Vibrations on "Free" Strings

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1 Introduction

We simulate vibrations on an elastic string in the absence of gravity and without the constraint of fixed endpoints. We consider both the case in which the endpoints undergo some prescribed periodic motion and that in which endpoints are nonexistent because the string forms a closed loop. The general procedure for both cases involves a discretization of the string into equally spaced nodes of equal mass, which we refer to as a uniform discretization.

2 Arc Length at Rest

We say that a string is at rest if there is no tension present within it. Before considering any distortions of the string, we must determine its resting state. To that end, we define the resting arc length l between adjacent nodes to be the arc length between them for any arbitrary resting configuration of the string. Since an infinitesimally thin string cannot be compressed (it will just bend over itself under compressive force), l is actually the minimum possible arc length between adjacent nodes under any configuration. Whenever the distance between adjacent nodes is less than l, the segment joining them will be curved with arc length l. Hence l is an intrinsic property of the string, and so our definition is justified.

Now, if the distance between adjacent nodes exceeds the resting arc length, then the segment joining them will experience tension. Since tension will pull the segment into a straight line, the arc length is now equal to the distance between the nodes. We may therefore compute the excess arc length as the distance minus the resting arc length l. By Hooke's law, the magnitude of the force of tension between the adjacent nodes is exactly proportional to this excess arc length.

3 Tension Formula

For a pair of adjacent nodes with positions $\mathbf{x_1}$ and $\mathbf{x_2}$, the excess length is given by

$$\begin{aligned} E &= max\{\|\mathbf{x_1} - \mathbf{x_2}\|, l\} - l \\ &= max\{\|\mathbf{x_1} - \mathbf{x_2}\| - l, 0\} \end{aligned}$$

It follows that the force of tension between them is $F = TE$, where T is the constant of proportionality (tension per unit excess length).

4 Equations of Motion

For each interior point i, we denote the difference in position with its neighbors by

$$\begin{aligned} D_1 &= \mathbf{x}_{i-1}(t) - \mathbf{x}_i(t) \\ D_2 &= \mathbf{x}_{i+1}(t) - \mathbf{x}_i(t) \end{aligned}$$

Then the excess lengths are

$$\begin{aligned} E_1 &= max\{\|D_1\| - l, 0\} \\ E_2 &= max\{\|D_2\| - l, 0\} \end{aligned}$$

and the corresponding forces acting on node i are thus

$$\begin{aligned} F_1 &= TE_1 \frac{D_1}{\|D_1\|} \\ F_2 &= TE_2 \frac{D_2}{\|D_2\|} \end{aligned}$$

Lastly, the acceleration experienced by node i is

$$\dot{\mathbf{v}} = \frac{1}{m}(F_1 + F_2)$$

In the above equations, all dependencies on time and index were kept implicit for the sake of legibility.

5 Numerical Method

We use the modified Euler scheme is which velocities are updated first and then the updated velocities are used to compute positions.

$$\begin{aligned} \mathbf{v}(t + \Delta t) &= \mathbf{v}(t) + \Delta t \cdot \frac{1}{m}\big(F_1 + F_2\big) \\ \mathbf{x}(t + \Delta t) &= \mathbf{x}(t) + \Delta t \cdot \mathbf{v}(t + \Delta t) \end{aligned}$$

6 Procedural Differences

Open String: We impose a rule such as uniform circular motion on one or both of the endpoints. Their positions are then updated according to the rule, independently of any tension.

Closed String: We must identify the first and last nodes as being adjacent. To do this, we define the neighbor indices

$$n_1 = mod(i, N) + 1$$
$$n_2 = mod(i - 2, N) + 1$$

7 Initialization

Closed Configurations: The particular shapes we test are a square, circle, and tetracuspid. A tetracuspid is the shape enclosed by four mutually tangent circles, and presents a convenient example of a concave shape. In all cases, we choose l greater than or equal to the initial distance between nodes so that the string begins at rest. To perturb the string, we assign some of the nodes a nonzero velocity. We apply symmetric perturbations so that the string itself has no momentum. This is convenient for plotting, as it allows us to keep the axes fixed. As for the form of the perturbations, we experiment with both distributed and single point perturbations, with the distributed version following a quadratic rule within a particular range. Finally, we note that our model does not account for self intersection, so the simulation results are no longer physically meaningful once self intersection occurs. Regardless, we continue to record after intersection occurs, as the emergent patterns may be of interest.

Open Configurations: We choose l less than the initial distance between nodes so that the string begins under tension. Rather then apply an external perturbation, we observe the vibrations induced by the motion of the endpoints. Varying T does not appear to have much impact on the vibration patterns within the string, but it does amplify them. We therefore select relatively small values of T in order to increase visibility.

Variant 1: We fix one endpoint and move the other.

Variant 2: We move both endpoints in the same direction.

Variant 3: We move the endpoints in opposite directions.

8 Results

`https://drive.google.com/drive/folders/1NqPBAV5YdBkSiynMBJhicnRl-cGvd7MG?usp=drive_link`

9 Code

https://github.com/jordankaisman/StringVibrations

10 Discussion

Closed: Among the closed variants, the square exhibited the greatest sensitivity to perturbations. Although all variants eventually contract, the square experienced the most rapid change of shape and the sharpest patterns. Despite initiating the wave with a transverse perturbation, the direction changes as the wave propagates along the edge. By the time it reaches the upper corner, it is nearly longitudinal, and so the corner is pulled downward. The only resistance to this downward motion comes from the node on the adjacent edge, resulting in diagonal movement. The circle, by contrast, appeared more robust to perturbations. With no corners to impede the transfer of waves, perturbations dissipated throughout the string. As for the tetracuspid, the cusps produced some sensitivity but not to the same extent as corners in the square. We attribute this lesser effect to the fact that cusps are in a sense 180° corners. They act as dead ends, and do not facilitate a transfer between horizontal and vertical vibrations.

Open: We observed consistent oscillations among all variants. For the first variant, we saw an S shape that would expand, contract, reverse, and expand again in the opposite direction. For the second and third variants, our choice to move the endpoints in smaller circles resulted in many coils within the string. For each coil, however, similar oscillations were seen. For variant 2, it initially seems as though the oscillations of each coil are synchronized. However, after running the simulation a while we see that they eventually separate. This suggests that the oscillations are actually quasi periodic.

11 Validation

To ensure that the results do not depend on the choice of Δt, we can halve the time step and compare the resulting simulation side by side with the original. To make the comparison easier, we only plot the results at even time steps. Since every two time steps in the new simulation correspond to a single time step in the original, this ensures that we can play both videos simultaneously.

Similarly, to verify that the spatial discretization is sufficiently detailed, we can double the number of nodes and compare the results with the original. Unfortunately, the time of the dynamics does not appear to scale linearly with the number of nodes, so a side by side comparison is not particularly useful. Instead, we use the new simulation to check that macroscopic patterns hold.

We include two examples of time validation among the results, one for square vibrations and another for the open variant 2.

Square: We change time step from 0.025 to 0.005.

Open V2: We change time step from 0.05 to 0.01.