Homework Assignment 1

Problem. *Let* $a_0 = a_1 = 1$, *and let* $a_n = a_{n-1} + 5a_{n-2}$ *for* $n \geq 2$, *then* $a_n \leq 3^n$ *for all* $n \geq 0$.

Proof. By Strong Induction
Basis:
For the base cases, let:
$a_0 = 1 \leq 1 = 3^0$
$a_1 = 1 \leq 3 = 3^1$
Induction:
Assume that a_m holds for all $0 \leq m \leq n$, thus $a_m \leq 3^m$. Now,

$$\begin{aligned} a_{n+1} &= a_n + 5a_{n-1} \\ &\leq 3^n + 5(3^{n-1}) \\ &= 3^{n-1}(3+5) \\ &\leq 3^{n-1}(3^2) \\ &= 3^{n+1} \end{aligned}$$

Thus, it holds that $a_n \leq 3^n$ for all $n \geq 0$, by strong induction. □

Problem. *For all positive integers* n,

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2.$$

Proof. Let there be n males and n females. Form a committee size n with a female president. We will show that both the LHS and RHS counts the committees. Now, for the LHS, one person is chosen, n, and then we choose a group size $n-1$ from the remaining $2n-1$. Therefore, we have a total of n people on the committee.

$$n\binom{2n-1}{n-1} = \frac{(2n-1)!}{(n-1)!(n-1)!} * n$$

For the RHS, from each of the men and women groups, both size n, and for some fixed k, we will first choose k women and then a president. This leaves $n-k$ men to be picked. Finally we sum that over k to find how many ways we can select this committee

$$\sum_{k=1}^{n} k\binom{n}{k}^2$$

Therefore, the LHS and RHS are both different ways of counting the same committee, thus they are equal. □

Problem. *Let $F(n,k)$ be the number of partitions of the set $[n]$ into exactly k blocks in which each block contains two or more elements. $F(n,k)$ is in terms of the Stirling numbers of the second kind.*

Proof. We want to show that $F(n,k) = S(n,k) - |A_1 \cup ... \cup A_i|$.
For $1 \leq i \leq n$, define A_i to be the collection of partitions in which element i is a singleton.
By the Sieve formulat, $|A_1 \cup \cdots \cup A_n| = \sum_{j=1}^{n}(-1)^{j-1}|A_{i1} \cap ... \cap A_{ij}|$.
Note: $|A_1| = S(n-1,k-1)$, since 1 ball is in a box and the rest of the $n-1$ balls must be in the $k-1$ boxes. Meaning $|A_2| = S(n-2,k-1)$.
Therefore $|A_1 \cup \cdots \cup A_n| = \sum_{j=1}^{n} S(n-j,k-1)$, which we can deduce $F(n,k) = S(n,k) - |A_1 \cup ... \cup A_i|$. □

Problem. *Find a closed formula for $S(n,n-3)$ for all $n \geq 3$.*

Proof. Let $S(n,n-3)$ be Sterling equation that partitions $[n]$ into $n-3$ blocks. This will create three different possibilities:

- **Case 1: One block length 4 and the rest, $n-4$, blocks of singletons.**
  In this case, the partition of the set is purely based by the block that is of length 4 since the remaining elements will be put into singletons. With that being said, the number of partitions for this case is $\binom{n}{4}$.
- **Case 2: One block length 3, another block length 2 and the rest, $n-5$, blocks of singletons.**
  In this case, the partition of the set is based by the block of length 3 and the block of length 2, since the remaining elements will be put into singletons. Which means for this case, the number of partitions is $\binom{n}{3}\binom{n-3}{2}$.
- **Case 3: Three blocks length 2 and the rest, $n-6$, blocks of singletons.**
  In this case, the partition of the set is based by the three blocks of length 2. Note that in this case, the three blocks of length 2 are not the same and the order of which the boxes are place does not matter. This means the number of partitions for this case is $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\frac{1}{6}$.

By putting these three cases together, we have that the closed formula for

$$S(n,n-3) = \binom{n}{4} + \binom{n}{3}\binom{n-3}{2} + \binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\frac{1}{6}$$

□