Matrix Tree Theorem

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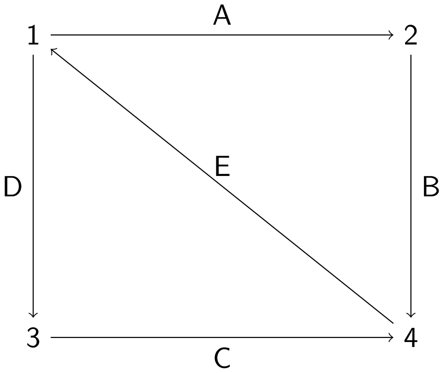
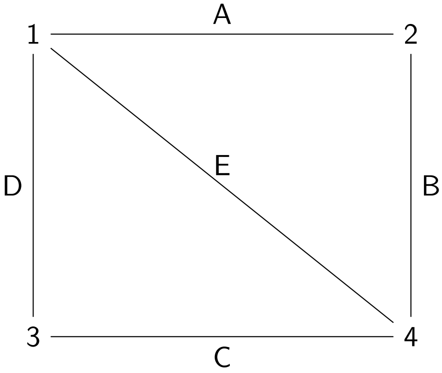


Directed and Undirected Graphs

An **undirected graph** is a graph which no edges are assigned a direction.



Trees

Definition

A connected simple graph G that satisfies either (and therefore both since they are equivalent) of the following conditions is called a **tree**. *(Note: all trees on n vertices have exactly $n-1$ edges)*

- *G is a minimally connected, meaning if any edge of G is removed, then the obtained graph, G', will not be connected.*
- *G does not contain a cycle.*

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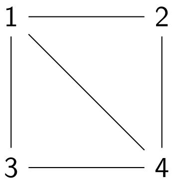

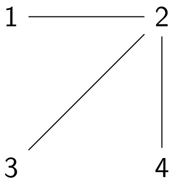

Spanning Trees

Definition

If G is a connected graph, we say that T is a **spanning tree** *of G, if G and T have the same vertex set and each edge of T is also an edge of G.*

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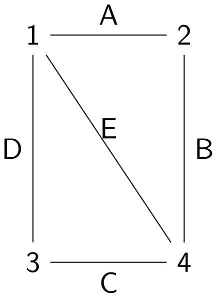

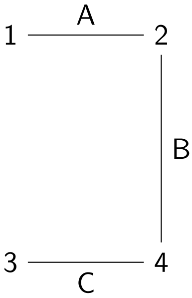

Adjacency Matrix

Definition

Let G be an undirected graph on n labeled vertices and define an $n \times n$ matrix $A = A_G$ by setting $A_{i,j}$ equal to the number of edges between vertices i and j. Then A is called the **adjacency matrix** *of G.*

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

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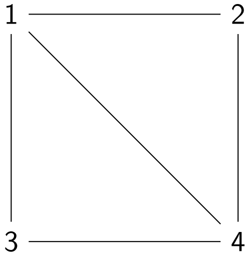

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$$

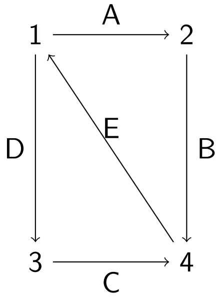
Incidence Matrix

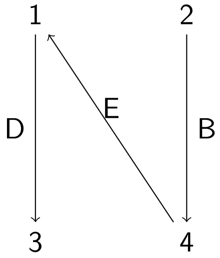
Definition

Let G be a directed graph. Let $\{v_1, v_2, ..., v_n\}$ denote the vertices of G and let $\{e_1, e_2, ..., e_m\}$ denote the edges of G. Then the **incidence matrix** *of G is the $n \times m$ matrix A defined by:*

- $a_{i,j} = 1$ *if v_i is the head of e_j*
- $a_{i,j} = -1$ *if v_i is the tail of e_j*
- $a_{i,j} = 0$ *otherwise*

Incidence Matrix Example





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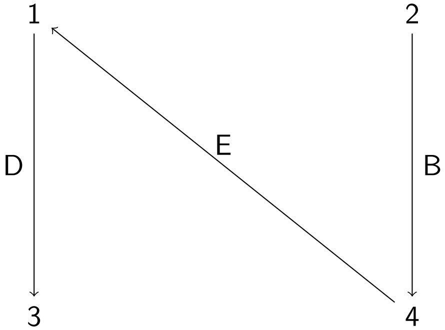

$$\begin{array}{cc} & \begin{array}{ccc} B & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

Spanning Trees for Directed Graphs

Theorem

Let G be a directed graph without loops and let A be the incidence matrix of G. Remove any row from A and let A_0 be the remaining matrix. Then the number of spanning trees of G is $\det A_0 A_0^T$.

Proof.

Assume, without loss of generality, that the last row of A is removed and call that A_0. Let B be an $(n-1) \times (n-1)$ submatrix of A_0.

Now we will claim that the $\det B = 1$ if and only if the subgraph G' corresponding to the columns of B is a spanning tree.

Otherwise, $\det B = 0$. □

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Proof Cont.

Let's assume there is a vertex $v_i (i \neq n)$ of degree one in G'. Then the ith row of B contains exactly one nonzero element, which is either 1 or -1.

When we expand the det B by this row, the claim follows: G' is a spanning tree of G if and only if $G' - v_i$ is also a spanning tree of $G - v_i$.

Therefore det $B = 1$. $\square$

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Proof Cont.

Now, let's assume that G' has no vertices of degree one. Then G' is not a spanning tree.

Since G' has $n-1$ edges and it is not a spanning tree, there must be a vertex in G' that has degree 0.

If this vertex is not v_n, then B has a zero row and the det $B = 0$.

If this vertex is v_n, then each column of B are linearly dependent and the det $B = 0$.

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So, we now know that det $B = 1$ if the subgraph G' corresponding to the columns of B is a spanning and det $B = 0$ otherwise.

By the Binet-Cauchy formula, $\sum(det B)^2 = det A_0 A_0^T$, where the summation ranges over all $(n-1) \times (n-1)$ submatrices B of A_0. Thus, det $A_0 A_0^T$ equals the number of spanning trees of G. $\square$

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Matrix Tree Theorem: Spanning Trees for Undirected Graphs

Theorem

Let U be a simple undirected graph. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of U. Define the $(n-1) \times (n-1)$ matrix L_0 by:

$$l_{i,j} = \begin{cases} \textit{the degree of } v_i \textit{ if } i = j \\ -1 \textit{ if } i \neq j, \textit{ and } v_i \textit{ and } v_j \textit{ are connected} \\ 0 \textit{ otherwise} \end{cases}$$

where $1 \leq i, j \leq n-1$. Then U has exactly $\det L_0$ spanning trees.

Matrix Tree Theorem: Spanning Trees for Undirected Graphs

Proof.

First, turn U into a directed graph G, by each edge going in both directions. Let A_0 be the incidence matrix of G.

The entry of $A_0A_0^T$ in position (i,j) is the scalar product of ith and jth row of A_0.

If $i = j$, then every edge that starts or ends at v_i contributes 1 to the product.

Therefore, the entry of $A_0A_0^T$ in position (i,i) is the degree of v_i in G, which is twice the degree of v_i in U. □

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If $i \neq j$, then every edge that starts at v_i and ends at v_j and every edge that starts at v_j and ends at v_i contributes -1 to the product.
Therefore, the entry of $A_0A_0^T$ in position (i, j) is -2 if v_iv_j is an edge of U and 0 otherwise.
So $A_0A_0^T = 2L_0$.
This implies that 2^{n-1} det $L_0 =$ det $(A_0A_0^T)$. Note that each spanning tree of U are 2^{n-1} different spanning trees of G.
Thus, it follows that U has exactly det L_0 spanning trees. □

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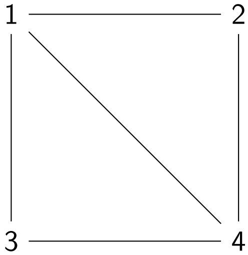

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