# Retrograde Capture

Jordan Kaisman

May 2024

## 1 Introduction

We simulate a simplified 3-body problem in which two bodies orbit a fixed star in opposite directions. Any body revolving around the star in the same direction as the star rotates is said to be in prograde orbit. Conversely, any body revolving around the star in the direction opposite the star's own rotation is said to exhibit a retrograde orbit. In this simulation we treat all bodies as point masses (ignore rotation) and simply designate clockwise motion as being prograde.

By the law of conservation of angular momentum, all planets present at the formation of a single star system must have a prograde orbit. Hence, retrograde orbits can only occur as a result of special circumstances, such as the capture of a rogue mass (asteroid) which has been ejected from another system. Therefore, in order to simulate a moderately realistic retrograde orbit, we must include the process by which the asteroid is captured. We will assume that the native planet initially follows a circular orbit around the star.

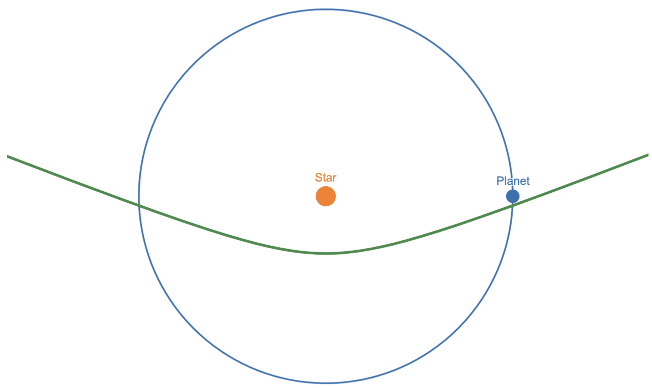
## 2 Capture

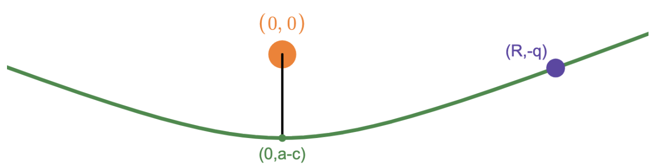
For the asteroid to fall into orbit, it must transfer energy to the native planet. Such an exchange will only happen if the planet and asteroid are in close proximity to one another. Thus we must start the asteroid on a path that very nearly collides with the planet. To construct such a path, we utilize two facts. First, by the symmetry of the planet's circular orbit we can assume without any loss of generality that the near intersection occurs when $\theta = 0$. That is, when the planet has position $(R, 0)$. Then our target position for the asteroid at this time will be $(R, -q)$ where $q$ is relatively small. In the code, we use $q = 0.01R$. This means that the asteroid passes right underneath the planet.

Second, we observe that since the planet and asteroid begin at a significant distance from one another, the gravitational force between them will be negligible compared to the force from the star. Therefore, the motion of both of these bodies reduces to that of a single body orbiting a star, and so they will follow approximately Keplerian orbits. The planet's trajectory will be roughly circular, and the asteroid's roughly hyperbolic.

## 3 Hyperbolic Trajectory

We now derive the equation for the asteroid's approximate hyperbolic trajectory. By trial and error, the top branch of a vertical hyperbola is a suitable choice.




A hyperbola is specified by two parameters. Since $(R, -q)$ must lie on the hyperbola, we have one free parameter. There a few different ways to express this freedom, and we choose the eccentricity $e = \frac{c}{a}$. We break with convention and denote the vertical semimajor axis by $a$ and the horizontal semiminor axis by $b$. Then

$$\begin{aligned}
\frac{(y+c)^2}{a^2} - \frac{x^2}{b^2} &= 1 \\
\frac{(-q+c)^2}{a^2} - \frac{R^2}{b^2} &= 1 \\
\frac{(c-q)^2}{a^2} &= 1 + \frac{R^2}{b^2} \\
(c-q)^2 &= a^2 + \frac{a^2R^2}{b^2} \\
(c-q)^2 &= a^2 + \frac{a^2R^2}{a^2(e^2-1)} \\
(c-q)^2 &= a^2 + \frac{R^2}{e^2-1} \\
c^2 - 2cq + q^2 &= a^2 + \frac{R^2}{e^2-1} \\
a^2e^2 - 2aeq + q^2 &= a^2 + \frac{R^2}{e^2-1} \\
a^2(e^2-1) - 2aeq &= -q^2 + \frac{R^2}{e^2-1} \\
(e^2-1)a^2 - 2eqa + \left(q^2 - \frac{R^2}{e^2-1}\right) &= 0
\end{aligned}$$

This equation is quadratic in $a$, so

$$\begin{aligned}
a &= \frac{1}{2(e^2-1)}\left(2eq + \sqrt{4e^2q^2 - 4(e^2-1)(q^2 - \frac{R^2}{e^2-1})}\right) \\
&= \frac{1}{2(e^2-1)}\left(2eq + 2\sqrt{e^2q^2 - (e^2-1)(q^2 - \frac{R^2}{e^2-1})}\right) \\
&= \frac{1}{(e^2-1)}\left(eq + \sqrt{e^2q^2 - (e^2-1)(q^2 - \frac{R^2}{e^2-1})}\right) \\
&= \frac{1}{(e^2-1)}\left(eq + \sqrt{q^2+R^2}\right) \\
&= \frac{eq + \sqrt{q^2+R^2}}{e^2-1}
\end{aligned}$$

Since $c = ea$ and $b = c^2 - a^2 = a^2(e^2-1)$, we can express all relevant information in terms of $e$ and $q$.

# 4 Hyperbolic Velocity

**Magnitude:** By the vis-viva equation,

$$\|\mathbf{v}\|^2 = \mu \left( \frac{2}{\|\mathbf{x}\|} + \frac{1}{a} \right)$$

where $\mu = GM$ is the standard gravitational parameter.

**Direction:** To determine the velocity components, we implicitly differentiate the hyperbolic equation.

$$\begin{aligned} \frac{d}{dy}\left(\frac{(y-c)^2}{a^2} - \frac{x^2}{b^2}\right) &= \frac{d}{dy}(1) \\ \frac{1}{a^2} \cdot 2(y-c) - \frac{1}{b^2} \cdot 2x \cdot \frac{dx}{dy} &= 0 \\ \frac{y-c}{a^2} &= \frac{x}{b^2} \cdot \frac{dx}{dy} \\ \frac{y-c}{x} \cdot \frac{b^2}{a^2} &= \frac{dx}{dy} \end{aligned}$$

This tells us the ratio of the different velocity components.

Let $h = \frac{y-c}{x} \cdot \frac{b^2}{a^2}$. Then

$$\begin{aligned} \hat{\mathbf{v}} &= (h, 1) \cdot \frac{1}{\sqrt{h^2 + 1^2}} \\ \mathbf{v} &= \|\mathbf{v}\| \cdot \hat{\mathbf{v}} \\ &= \|\mathbf{v}\| \cdot (1, h) \cdot \frac{1}{\sqrt{h^2 + 1}} \end{aligned}$$

# 5 Governing Equations

The evolution of the system over time is determined by Newton's law of universal gravitation. Let us refer to the native planet as planet 1 and the asteroid as planet 2. Then we have

$$\dot{\mathbf{v}}_1(t) = G\left(\frac{m_2(\mathbf{x_2}(t) - \mathbf{x_1}(t))}{\|\mathbf{x_1}(t) - \mathbf{x_2}(t)\|^3} - \frac{M\mathbf{x_1}(t)}{\|\mathbf{x_1}(t)\|^3}\right)$$

$$\dot{\mathbf{v}}_2(t) = G\left(\frac{m_1(\mathbf{x_1}(t) - \mathbf{x_2}(t))}{\|\mathbf{x_1}(t) - \mathbf{x_2}(t)\|^3} - \frac{M\mathbf{x_2}(t)}{\|\mathbf{x_2}(t)\|^3}\right)$$

# 6 Numerical Method

We use the modified Euler scheme is which velocities are updated first and then the updated velocities are used to compute positions.

$$\mathbf{v_1}(t+\Delta t) = \Delta t \cdot G\left(\frac{m_2(\mathbf{x_2}(t) - \mathbf{x_1}(t))}{\|\mathbf{x_1}(t) - \mathbf{x_2}(t)\|^3} - \frac{M\mathbf{x_1}(t)}{\|\mathbf{x_1}(t)\|^3}\right) + \mathbf{v_1}(t)$$

$$\mathbf{v_2}(t+\Delta t) = \Delta t \cdot G\left(\frac{m_1(\mathbf{x_1}(t) - \mathbf{x_2}(t))}{\|\mathbf{x_1}(t) - \mathbf{x_2}(t)\|^3} - \frac{M\mathbf{x_2}(t)}{\|\mathbf{x_2}(t)\|^3}\right) + \mathbf{v_2}(t)$$

$$\mathbf{x_1}(t+\Delta t) = \mathbf{x_1}(t) + \Delta t \cdot \mathbf{v_1}(t+\Delta t)$$

$$\mathbf{x_2}(t+\Delta t) = \mathbf{x_2}(t) + \Delta t \cdot \mathbf{v_2}(t+\Delta t)$$

# 7 Coordinating the Collision

Now that we know the shape of the hyperbolic trajectory, we can easily choose a starting point somewhere along it and determine the appropriate velocity. However, we need to be sure that the asteroid reaches $(R, -q)$ at the same time as the planet arrives at $(R, 0)$. The half period of the planet's circular orbit is $T = \pi\sqrt{\frac{R^3}{GM}}$. If we begin our simulation when the planet is at $(-R, 0)$, then the planet will arrive in a $T$ seconds. This means we must find the unique point on the hyperbola that reaches $(R, -q)$ in $T$ seconds as well. An analytic solution would require parameterizing the hyperbola by time. We opt for a numerical solution instead. Since trajectories are time reversible, we can just simulate the trajectory with a function *getIC*. We place the asteroid at $(R, -q)$, give it the proper velocity, and run the 2 body simulation for $T$ seconds. When time is up, we return the position and velocity which are then used to initialize the asteroid in our main program. More precisely, we return the position and negative velocity.

# 8 Code

https://github.com/jordankaisman/RetroCapture/blob/main/Capture.m

# 9 Results

```
https://drive.google.com/drive/folders/1jGWRWqXVVf2lvEPMDcyw3e6Secf6I59h?
usp=drive_link
```

We let $q = \frac{R}{100}$ and vary the eccentricity $e$. Note that for any hyperbola we must have $e > 1$, so all values are within this range. In section 3, we found that

$$a = \frac{eq + \sqrt{q^2 + R^2}}{e^2 - 1}$$

Thus larger values of $e$ correspond to smaller values of $a$. Moreover, if we reinspect the vis-viva equation,

$$\|\mathbf{v}\|^2 = \mu \left( \frac{2}{\|\mathbf{x}\|} + \frac{1}{a} \right)$$

we see that smaller values of $a$ correspond to larger velocities. Hence increasing $e$ will increase the velocity (and thereby kinetic energy) of the asteroid at the point of near collision. Since capture relies on the asteroid having a sufficiently low velocity (less than escape velocity), we expect that capture will only occur for small values of $e$. Indeed, that is what we observe. For values above 2, the asteroid escapes. On the flip side, values like 1.001 result in such low energy that the asteroid falls almost directly into the star. Although it is actually in orbit, the numerical instability produced by its small distance from the star makes simulation impractical. Avoiding either extreme, we consider 5 intermediate values: 1.1, 1.15, 1.3, 1.4, and 1.5.

In all 5 cases, the original near collision forces the asteroid into a retrograde orbit. However, after 3 complete orbits it once again comes into close contact with the planet. For $e = 1.1$ and $e = 1.15$ this disruption causes the asteroid to reverse direction, taking on a prograde orbit. For the other 3 cases, however, the retrograde orbit is maintained. Since the lower values of $e$ cause the asteroid to go under the star and higher values cause it to go above the star, there must be a trajectory somewhere in between that leads directly into the star. After trying a few values between 1.15 and 1.3, we estimate this threshold to occur around $e = 1.23$. The threshold simulation is included in the results, but only to provide a visualization of the degenerate trajectory. As the asteroid approaches the star, numerical instability kicks in and the results cease to be meaningful.

## 10 Validation

To ensure that the results do not depend on the choice of $\Delta t$, we halve the time step and compare the resulting simulation side by side with the original. Since every 2 time steps in the new simulation correspond to a single time step in the original, we only plot every second frame.

## 11 Conclusion

Although there are no guarantees on long term stability, we were able to demonstrate a system in which retrograde capture occurs and is sustained for at least a few complete orbits. Since our objective was only to find the initial conditions necessary for short term retrograde orbits to occur, numerical stability after the first couple orbits was not a concern. We see numerical instability take hold towards the end of the simulations for $e = 1.3$ and $e = 1.4$. Although that is technically indication that a smaller time step is needed, we are content with the first few orbits.

## 12 References


"Two-Body Relative Numerical Solution." https://orbital-mechanics.space/the-n-body-problem/two-body-relative-numerical-solution.html(accessed February 15, 2024).