# Homework Assignment 2

**Problem.** *The $n$-dimensional hypercube $Q_n$ is a simple graph whose vertex set is $\{(x_1, \ldots, x_n) \mid x_i \in \{0,1\}\}$. Two vertices are adjacent in this graph if and only if they agree in exactly $n-1$ coordinates. Then $Q_n$ has a Hamiltonian cycle for $n \geq 2$*

*Proof. By Induction*

**Basis:**

For the base case, look at the 2-Dimension square, $Q_1 = C_{2,1} = (n_0, ..., n_j)$, which has a Hamiltonian Cycle.

**Induction:**

Assume that it holds for $Q_k$ (want to show that it then holds for $Q_{k+1}$). Now, let $C_{k,1} = (u_0, ..., u_j)$ and $C_{k,2} = (v_0, ..., v_j)$ Then, create a hypercube with $C_{k,1}$ and $C_{k,2}$ by following vertex $u_0$ to $u_j$, then from $u_j$ to $v_j$, then follow $C_{k,2}$ backwards from $v_j$ to $v_0$ and then connect it back to $u_0$. This will create a Hamiltonian cycle with the hypercube $Q_k$. Now if we add the base case, we will follow similar steps to connecting $C_{2,1}$ with $C_{k,1}$ and $C_{k,2}$. This will create hypercube $Q_{k+1}$ with a Hamiltonian cycle. Thus, it holds that $Q_n$ has a Hamiltonian cycle for $n \geq 2$, by induction. $\square$

**Problem.** *The two longest paths in any tree must cross each other.*

*Proof.* Let $P_1 = (n_0, ..., n_i)$ and $P_2 = (m_0, ..., m_i)$ be the two longest paths in any tree, $T$. For the sake of contradiction, assume that $P_1$ and $P_2$ do not share any vertices. Since both paths exist within the same tree $T$, there must be a path, $P_3$ that connects $P_1$ and $P_2$. Define this path such that it is the shortest path from $P_1$ to $P_2$, and only shares one vertex with each path. Then we can construct $P_3 = (l_0 = n_p, l_1, ..., l_k = m_q)$, where $n_p \in (n_0, ..., n_i)$ and $m_q \in (m_0, ..., m_i)$. Now, consider the path $P_4 = (n_0, ..., n_p, l_1, ..., l_{k-1}, m_q, m_{q-1}, ..., m_0)$. With certain restrictions on where $n_p$ and $m_q$ lie within their respected paths, we know that the length of $P_4$ is greater than either path $P_1$ or $P_2$, contradicting that $P_1$ and $P_2$ are the two longest paths. Thus the two longest paths within any tree, T, share a vertex. $\square$

**Problem.** *The chromatic polynomial $p(x)$ of a simple graph $G$ is the number of ways to properly color $G$ using up to $x$ colors, where $x$ is a positive integer. Let $K_{m,n}$ be the bipartite graph with vertex set $[m] \cup [n]$ obtained by connecting each vertex of $[m]$ to each vertex of $[n]$. Find the chromatic polynomial of $K_{3,3}$.*

*Proof.* Considering the chromatic polynomial $K_{3,3}$, we will have three cases it must follow.

**Case 1: Vertex set $[m]$ will all be one color.**

In this case, vertex set $[m]$ only uses one color, meaning that there are $[x-1]$ colors left to choose from to color the 3 vertices in vertex set $[n]$. Therefore, in this case, we have $x(x-1)^3$.

**Case 2: Vertex set $[m]$ will have two different colors.**
In this case, vertex set $[m]$ uses two colors, meaning that two of the vertices must share a same color, which there are $\binom{3}{2}$ ways to do. Also since vertex set $[m]$ has two colors, we know it can be represented by $(x)(x-1)$. For vertex set $[n]$ there are $(x-2)$ colors to choose from. Therefore, in this case, we have $\binom{3}{2}x(x-1)(x-2)^3$.

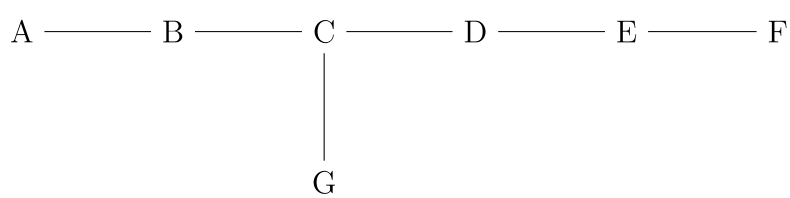
**Case 3: Vertex set $[m]$ will have three different colors.**
In this case, since there are only three vertices in set $[m]$, all vertices will be different colors, meaning it vertex set $[m]$ can be represented by $x(x-1)(x-2)$. For vertex set $[n]$ there are $(x-3)$ colors to choose from. Therefore, in this case, we have $x(x-1)(x-2)(x-3)^3$. Thus, combining the three cases, we get:

$$K_{3,3} = x(x-1)^3 + \binom{3}{2}x(x-1)(x-2)^3 + x(x-1)(x-2)(x-3)^3$$

□

**Problem.** *Find the smallest tree with at least one edge that has no non-trivial automorphisms. Prove your tree is indeed the smallest possible*

*Proof.* The following tree, $T$, is the smallest tree that has no non-trivial automorphisms.




To further prove this claim, we will look at cases that will remove certain vertices and will show that the resulting graph, $T'$ is a tree with trivial automorphisms or that it is no longer a tree.

**Case 1: We remove vertex $A$.**
Looking at this case, if we remove vertex $A$, the resulting tree $T'$ has a resulting trivial automorphism since each vertex is the same distance from the middle.

**Case 2: We remove vertex $F$.**
This case follows from the first case.

**Case 3: We remove vertex $G$.**
This case follows from the two previous cases.

**Case 4: We remove any one vertex from the remaining vertices: $\{B, C, D, E\}$.**
Looking at this case, if we remove any of those vertices, then the resulting graph $T'$ is no longer a tree and thus is not the smallest tree.

Since the tree $T$ given has no non-trivial automorphisms, and we showed that if we try to make the tree smaller by one vertex you will no longer have a tree with non-trivial automorphisms or you will no longer have a tree in general, the tree $T$ must be the smallest tree with no non-trivial automorphisms. $\square$