

# Exam 1 practice solutions

Jordan Hoffart

February 11, 2026

1. Consider the fixed-point iteration for

$$12x = x^7 - x - 1.$$

For which of the two intervals below will the method converge for any initial value  $x_0$  in that interval? For the intervals that give a convergent method, estimate the number of iterations necessary to obtain approximations accurate to within  $10^{-5}$ , and perform the calculations.

- (a)  $[-0.5, 1]$
- (b)  $[0.5, 4.2]$

*Solution.* See Theorem 2.4 in the textbook. The fixed-point iteration is

$$12x_{n+1} = x_n^7 - x_n - 1.$$

Let  $g(x) = (x^7 - x - 1)/12$ , so that the iteration is in the form  $x_{n+1} = g(x_n)$ . Then,  $g'(x) = (7x^6 - 1)/12$ . We have that

$$|g'(x)| < 1 \iff |7x^6 - 1| < 12 \iff -12 < 7x^6 - 1 < 12 \iff |x| < (13/7)^{1/6}.$$

Therefore, from Theorem 2.4,  $g$  has a unique fixed point, and any initial condition from interval (a) gives a convergent method, but not from interval (b).

On interval (a), if  $-0.5 \leq x \leq 1$ , then

$$-0.5 < \frac{(-0.5)^7 - 2}{12} \leq g(x) \leq \frac{0.5}{12} < 1.$$

Therefore, by the same theorem, the fixed point  $p$  must be in the interval (a) itself, and the error after the  $n$ th iteration is bounded as

$$|x_n - p| \leq k^n \max\{x_0 - a, b - x_0\},$$

where  $a = -0.5$ ,  $b = 1$ ,  $k := \max_{x \in [a, b]} |g'(x)|$ , and  $x_0$  is the initial condition from  $[a, b]$ . See Corollary 2.5 in the textbook.

By examining  $g'(x)$  on  $[a, b]$ , we see that  $k = 0.5$ . The optimal initial condition is  $x_0 = (a + b)/2 = 0.25$ . Therefore, for this problem, the error is bounded as

$$|x_n - p| \leq 0.25(0.5)^n = (0.5)^{n+2}.$$

To obtain an error within  $10^{-5}$ , we estimate at least 15 iterations. Thankfully, the actual iterations required is only 6.

0	0.250000
1	-0.104161
2	-0.074653
3	-0.077112
4	-0.076907
5	-0.076924
6	-0.076922

□

2. Solve

$$2x - \sin(x) = 1$$

using 3 iterations of Newton's method with initial guess  $x_0 = -1$ .

*Solution.* Let  $f(x) = 2x - \sin(x) - 1$ . Then, Newton's method reads

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - \sin(x_n) - 1}{2 - \cos(x_n)}.$$

$n$	$x_n$
0	-1
1	0.478750
2	0.931066
3	0.888390

□

3. Consider the equation

$$x = x^3 - 6,$$

with fixed-point solution  $p = 2$ .

(a) Perform 3 fixed-point iterations with  $x_0 = 3$ . Why does the method not converge?

*Solution.* The method is  $x_{n+1} = g(x_n)$  with  $g(x) = x^3 - 6$ .  $x_1 = 27 - 6 = 21$ .  $x_2 = 21^3 - 6 \approx 21^3$ .  $x_3 = x_2^3 - 6 \approx 21^9 - 6 \approx 21^9$ . The method fails to converge because the derivative  $g'(x) = 3x^2$  is larger than 1 at the solution  $x = p$ .  $\square$

(b) Now, perform 3 iterations of the related Newton method for this problem with  $x_0 = 3$ . Does the method converge?

*Solution.* The method is  $x_{n+1} = x_n - f(x_n)/f'(x_n)$ , where  $f(x) = x - x^3 + 6$ .

$n$	$x_n$
0	3.000000
1	2.307692
2	2.041819
3	2.000924

The method converges.  $\square$

4. Let  $x_0 = 0, x_1 = 1, x_2 = 1.5$ .

(a) Construct the Lagrange interpolating polynomials of minimal degree with respect to these points.

*Solution.* The general form of a Lagrange interpolating polynomial is

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Applied to these points:

$$\begin{aligned} L_0(x) &= \frac{(x-1)(x-1.5)}{(0-1)(0-1.5)}, \\ L_1(x) &= \frac{(x-0)(x-1.5)}{(1-0)(1-1.5)}, \\ L_2(x) &= \frac{(x-0)(x-1)}{(1.5-0)(1.5-1)}. \end{aligned}$$

□

(b) Find using the previous part, find the interpolation of  $f(x) = x^7$ .

*Solution.* To interpolate a function  $f(x)$ :

$$p(x) = \sum_i f(x_i) L_i(x).$$

For  $f(x) = x^7$ :

$$p(x) = 0^7 L_0(x) + 1^7 L_1(x) + (1.5)^7 L_2(x).$$

□

(c) Using the previous part, estimate  $f(0.5)$ .

*Solution.* We estimate with the interpolating polynomial:

$$\begin{aligned} L_1(0.5) &= \frac{(0.5-0)(0.5-1.5)}{(1-0)(1-1.5)} = \frac{0.5(-1)}{-0.5} = 1, \\ L_2(0.5) &= \frac{(0.5-0)(0.5-1)}{(1.5-0)(1.5-1)} = \frac{0.5(-0.5)}{(1.5)(0.5)} = -\frac{1}{3}, \\ p(0.5) &= L_1(0.5) + (1.5)^7 L_2(0.5) = 1 + (1.5)^7/3. \end{aligned}$$

□

- (d) Find an upper bound for the maximum error between  $f(x)$  and  $p(x)$  on  $[0, 1.5]$ .

*Solution.* We recall the error representation formula:

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x - x_i),$$

where  $\xi(x) \in [0, 1.5]$ . For this problem:

$$\begin{aligned} f(x) - p(x) &= \frac{7(6)(5)\xi(x)^4}{6} (x-0)(x-1)(x-1.5) \\ &= \frac{35}{2} \xi(x)^4 x(x-1)(2x-3) \\ &= \frac{35}{2} \xi(x)^4 (2x^3 - 5x^2 + 3x). \end{aligned}$$

Since  $0 \leq \xi(x) \leq 1.5$ ,  $\xi(x)^4 \leq (1.5)^4$ . Let  $q(x) = 2x^3 - 5x^2 + 3x$ . Then,  $q'(x) = 6x^2 - 10x + 3 = 0$  when

$$x = \frac{10 \pm \sqrt{100 - 4(6)(3)}}{12} = \frac{5 \pm \sqrt{7}}{6} = x_0, x_1,$$

where  $x_0 = \frac{5-\sqrt{7}}{6}$  and  $x_1 = \frac{5+\sqrt{7}}{6}$ . Inserting these into  $q(x)$ :

$$\begin{aligned} q(x_0) &= 2 \frac{(5-\sqrt{7})^3}{6^3} - 5 \frac{(5-\sqrt{7})^2}{36} + 3 \frac{5-\sqrt{7}}{6} \approx 0.264077, \\ q(x_1) &= 2 \frac{(5+\sqrt{7})^3}{6^3} - 5 \frac{(5+\sqrt{7})^2}{36} + 3 \frac{5+\sqrt{7}}{6} \approx -0.078892. \end{aligned}$$

Therefore,

$$\max_{x \in [0, 1.5]} |f(x) - p(x)| \leq \frac{35}{2} (1.5)^4 (0.264077) \approx 23.395526.$$

□

5. (a) Find the Hermite interpolation to  $f(x) = x^4 - x + 1$  using points  $x_0 = 1, x_1 = 2$ . Use divided differences.

*Solution.* We compute

$$f'(x) = 4x^3 - 1.$$

Therefore,

$$f(1) = 1, f'(1) = 3, f(2) = 15, f'(2) = 31.$$

We use this data in the divided difference algorithm as follows. First, we set  $z_0 = z_1 = x_0 = 1, z_2 = z_3 = x_1 = 2$ . As notation, we let  $f_i = f(z_i), f_{i,j} = f[z_i, z_j]$ , etc. Then

$$\begin{aligned} f_{0,1} &= f'(1) = 3 & f_{0,1,2} &= \frac{f_{1,2} - f_{0,1}}{z_2 - z_0} = 11 & f_{0,1,2,3} &= \frac{f_{1,2,3} - f_{0,1,2}}{z_3 - z_0} = 6 \\ f_{1,2} &= \frac{f_2 - f_1}{z_2 - z_1} = 14 & f_{1,2,3} &= \frac{f_{2,3} - f_{1,2}}{z_3 - z_1} = 17 \\ f_{2,3} &= f'(2) = 31 \end{aligned}$$

Therefore, the Hermite interpolating polynomial is

$$\begin{aligned} p(x) &= f_0 + f_{0,1}(x - z_0) + f_{0,1,2}(x - z_0)(x - z_1) + f_{0,1,2,3}(x - z_0)(x - z_1)(x - z_2) \\ &= 1 + 3(x - 1) + 11(x - 1)^2 + 6(x - 1)^2(x - 2). \end{aligned}$$

□

- (b) Using the previous part, estimate  $f(1.5)$ .

*Solution.*

$$f(1.5) \approx p(1.5) = 9/2.$$

□

6. A natural cubic spline on  $[0, 2]$  is defined by

$$S(x) = \begin{cases} S_0(x) := a + bx + cx^3 & 0 \leq x \leq 1, \\ S_1(x) := 2 + d(x-1) + e(x-1)^2 + f(x-1)^3 & 1 \leq x \leq 2. \end{cases}$$

Find the coefficients  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$ .

*Solution.* A natural cubic spline is twice continuously differentiable on  $[0, 2]$  and has vanishing second derivatives at  $x = 0$  and  $x = 2$ . This implies that

$$\begin{aligned} S_0(1) &= S_1(1), \\ S'_0(1) &= S'_1(1), \\ S''_0(1) &= S''_1(1), \\ S''_0(0) &= 0, \\ S''_1(2) &= 0. \end{aligned}$$

Substituting for the coefficients:

$$\begin{aligned} a + b + c &= 2, \\ b + 3c &= d, \\ 6c &= 2e, \\ 2e + 6f &= 0. \end{aligned}$$

This system is underdetermined: there are more unknowns than equations. The best we can do is reduce the system to as few unknowns as possible. Writing in matrix-vector form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since this matrix is in echelon form, the nonzero entries of each row imply that we can solve for  $a, b, c, e$  in terms of  $d$  and  $f$ :

$$\begin{aligned} e &= -3f, \\ c &= e/3 = -f, \\ b &= d - 3c = d + f, \\ a &= 2 - b - c = 2 - d. \end{aligned}$$

Thus, the reduced form of the cubic spline is

$$S(x) = \begin{cases} 2 - d + (d + f)x - fx^3 & 0 \leq x \leq 1, \\ 2 + d(x-1) - 3f(x-1)^2 + f(x-1)^3 & 1 \leq x \leq 2. \end{cases}$$

□