Mu Alpha Theta Sequences & Series Review

1 Introduction

Sequences & Series is a somewhat more confined topic than Number Theory, but this just means that the questions will focus more on application than knowledge. So practice makes perfect. Also, the subject is for the most part divided into two parts: pre-calculus and calculus. If you are taking S&S Theta or Alpha, you do not need to worry about the second half of this review.

2 Types of Sequences

2.1 Arithmetic Sequences

These are the easiest type, but they will probably be abundant on the test, so get to know them very well. An arithmetic sequence is a sequence $a_1, a_2, \ldots, a_k$ in which $a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} = d$, the common difference. An example is $2, 5, 8, 11, \ldots$.

2.1.1 Finding Terms

You are given that the first term of an arithmetic sequence is 1 and the 41st term is 381. What is the 43rd term? The neat thing about arithmetic sequences is that the difference between a_i and a_j is $d \cdot (j - i)$. Let's try it out. Suppose we have the sequence above, $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11, a_5 = 14, a_6 = 17$. We see that

$$a_2 - a_1 = 3(2-1) \quad \text{and} \quad a_3 - a_1 = 3(3-1) \quad \text{and} \quad a_6 - a_2 = 3(6-2)$$

and so on. How can we use this to solve the given problem? Well since we know $a_1 = 1$ and $a_{41} = 381$, we have $a_{41} - a_1 = 380 = 40d$. So

$$a_{43} - a_{41} = 2d = \frac{40d}{20} = \frac{380}{20} = 19 \Rightarrow a_{43} = 381 + 19 = 400.$$

2.1.2 Finding the Sum

An infinite arithmetic sequence will always diverge, so we won't worry about that. If we wanted to find $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$, however, there is a nice way to do this. It is the average of the terms multiplied by the number of terms, which makes sense. So we have

$$a_i + a_{i+1} + a_{i+2} + \cdots + a_j = \frac{a_i + a_j}{2} \cdot (j - i + 1).$$

Putting this into practice, suppose we want to find $2 + 5 + 8 + 11 + 14 + 17 + 20 + 23$. Well, using the formula and seeing that there are 8 terms, we get $\frac{25}{2} \cdot 8 = 100$. Note that if you every get a fractional sum from an arithmetic sequence of integers, you probably did something wrong!

2.2 Geometric Sequences

These are slightly harder, but in general still on the easy side. A geometric sequence is a sequence $g_1, g_2, \dots, g_k$ in which $g_2/g_1 = g_3/g_2 = \cdots = g_k/g_{k-1} = r$, the common ratio. It is similar to an arithmetic sequence only with division instead of subtraction. An example would be $\frac{1}{3}, 1, 3, 9, 27, \dots$ which has common ratio 3.

2.2.1 Finding Terms

You are given that the first term of a geometric sequence of positive integers is 1 and the 11th term is 243. How can you find the 13th term? Well, this is very similar to an arithmetic sequence except we have $\frac{g_j}{g_i} = r^{j-i}$ instead of the difference. For example, with $g_1 = \frac{1}{3}, g_2 = 1, g_3 = 3, g_4 = 9, g_5 = 27$ we have

$$\frac{g_2}{g_1} = 3^{2-1} \quad \text{and} \quad \frac{g_3}{g_1} = 3^{3-1} \quad \text{and} \quad \frac{g_5}{g_2} = 3^{5-2}.$$

Applying this, we know $g_1 = 1$ and $g_{11} = 243$ so $\frac{g_{11}}{g_1} = 243 = r^{10}$. This allows us to say

$$\frac{g_{13}}{g_1} = r^{12} = (r^{10})^{\frac{6}{5}} = 243^{\frac{6}{5}} = 729 \Rightarrow g_{13} = 729.$$

If that doesn't all make sense, read it over again and maybe review your exponent rules, as they will be crucial for geometric sequences.

2.2.2 Finding the Sum

Here, we have both the infinite and finite cases, but they are quite similar. If we have a geometric sequence $g_1, g_2, g_3, \dots$ and the common ratio r satisfies $|r| < 1$, the infinite sum is

$$g_1 + g_2 + g_3 + \cdots = \frac{g_1}{1-r}.$$

This is very important. If, on the other hand, $|r| \geq 1$, the sum diverges (goes to positive or negative infinity). Now for the finite case, it does not matter what r is because it will never go to infinity:

$$g_1 + g_2 + \cdots + g_k = \frac{g_1 - r g_k}{1-r}.$$

So if we wanted to find $1 + 3 + 9 + 27 + 81$, we would have $\frac{1 - 3 \cdot 81}{1-3} = 121$, whereas if we needed $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$, it is $\frac{1/3}{1-(-1/3)} = \frac{1}{4}$.

2.3 Harmonic Sequences

These are probably a lot less important, but it's good to know what they are. A harmonic sequence is a sequence $h_1, h_2, \dots, h_k$ such that $\frac{1}{h_1}, \frac{1}{h_2}, \dots, \frac{1}{h_k}$ is an arithmetic sequence. For example, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$ and $2, \frac{4}{5}, \frac{1}{2}, \frac{4}{11}, \dots$ are harmonic sequences.

3 Techniques for Evaluating Series

In the case that a series is not arithmetic or geometric, there are certain types that can be evaluated.

3.1 Telescoping

Telescoping series are series in which terms cancel parts of each other out, so it makes finding the sum easier. The classic example is

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

This would be a pain to add up especially if n is big, so there is a neat trick we can do. Find the partial fraction decomposition of $\frac{1}{i(i+1)}$, which is $\frac{1}{i} - \frac{1}{i+1}$. So our summation is

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

But wait, if we look at the second part of each set of parentheses and the first part of the next set, we see that they are exactly negatives of each other. So

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

We can do this for a variety of different types of series, such as $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2+4i+3}$. Also, if we take $n \to \infty$, it is not hard to find the sum of an infinite telescoping series either.

3.2 Nested Radicals

These are not really series, but that's ok since they show up at Mu Alpha Theta. Suppose we want to find the value of the expression $2 + \sqrt{2 + \sqrt{2 + \cdots}}$, where the radical repeats infinitely. The standard way to go about this is to let $x = 2 + \sqrt{2 + \sqrt{2 + \cdots}}$. But notice that what is inside the first radical looks exactly like what the whole thing is. So in fact we have the equation

$$x = 2 + \sqrt{x} \Rightarrow (x-2)^2 = x \Rightarrow x^2 - 5x + 4 = (x-4)(x-1) = 0 \Rightarrow x = 1, 4.$$

Two answers? That cannot be possible! So we try to eliminate one of them. Since the square root is always positive and our expression is $2 + \sqrt{\text{blah}}$, it must be the case that $x > 2$ and thus $x = 4$ is the correct value. This method basically takes care of any infinite radical; try it out on $2\sqrt{2\sqrt{2\ldots}}$.

3.3 Continued Fractions

The method to solve these is identical to that of nested radicals. If we want to evaluate

$$x = \frac{2}{1 + \frac{2}{1 + \frac{2}{\cdots}}},$$

we just substitute x appropriately into the right side to get

$$x = \frac{2}{1+x} \Rightarrow x^2 + x - 2 = (x+2)(x-1) = 0 \Rightarrow x = 1.$$

4 Common Summations

Here are a bunch of formulas that you should know:

- Sum of the first n positive integers: $1+2+\cdots+n = \dfrac{n(n+1)}{2}$.
- Sum of the first n odd integers: $1+2+\cdots+(2n-1) = n^2$.
- Sum of the first n even integers: $1+2+\cdots+2n = n(n+1)$.
- Sum of the first n perfect squares: $1+4+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
- Sum of the first n perfect cubes: $1+8+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$.

5 Taylor Series (Mu Only)

5.1 Definition

The Taylor series of an infinitely differentiable function $f(x)$ centered at $x = c$ is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)(x-c)^k}{k!} = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f^{(3)}(c)(x-c)^3}{6} + \cdots.$$

Note that a Maclaurin series is simply a Taylor series centered at $x = 0$. A neat thing about Taylor series is that if we wanted $f(g(x))$ we can simply plug $g(x)$ as if it were x into the Taylor series - no worrying about repercurssions like with the chain rule. Anyway, you basically need to know how to write the first few terms of the Taylor series of a function as well as recognize the Taylor series when it is given.

5.2 Common Taylor Series

Here is a list of Taylor series that show up a lot; unfortunately, not very many will show up on the Mu Alpha Theta test, but you don't know which ones will, so yeah. Sometimes they may plug in a value of x and then tell you to identify it, so look out for those as well.

- $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$.
- $\sin x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$.
- $\cos x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$.
- $\ln(1+x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$.

6 Convergence (Mu Only)

This is a big part of the Mu Sequences & Series test as well as the Calculus BC test. So if you are taking either of these you should give this section some attention.

6.1 Definition

A series $a_1 + a_2 + a_3 + \cdots$ is said to converge if $\lim_{k\to\infty} \sum_{i=1}^{k} a_i$ is defined, i.e. the partial sums of the sequence do not tend to infinity. If the limit is not defined, we say that the series diverges.

6.2 Determining Convergence/Divergence

This is a list of the useful tests that you can use to determine whether a series converges or diverges. In most cases, if a series fails the convergence test, it diverges.

- n-th term test: If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{i=1}^{\infty} a_i$ must diverge. Example: the series $(-1) + 1 + (-1) + 1 + \cdots$ diverges.
- p-series test: The series $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges for all $p > 1$ and diverges for all $p \leq 1$. Example: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$ diverges, but $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ converges.
- Direct comparison test: If $\{a_i\}$ and $\{b_i\}$ are sequences with positive terms with $a_i \leq b_i$ for all $i \geq N$ for some positive integer N and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges as well. Example: since $\frac{1}{n^2+1} \leq \frac{1}{n^2}$, the series $\frac{1}{1^2+1} + \frac{1}{2^2+1} + \frac{1}{3^2+1} + \cdots$ converges.
- Limit comparison test: If we replace the condition $a_i \leq b_i$ in the previous test by the condition $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, the same conclusion holds. Example: Since $\lim_{n\to\infty} \frac{3n^2-2n+1}{n^2+1} = 3$, we know $\frac{1}{3(1)^2-2(1)+1} + \frac{1}{3(2)^2-2(2)+1} + \frac{1}{3(3)^2-2(3)+1} + \cdots$ converges.
- Alternating series test: The series $\sum_{i=1}^{\infty} (-1)^i a_i$ converges if (1) $a_i \geq 0$; (2) $\lim_{n\to\infty} a_n = 0$; and (3) a_n is strictly decreasing. Example: The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.
- Integral test: If $f(x)$ is a decreasing function, $f(i) = a_i$ for all i, and $\int_N^{\infty} f(x)$ converges for some N, the series $\sum_{i=1}^{\infty} a_i$ converges as well. Example: $\int_2^{\infty} \frac{1}{x \ln x} dx = \infty$ so the series $\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots$ diverges.

- Ratio test: This is the ultimate test, and it works for tons of series. Let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $|L| < 1$, then the series $\sum_{i=1}^{\infty} a_i$ converges. If $|L| > 1$, the series diverges. Unfortunately, when $|L| = 1$, no conclusion can be drawn. The next section will be based entirely on the ratio test.

6.3 Radius and Interval of Convergence

If $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ is a power series centered at $x = c$, it is always the case that f converges when $|x - c| < R$. We call R the radius of convergence. Then, if I is an interval such that $f(x)$ converges for all $x \in I$, we call I the interval of convergence.

6.3.1 Finding the Radius of Convergence

Here comes the ratio test. Using that, we know that f converges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n}(x-c) \right| < 1.$$

Then if we rewrite to get $|x - c| < R$, we have found the radius of convergence. So, for example, if we had $f(x) = \sum_{k=0}^{\infty} \frac{2^k(x-2)^k}{k^2}$ we would use the ratio test to get that f converges if

$$\lim_{n\to\infty} \left| \frac{\frac{2^{k+1}}{(k+1)^2}}{\frac{2^k}{k^2}}(x-2) \right| = \lim_{n\to\infty} \left| \frac{2k^2}{(k+1)^2}(x-2) \right| = 2|x-2| < 1.$$

Since this is the same as $|x - 2| < \frac{1}{2}$, we know the radius of convergence is $\frac{1}{2}$.

6.3.2 Finding the Interval of Convergence

In general, we know that the interval of convergence will surely contain $(c - R, c + R)$. The only thing we have left to do is find out if $x = c - R$ and $x = c + R$ converge, which we can do by direct substitution and the tests mentioned above. So for our previous example, we plug in $x = 2 \pm \frac{1}{2}$. The resulting series are

$$\sum_{k=0}^{\infty} \frac{2^k\left(\frac{1}{2}\right)^k}{k^2} = \sum_{k=0}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{2^k\left(-\frac{1}{2}\right)^k}{k^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2},$$

both of which converge. So we extend the interval of convergence from $\left(\frac{3}{2}, \frac{5}{2}\right)$ to $\left[\frac{3}{2}, \frac{5}{2}\right]$. In some cases, only one may endpoint may converge or neither of them.