x_a}^{x_b} r(x)h(x)dx \quad \text{or} \quad 2\pi \int_{y_a}^{y_b} r(y)h(y)dy.$$

Here, r is the distance away from the axis of rotation (usually x or y), and h is the height of the cylindrical shell when f is at a distance r away from the axis of rotation. h is usually either f or f^{-1}.

4.3.5 Surface Area of Revolution

We can also find the surface area created by rotating a graph. Again, we need the function $r(x)$, the distance away from the axis of revolution - generally, $r(x) = f(x)$. Then it is very similar to the arc length formula:

$$2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx.$$

4.3.6 Other Volumes

Sometimes, they will ask you to find the volume of a solid with cross-section semicircle, square, triangle, etc. In this case, we find the bounds and write $f(x)$ (or $f(y)$) to be the function of the area of the cross-section at any point x (or y). Then we just integrate it from the lower bound to the upper bound. Just make sure your bounds correspond to the same variable as your function.

5 Parametric Functions (BC Only)

We say a function is defined parametrically if x and y are functions of t, like $x = \ln t$ and $y = t^2$ or anything of that sort. There are several properties of these functions that you need to know.

5.1 Derivatives

The first derivative with respect to x, $\frac{dy}{dx}$ is very easy. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$$

in the example above. The second derivative usually tricks people though, since we cannot just divide $y''(t)$ by $x''(t)$. Instead, we need

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[dy/dx]}{dx/dt} = \frac{4t}{1/t} = 4t^2.$$

Just remember these and you should be fine.

5.2 Arc Length

This is probably the easiest arc length formula to memorize, though every arc length formula can be derived from every other one, so whatever. Simply

$$\int_{t_a}^{t_b} \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

6 Polar Coordinates and Functions (BC Only)

Instead of regular (x, y) coordinates, sometimes we use polar coordinates (r, θ). All this means is a point is r units away from the origin at an angle of θ away from the positive x-axis. To convert from one to another, we have $x = r\cos\theta$ and $y = r\sin\theta$ (you should be able to get the reverse conversion from this).

6.1 Derivatives

In fact, polar function derivatives behave exactly like parametric function derivatives. If we want to find $\frac{dy}{dx}$ from the function $r(\theta) = 1 + \theta$, we just let

$$x(\theta) = r(\theta)\sin\theta = \sin\theta + \theta\sin\theta$$

$$y(\theta) = r(\theta)\cos\theta = \cos\theta + \theta\cos\theta$$

Then, use the same rules as with parametric differentiation, replacing t with θ.

6.2 Area in a Polar Graph

This is probably the most tricky polar thing. First, you must graph your function(s) and find out the bounds of integration. Make sure you don't count areas twice or something like that because it is very easy to do so. After we get the bounds θ_1 and θ_2, the area in the graph is given by

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 d\theta.$$

6.3 Arc Length

Again, you can derive this:

$$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + [r'(\theta)]^2} d\theta.$$