The analysis of two Coupled Pendula

J.J. Whittaker (z5363798)[1,2]
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This experiment aims to verify the mathematical representation of two coupled pendula, predicting their frequency and nature, and showing the superimposed normal modes within beating motion. The frequency of the normal modes, ω_1 & ω_2 were predicted with an average error of $1.386 \pm 3.2 \times 10^{-3}$ % and $1.955 \pm 1.29 \times 10^{-2}$ % respectively, the nature of their motion was successfully shown and within beating motion, the superposition of the normal modes was evident.

INTRODUCTION

A single pendulum has a trivial mathematical derivation with its angular frequency being constant for any state of the pendulum, only being influenced by gravity and the length of the pendulum seen in formula (1),

$$\omega = \sqrt{\frac{g}{L}}. \tag{1}$$

A single pendulum with a spring attached at a constant position (see Fig. (b) for general representation), can now be influenced by the force the spring applies, where the force is applied and the mass of the pendulum. But still, the angular frequency remains constant for any state of the pendulum with formula (2),

$$\omega = \sqrt{\frac{g}{L} + \frac{2l^2k}{mL^2}}. \tag{2}$$

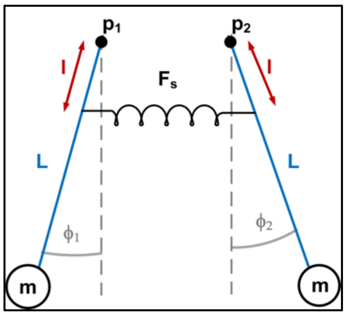

Figure (a) Diagram of the Coupled pendula experimental

Now consider two pendula where the spring is attached at the same point (seen in Fig. (a)), the state of one pendulum affects the state of the other. The pendula are not independent and thus the pendula are 'Coupled'. This allows a continuous spectrum of angular frequencies, and what will be shown later, all the possibilities of frequency are just the superposition of the 'normal modes' of the coupled pendula. These 'normal modes' are the two angular frequencies associated with the two scenarios above, (1) and (2).

To be able to investigate these coupled pendula, a mathematical representation of the scenario must be derived. The torques acting on the pendula are summated at the fulcrum, equation (3), the force from the spring and the tangential component of gravity. (As the pendula are symmetrical, the derivation for one pendulum is valid for the other). Assume anticlockwise angular displacement is positive.

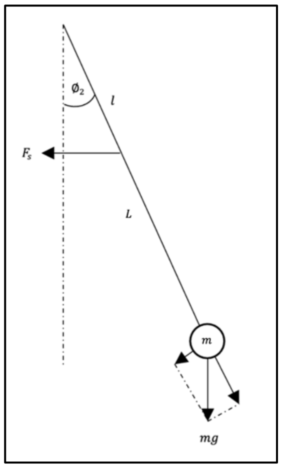

Figure (b) Free body diagram of the right pendula.

$$\tau = F_s\, l \cos(\emptyset_2(t)) - mg_t L, \tag{3}$$

$$\tau = \alpha_2(t) I = \ddot{\emptyset}_2(t) I, \quad (4) \qquad F_s = -lk\big(\sin(\emptyset_1(t)) - \sin(\emptyset_2(t))\big), \quad (5) \qquad mg_t = mg\sin(\emptyset_2(t)). \quad (6)$$

Substitute (4), (5) & (6) in (3),

$$\therefore\ \ddot{\emptyset}_2(t) I = -l^2k\big(\sin(\emptyset_1(t)) - \sin(\emptyset_2(t))\big)\cos(\emptyset_2(t)) - mg\sin(\emptyset_2(t))\,L, \tag{7}$$

$$I_{Point\ Mass} = mr^2 = mL^2. \tag{8}$$

Substitute (8) in (7),

$$\therefore\ \ddot{\emptyset}_2(t) mL^2 = -l^2k\big(\sin(\emptyset_1(t)) - \sin(\emptyset_2(t))\big)\cos(\emptyset_2(t)) - mg\sin(\emptyset_2(t))\,L. \tag{9}$$

Using small angle approximation, substitute $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ in (9),

$$\ddot{\emptyset}_2(t) mL^2 = -l^2k\big(\emptyset_1(t) - \emptyset_2(t)\big) - mg\emptyset_2(t)L, \tag{10}$$

$$\ddot{\emptyset}_2(t) + \frac{g}{L}\emptyset_2(t) = \frac{l^2k}{mL^2}\big(\emptyset_2(t) - \emptyset_1(t)\big). \tag{11}$$

Let,

$$A^2 = \frac{g}{L} \quad (12) \qquad\qquad B^2 = \frac{l^2 k}{mL^2}. \quad (13)$$

Substitute (12) & (13) in (11),

$$\ddot{\emptyset}_2(t) + A^2\emptyset_2(t) = B^2\big(\emptyset_2(t) - \emptyset_1(t)\big), \tag{14}$$

and by symmetry,

$$\ddot{\emptyset}_1(t) + A^2\emptyset_1(t) = -B^2\big(\emptyset_2(t) - \emptyset_1(t)\big). \tag{15}$$

When solving the pair of differential equations (14) & (15), there are **3** scenarios to consider which will be the focus of the experiment. In phase, out of phase and beating pendula.

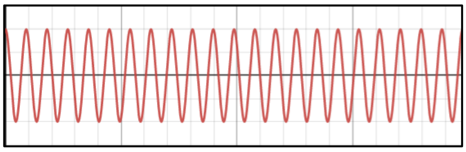

Figure (c) Graph showing the theoretical nature of an arbitrary state of an in phase coupled

1. When $\emptyset_2(t) = \emptyset_1(t)$, the solution is,

$$\emptyset_2(t) = \emptyset_1(t) = \emptyset_{max}\cos(At). \tag{16}$$

In this scenario, the Pendula are **in phase** (nature shown in Fig. (c)) with angular frequency $\omega_1 = \sqrt{\frac{g}{L}}$ (1).

As $\emptyset_2(t) - \emptyset_1(t) = 0$, the component governed by the spring is eliminated and is identical to if the pendula were not coupled. As the spring has no effect on this state, the frequency of the pendula in phase is constant throughout all spring positions.

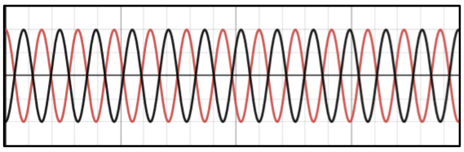

Figure (d) Graph showing the theoretical nature of an arbitrary state of an out of phase coupled pendula.

2. When $\emptyset_2(t) = -\emptyset_1(t)$, the solution is,

$$\emptyset_1(t) = \emptyset_{max}\cos(\sqrt{A^2 + 2B^2}\,t), \tag{17}$$

$$\emptyset_2(t) = -\emptyset_{max}\cos(\sqrt{A^2 + 2B^2}\,t). \tag{18}$$

In this scenario, the Pendula are **out of phase** (nature shown in Fig. (d)) with angular frequency $\omega_2 = \sqrt{\frac{g}{L} + \frac{2l^2k}{mL^2}}$ (2).

As $\emptyset_2(t) - \emptyset_1(t) = 2\emptyset_2(t)$, the component governed by the spring is maximised and thus, when the Pendula are out of phase, they are maximally coupled. ω_2 is the highest frequency of any state of the coupled pendula.

3. When both Pendula are initially stationary, with $\emptyset_2(0) = 0\ \ \&\ \ \emptyset_1(0) = \emptyset_{max}$, the solution is,

$$\emptyset_1(t) = \emptyset_{max}\cos\left(\frac{\sqrt{A^2 + 2B^2} - A}{2}\,t\right)\cos\left(\frac{\sqrt{A^2 + 2B^2} + A}{2}\,t\right), \tag{19}$$

$$\emptyset_2(t) = -\emptyset_{max}\sin\left(\frac{\sqrt{A^2 + 2B^2} - A}{2}\,t\right)\sin\left(\frac{\sqrt{A^2 + 2B^2} + A}{2}\,t\right). \tag{20}$$

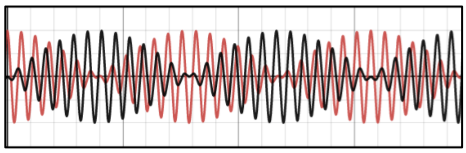

Figure (e) Graph showing the theoretical nature of an arbitrary state of beat motion in coupled pendula.

This scenario produces **beat motion** (nature shown in Fig. (e)), the process of the pendula transferring their energy to one another. The 1st component of both $\emptyset_1(t)$ & $\emptyset_2(t)$ is the modulating component, modelling how much energy either pendulum have. As the modulation is shifted relative to one another by half a wavelength, when one's energy is zero, the other is at its maximum. This creates a cycle of one pendulum slowly transferring all its energy to the other until it completely stops for then the cycle to repeat. Beat motion is the superposition of the normal modes of the pendula. These normal modes are isolated when the pendula are perfectly in and out of phase. Thus, beat motion is the combination of the normal modes, $\omega_1 = \sqrt{\frac{g}{L}}$ which is constant for all setups and $\omega_2 = \sqrt{\frac{g}{L} + \frac{2l^2k}{mL^2}}$ which depends on the spring setup. This can be shown mathematically by rearranging $\emptyset_1(t)$ & $\emptyset_2(t)$ (derivation in annex (1)) to,

$$\emptyset_1(t) = \frac{\emptyset_{max}}{2}(\cos(\omega_2) + \cos(\omega_1)), \tag{21}$$

$$\emptyset_2(t) = \frac{\emptyset_{max}}{2}(\cos(\omega_2) - \cos(\omega_1)), \tag{22}$$

the linear combination of two 'normal modes', ω_1 & ω_2. This will be one of the main investigations throughout the report.

AIM

To verify the theoretical calculations of frequency, the nature of motion throughout all spring positions and setups, and to show the nature of beat motion, being the superposition of two normal modes.

METHOD

The two variations throughout the experiment are coupling strength, and the initial conditions. The coupling strength will be varied by changing the value of l, the distance from the fulcrum to the position of the spring and the initial conditions, starting the pendula in phase, out of phase or in beat motion.

The strength of the spring is initially unknown. To determine the spring constant;

1. Hang the spring vertically and measure its equilibrium length from a datum.
2. Attach $60g$ of mass on the end of the spring and record its new equilibrium position from the datum.
3. Find Δy, the change in displacement from the two equilibrium positions.
4. Enter the value Δy into (23) and solve for k,

$$mg = k\Delta y, \tag{23}$$

$$(0.06)(9.797) = k(0.195 \pm 0.0005) \tag{24}$$

$$k = 3.01 \pm 7.8 \times 10^{-3}\ Nm^{-1} \tag{25}$$

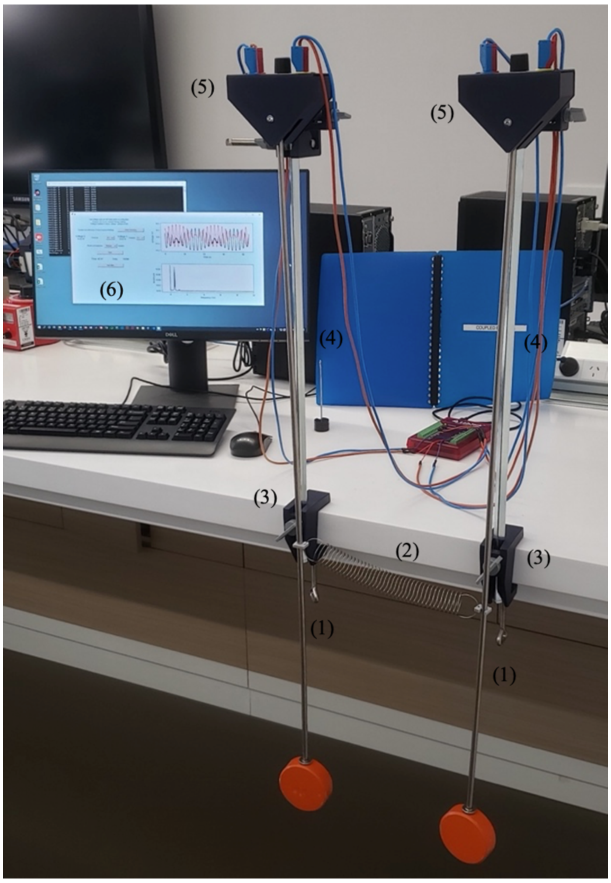

Figure (f) Shows the experimental setup for the Coupled Pendula. (1) the pendula, (2) coupling spring, (3) bench clamps, (4) support rods, (5) recorder connections of the pendula & (6) recording software.

Method for **initial setup**;

1. Ensure that angular displacement from both pendula is read by the software.
2. Place the pendula in their equilibrium position and ensure they're stationary.
3. Turn the dial on the recorder connections of the pendula until 0V is read for both pendula.

Method for **experimentation**;

1. Initially, setup the pendula without the spring (no coupling).
2. Test the first scenario - In Phase: Raise both pendula to the same small angular displacement and release simultaneously.
3. Start recording for 2 minutes then save results.
4. Test the second scenario - Out of Phase: Raise both pendula to opposite small angular displacement such they mirror each other and release simultaneously.
5. Start recording for 2 minutes then save results.
6. Test the third scenario - Beating motion: Raise one of the pendula to a small angular displacement while holding the other at its initial position and release simultaneously.
7. Start recording for 2 minutes then save results.
8. Attach the spring at its lowest position, the furthest from the fulcrum (the highest coupling strength) and repeat steps 2 – 7.
9. Decrease the distance to the fulcrum by 15 cm and repeat steps 2 – 7.
10. Repeat step 9 until the spring is at the closest possible distance to the fulcrum (the lowest coupling strength).

As the derivation for the mathematical representation of the pendula used a small angle approximation, it is important that the angular displacement for all scenarios is small.

RESULTS & ANALYSIS

The software records two variables that will be used for analysis, voltage as a relative representation of angular displacement over time, and the relative intensity of frequency through a Fourier transform. The plots of the voltage time graphs will be used to qualitatively verify the nature of the motion of the three scenarios. The plots of the Fourier transform will be used to quantitatively analyse the frequency of the pendula and verify the nature of the superimposed normal modes within beating motion.

Part 1: Frequency Analysis

The angular frequencies that will be analysed are,

$$\omega_1 = \sqrt{\frac{g}{L}} \quad (1) \quad \& \quad \omega_2 = \sqrt{\frac{g}{L} + \frac{2l^2k}{mL^2}} \quad (2)$$

ω_1 is constant through all spring setups. $L = 1 \pm 0.0005\ m$ (All measurements of distance were taken to an accuracy of $\pm 0.5\ mm$) and $g = 9.797\ ms^{-2}$ [1], taking an accurate value for the acceleration due to gravity in Sydney.

$$f_1 = \frac{\omega_1}{2\pi} = \frac{\sqrt{\frac{g}{L}}}{2\pi} \tag{26}$$

$$= \frac{\sqrt{\frac{9.797}{1 \pm 0.0005}}}{2\pi} \tag{27}$$

$$f_{1_{theo}} = 0.4980 \pm 1.3 \times 10^{-4}\ s^{-1} \tag{28}$$

The pendula were tested over 8 coupling strengths (varying values of l), uncoupled, 2cm to fulcrum (minimum distance), and increments of 15cm to maximum coupling at 90cm from the fulcrum. The values of frequency were taken from the Fourier transforms of their in phase (seen in Fig. (g)) and beating motion (seen in Fig. (k)) trials as both have distinguishable values for f_1. The uncertainty in data is calculated from the average of the difference of the in phase and beat motion values of $f_{1_{exp}}$. (The full dataset and calculations can be found in annex (2)). As coupling strength decreases, the Fourier transform's ability to distinguish between ω_1 and ω_2 reduces. This is due to the two frequencies approaching each other as coupling strength goes to zero and thus low coupling strength beat motion values were invalid. The average value/uncertainty and error for $f_{1_{exp}}$ is;

$$f_{1_{exp}} = 0.5049 \pm 5.25 \times 10^{-4}\ s^{-1}, \tag{29}$$

with an error of,

$$|\text{Error}| \text{ to } f_{1_{theo}} = 1.386 \pm 3.2 \times 10^{-3}\ \% \tag{30}$$

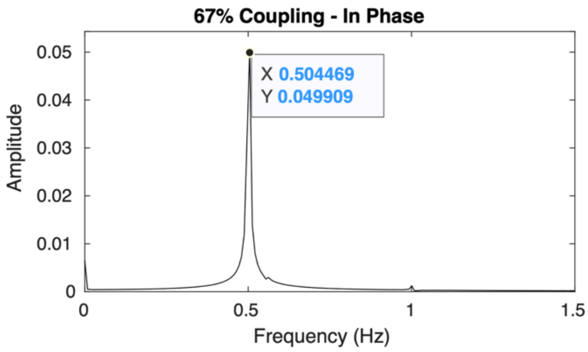

Figure (g) The Fourier transform of the pendula in phase with spring position at 60cm from the fulcrum (67% Coupled).

The value of ω_2 is dependent on the position of the spring and thus has a different value for all coupling strengths. (The full dataset and calculations can be found in annex (3)). All other analysis is done in the same way as $f_{1_{exp}}$. The average uncertainty and error in $f_{2_{exp}}$ are,

$$\text{Uncertainty}_{ave} = 0.638\% \tag{31}$$

$$|\text{Error}|_{\text{ave}} \text{ to } f_{2_{theo}} = 1.955 \pm 1.29 \times 10^{-2}\ \% \tag{32}$$

Part 2: Nature of Motion

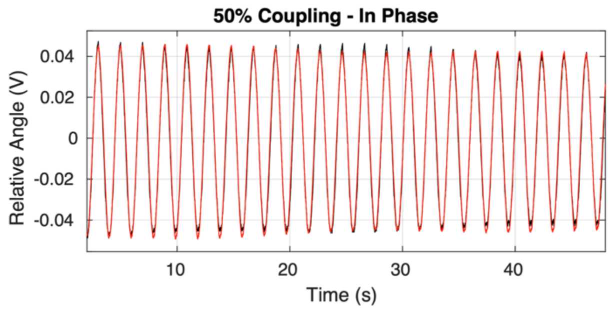

Figure (h) The Relative Voltage time graph of the pendula in phase with spring position at 45cm from the fulcrum (50% Coupled).

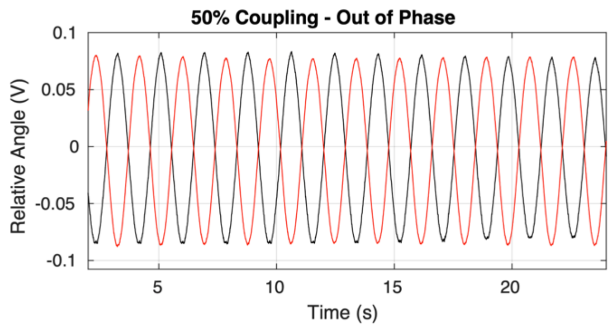

Figure (i) The Relative Voltage time graph of the pendula out of phase with spring position at 45cm from the fulcrum (50% Coupled).

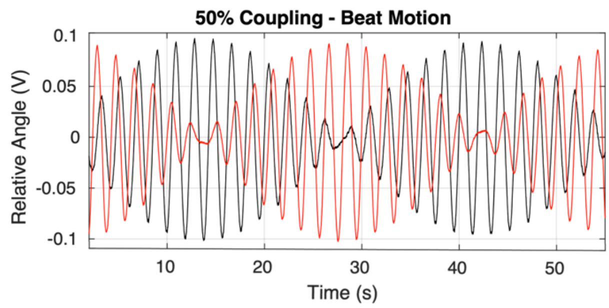

Figure (j) The Relative Voltage time graph of the pendula in beat motion with spring position at 45cm from the fulcrum (50% Coupled).

The motion exhibited by the pendula in all 3 initial conditions (seen in Fig. (h), (i) & (j)) match the theoretical motion well.

There are 2 main observations that arise from comparing the experimental nature of motion to their theoretical,

1. **Damping** – The pendula in reality have some friction opposing its motion and thus their net amplitude decreases over time compared to their theoretical motion in Fig. (c), (d) & (e) respectively.
2. **Persisting Inconsistencies** – When conducting the experiment, creating perfectly in and out of phase pendula proved to be difficult as error was introduced from differences in initial angular displacement and differences in the release instant of the pendula. This was an expected outcome, but the steady state nature of these errors was not. Instead of these errors being slowly eliminated, the errors persisted as a

background frequency seen in the slight modulating nature of the in-phase amplitude in figure (h). This is due to the slight difference in the release time or displacement of the pendula, introducing a similar modulating affect seen in beat motion (figure (j)). This can also be seen in the amplitude difference in the out of phase pendula (figure (i)). These observations seem to state that any configuration of the pendula can be expressed as beat motion, just within perfect in and out of phase pendula, the component affecting the modulation is 0.

Part 3: Superposition of Normal Modes

The Fourier Transforms of the pendula in beat motion have 2 clear peaks representing the normal modes of the pendula (seen in figure (k)). These peaks, shown in Part 1 of analysis are within the theoretical values of ω_1 and ω_2 by 1.386 % and 1.955 % respectively.

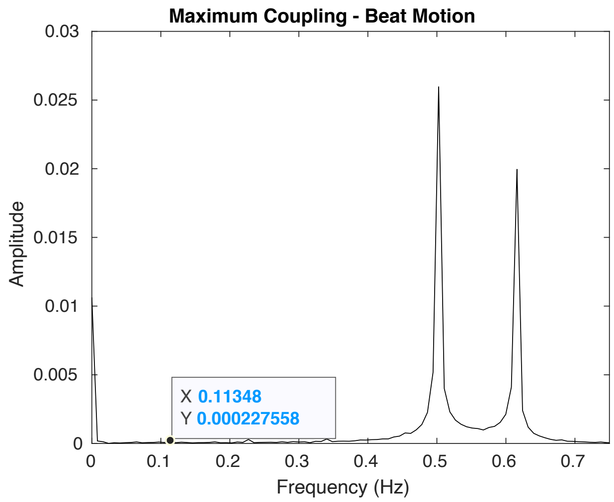

Figure (k) The Fourier transform of the pendula in beat motion with spring position at 90 cm from the fulcrum (Maximum Coupling).

The point marked on Fig. (k) is a harmonic of the frequency of the modulating component. From equations (19) and (20), modulating frequency is,

$$f_{mod} = \frac{\sqrt{A^2 + 2B^2} - A}{2}. \tag{33}$$

This frequency is the difference between the two normal modes and the frequency shown in Fig. (k) is the first harmonic of this frequency. Using the experimental values of the normal modes obtained at maximum coupling (data in annex (2)), we can determine the fundamental modulating frequency,

$$2f_{mod_{exp}} = 0.11348 \tag{34}$$

$$2f_{mod_{exp}} = \omega_2 - \omega_1 \tag{35}$$

$$= 0.616032 - 0.502552 \tag{36}$$

$$= 0.11348 \tag{37}$$

The calculated modulating frequency is exactly what is seen in Fig. (k) to 5 significant figures.

Looking at Fig. (g), there is a small peak just after the main peak at about 0.52 Hz. This is the approximate value of ω_2. Even though the pendula are in phase, the small initial imperfections outlined in Part 2 of analysis introduce a small component of 2nd normal mode and thus, a small peak is seen.

CONCLUSIONS

The experiment successfully predicted the frequency of the normal modes, ω_1 & ω_2 were predicted with an average error of $1.386 \pm 3.2 \times 10^{-3}$ % and $1.955 \pm 1.29 \times 10^{-2}$ % respectively, the nature of their motion was shown and the superposition of the normal modes was evident, not only in beating motion, but it was shown that all states of the pendula can be represented by their respective components of the two normal modes.

References

[1] G A Bell *et al* 1973 *Metrologia* **9** 47 - An Absolute Determination of the Gravitational Acceleration at Sydney, Australia.

APPENDIX

(1)

$$\emptyset_1(t) = \emptyset_{max}\cos\left(\frac{\sqrt{A^2+2B^2}-A}{2}\ t\right)\cos\left(\frac{\sqrt{A^2+2B^2}+A}{2}\ t\right), \tag{1}$$

$$\emptyset_2(t) = -\emptyset_{max}\sin\left(\frac{\sqrt{A^2+2B^2}-A}{2}\ t\right)\sin\left(\frac{\sqrt{A^2+2B^2}+A}{2}\ t\right). \tag{2}$$

Let,

$$a = \frac{\sqrt{A^2+2B^2}-A}{2}, \tag{3}$$ $$b = \frac{\sqrt{A^2+2B^2}+A}{2}, \tag{4}$$

then,

$$a - b = \frac{\sqrt{A^2+2B^2}-A}{2} - \frac{\sqrt{A^2+2B^2}+A}{2} = -A, \tag{5}$$

$$a + b = \frac{\sqrt{A^2+2B^2}-A}{2} + \frac{\sqrt{A^2+2B^2}+A}{2} = \sqrt{A^2+2B^2}. \tag{6}$$

From product to sum formulae,

$$\cos(a)\cos(b) = \left(\frac{\cos(a+b)+\cos(a-b)}{2}\right), \tag{7}$$

$$\sin(a)\sin(b) = \left(\frac{\cos(a-b)-\cos(a+b)}{2}\right), \tag{8}$$

(1) and (2), substituting (5) & (6), can be written as,

$$\emptyset_1(t) = \frac{\emptyset_{max}}{2}\left(\cos\left(\sqrt{A^2+2B^2}\right)+\cos(A)\right), \tag{9}$$

$$\emptyset_2(t) = \frac{\emptyset_{max}}{2}\left(\cos\left(\sqrt{A^2+2B^2}\right)-\cos(A)\right). \tag{10}$$

Let,

$$\omega_1 = A, \tag{11}$$ $$\omega_2 = \sqrt{A^2+2B^2}, \tag{12}$$

then by substituting (11) & (12) in (9) & (10),

$$\emptyset_1(t) = \frac{\emptyset_{max}}{2}(\cos(\omega_2)+\cos(\omega_1)), \tag{13}$$

$$\emptyset_2(t) = \frac{\emptyset_{max}}{2}(\cos(\omega_2)-\cos(\omega_1)). \tag{14}$$

(2)

Distance to Fulcrum, l (cm)	In Phase		Beat Motion		Uncertainty: In Phase – Beat Motion (%)
	Frequency (s^{-1})	|Error| to $f_{1_{theo}}$ (%)	Frequency (s^{-1})	|Error| to $f_{1_{theo}}$ (%)	
No coupling	0.507901	2.0	-	-	-
2	0.507326	1.9	-	-	-
15	0.505508	1.5	-	-	-
30	0.501084	0.62	0.499575	0.32	0.30
45	0.507901	2.0	0.507901	2.0	0
60	0.504469	1.3	0.503336	1.1	0.22
75	0.507901	2.0	0.507901	2.0	0

90	0.502552	0.91	0.502552	0.91	0
Average	0.505580	1.529	0.504253	1.266	0.104

$$f_{1_{exp}} = \frac{0.505580 + 0.504253}{2} s^{-1} \pm 0.104\ \% \quad (1)$$

$$= 0.5049 \pm 5.25 \times 10^{-4}\ s^{-1} \quad (2)$$

$$|\text{Error}| \text{ to } f_{1_{theo}} = \frac{0.5049 \pm 5.25 \times 10^{-4} - (0.4980 \pm 0.0005)}{0.4980 \pm 0.0005}\ \% \quad (3)$$

$$= 1.386 \pm 3.2 \times 10^{-3}\ \% \quad (4)$$

(3)

Distance to Fulcrum, $l\ (cm)$	$f_{2_{theo}}$ (s^{-1})	Uncertainty in $f_{2_{theo}}$ $(\times 10^{-4} s^{-1})$	Out of Phase		Beat Motion		Uncertainty: Out of Phase – Beat Motion (%)
			$f_{2_{exp}}$ (s^{-1})	|Error| to $f_{2_{theo}}$ (%)	$f_{2_{exp}}$ (s^{-1})	|Error| to $f_{2_{theo}}$ (%)	
No coupling	0.4980	1.3	0. 507901	2.0	-	-	-
2	0.4982	1.3	0.506653	1.7	-	-	-
15	0.5016	1.6	0.511063	1.9	-	-	-
30	0.5117	2.1	0.519838	1.6	0.516227	0.88	0.69
45	0.5282	2.9	0.541206	2.5	0.541206	2.5	0
60	0.5505	3.8	0.554089	0.65	0.561096	1.9	1.3
75	0.5779	4.7	0.591164	2.3	0.591164	2.3	0
90	0.6095	5.9	0.618764	1.5	0.616032	1.3	0.27
Average				1.89	-	2.06	0.638

$$f_{2_{theo}} = \frac{\sqrt{\frac{9.797}{1 \pm 0.0005} + \frac{2(l \pm 0.0005)^2(3.01 \pm 0.0078)}{(1)(1 \pm 0.0005)^2}}}{2\pi} \quad (1)$$

$$\text{Uncertainty}_{ave} = 0.638\% \quad (2)$$

$$|\text{Error}|_{\text{ave}} \text{ to } f_{2_{theo}} = \frac{(8)(1.89 \pm 0.638\%) + (5)(2.06 \pm 0.638\%)}{13} \quad (3)$$

$$= 1.955 \pm 1.29 \times 10^{-2}\ \% \quad (4)$$