MATH 437 Homework 3 (20 points)

1. The composite right rectangular rule with uniform spacing is

$$\int_a^b f(x)\,\mathrm{d}x \approx h\sum_{i=1}^n f(x_i).$$

   (a) (1 point) Assuming f is an analytic function, derive this formula.

   (b) (1 point) Write a computer program that implements this composite rule to integrate $f(x) = x^2 e^{-x^2}$ on $[0,1]$ with $h = 0.1$ and $h = 0.05$.

   Solution. (a) Since f is analytic, on each subinterval $[x_{i-1}, x_i]$ of length h:
   $$f(x) = f(x_i) + f'(x_i)(x - x_i) + \mathcal{O}(h^2).$$

   Integrating over $[x_{i-1}, x_i]$:
   $$\int_{x_{i-1}}^{x_i} f(x)\,dx = hf(x_i) + \mathcal{O}(h^2).$$

   Summing over i and accounting for the fact that $n \sim 1/h$:
   $$\int_a^b f(x)\,dx = h\sum_{i=1}^n f(x_i) + \mathcal{O}(h).$$

	h	integral
(b)	0.1	0.20786611213159514
	0.05	0.1986693190548683

   □

2. (3 points) Write a compute program that evaluates

$$\int_0^2 e^{2x}\sin(3x)\,dx$$

   using the composite trapezoidal rule with $h = 0.1$ and $h = 0.05$.

	h	integral
Solution.	0.1	-14.111007098611367
	0.05	-14.18820404213729

   □

3. (3 points) Approximate

$$\int_0^1 x^2 e^{-x}\, dx$$

using Gaussian quadrature with $n = 2$.

Solution. The integral is approximately 0.15941043096637894. □

4. (3 points) Find a, b, c, d such that the quadrature formula

$$\int_{-1}^1 f(x)\, dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

has degree of precision 3.

Solution. We set up the following system of equations:

$$2 = \int_{-1}^1 1\, dx = a + b + c + d,$$

$$0 = \int_{-1}^1 x\, dx = -a + b + c + d,$$

$$2/3 = \int_{-1}^1 x^2\, dx = a + b - 2c + 2d,$$

$$0 = \int_{-1}^1 x^3\, dx = -a + b + 3c + 3d.$$

Solving it yields $a = 1$, $b = 1$, $c = 1/3$, and $d = -1/3$. □

5. (a) (1 point) Write the explicit and implicit Euler schemes for

$$\frac{dy}{dt} = -100y, \ y(0) = 1.$$

(b) (2 points) The solution is $y(t) = e^{-100t}$. Use $h = 0.01$ and $h = 0.001$ for 3 timesteps to compute the solution $y(0.03)$ and $y(0.003)$ and the error respectively using both methods. Which method is a better approximation?

Solution. (a) Explicit Euler:

$$y^{n+1} = y^n - 100hy^n.$$

Implicit Euler:
$$y^{n+1} = y^n - 100hy^{n+1}.$$

Values and errors:

h = 0.01
exact: 0.049787068367863944
explicit Euler (error): 0.0 (0.049787068367863944)
implicit Euler (error): 0.125 (0.07521293163213605)

h = 0.001
exact: 0.7408182206817179
explicit Euler (error): 0.7290000000000001 (0.011818220681717784)
implicit Euler (error): 0.7513148009015777 (0.010496580219859775)

While the explicit Euler method gives a lower error for $h = 0.01$, it has the wrong qualitative behavior, since $e^{-100t} > 0$ for all t but the method produces $y = 0$ for all t. The implicit Euler method always has the correct qualitative behavior for all $h > 0$ and gives a better approximation than the explicit Euler method for $h = 0.001$. Therefore, for this test case, we prefer the implicit Euler method.

$\square$

6. (3 points) Consider the following ODE problem:

$$\frac{dy}{dt} = -e^y, \ y(0) = 0.$$

Write a computer program that solves this problem using the implicit Euler method for $0 \le t \le 2$. For the nonlinear solve, use Newton's method. Report the error at the final time

$$e_N := |y(t_N) - y_{h,N}|,$$

where $y(t_N)$ is the exact solution at the final time time $t_N = 2$ and $y_{h,N}$ is the discrete solution at time t_N. Do all of this for timesteps (a) $h = 0.2$ and (b) $h = 0.1$. The exact solution is $y(t) = -\log(t+1)$.

	h	error
Solution.	0.2	0.034764531045191394
	0.1	0.01786509930941249

$\square$

7. (a) (1 point) Write a computer program for solving

$$\frac{dy}{dt} = \frac{1+t}{1+y}, \ y(1) = 2, \ h = 0.02, \ 1 \le t \le 2$$

using the explicit Euler scheme.

(b) (1 point) Write a computer program to solve the same problem from the previous part using a fourth-order explicit Runge-Kutta scheme.

(c) (1 point) Compare your solutions from the previous parts to the exact solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$. For both approximations, report the error at the final time $t_N = 2$:

$$e_N := |y(t_N) - y_{h,N}|.$$

Solution.

method	error
explicit Euler	0.001203385963981951
RK4	3.7623237858497305e-12

□

4