Uncountable Fort Space Topology

Theorem. *Fix a point p in an uncountable set X and define $U \subset X$ open if and only if $X \setminus U$ is finite or $p \notin U$.*

Proof. Let τ be the collection of all open sets $U \subset X$.

Since $X \setminus X = 0$, which is finite, $X \in \tau$. Also since we know that $p \notin \emptyset$, thus $\emptyset \in \tau$. Showing that the first axiom of a topological space is met.

Now, let $\mathcal{A}$ be a subcollection of τ, two cases arise. In the first case, we have no $A \in \mathcal{A}$ have a p. Thus, $p \notin \bigcup\limits_{A \in \mathcal{A}} A \in \tau$. In the second case we have p is in at least one A. Without loss of generality, let $p \in A_1$. So, $U = \bigcup\limits_{A \in \mathcal{A}} A$. Then, $X \setminus U \subset X \setminus A_1$. Since $p \in A_1$ and A_1 is infinite, then $X \setminus A_1$ is finite, thus $U \in \tau$. This shows that the second axiom of a topological space is met.

Finally, let $\mathcal{A}$ be a subcollection, two cases arise. In the first case some $A \in \mathcal{A}$ does not have p in it. Then, $p \notin \bigcap\limits_{A \in \mathcal{A}} A$. Thus $p \notin U, \in \tau$. In the second case $p \in A$ for all $A \in \mathcal{A}$. We know that $X \setminus A$ is finite for all $A \in \mathcal{A}$. So, $p \in (\bigcap\limits_{A \in \mathcal{A}} A)$ Now consider $(\bigcap\limits_{A \in \mathcal{A}} A)^c = \bigcup\limits_{A \in \mathcal{A}} A^c$, by DeMorgan's. Thus, it is finite, because a union of a finite set is finite. Showing that the third axiom of a topological space is met.

$\square$

Fortissimo Space Topology

Theorem. *Let X be uncountable and $p \in X$. Define $U \subset X$ open if and only if $X \setminus U$ is countable or $p \notin U$.*

Proof. Let τ be the collection of all open sets $U \subset X$.

Since $X \setminus X = 0$, which is countable, $X \in \tau$. Also since we know that $p \notin \emptyset$, thus $\emptyset \in \tau$. Showing that the first axiom of a topological space is met.

Now, let $\mathcal{A}$ be a subcollection of τ, two cases arise. In the first case, we have no $A \in \mathcal{A}$ have a p. Thus, $p \notin \bigcup\limits_{A \in \mathcal{A}} A \in \tau$. In the second case we have p is in at least one A. Without loss of generality, let $p \in A_1$. So, $U = \bigcup\limits_{A \in \mathcal{A}} A$. Then, $X \setminus U \subset X \setminus A_1$. Since $p \in A_1$ and A_1 is uncountable, then $X \setminus A_1$ is countable, thus $U \in \tau$. This shows that the second axiom of a topological space is met.

Finally, let $\mathcal{A}$ be a subcollection, two cases arise. In the first case some $A \in \mathcal{A}$ does not have p in it. Then, $p \notin \bigcap\limits_{A \in \mathcal{A}} A$. Thus $p \notin U, \in \tau$. In the second case $p \in A$ for all $A \in \mathcal{A}$. We know that $X \setminus A$ is finite for all $A \in \mathcal{A}$. So, $p \in (\bigcap\limits_{A \in \mathcal{A}} A)$ Now consider $(\bigcap\limits_{A \in \mathcal{A}} A)^c = \bigcup\limits_{A \in \mathcal{A}} A^c$, by DeMorgan's. Thus, it is countable, because a union of a countable set is countable. Showing that the third axiom of a topological space is met.

$\square$

Countable Fort Space

Theorem. *Fix a point p in a countable set X. Define $U \subset X$ to open provided $X \setminus U$ is finite or $p \notin U$.*

Proof. Let τ be the collection of all open sets $U \subset X$.

Then X is an element of τ since $X \setminus X = \emptyset$. Also, the empty set is an element of τ since $p \notin \emptyset$.

Now, let $\mathcal{A}$ be a subcollection of elements from τ, two cases arise. The first case is that no $A \in \mathcal{A}$ contains p. Thus, $\bigcup\limits_{A \in \mathcal{A}} A \in \tau$. The second case is that p is in at least one element of $\mathcal{A}$. Without loss of generality, let $p \in A_1$. Denote $U = \cup_{A \in \mathcal{A}} A$. Now, $X \setminus U \subset X \setminus A_1$. Since, $p \in A_1$ but A_1 is still open, it must be that $X \setminus A_1$ is finite. Therefore, $X \setminus U$ is finite, so $U \in \tau$.

Finally, let $\mathcal{A}$ be a subcollection, two cases arise. In the first case some $A \in \mathcal{A}$ does not have p in it. Then, $p \notin \bigcap\limits_{A \in \mathcal{A}} A$. Thus $p \notin U, \in \tau$. In the second case $p \in A$ for all $A \in \mathcal{A}$. We know that $X \setminus A$ is finite for all $A \in \mathcal{A}$. So, $p \in (\bigcap\limits_{A \in \mathcal{A}} A)$ Now consider $(\bigcap\limits_{A \in \mathcal{A}} A)^c = \bigcup\limits_{A \in \mathcal{A}} A^c$, by DeMorgan's. Thus, it is countable, because a union of a countable set is countable. Showing that the third axiom of a topological space is met.

$\square$

Countable Complement Topology

Theorem. *Let X be an uncountable space. Define the open sets on X by a letting a set $U \subset X$ be open iff its complement is is countable. Taking the collection of all such sets, U, together with both the $\emptyset$ and X yields a topology on X.*

Proof. Let $\tau = \{\text{Any countable set}\}$. And let X be an uncountable space.
Frist we know that $X^c = \emptyset$, which is countable. Also $\emptyset^c = X$, which is explicitly allowed, showing that both X and $\emptyset$ are in τ. Now let $\{U_i | i \in \mathbf{I}\}$ be a sub collection of X. (Show $\bigcup\limits_{i \in \mathbf{I}} U_i \in X$) We know $(\bigcup\limits_{i \in \mathbf{I}} U_i \in X)^c$ is countable. So $(\bigcup\limits_{i \in \mathbf{I}} U_i)^c = \bigcap\limits_{i \in \mathbf{I}} U_i^c$, by the DeMorgan's Law. We know that $\bigcap\limits_{i \in \mathbf{I}} U_i^c \subseteq U_j^c$ for any $j \in \mathbf{I}$, which is countable. Now let $\mathcal{A} = \{U_i | i \in [n]\}$ be a

sub collection of open sets in X. Let $\bigcap_{i=1}^{n} U_i$, where $U_i \in \mathcal{A}$. We know that $(\bigcap_{i=1}^{n} U_i)^c = \bigcup_{i=1}^{n} U_i^c$ by DeMorgan's Law. Since a countable union of countable sets is countable, it is countable.

□

Finite Complement Topology

Theorem. *$U \subset X$ is open if and only if $X \setminus U$ is finite or $U = \emptyset$.*

Proof. When know that $U = \emptyset$ is open by definition. Now let $U = X$. This implies $X/U = X/X = \emptyset$. Now let A be a collection of open sets in X. Let $U = \bigcup_{i=1}^{\infty} a_i$ where $a_i \in U$ (show that X/U is finite or $\emptyset$). So, $X/U = X/\bigcup_{i=1}^{\infty} a_i = (X/a_i) \cap (X/a_{i+1}) \cap \ldots$. An arbitrary intersection of finite sets is finite. Now let A be a collection of open sets in X. Let $U = \bigcap_{i}^{j} a_i$ where $a_i \in A$. So, $X/U = X/\bigcap_{i}^{j} a_i = (X/a_i) \cup (X/a_{i+1}) \cup \cdots \cup (X/a_j)$. A finite union of finite sets is finite.

□

Odd-Even Topology

Theorem. *Define a topology on $\mathbb{N}$ by taking as a basis all sets of the form $\{\{2k-1, 2k\} \mid k \in \mathbb{N}\}$.*

Proof. Let $X = \{\{2k-1, 2k\} | k \in \mathbb{N}\}$. Also, let $\tau = \{\text{Collection of all subsets, } B\text{, of } X\}$. Finally, let $\mathcal{B} = \{\text{collection of all } B\}$ Now, for any $k \in \mathbb{N}$, $\{2k-1, 2k\} \in X$. Since $B \subseteq X$, we know that for any $\{2k-1, 2k\}$ chosen, it is in an arbitrary B, that it is in at least one B. Without loss of generality, let $\{2k-1, 2k\}$ be in B_1 and B_2. Let $B_1 \cap B_2 = \{2k-1, 2k\}$. Again, without the loss of generality, let there be a B_3 such that $\{2k-1, 2k\} \in B_3$. This means that $B_3 \subset B_1 \cap B_2$, which is vacuously true.

□

Banach-Mazur: Real Ordered Topology

Theorem. *Player One has a winning strategy for the Banach-Mazur topological game in the Real Ordered Topology*

Proof. First the Real Ordered Topology is $\{(a, b)\}$. And then define the first term as (x_1, y_1), followed by the second as (x_2, y_2) and let the n^{th} turn be $\{(x_n, y_n) | n \in \mathbb{N}\}$. Now create a sequence using just the x's and just the y's. The x sequence will look as followed $(x_n) =$

$\{x_1, x_2, ..., x_n\}$ and the y sequence, $(y_n) = \{y_1, y_2, ..., y_n\}$. By the Monotone Convergence Theorem, since both sequences are bounded and monotone, they converge to some number. In this case, let's say $(x_n) \to L$ and $(y_n) \to M$. We know that $L \leq M$. Thus, if $L < M$: $(L, M) \in \bigcap_{n=1}^{\infty}(x_n, y_n)$, otherwise, if $L = M$: $L = M \in \bigcap_{n=1}^{\infty}(x_n, y_n)$. In both scenarios, the intersection of the countably infinite amount of turns is non empty, showing Player One always has a winning strategy.

$\square$

Banach-Mazur: Right Ordered Topology

Theorem. *Player Two has a winning strategy for the Banach-Mazur topological game in the Right Ordered Topology*

Proof. First the Right Ordered Topology is $\{(x, \infty)\}$. Now lets define the first term (the first turn of the game) as (x_1, ∞), followed by the second as (x_2, ∞) and let the n^{th} turn be $\{(x_n, \infty)|n \in \mathbb{N}\}$ Now create a sequence using just the x's and define it as $(x_n) = \{x_1, x_2, ..., x_n\}$. Since the sequence is strictly getting bigger it will converge to infinity. Since we know the sequence approaches infinity, $\bigcap_{n=1}^{\infty}(x_n, \infty) = \emptyset$. Therefore, the intersection of the countably infinite amount of turns is empty, showing Player Two always has a winning strategy.

$\square$

Banach-Mazur: Countable Complement

Theorem. *Player Two has a winning strategy for the Banach-Mazur topological game in the Countable Complement Topology*

Proof. First the Countable Complement Topology is for any set X that is uncountable, X^c is countable. So then for the first turn, an uncountable space will be chosen, define as X_1. This means then that the complement X_1^c is countable. Similarly, the second turn will be defined as X_2 and its complement X_2^c is countable. So define the n^{th} turn as X_n and its complement as X_n^c. As the game goes on for a countably infinite amount of turns, the set of X's stay uncountable, meaning that their complements stay countable. Without loss of generality, for player two to always have a winning strategy, the set X that is chosen needs to be one such that its complement is the empty set. Meaning that the cardinality of any, at least one, set X_{2n} must be uncountable such that its complement $X_{2n}^c = \emptyset$. Since one of the complements is the empty set, $\bigcap_{n=1}^{\infty}(X_n^c) = \emptyset$. Therefore, since the intersection of all the countable complements is the empty set, player two has a winning strategy.

$\square$