Banach-Mazur Game

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Banach-Mazur Instructions

Banach Mazur is a topology game that involves two players who take turns choosing open subsets.

- Decide on a topological space
- Player One - Picks a nonempty open subset of the topological space
- Player Two - Picks a nonempty open subset of the previous nonempty open subset chosen
- Then player one picks a nonempty open subset of the previous nonempty open subset picked by player two and it repeats
- This continues for a countably-infinite amount of turns

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Banach-Mazur Winning Strategies

Player One - Wins if the intersection of all the subsets chosen is nonempty

Player Two - Wins if the intersection of all the subsets chosen is empty

Real Ordered Topology is $\{(a, b)\}$

- Player One chooses an interval on the real line.
- Player Two then must (by the rules of the game) choose either the same interval or an interval within the previous one.
- No matter what Player Two chooses, Player One will choose an interval smaller (different endpoints) than the previous interval Player Two chose.

Banach-Mazur: Real Line (Player 1 Winning Strategy)

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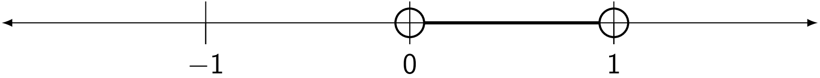

Player Two:

Player One:

Many Turns Later:

The interval will continue to collapse.

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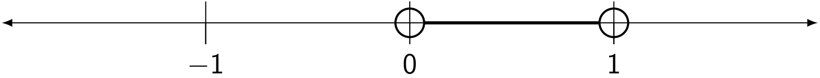

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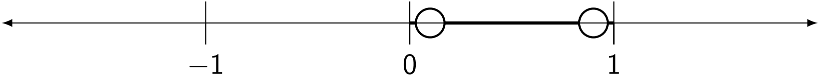

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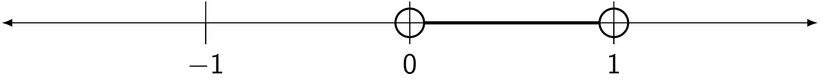

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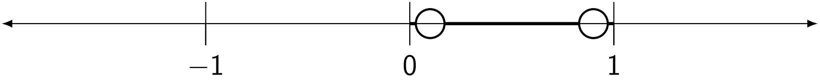

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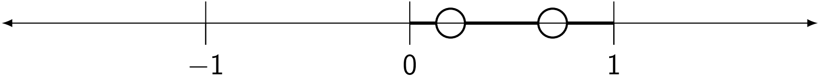

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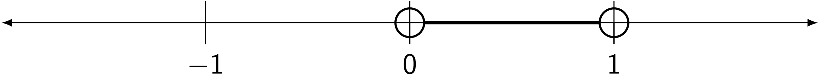

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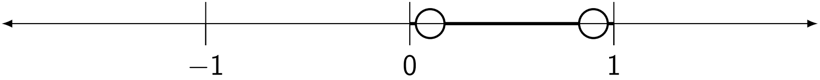

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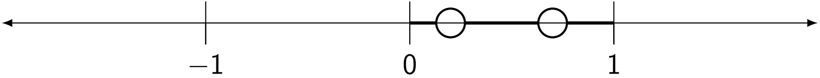

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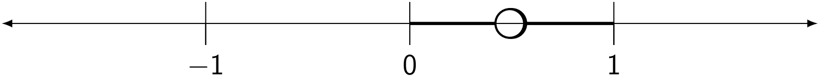

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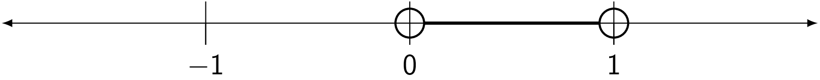

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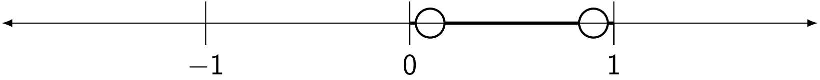

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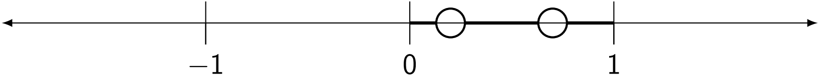

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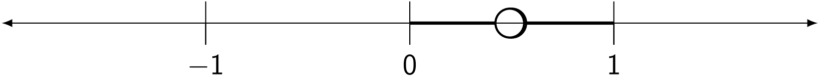

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- n^{th} Turn: (x_n, y_n)

$(x_n) = \{x_1, x_2, x_3, ..., x_n\} \to L$ (By Monotone Convergence)
$(y_n) = \{y_1, y_2, y_3, ..., y_n\} \to M$ (By Monotone Convergence)
We know that $L \leq M$.

Thus, if $L < M$: $(L, M) \in \bigcap_{n=1}^{\infty} (x_n, y_n)$,

otherwise, if $L = M$: $L = M \in \bigcap_{n=1}^{\infty} (x_n, y_n)$.

In both scenarios, the intersection of the countably infinite amount of turns is non empty, showing Player One always has a winning strategy.

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Banach-Mazur: Right Order (Player 2 Winning Strategy)

Right Ordered Topology is $\{(x, \infty)\}$

- Player One must choose (by the rules of the game) any real number as the left endpoint.
- Player Two then chooses another real number as the left endpoint that is bigger than the previous.
- This will continue, and the players will always choose a bigger left endpoint than the previous left endpoint.

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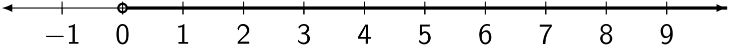

Player Two $(3, \infty)$:

Player One $(5, \infty)$:

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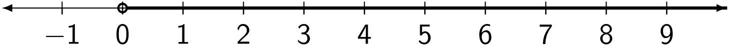

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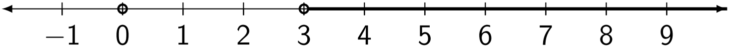

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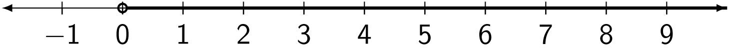

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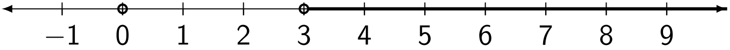

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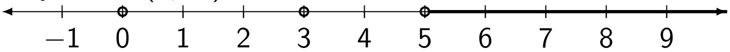

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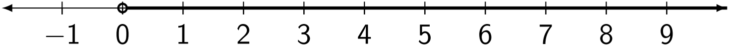

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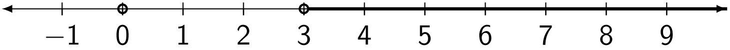

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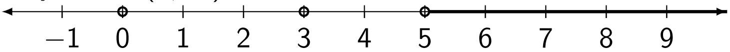

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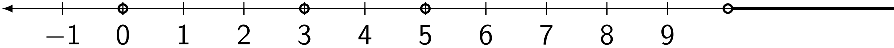

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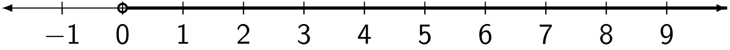

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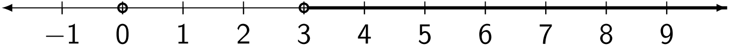

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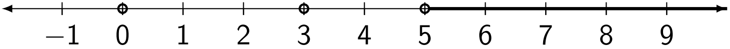

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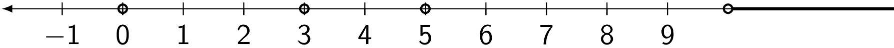

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- n^{th} Turn: (x_n, ∞)

$(x_n) = \{x_1, x_2, x_3, ..., x_n\} \to \infty$
We know that the sequence (x_n) approaches infinity,
Thus, $\bigcap_{n=1}^{\infty} (x_n, \infty) = \emptyset$.
Therefore, the intersection of the countably infinite amount of turns is empty, showing Player Two always has a winning strategy.

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$(x_n) = \{x_1, x_2, x_3, ..., x_n\} \to \infty$
We know that the sequence (x_n) approaches infinity,
Thus, $\bigcap_{n=1}^{\infty} (x_n, \infty) = \emptyset$.
Therefore, the intersection of the countably infinite amount of turns is empty, showing Player Two always has a winning strategy.

Banach-Mazur: Right Order (Player 2 Winning Strategy)

- First Turn (Player One): (x_1, ∞)
- Second Turn (Player Two): (x_2, ∞)
- Third Turn (Player One): (x_3, ∞)
- n^{th} Turn: (x_n, ∞)

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