Exercise 1. *Give a real-world example of a joint distribution $P(x, y)$ where x is discrete and y is continuous. Do not use examples involving coins and dice. 1 point*

Solution. A real-world example of a join distribution that has a discrete x and a continuous y is the probability of the number of games an NBA player has played in the regular season, and the number of hours spent practicing. The number of games played is discrete because it is a countable number and is limited between 0 and the maximum number of games you can play in a regular season, which is 82, and the number of hours spent practicing is continuous because it can be any real number. □

Exercise 2. *What remains if I marginalize a joint distribution $P(v, w, x, y, z)$ over five variables with respect to variables w and y? What remains if I marginalize the resulting distribution with respect to v? 1 point*

Solution. When you marginalize $P(v, w, x, y, z)$ over five variables with respect to w and y, you are left with $P'(v, x, z) = \sum_w \sum_y P(v, w, x, y, z)$ because you are summing over all possible values of w and y. When you marginalize $P'(v, x, z)$ with respect to v, you are left with $P''(x, z) = \sum_v P'(v, x, z)$ because you are summing over all possible values of v. □

Exercise 3. *If variables x and y are independent and variables x and z are independent, does it follow that variables y and z are independent? 1 point*

Solution. No, it does not necessarily follow that the variables y and z are independent. The independence of x and y implies only that the information of x does not provide any information about the value of y, and the independence of x and z implies only that the information of x does not provide any information about the value of z. However, the independence of x and y and the independence of x and z does not imply that the information of y does not provide any information about the value of z, and vice versa. Independence between certain pairs of variables does not necessarily guarantee independence between all variables. □

Exercise 4. *Show that the following relation is true (1 point):*

$$P(w, x, y, z) = P(x, y)P(z \mid w, x, y)P(w \mid x, y) \tag{1}$$

Solution. To show that the above relation is true, we need to use the definition of conditional probability as well as the multiplication rule of probability. The definition of conditional probability states that $P(A \mid B) = \frac{P(A \bigcap B)}{P(B)}$, and the multiplication rule of probability states that $P(A \bigcap B) = P(A \mid B)P(B)$. We can then extend the multiplication rule of probability to three variables, which states that $P(A \bigcap B \bigcap C) = P(C \mid A \bigcap B)P(B \mid A)P(A)$. Using these two rules, we can show that the above relation is true. We can make the following associations: $P(A) = P(x, y)$, $P(B) = P(w)$, and $P(C) = P(z)$. Given those associations, we can rewrite the above relation as follows: $P(A \bigcap B \bigcap C) = P(w, x, y, z)$,

$P(C \mid A \cap B) = P(z \mid w, x, y)$, $P(B \mid A) = P(w \mid x, y)$, and $P(A) = P(x, y)$. Therefore, using the multiplication rule of probability, we can assume that the above relation is true. This is because as per the multiplication rule of probability, $P(A \cap B \cap C) = P(C \mid A \cap B)P(B \mid A)P(A)$, we can substitute the associations we made earlier to get the following: $P(w, x, y, z) = P(x, y)P(z \mid w, x, y)P(w \mid x, y)$. $\square$

Exercise 5. *In my pocket there are two coins. Coin 1 is a fair coin, so the probability $P(h = 1 \mid c = 1)$ of getting heads is 0.5 and the likelihood $P(h = 0 \mid c = 1)$ of getting tails is also 0.5. Coin 2 is biased, so the probability $P(h = 1 \mid c = 2)$ of getting heads is 0.8 and the probability $P(h = 0 \mid c = 2)$ of getting tails is 0.2. I reach into my pocket and draw one of the coins at random. I assume there is an equal chance I might have picked either coin. Then I flip that coin and observe a head.*

Think about the Bayesian framework and describe what is the prior, what is the likelihood in this case. 1 point

Use Bayes' rule to compute the posterior probability that I chose coin 2. 3 points

    Solution. When considering the Bayesian framework, the prior probability is our initial belief or probability about a hypothesis before considering new evidence. In this case, our prior probability of choosing coin 1 is $P(c = 1) = 0.5$, while the prior probability of choosing coin 2 is $P(c = 2) = 0.5$. The likelihood is the probability of the evidence given the hypothesis, and in this case the evidence is flipping a coin and getting a head. For coin 1, $P(h = 1 \mid c = 1) = 0.5$, and for coin 2, $P(h = 1 \mid c = 2) = 0.8$. Using Bayes' rule and the law of total probability, we can compute the posterior probability of choosing coin 2. The law of total probability in this case will be represented by $P(h = 1 \mid c = 1)P(c = 1) + P(h = 1 \mid c = 2)P(c = 2)$. Using Bayes' rule, we can compute the posterior probability of choosing coin 2 as follows:

$$P(c = 2 \mid h = 1) = \frac{P(h = 1 \mid c = 2)P(c = 2)}{P(h = 1 \mid c = 1)P(c = 1) + P(h = 1 \mid c = 2)P(c = 2)} \tag{2}$$

When plugging in the values, we get the following:

$$P(c = 2 \mid h = 1) = \frac{0.8(0.5)}{0.5(0.5) + 0.8(0.5)} = \frac{0.4}{0.65} = 0.615 \tag{3}$$

Therefore, the posterior probability of choosing coin 2 is 0.615. $\square$

Exercise 6. *Consider a biased die where the probabilities of rolling sides $\{1,2,3,4,5,6\}$ are $\{1/12,1/12,1/12,1/12,1/6,1/2\}$, respectively. What is the expected value of the outcome? If I roll the die twice, what is the expected value of the sum of the two rolls? 2 points*

    Solution. The expected value of the outcome is the sum of the product of the probability of each outcome and the value of each outcome. The following represents the expected value of the outcome in this case:

$$E(x) = \sum_{i=1}^{6} P(x = i)i = \frac{1}{12}(1) + \frac{1}{12}(2) + \frac{1}{12}(3) + \frac{1}{12}(4) + \frac{1}{6}(5) + \frac{1}{2}(6) = 14/3 = 4.\overline{6} \tag{4}$$

Therefore, the expected value of the outcome is $4.\overline{6}$. If you roll the die twice, the expected value of the sum of the two rolls is $2(4.\overline{6}) = 9.\overline{3}$. $\square$