Group Paper

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December 12, 2016

Introduction

My group is the following 2x2 matrix under multiplication:

$$\begin{bmatrix} Cos(x) & Sin(x) \\ -Sin(x) & Cos(x) \end{bmatrix}$$

where $x \in \mathbb{R}$

In this paper I will be covering the following:

- The proof which shows this is in fact a group.
- That this group is commutative, thus its abelian.
- That this group is not cyclic.
- The order of this group.
- The proper subgroups of this group.
- That the Lagrange's Theorem holds for each of this group's subgroups.
- The cosets of each of the group's subgroups.

To prove that this matrix under multiplication is a group, we must show:

- The matrix is closed under multiplication.
- The matrix is associative under multiplication.
- The matrix contains the 2x2 identity matrix.
- The matrix multiplied by its inverse gives the identity element.

Before diving into this paper and proving that this matrix is a group under multiplication, a few trigonometry identities need to be stated, which will become key components to each proof.

Trigonometric Identities

Pythagorean Identity:

- $sin^2(x) + cos^2(x) = 1$

Even-Odd Identities:

- $sin(-x) = -sin(x)$
- $cos(-x) = cos(x)$

Double Angle Formulas:

- $sin(2x) = 2sin(x)cos(x)$
- $cos(2x) = cos^2(x) - sin^2(x)$

Product to Sum Formulas:

- $sin(x)sin(y) = 1/2(cos(x-y) - cos(x+y))$
- $cos(x)cos(y) = 1/2(cos(x-y) + cos(x+y))$
- $sin(x)cos(y) = 1/2(sin(x+y) + sin(x-y))$
- $cos(x)sin(y) = 1/2(cos(x+y) - sin(x-y))$

**Note these identities were found on S.O.S Mathematics Cyberboard published by Valdez-Sanchez

Proof of Group

Closure:

Let G be the 2x2 matrix defined by

$$\begin{bmatrix} Cos(x) & Sin(x) \\ -Sin(x) & Cos(x) \end{bmatrix}$$

where $x \in \mathbb{R}$.
Now, let both a and $b \in \mathbb{R}$
Then we will have two matrices in G:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}$$

To check if it is closed, we will multiple the two matrices.

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a)Cos(b) - Sin(a)Sin(b) & Cos(a)Sin(b) + Sin(a)Cos(b) \\ -Sin(a)Cos(b) - Cos(a)Sin(b) & -Sin(a)Sin(b) + Cos(a)Cos(b) \end{bmatrix}$$

****Note that from the Trigonometric Identity: Product-to-Sum formulas that we have the following:**
$Cos(a)Cos(b) - Sin(a)Sin(b) =$
$1/2(Cos(a-b) + Cos(a+b)) - 1/2(Cos(a-b) - Cos(a+b)) =$
$Cos(a+b)$

Similarly, we use Product-to-Sum formulas to get

$$\begin{bmatrix} Cos(a)Cos(b) - Sin(a)Sin(b) & Cos(a)Sin(b) + Sin(a)Cos(b) \\ -Sin(a)Cos(b) - Cos(a)Sin(b) & -Sin(a)Sin(b) + Cos(a)Cos(b) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Showing that when you multiple two matrix in G, then you get a matrix of the same format. Thus, it is closed under matrix multiplication.

Proof of Group Cont.

Associativity:

To prove that it is associative, we want to show that:

$$(\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}) * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * (\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix})$$

We know that this is inherited through multiplication of 2x2 matrices, however, we will prove its associative anyways.

Let a, b and $c \in \mathbb{R}$.
Then we will have three matrices in G:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix}$$

Now we will multiply the three matrices in this order:

****Note that we will again use the Product-to-Sum formulas**

$$(\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}) * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix} =$$

$$(\begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}) * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a+b+c) & Sin(a+b+c) \\ -Sin(a+b+c) & Cos(a+b+c) \end{bmatrix}$$

Likewise, we know that,

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * (\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(c) & Sin(c) \\ -Sin(c) & Cos(c) \end{bmatrix}) =$$

$$\begin{bmatrix} Cos(b+c+a) & Sin(b+c+a) \\ -Sin(b+c+a) & Cos(b+c+a) \end{bmatrix}$$

Since a, b and $c \in \mathbb{R}$ and the Reals are commutative under addition, we have:

$$\begin{bmatrix} Cos(b+c+a) & Sin(b+c+a) \\ -Sin(b+c+a) & Cos(b+c+a) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(a+b+c) & Sin(a+b+c) \\ -Sin(a+b+c) & Cos(a+b+c) \end{bmatrix}$$

Showing that it is associative under multiplication.

Proof of Group Cont.

Identity Element:

The identity element of a 2x2 matrix is, (Lay):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since we know that $0 \in \mathbb{R}$, we can create the following matrix that is in G:

$$\begin{bmatrix} Cos(0) & Sin(0) \\ -Sin(0) & Cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus it contains the identity element.

Proof of Group Cont.

Inverse:

Let $a \in \mathbb{R}$.
Then we have the following matrix that is in G:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}$$

To find this matrix's inverse, we must first find the determinant.
The determinant is found by $Cos(a) * Cos(a) - (-Sin(a) * Sin(a))$.
Showing the determinant is:
$Cos^2(a) - (-Sin^2(a)) =$
$Cos^2(a) + Sin^2(a) =$
1, by the Pythagorean Identity.

So the inverse matrix is:

$$\frac{1}{Determinant} = \frac{1}{1} * \begin{bmatrix} Cos(a) & -Sin(a) \\ Sin(a) & Cos(a) \end{bmatrix}$$

Next we need to show that the inverse is in G.

****Note that we will use the Even-Odd Identities**
$Sin(a) = Sin(-(-a)) = -Sin(-a)$
$-Sin(a) = Sin(-a)$
$Cos(a) = Cos(-a)$

Thus, showing us that:

$$\begin{bmatrix} Cos(a) & -Sin(a) \\ Sin(a) & Cos(a) \end{bmatrix} = \begin{bmatrix} Cos(-a) & Sin(-a) \\ -Sin(-a) & Cos(-a) \end{bmatrix}$$

Therefore, the inverse is in G.

Now, when you take the matrix and multiple it by its inverse you get:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(a) & -Sin(a) \\ Sin(a) & Cos(a) \end{bmatrix} =$$

$$\begin{bmatrix} Cos^2(a) + Sin^2(a) & -Cos(a)Sin(a) + Cos(a)Sin(a) \\ -Cos(a)Sin(a) + Cos(a)Sin(a) & Cos^2(a) + Sin^2(a) \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus it has an inverse and when you multiple a matrix by its inverse, you get the identity element. Since the four axioms of a group are met, it is a group. QED.

Proof of Abelian

Commutativity:

To prove that it is abelian, we must show that it is commutative, meaning:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} =$$

$$\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}$$

Let a and $b \in \mathbb{R}$.
Then we will have two matrices in G:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix}$$

We know that,

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} = \begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Likewise, we know that,

$$\begin{bmatrix} Cos(b) & Sin(b) \\ -Sin(b) & Cos(b) \end{bmatrix} * \begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} = \begin{bmatrix} Cos(b+a) & Sin(b+a) \\ -Sin(b+a) & Cos(b+a) \end{bmatrix}$$

Since a and $b \in \mathbb{R}$ and the Reals are commutative under addition, we have:

$$\begin{bmatrix} Cos(b+a) & Sin(b+a) \\ -Sin(b+a) & Cos(b+a) \end{bmatrix} = \begin{bmatrix} Cos(a+b) & Sin(a+b) \\ -Sin(a+b) & Cos(a+b) \end{bmatrix}$$

Showing that it is commutative under multiplication, thus it is abelian. QED.

Not Cyclic

Let $a \in \mathbb{R}$.
Then we have a matrix in G:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}$$

If it were cyclic then we can compose

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}^n$$

which would generate my entire group.
We know that:

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}^n =$$

$$\begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * \begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix} * ... * \begin{bmatrix} Cos(a) & Sin(a) \\ -Sin(a) & Cos(a) \end{bmatrix}$$

From the Product-to-Sum formulas, that would equal:

$$\begin{bmatrix} Cos(a+a+...+a) & Sin(a+a+...+a) \\ -Sin(a+a+...+a) & Cos(a+a+...+a) \end{bmatrix} = \begin{bmatrix} Cos(na) & Sin(na) \\ -Sin(na) & Cos(na) \end{bmatrix}$$

However, because of the density of the rationals in the reals, we know that for any na and $(n+1)a$ there exists a $b \in \mathbb{Q}$ such that $na < b < (n+1)a$, (Abbott). Therefore, there will be rational holes between any two values.

Order of the Group

Since the group:

$$\begin{bmatrix} Cos(x) & Sin(x) \\ -Sin(x) & Cos(x) \end{bmatrix}, \text{where } x \in \mathbb{R}$$

is based on trigonometric cosines and sines, the group is based around the unit circle. And since each $x \in \mathbb{R}$, and the Reals have an infinite order, so does my group.

Subgroups

The subgroups of my group would be any subgroup of the Reals. Since the Integers and Rationals are subgroups of the Reals, the following matrices would be subgroups of my group:

$$\begin{bmatrix} Cos(y) & Sin(y) \\ -Sin(y) & Cos(y) \end{bmatrix} \text{where } y \in \mathbb{Z}, \text{and} \begin{bmatrix} Cos(z) & Sin(z) \\ -Sin(z) & Cos(z) \end{bmatrix} \text{where } z \in \mathbb{Q}$$

Langrange's Theorem

Since my subgroups also have infinite order, the Langrange's theorem does not apply to my subgroups.

Cosets

The cosets of the integer subgroup is:

$$\begin{bmatrix} Cos(n) & Sin(n) \\ -Sin(n) & Cos(n) \end{bmatrix} * \begin{bmatrix} Cos(y) & Sin(y) \\ -Sin(y) & Cos(y) \end{bmatrix} \text{where } n, y \in \mathbb{Z} \text{ and n is fixed.}$$

Which equals:

$$\begin{bmatrix} Cos(n+y) & Sin(n+y) \\ -Sin(n+y) & Cos(n+y) \end{bmatrix}$$

We also have the coset for rationals:

$$\begin{bmatrix} Cos(m) & Sin(m) \\ -Sin(m) & Cos(m) \end{bmatrix} * \begin{bmatrix} Cos(z) & Sin(z) \\ -Sin(z) & Cos(z) \end{bmatrix} \text{where } n \in \mathbb{Z} \text{ and } z \in \mathbb{Q} \text{ and n is fixed.}$$

Which equals:

$$\begin{bmatrix} Cos(m+z) & Sin(m+z) \\ -Sin(m+z) & Cos(m+z) \end{bmatrix}$$

Conclusion

In this paper we have confirmed my group is in fact a group, that it is abelian and it is not cyclic. On top of that we looked at its order and some subgroups and cosets of the group.

Bibliography

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