MATH 437 Homework 5 (20 points)

1. Consider $Ax = b$ with

$$A = \begin{bmatrix} 7 & -1 & 1 & 1 \\ 3 & 9 & 9 & 1 \\ 3 & 3 & 15 & 1 \\ 3 & 3 & 5 & 14 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) (1 point) Solve this equation using the Jacobi iterative method with tolerance 10^{-5}. Report the residuals and the number of iterations.

   (b) (1 point) Solve this equation using the Gauß–Seidel iterative method with tolerance 10^{-5}. Report the residuals and the number of iterations.

   (c) (1 point) Solve this equation using the SOR iterative method with tolerance 10^{-5} and $\omega = 1.0911$. Report the residuals and the number of iterations.

   (d) (1 point) Compare the spectral radius of the iteration matrix for these methods.

Solution. We recall that all of the iterative methods are of the form

$$Ex^{n+1} + Bx^n = b,$$

For Jacobi, $E := D$, the diagonal of A. For Gauß–Seidel, $E := D + L$, where L is the strictly lower-triangular part of A. For SOR, $E := \omega^{-1}D + L$. In all 3 cases, $B := A - E$.

The iteration matrix is

$$T := E^{-1}B.$$

The spectral radius of T is then

$$\rho(T) := \max\{|\lambda| : \lambda \text{ eigenvalue of } T\}.$$

Thus, we find the eigenvalues of T numerically and take the largest in magnitude.

For the implementation, see `problem_1.py`.

```
(a) Jacobi
1 1.762021e+00
2 9.933272e-01
3 5.413245e-01
4 3.000965e-01
5 1.659671e-01
6 9.183643e-02
7 5.078943e-02
8 2.810307e-02
9 1.554330e-02
10 8.599927e-03
11 4.756731e-03
12 2.631720e-03
13 1.455697e-03
14 8.053547e-04
15 4.454832e-04
16 2.464544e-04
17 1.363295e-04
18 7.542017e-05
19 4.172030e-05
20 2.308019e-05
21 1.276745e-05
22 7.063045e-06

(b) Gauß-Seidel
1 2.481816e-01
2 5.353999e-02
3 1.385446e-02
4 3.516543e-03
5 9.088170e-04
6 2.334454e-04
7 6.012359e-05
8 1.546755e-05
9 3.981011e-06

(c) SOR
1 2.994829e-01
2 5.136698e-02
3 6.328637e-03
4 8.146517e-04
5 1.569048e-04
6 3.725886e-05
7 8.266490e-06

(d) Spectral radii
Jacobi:      5.531937e-01
Gauß-Seidel: 2.573430e-01
SOR:         2.262987e-01
```

□

2. (4 points) Let $n > 0$, and let A be the $n \times n$ tridiagonal matrix with entries -2 on the main diagonal and 1 on the off-diagonals. Let b be the n-dimensional vector with first and last entry 1 and all other entries 0. Consider the linear equation

$$Ax = b.$$

For $n = 10$, $n = 20$, and $n = 40$, solve this system using the Gauß–Seidel method with tolerance 10^{-5} and report the number of iterations along with the residual at the last iteration. How does increasing n affect the number of iterations?

Solution. See `problem_2.py`.

```
 n iters        resid
10   124 9.307666e-06
20   408 9.902382e-06
40  1385 9.942097e-06
```

Doubling n approximately quadruples the number of iterations. □

3. (3 points) Find the condition number of

$$A = \begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}$$

with respect to the $\|\cdot\|_\infty$ norm to at least 6 digits of accuracy.

Solution. The condition number of a matrix A with respect to a matrix norm $\|\cdot\|$ is

$$K(A) := \|A\|\|A^{-1}\|.$$

For the $\|\cdot\|_\infty$ matrix norm,

$$\|A\|_\infty = \max_i \sum_j |A_{ij}|.$$

For A above, we conclude that

$$\|A\|_\infty = 7.$$

By directly computing A^{-1} with exact arithmetic (see, e.g. `problem_3.py`),

$$A^{-1} = \frac{1}{87}\begin{bmatrix} 2400 & -60 & 3 \\ -1000 & 170 & 6 \\ -100 & -70 & 18 \end{bmatrix}.$$

Thus,

$$\|A^{-1}\|_\infty = \frac{2463}{87} = \frac{821}{29},$$

so

$$K(A) = 7\frac{821}{29} = \frac{5747}{29} \approx 198.17241379310343.$$

□

4. (a) (2 points) Compute the ℓ_2 and ℓ_∞ norms of $x := (3, -4, 0, 2)$.

   (b) (1 point) Compute the ℓ_∞ norm of the matrix

$$A := \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 3 & 0 & 1 \end{bmatrix}$$

Solution.

(a)

$$\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{29},$$
$$\|x\|_\infty = \max_i |x_i| = 4.$$

(b)

$$\|A\|_\infty = \max_i \sum_j |A_{ij}| = 5.$$

□

5. (3 points) If the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix}$$

is nonsingular, compute its inverse using Gauß–Jordan elimination.

Solution. We proceed with the elimination process. If the matrix is singular, the process will fail at an intermediate step. If the matrix is nonsingular, the process will succeed.

$$\begin{aligned}
\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & 4 & 3 & 0 & 0 & 1 \end{pmatrix} &\xrightarrow{E_3 \leftarrow E_3 + E_1} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{pmatrix} \\
&\xrightarrow{E_3 \leftarrow E_3 - 6E_2} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -10 & 1 & -6 & 1 \end{pmatrix} \\
&\xrightarrow{E_3 \leftarrow -E_3/10} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/10 & 6/10 & -1/10 \end{pmatrix} \\
&\xrightarrow{E_2 \leftarrow E_2 - 2E_3} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/10 & -2/10 & 2/10 \\ 0 & 0 & 1 & -1/10 & 6/10 & -1/10 \end{pmatrix} \\
&\xrightarrow{E_1 \leftarrow E_1 + E_3} \begin{pmatrix} 1 & 2 & 0 & 9/10 & 6/10 & -1/10 \\ 0 & 1 & 0 & 2/10 & -2/10 & 2/10 \\ 0 & 0 & 1 & -1/10 & 6/10 & -1/10 \end{pmatrix} \\
&\xrightarrow{E_1 \leftarrow E_1 - 2E_2} \begin{pmatrix} 1 & 0 & 0 & 5/10 & 1 & -5/10 \\ 0 & 1 & 0 & 2/10 & -2/10 & 2/10 \\ 0 & 0 & 1 & -1/10 & 6/10 & -1/10 \end{pmatrix}.
\end{aligned}$$

Thus, the inverse is

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 5 & 10 & -5 \\ 2 & -2 & 2 \\ -1 & 6 & -1 \end{pmatrix}.$$

□

6. (a) (1 points) Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$$

(b) (2 points) Factor the above matrix into an LU decomposition with $L_{ii} = 1$ using Gaussian elimination.

Solution.

(a) Expanding along the first row:

$$\det(A) = 2\det\begin{bmatrix} 3 & 9 \\ 3 & 5 \end{bmatrix} + \det\begin{bmatrix} 3 & 9 \\ 3 & 5 \end{bmatrix} + \det\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = 3(15-27) = -36.$$

(b) Let $R_{c,i,j}$ denote the transformation that adds c times row i of a matrix to row j, where $i \neq j$. In matrix-form,

$$R_{c,i,j} = I + cE_{j,i},$$

where I is the identity matrix and $E_{j,i}$ is the square matrix with all entries 0 except a 1 in the (j,i) entry. For example, in a 3×3 matrix,

$$R_{2,1,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, we do Gaussian elimination on A to reduce it to an upper-triangular matrix U. The reductions we perform will be of the form $R_{c,i,j}$. Each reduction corresponds by multiplying on the left by such an $R_{c,i,j}$. Therefore,

$$\begin{aligned} A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix} &\longrightarrow R_{-3/2,1,3}R_{-3/2,1,2}A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 9/2 & 7/2 \end{bmatrix} \\ &\longrightarrow R_{-1,2,3}R_{-3/2,1,3}R_{-3/2,1,2}A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{bmatrix} = U. \end{aligned}$$

Since $A = LU$, we conclude that

$$\begin{aligned} L &= (R_{-1,2,3}R_{-3/2,1,3}R_{-3/2,1,2})^{-1} \\ &= (R_{-3/2,1,2})^{-1}(R_{-3/2,1,3})^{-1}(R_{-1,2,3})^{-1}. \end{aligned}$$

Now, for $i \neq j$,

$$(R_{c,i,j})^{-1} = R_{-c,i,j},$$

which corresponds to subtracting c times row i from row j. Therefore,

$$L = R_{3/2,1,2}R_{3/2,1,3}R_{1,2,3}.$$

Reading from right to left, this says add row 2 to row 3, then add 3/2 of row 1 to row 3, and then add 3/2 of row 1 to row 2. Applying this to the identity matrix, we end with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{bmatrix}.$$

□