Matrix Multiplication of Trig. Functions

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Department of Mathematics

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NEBRASKA WESLEYAN UNIVERSITY

Matrix Group

My group is the following 2x2 matrix under multiplication:

$$\begin{bmatrix} Cos(x) & Sin(x) \\ -Sin(x) & Cos(x) \end{bmatrix}$$

where $x \in \mathbb{R}$

I will show the following:

- That this is in fact a group under matrix multiplication.
- That it is commutative, thus its abelian.
- This it is not cyclic.

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Proof that it's a Group

To prove that this matrix under multiplication is a group, we must show:

- The matrix is closed under multiplication.
- The matrix is associative under multiplication.
- The matrix contains the 2x2 identity matrix.
- The matrix multiplied by its inverse gives the identity element.

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Pythagorean Identity:
$sin^2(x) + cos^2(x) = 1$

Even-Odd Identity:
$sin(-x) = -sin(x)$

Double Angle Formulas:
$sin(2x) = 2sin(x)cos(x)$
$cos(2x) = cos^2(x) - sin^2(x)$

Product to Sum Formulas:
$sin(x)sin(y) = 1/2(cos(x - y) - cos(x + y))$
$cos(x)cos(y) = 1/2(cos(x - y) + cos(x + y))$
$sin(x)cos(y) = 1/2(sin(x + y) + sin(x - y))$
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Trig. Identities

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Closure

Let G be the 2x2 matrix defined by

$$\begin{bmatrix} Cos(x) & Sin(x) \\ -Sin(x) & Cos(x) \end{bmatrix}$$

where $x \in \mathbb{R}$

Now, let both a and $b \in \mathbb{R}$

Then we will have two matrices in G:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}$$

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Closure Cont.

To check if it is closed, we will multiple the two matrices.

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a)Cos(b) - Sin(a)Sin(b) & Cos(a)Sin(b) + Sin(a)Cos(b) \\ -Sin(a)Cos(b) - Cos(a)Sin(b) & -Sin(a)Sin(b) + Cos(a)Cos(b) \end{bmatrix} =$$

NOTE*

If we take $Cos(a)Cos(b) - Sin(a)Sin(b) =$
$1/2(Cos(a-b) + Cos(a+b)) - 1/2(Cos(a-b) - Cos(a+b)) =$
$Cos(a+b)$

$$\begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Thus it is closed under multiplication.

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Associativity

Want to show:

$$(\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}) * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * (\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix})$$

Associativity Cont.

We know that this is inherited. However, here is a quick outline.

$$\left(\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}\right) * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a+b+c) & Sin(a+b+c) \\ -Sin(a+b+c) & Cos(a+b+c) \end{bmatrix}$$

Likewise, we know that,

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \left(\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix}\right) =$$

$$\begin{bmatrix} Cos(b+c+a) & Sin(b+c+a) \\ -Sin(b+c+a) & Cos(b+c+a) \end{bmatrix}$$

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We can then say, through the commutativity of the reals under addition,

$$\begin{bmatrix} Cos(b+c+a) & Sin(b+c+a) \\ -Sin(b+c+a) & Cos(b+c+a) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a+b+c) & Sin(a+b+c) \\ -Sin(a+b+c) & Cos(a+b+c) \end{bmatrix}$$

Thus, showing that it is associative.

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Thus, showing that it is associative.

Identity Element

The identity element of a 2x2 matrix is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we let $x = 0 \in \mathbb{R}$, we get the following matrix:

$$\begin{bmatrix} Cos(0) & Sin(0) \\ -Sin(0) & Cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Thus it contains the identity element.

Inverse

We have the matrix:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}$$

When you take the determinant, you get
$Cos^2(a) - (-Sin^2(a)) =$
$Cos^2(a) + Sin^2(a) = 1$

So the inverse matrix is:

$$1 * \begin{bmatrix} Cos(a) & -Sin(a) \\ Sin(a) & Cos(a) \end{bmatrix}$$

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Inverse Cont.

When you take a matrix and multiple it by its inverse you get:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(a) & -Sin(a) \\ Sin(a) & Cos(a) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} Cos^2(a) + Sin^2(a) & -Cos(a)Sin(a) + Cos(a)Sin(a) \\ -Cos(a)Sin(a) + Cos(a)Sin(a) & Cos^2(a) + Sin^2(a) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus it has an inverse and when you multiple a matrix by its inverse, you get the identity element.

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When you take a matrix and multiple it by its inverse you get:

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Commutivity

Want to show:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}$$

$=$

$$\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}$$

Commutativity Cont.

We know that (from the closure section),

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} = \begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Likewise, we know that,

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Through the commutativity of the Reals,

$$\begin{bmatrix} Cos(b+a) & Sin(b+a) \\ -Sin(b+a) & Cos(b+a) \end{bmatrix} = \begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Thus it is commutative under multiplication.

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$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} = \begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Likewise, we know that,

$$\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} = \begin{bmatrix} Cos(b+a) & Sin(b+a) \\ -Sin(b+a) & Cos(b+a) \end{bmatrix}$$

Through the commutativity of the Reals,

$$\begin{bmatrix} Cos(b+a) & Sin(b+a) \\ -Sin(b+a) & Cos(b+a) \end{bmatrix} = \begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Thus it is commutative under multiplication.

Not Cyclic

Since we know from the closure step that:

$$\begin{bmatrix} Cos(x) & Sin(x) \\ -Sin(x) & Cos(x) \end{bmatrix}^n = \begin{bmatrix} Cos(nx) & Sin(nx) \\ -Sin(nx) & Cos(nx) \end{bmatrix}$$

And it is known through the density of the rationals in the reals that for any nx and $(n+1)x$ there exists a $y \in \mathbb{Q}$ such that $nx < y < (n+1)x$. Therefore, there will be rational holes between any two values.