Chapter 1

The Serre Equations

1.1 The Equations

There are three primary ways in which the Serre equations have been derived from the Euler equations in the literature; by asymptotic expansion [1, 2], directed fluid sheets [3] and depth integration [4, 5]. In this thesis the depth integration view of the equations is taken, although the derivation is omitted given the extent of literature already available.

From the depth-integration approach the Serre equations describe a free surface defined by its height $h(x,t)$ above a stationary bed profile $b(x)$ and a depth average of its horizontal velocity $u(x,t)$ as in Figure 1.1. The derivation is similar to that of the Shallow Water Wave Equations (SWWE) [], except for the Serre equations we allow the vertical velocity to vary linearly with depth and so we get non-hydrostatic pressure terms and therefore dispersion.

By depth integrating the Euler equations [] with a no-slip condition at the bed and a free surface condition at the free surface we get a depth integrated approximation of the conservation of mass and momentum equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{1.1a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right) + \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0. \tag{1.1b}$$

Definition 1.1. The Φ and Ψ terms account for the non-hydrostatic part of the

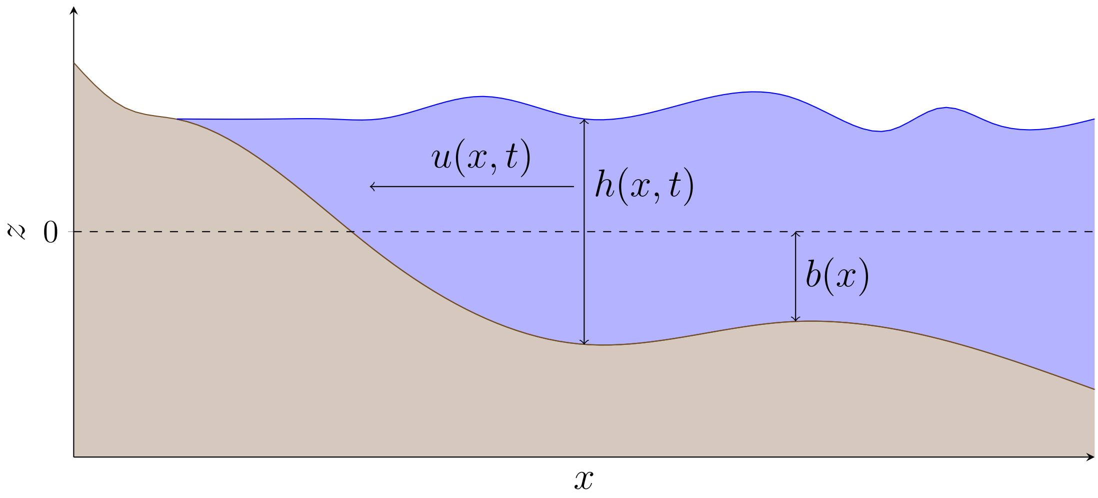

Figure 1.1: Diagram demonstrating the quantities used to describe the fluid (■) and the bed (■) for the Serre equations.

pressure and are

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial b}{\partial x}, \tag{1.2a}$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}. \tag{1.2b}$$

When $\Phi = \Psi = 0$ the Serre equations are equivalent to the SWWE where the vertical velocity is constant in depth, only the hydrostatic pressure is present and there is no dispersion. Due to the presence of the Φ and Ψ terms the Serre equations are much more difficult to solve analytically and numerically than the SWWE. The primary reason for this is that whilst the SWWE are hyperbolic for finite water depth, the Serre equations are neither hyperbolic nor parabolic. Furthermore the Serre equations are not in conservation law form due to the presence of temporal derivatives in Φ and Ψ, although they are derived from conservation equations.

For a horizontal bed $\partial b/\partial x = 0$, thus $\Psi = 0$ and all the source terms drop out and the Serre equations become

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{1.3a}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Phi\right) = 0. \tag{1.3b}$$

These equations are still neither hyperbolic nor parabolic and are still not in conservation law from as Φ contains a temporal derivative. As such even for horizontal beds the Serre equations are more difficult to solve analytically and numerically than the SWWE.

1.1.1 Alternative form of the Serre Equations

A major hurdle for developing numerical methods for the Serre equations is the presence of the mixed temporal and spatial derivative in Φ (1.2b). By rewriting the Serre equations and introducing a new conserved quantity G [6, 5, 7], the mixed temporal and spatial derivative can be removed and the Serre equations can be written in conservation law form.

Definition 1.2. The conserved quantity G is

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x}^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

The Serre equations (1.1) can then rewritten as conservation laws for the conserved variables h and G

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{1.4a}$$

$$\begin{gathered}\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}^2 + h^2u\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) \\ + \frac{1}{2}h^2u\frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.\end{gathered} \tag{1.4b}$$

The conserved quantity G resembles h multiplied by the irrotationality [8, 9] and its conservation equation is equivalent to the conservation equation for momentum (1.1b).

This conservation law form makes the Serre equations well suited for a finite volume method for the conservation of mass and G equations, provided one can solve for u given h and G. Even in conservation law from these equations are still considerably more difficult to solve numerically and analytically than the SWWE.

For a horizontal bed $\partial b/\partial x = 0$ the conservation law form of the Serre equations is

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{1.5a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \frac{\partial u}{\partial x}^2 \right) = 0, \tag{1.5b}$$

$$G = hu - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3 \frac{\partial u}{\partial x} \right). \tag{1.5c}$$

These equations are much simpler to solve than their counterparts that allow for variations in bathymetry, although they still present a number of challenges.

1.2 Properties of the Serre Equations

The Serre equations are significantly more complex than their dispersionless counterparts the SWWE and so fewer analytic solutions can be found in the literature. This increases the need to utilise the known properties of the Serre equations and construct forced solutions to assess our numerical methods. The relevant properties of the Serre equations, analytic solutions and forced solutions are thus presented here, beginning with the conservation properties of the Serre equations.

1.2.1 Conservation Properties

Conservation of a quantity means that in a closed system the total amount of a quantity q remains constant in time.

Definition 1.3. The total amount of a quantity q in a system occurring on the interval $[a, b]$ at time t is

$$\mathcal{C}_q(t) = \int_a^b q(x,t)\, dx.$$

Conservation of a quantity q means that $\mathcal{C}_q(0) = \mathcal{C}_q(t)\ \forall t$. Given that the Serre equations (1.1) are conservation equations for mass and momentum and that the conservation of momentum equation can be rewritten as a conservation equation for G (1.4), the Serre equations conserve these quantities. Additionally owing to the Hamiltonian structure of the Serre equations, the Serre equations possess a Hamiltonian $\mathcal{H}$ which acts as an energy and is therefore also conserved.

Definition 1.4. The Hamiltonian of the Serre equations is

$$\mathcal{H}(x,t) = \frac{1}{2}\left(hu^2 + \frac{h^3}{3}\left(\frac{\partial u}{\partial x}\right)^2 + gh^2 + 2ghb + u^2h\frac{\partial b}{\partial x} - uh^2\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right).$$

For the system to be closed the flux terms of the conservation of mass and momentum equations at both boundaries must with cancel and the integral of the source terms over the domain must be zero.

1.2.2 Dispersion Properties

The dispersion properties of wave equations are primarily studied through linearising the equations, assuming periodic wave solutions and then deriving a relationship between the frequency ω and wave number k of these solutions. For the Serre equations the dispersion relation [] is

$$\omega = Uk \pm k\sqrt{gH}\sqrt{\frac{3}{(kH)^2 + 3}}. \tag{1.6}$$

Barthélemy [10] compared this dispersion relation to that of the linear theory of water waves and demonstrated its utility when k is large. However when k is small the difference between the dispersion relation of the Serre equations and that of water wave theory increases. The dispersion relation of the Serre equations can be modified by introducing terms to reduce this difference [10], but such modifications are beyond the scope of this thesis.

From the dispersion relation (1.6) the phase velocity $v_p = \omega/k$ and the group velocity $v_g = \partial\omega/\partial k$ can be written in terms of wave number as

$$v_p = U \pm \sqrt{gH}\sqrt{\frac{3}{(kH)^2 + 3}}, \tag{1.7a}$$

$$v_g = U \pm \sqrt{gH}\left(\sqrt{\frac{3}{(kH)^2 + 3}} \mp (kH)^2\sqrt{\frac{3}{\left((kH)^2 + 3\right)^3}}\right). \tag{1.7b}$$

Since both the phase and group velocities depend on the wave number, waves of different spatial frequencies travel at different speeds meaning the Serre equations describe dispersive waves.

Fortunately, the phase velocity and the group velocity of waves are bounded, since as $k \to 0$ then $v_p, v_g \to U$ and as $k \to \infty$ then $v_p, v_g \to U \pm \sqrt{gH}$. Therefore we have that

$$U - \sqrt{gH} \leq v_p \leq U + \sqrt{gH}, \tag{1.8a}$$
$$U - \sqrt{gH} \leq v_g \leq U + \sqrt{gH}. \tag{1.8b}$$

1.2.3 Analytic Solutions

Due to the complexity of the Serre equations, few analytic solutions have been discovered. In particular there is a travelling wave solution for horizontal beds and a lake at rest solution for any bathymetry. Both analytic solutions will be used to assess the capabilities of the numerical methods and so we present them here.

Solitary Travelling Wave Solution

The Serre equations admit a travelling wave solution that propagates at a constant speed without deformation due to a balance between nonlinear and dispersive effects. Unlike the Euler equations this travelling wave solution has a closed form

$$h(x,t) = a_0 + a_1 \text{sech}\left(\kappa\left(x - ct\right)\right), \tag{1.9a}$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right), \tag{1.9b}$$
$$b(x) = 0 \tag{1.9c}$$

with

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{(a_0+a_1)}},$$

$$c = \sqrt{g(a_0+a_1)}.$$

This wave is has an amplitude of a_1, an infinite wavelength and propagates on water a_0 deep. These solitary waves are not true solitons however, due to their inelastic collisions with one another [11]. These solitary waves can be generalised to a family of periodic travelling wave solutions [12].

From (1.9) we can derive G for the solitary wave solution in terms of h

$$G = c(h - a0) - \frac{ca_0}{3}\left(h\frac{\partial^2 h}{\partial x^2} + \left[\frac{\partial h}{\partial x}\right]\right) \tag{1.10}$$

where the derivatives of h are

$$\begin{aligned}
\frac{\partial h}{\partial x} &= -2a_1\kappa \text{sech}^2\left(\kappa\left(x-ct\right)\right)\tanh\left(\kappa\left(x-ct\right)\right),\\
\frac{\partial^2 h}{\partial x^2} &= 2a_1\kappa^2\left(\cosh\left(2\kappa\left(x-ct\right)\right)-2\right)\text{sech}^4\left(\kappa\left(x-ct\right)\right).
\end{aligned}$$

As we would like to assess the conservation properties of our numerical methods we will require the total mass, momentum, G and Hamiltonian for this solution. In particular we require these totals at the initial time $t = 0$, to allow for various domains we present the integrals in indefinite form

$$\int h(x,0)\;dx = a_0 x + \frac{a_1}{\kappa}\tanh\left(\kappa x\right) + \text{constant}, \tag{1.11a}$$

$$\int u(x,0)h(x,0)\;dx = \frac{a_1 c}{\kappa}\tanh\left(\kappa x\right) + \text{constant}, \tag{1.11b}$$

$$\begin{aligned}
\int G(x,0)\;dx = \frac{ca_1}{3\kappa}\Bigg(&3 + 2a_0^2\kappa^2\text{sech}^2\left(\kappa x\right)\\
&+ 2a_0a_1\kappa^2\text{sech}^4\left(\kappa x\right)\Bigg)\tanh\left(\kappa x\right) + \text{constant},
\end{aligned} \tag{1.11c}$$

$$\begin{aligned}
\int \mathcal{H}(x,0)\;dx = \frac{1}{2}\Bigg(&\int g\left[h(x,0)\right]^2\;dx + \int h(x,0)\left[u(x,0)\right]^2\;dx\\
&+\int \left[h(x,0)\right]^3\left[\frac{\partial u(x,0)}{\partial x}\right]^2\;dx\Bigg)
\end{aligned} \tag{1.11d}$$

where the integrals of the Hamiltonian are

$$\int g\left[h(x,0)\right]^2 \, dx = \frac{g}{12\kappa}\operatorname{sech}^3(\kappa x)\left[9a_0^2\kappa x\cosh(\kappa x) + 3a_0^2\kappa x\cosh(3\kappa x)\right.$$
$$\left. + 4a_1\left(3a_0 + 2a_1 + (3a_0+a_1)\cosh(2\kappa x)\right)\sinh(\kappa x)\right]$$
$$+ \text{constant},$$

$$\int h(x,0)\left[u(x,0)\right]^2 \, dx = \frac{\sqrt{a_1}c^2}{\kappa}\left(-\frac{a0}{\sqrt{a_0+a_1}}\operatorname{arctanh}\left(\frac{\sqrt{a1}\tanh(\kappa x)}{\sqrt{a_0+a1}}\right)\right.$$
$$\left. + \frac{\sqrt{a_1}}{\kappa}\tanh(\kappa x)\right) + \text{constant},$$

$$\int \left[h(x,0)\right]^3\left[\frac{\partial u(x,0)}{\partial x}\right]^2 \, dx = \frac{2a_0^2c^2\kappa}{9\sqrt{a_1}\left(a_0 + a_1\operatorname{sech}^2(\kappa x)\right)}$$
$$\times\left(a_0 + 2a_1 + a_0\cosh(2\kappa x)\right)\operatorname{sech}^2(\kappa x)$$
$$\times\left[-3a_0\sqrt{a_0+a_1}\operatorname{arctanh}\left(\frac{\sqrt{a1}\tanh(\kappa x)}{\sqrt{a_0+a1}}\right)\right.$$
$$\left. + \sqrt{a_1}\left(3a_0 + a_1 - a_1\operatorname{sech}^2(\kappa x)\right)\tanh(\kappa x)\right] + \text{constant}.$$

Therefore, we have the analytic values of our physical variables h, u, G and b for all t as well as the total amounts of our conserved quantities for the initial conditions when $t = 0s$, as desired.

Lake at Rest

The lake at rest solution is a rudimentary stationary solution of the Serre equations that exists for all bathymetry $b(x)$, due to a balance between hydrostatic pressure and the forcing of the bed slope. The lake at rest solution is

$$h(x,t) = \max\left\{a_0 - b(x), 0\right\}, \tag{1.12a}$$
$$u(x,t) = 0, \tag{1.12b}$$
$$G(x,t) = 0. \tag{1.12c}$$

It represents a quiescent body of water with a horizontal water surface or stage $w(x,t) = h(x,t) + b(x)$ over any bathymetry. The maximum function is included for the water depth to allow for dry regions of the bed when $b(x) > a_0$. We write these solutions in terms of $b(x)$ as this solution holds for all bed profiles.

For these quantities (1.12) the Serre equations (1.4) reduce to

$$\frac{\partial h}{\partial t} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(\frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} = 0.$$

Since we have that $\partial h/\partial x = -\partial b/\partial x$ when $h \neq 0$, then G and h are constant in time and therefore so is u and thus we possess a stationary solution.

The total momentum and G in our system is straightforward to calculate as both are zero everywhere and so we have

$$\int u(x,0)h(x,0)\ dx = 0 + \text{constant}, \tag{1.13}$$

$$\int G(x,0)\ dx = 0 + \text{constant}. \tag{1.14}$$

To calculate the total mass and Hamiltonian in our system we must break up our domain into wet regions where $b(x) < a_0$ and dry regions where $b(x) \geq a_0$. For the dry regions the total mass and energy are 0 and so we have

$$\int h(x,0)\ dx = 0, \tag{1.15a}$$

$$\int \mathcal{H}(x,0)\ dx = 0 \tag{1.15b}$$

whilst in a wet region we have

$$\int h(x,0)\ dx = a_0 x - \int b(x)\ dx, \tag{1.16a}$$

$$\int \mathcal{H}(x,0)\ dx = \frac{g}{2}\left(a_0^2 x - 2a_0 \int b(x)\ dx + \int b(x)^2\ dx\right). \tag{1.16b}$$

Therefore as desired we have expressions for all the quantities in terms of the bed profile $b(x)$.

1.2.4 Forced Solutions

To account for the small number of analytic solutions to the Serre equations we also constructed forced solutions to assess our numerical methods. These work by introducing a source term S for both the conservation of mass and G equations that balances these equation to force a solution. For instance by adding the source terms S_{mass} and S_G into (1.4) we obtain the forced Serre equations in conservation law form

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + S_{\text{mass}} = 0, \tag{1.17a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}^2 + h^2u\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right)$$
$$+\frac{1}{2}h^2u\frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} + S_G = 0. \tag{1.17b}$$

To construct the forced solution we choose some functions for h, u and b for all x and t, which will be our forced analytic solutions to the forced Serre equations. From these chosen functions we determine G, S_{mass} and S_G so that these equations (1.17) possess our chosen h, u and b as solutions.

We demonstrate this for the particular forced solution we employ, a travelling Gaussian wave over a periodic bed profile which has

$$h(x,t) = a_0 + a_1 \exp\left(-\frac{\left((x - a_2t) - a_3\right)^2}{2a_4}\right), \tag{1.18a}$$

$$u(x,t) = a_5 \exp\left(-\frac{\left((x - a_2t) - a_3\right)^2}{2a_4}\right), \tag{1.18b}$$

$$b(x) = a_6 \sin\left(a_7 x\right). \tag{1.18c}$$

From these definitions G can be calculated using Def 1.2 while S_{mass} and S_G can be calculated using

$$S_{\text{mass}} = -\frac{\partial h}{\partial t} - \frac{\partial (uh)}{\partial x},$$

$$S_G = -\frac{\partial G}{\partial t} - \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\frac{\partial u}{\partial x}^2 + h^2u\frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right)$$
$$-\frac{1}{2}h^2u\frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} + hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} - gh\frac{\partial b}{\partial x}$$

with h, u and b from (1.18) and the corresponding G.

With these values the forced Serre equations (1.17) admit the analytic solutions (1.18). This allows us to assess our numerical solutions for a larger range of scenarios than possible with the current analytic solutions of the Serre equations.

1.2.5 Asymptotic Results

Beyond analytic solutions to the Serre equations there have also been studies of the long term behaviour of the Serre equations for situations that are difficult to treat analytically. One particular scenario of interest is the evolution of a moving discontinuous jump in h known as a bore, which can be observed naturally, for example tidal bores [].

For the non-dispersive SWWE bores propagate at a fixed speed and have a constant shape []. For the Serre equations dispersion causes bores to break up into wave train, which are referred to as undular bores []. This process is more difficult to treat analytically particularly over short time spans and so we do not possess analytic solutions for the Serre equations for bores.

To gain some insight into the behaviour of bores for long time spans Whitham modulation techniques were applied to the Serre equations as $t \to \infty$ [12]. These techniques provided an estimate of the speed S^+ and amplitude A^+ of the front of a bore

$$\frac{\Delta}{\left(A^+ + 1\right)^{1/4}} - \left(\frac{3}{4 - \sqrt{A^+ + 1}}\right)^{21/10} \left(\frac{2}{1 + \sqrt{A^+ + 1}}\right)^{2/5} = 0 \tag{1.19a}$$

$$S^+ = \sqrt{g\left(A^+ + 1\right)} \tag{1.19b}$$

where $\Delta = h_b/h_0$, h_b is the height of the bore and h_0 is the depth of still water in front of the bore. These estimates agreed well with numerical simulations provided that $\Delta < 1.43$ [12].

Bibliography

[1] F. Serre. Contribtion à l'étude des écoulements permanents et variables dans les canaux. *La Houille Blanche*, 6:830–872, 1953.

[2] D. Lannes and P. Bonneton. Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation. *Physics of Fluids*, 21(1):16601–16610, 2009.

[3] A. E. Green and P. M. Naghdi. A derivation of equations for wave propagation in water of variable depth. *Journal of Fluid Mechanics*, 78(2):237–246, 1976.

[4] C. H. Su and C. S. Gardner. Korteweg-de Vries equation and generalisations. III. Derivation of the Korteweg-de Vries equation and Burgers equation. *Journal of Mathematical Physics*, 10(3):536–539, 1969.

[5] C. Zoppou. *Numerical Solution of the One-dimensional and Cylindrical Serre Equations for Rapidly Varying Free Surface Flows.* PhD thesis, Australian National University, Mathematical Sciences Institute, College of Physical and Mathematical Sciences, Australian National University, Canberra, ACT 2600, Australia, 2014.

[6] O. Le Métayer, S. Gavrilyuk, and S. Hank. A numerical scheme for the Green-Naghdi model. *Journal of Computational Physics*, 229(6):2034–2045, 2010.

[7] M. Li, P. Guyenne, F. Li, and L. Xu. High order well-balanced CDG-FE methods for shallow water waves by a GreenNaghdi model. *Journal of Computational Physics*, 257(1):169–192, 2014.

[8] W. Choi and R. Camassa. Fully nonlinear internal waves in a two-fluid system. *Journal of Fluid Mechanics*, 396:1–36, 1999.

[9] J.D Carter and R. Cienfuegos. The kinematics and stability of solitary and cnoidal wave solutions of the serre equations. *European Journal of Mechanics-B/Fluids*, 30(3):259–268, 2011.

[10] E. Barthélemy. Nonlinear shallow water theories for coastal waves. *Surveys in Geophysics*, 25(3):315–337, 2004.

[11] D. Dutykh, D. Clamond, P. Milewski, and D. Mitsotakis. Finite volume and pseudo-spectral schemes for the fully nonlinear 1D Serre equations. *European Journal of Applied Mathematics*, 24(5):761–787, 2013.

[12] G.A. El, R. H. J. Grimshaw, and N. F. Smyth. Unsteady undular bores in fully nonlinear shallow-water theory. *Physics of Fluids*, 18(2):027104, 2006.

[13] P.L Roe. Characteristic-based schemes for the euler equations. *Annual Review of Fluid Mechanics*, 18(1):337–365, 1986.

[14] B. Van Leer. Towards the ultimate conservative difference scheme. iv. a new approach to numerical convection. *Journal of Computational Physics*, 23(3): 276–299, 1977.

[15] L.C. Evans and American Mathematical Society. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, 1998.

[16] A. Kurganov, S. Noelle, and G. Petrova. Semidiscrete central-upwind schemes for hyperbolic conservation laws and Hamilton-Jacobi equations. *Journal of Scientific Computing, Society for Industrial and Applied Mathematics*, 23(3):707–740, 2002.

[17] E. Audusse, F. Bouchut, M. Bristeau, R. Klein, and B. Perthame. A fast and stable well–balanced scheme with hydrostatic reconstruction for shallow water flows. *Journal of Scientific Computing, Society for Industrial and Applied Mathematics*, 25(6):2050–2065, 2004.

[18] J. Pitt. *A Second Order Well Balanced Hybrid Finite Volume and Finite Difference Method for the Serre Equations*. PhD thesis, Australian National University, Mathematical Sciences Institute, College of Physical and Mathematical Sciences, Australian National University, Canberra, ACT 2600, Australia, 2014.

[19] S. Gottlieb, C. Shu, and E. Tadmor. Strong stability-preserving high-order time discretization methods. *Society for Industrial and Applied Mathematics Review*, 43(1):89–112, 2001.

[20] R. Courant, K. Friedrichs, and H. Lewy. On the partial difference equations of mathematical physics. *IBM Journal of Research and Development*, 11(2): 215–234, 1967.

[21] C. Zoppou, J. Pitt, and S. Roberts. Numerical solution of the fully non-linear weakly dispersive serre equations for steep gradient flows. *Applied Mathematical Modelling*, 48:70–95, 2017.

[22] W.H Press, S.A Teukolsky, W.T Vetterling, and B.P Flannery. *Numerical Recipes in C*, volume 2. Cambridge University Press, 1996.

[23] P. Lax and R. Richtmyer. Survey of the stability of linear finite difference equations. *Communications on Pure and Applied Mathematics*, 9(2):267–293, 1956.

[24] A. G. Filippini, M. Kazolea, and M. Ricchiuto. A flexible genuinely nonlinear approach for nonlinear wave propagation, breaking and run-up. *Journal of Computational Physics*, 310:381–417, 2016.

[25] S Beji and J.A. Battjes. Experimental investigation of wave propagation over a bar. *Coastal Engineering*, 19(1):151–162, 1993.

[26] S Beji and J.A. Battjes. Numerical simulation of nonlinear wave propagation over a bar. *Coastal Engineering*, 23(1):1–16, 1994.

[27] D. Lannes. *The Water Waves Problem: Mathematical Analysis and Asymptotic*, volume 1 of *American Mathematical Society*. Mathematical Surveys and Monographs, 2013.

[28] J. Pitt, C. Zoppou, and S.G Roberts. Importance of dispersion for shoaling waves. *Modelling and Simulation Society of Australia and New Zealand*, 22 (1):1725–1730, 2017.

[29] Y. Zhang, A.B. Kennedy, N. Panda, C. Dawson, and J.J. Westerink. Boussinesq-green-naghdi rotational water wave theory. *Coastal Engineering*, 73(1):13–27, 2013.