Numerically solving the 1D Serre Equations In the Presence of Discontinuities

Jordan Pitt, Stephen Roberts and Christopher Zoppou

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Serre Equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\underbrace{\underbrace{\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}}}_{\text{Serre Equations}} = 0$$

Conservation Law Form

New conserved quantity

$$G = uh - h^2 \frac{\partial h}{\partial x}\frac{\partial u}{\partial x} - \frac{h^3}{3}\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

Reformulated equations

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{3a}$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(Gu + \frac{gh^2}{2} - \frac{2h^3}{3}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right) = 0 \tag{3b}$$

Basic Overview

Vector of conserved quantities:

$$\boldsymbol{U} = \begin{bmatrix} h \\ G \end{bmatrix}$$

Algorithm:

$$\mathcal{H}\left(\bar{\boldsymbol{U}}^n, \Delta x, \Delta t\right) = \begin{cases} \boldsymbol{U}^n & = & \mathcal{M}\left(\bar{\boldsymbol{U}}^n\right) \\ \boldsymbol{u}^n & = & \mathcal{A}\left(\boldsymbol{U}^n, \Delta x\right) \\ \bar{\boldsymbol{U}}^{n+1} & = & \mathcal{L}\left(\bar{\boldsymbol{U}}^n, \boldsymbol{u}^n, \Delta x, \Delta t\right) \end{cases} .$$

$\mathcal{A}$: elliptic equation for G

Subscripts represent the spatial cell centre and superscripts represent the time step.

$$\boldsymbol{u}^n = [u_0^n, u_1^n, \dots, u_m^n]$$

where $x_i - x_{i-1} = \Delta x$ for all i.

$$\boldsymbol{u}^n = \mathcal{A}\left(\boldsymbol{U}^n, \Delta x\right)$$

In this version of the scheme this represents an appropriate order centered finite difference approximation to (2).

$\mathcal{L}$: conservative update

Bar represents the average over the cell so for example

$$\bar{u}_i^n = \frac{1}{\Delta x} \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} u(x, t^n) dx$$

$$\bar{\boldsymbol{U}}^{n+1} = \mathcal{L}\left(\bar{\boldsymbol{U}}^n, \boldsymbol{u}^n, \Delta x, \Delta t\right)$$

This represents an appropriate order Godunov type finite volume method. In particular we use the the method by Kurganov (Kurganov et al., 2002).

$\mathcal{M}$: nodal values to cell averages

$$\boldsymbol{U}^n = \mathcal{M}\left(\bar{\boldsymbol{U}}^n\right)$$

For first and second order versions this is the identity map.

$\mathcal{H}$: Euler step

$$\mathcal{H}\left(\bar{\boldsymbol{U}}^n, \Delta x, \Delta t\right) = \begin{cases} \boldsymbol{U}^n & = & \mathcal{M}\left(\bar{\boldsymbol{U}}^n\right) \\ \boldsymbol{u}^n & = & \mathcal{A}\left(\boldsymbol{U}^n, \Delta x\right) \\ \bar{\boldsymbol{U}}^{n+1} & = & \mathcal{L}\left(\bar{\boldsymbol{U}}^n, \boldsymbol{u}^n, \Delta x, \Delta t\right) \end{cases}.$$

Need to use strong stability preserving Runge-Kutta (Gottlieb et al., 2009) steps to increase the order of accuracy in time.

Soliton

$$h(x,t) = a_0 + a_1 \text{sech}^2\left(\kappa\left(x - ct\right)\right) \tag{4a}$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right) \tag{4b}$$

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{a_0 + a_1}} \tag{4c}$$

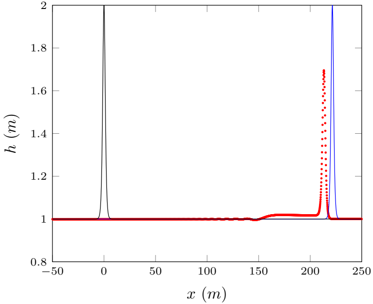
$$c = \sqrt{g\left(a_0 + a_1\right)} \tag{4d}$$

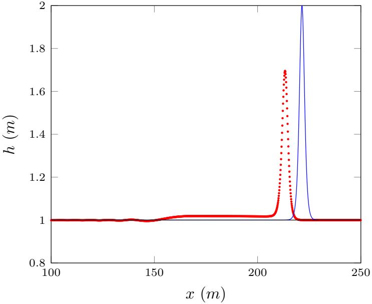
Nonlinearity (η):

$$\eta = \frac{a_1}{a_0}$$

Experiment 1

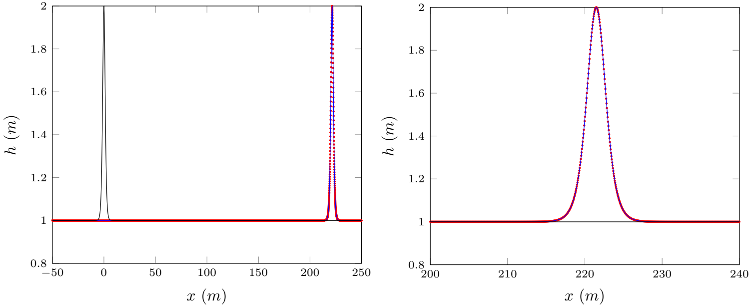
$a_0 = 1.0\text{m}$, $a_1 = 1.0\text{m}$ $(\eta = 1.0)$, $\Delta x = 100/2^{12}\text{m}$,
$\Delta t = \frac{0.5}{\sqrt{g(a_0+a_1)}} \Delta x$ s
First Order:





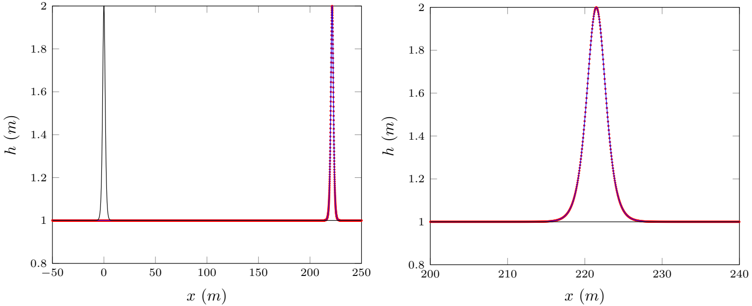
Experiment 1

Second Order:



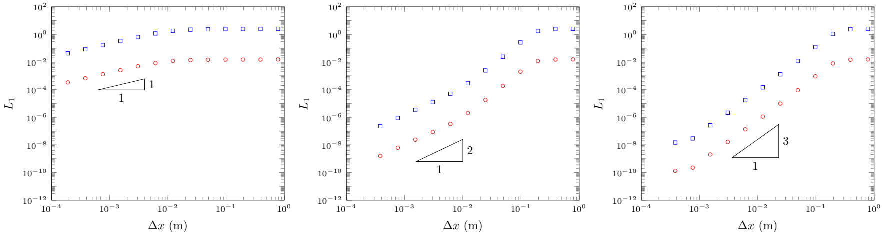
Experiment 1

Third Order:

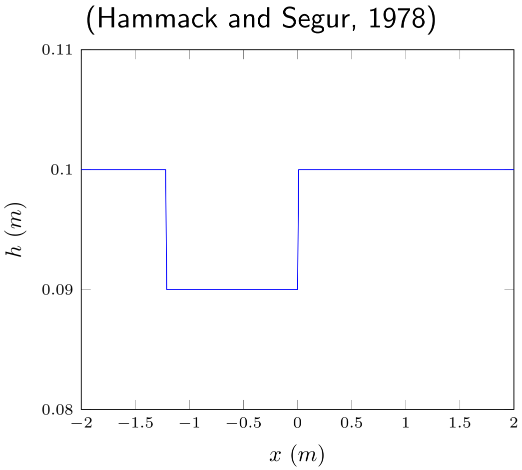


Experiment 1

Convergence:



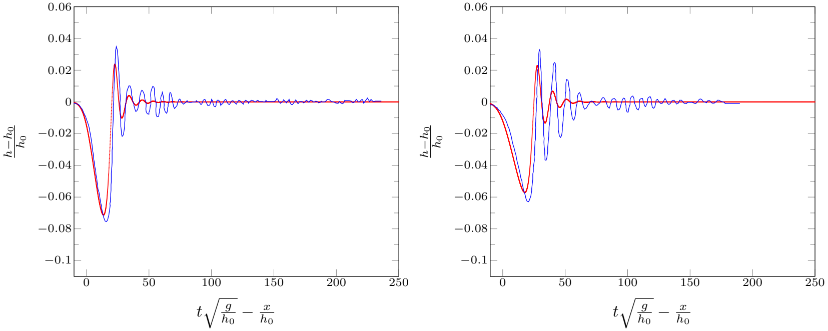
Hammack and Segur



Segur and Hammack

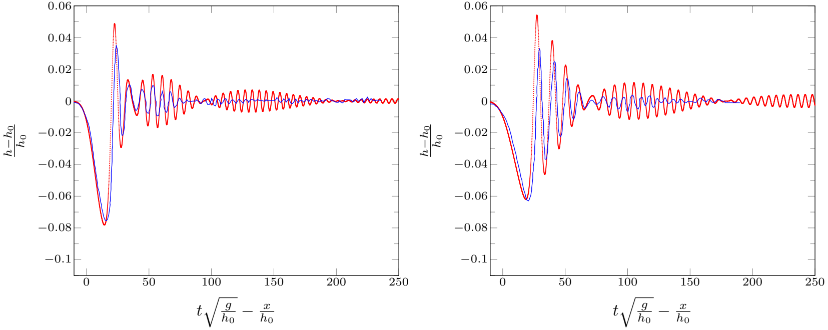
$\Delta x = 0.1\text{m}$, $\Delta t = \frac{0.5}{\sqrt{g0.1}} \Delta x$ s

First Order at $x = 10$m and $x = 20$m



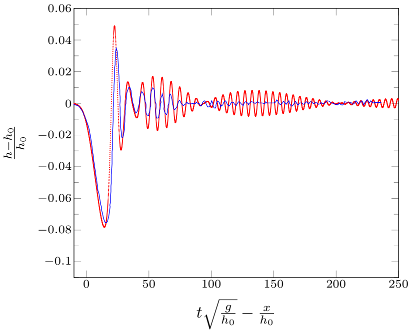
Segur and Hammack

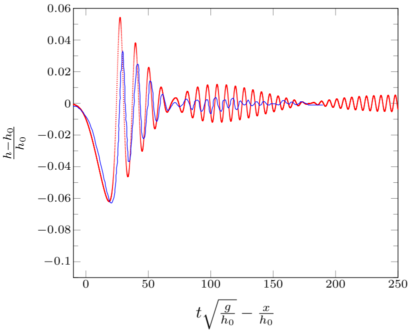
Second Order at $x = 10$m and $x = 20$m



Segur and Hammack

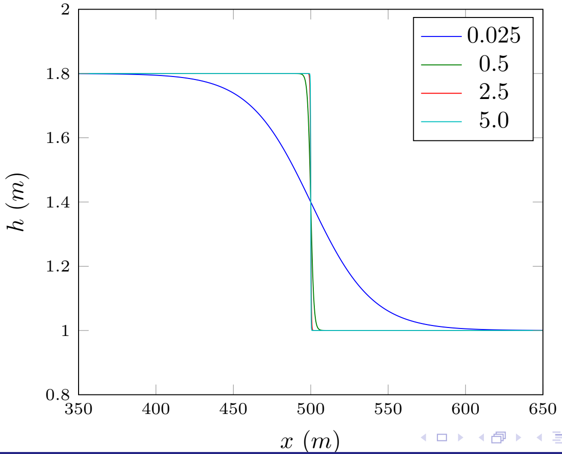
Third Order at $x = 10$m and $x = 20$m





Plot

$\Delta x = \frac{10}{2^6}\text{m}$



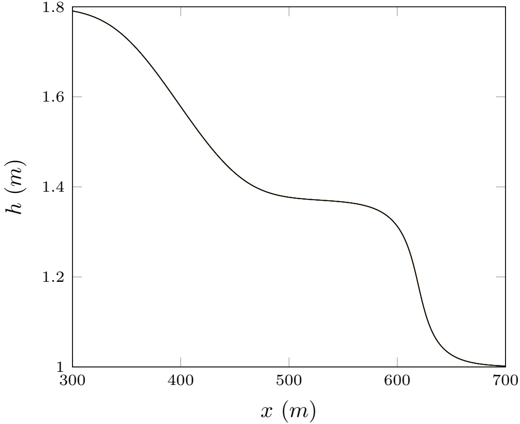
Equations

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(\alpha\left(x_0 - x\right)\right)\right) \text{ m}, \tag{5a}$$

$$u(x,0) = 0.0\text{m/s}. \tag{5b}$$

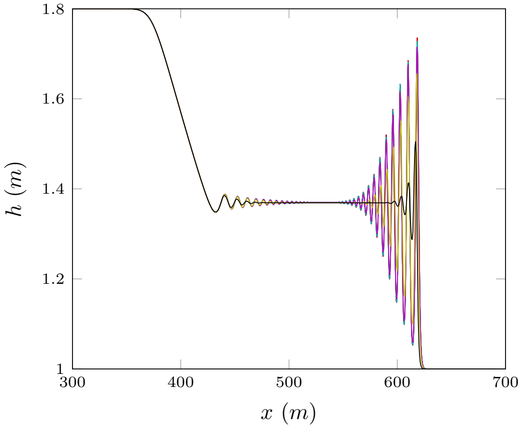
Problem of interest: $h_0 = 1$m , $h_1 = 1.8$m , $x_0 = 500$m, $\Delta t = 0.01\Delta x$

No oscillations



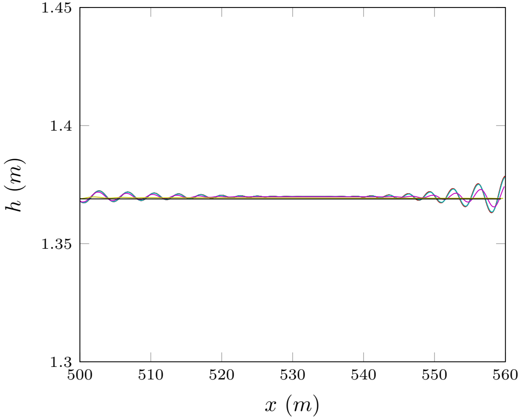
$\alpha = 0.025$ and $\Delta x = \frac{10}{2^k}$m with $k = [3, 4, \cdots, 9]$

Flat middle



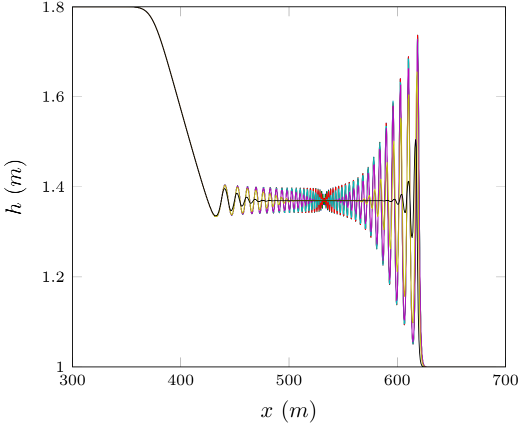
$\alpha = 0.5$ and $\Delta x = \frac{10}{2^k}$m with $k = [3, 4, \cdots, 9]$

Varying Δx Scenarios



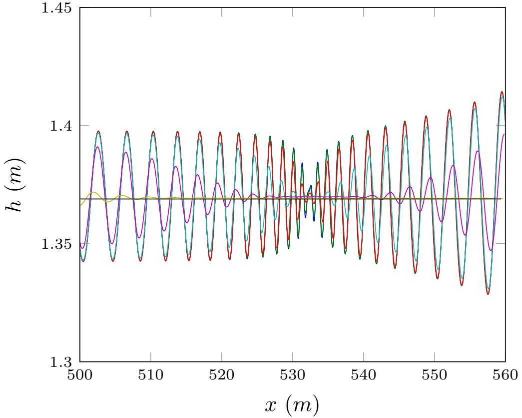
(Le Métayer et al., 2010)

Contact Discontinuity



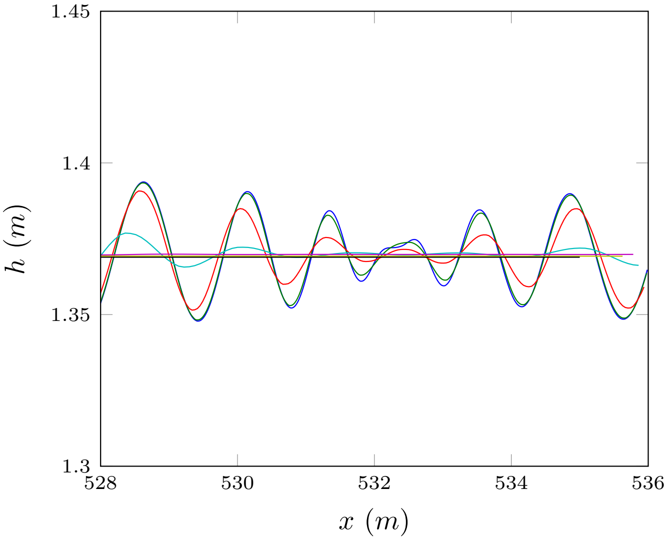
$\alpha = 2.5$ and $\Delta x = \frac{10}{2^k}$m with $k = [3, 4, \cdots, 9]$

Varying Δx Scenarios

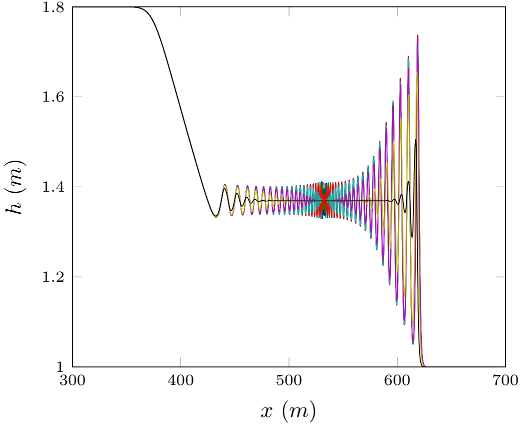


(El et al., 2006)

Varying Δx Scenarios

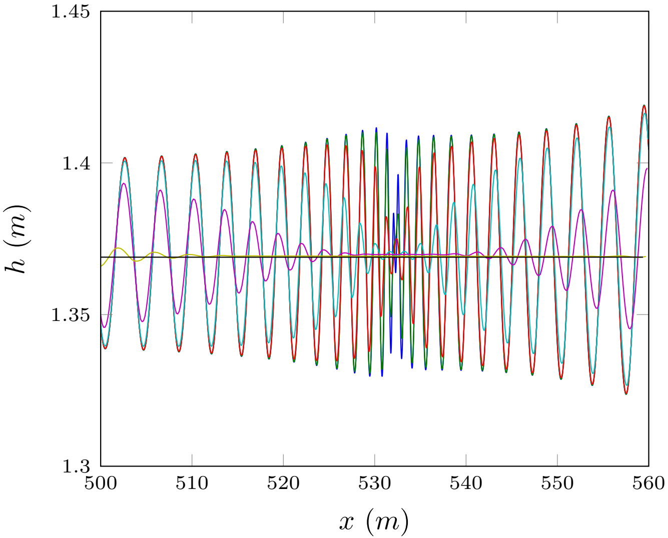


Bump

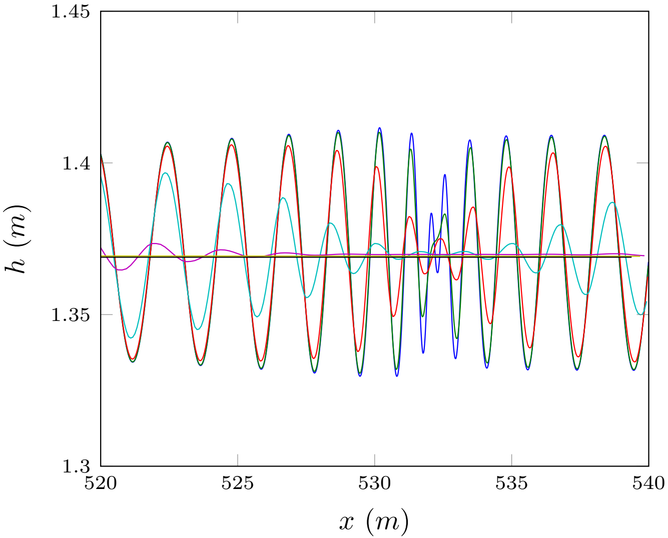


$\alpha = 5.0$ and $\Delta x = \frac{10}{2^k}$m with $k = [3, 4, \cdots, 9]$

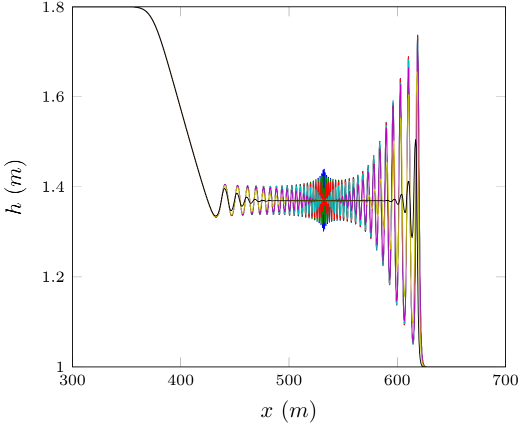
Varying Δx Scenarios



Varying Δx Scenarios

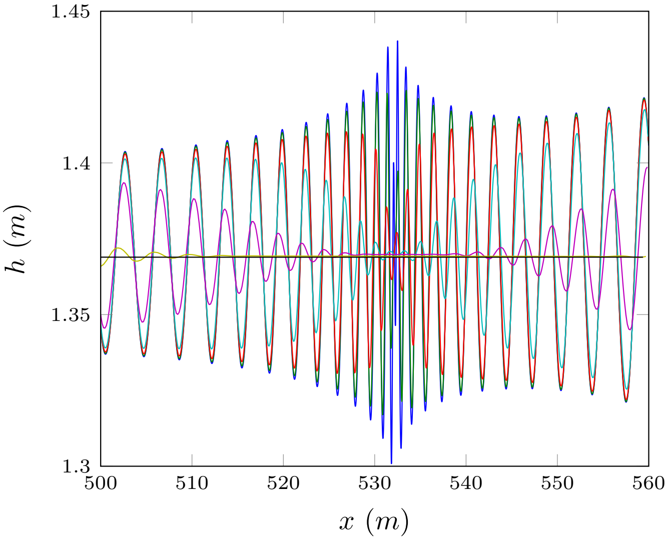


Larger Bump

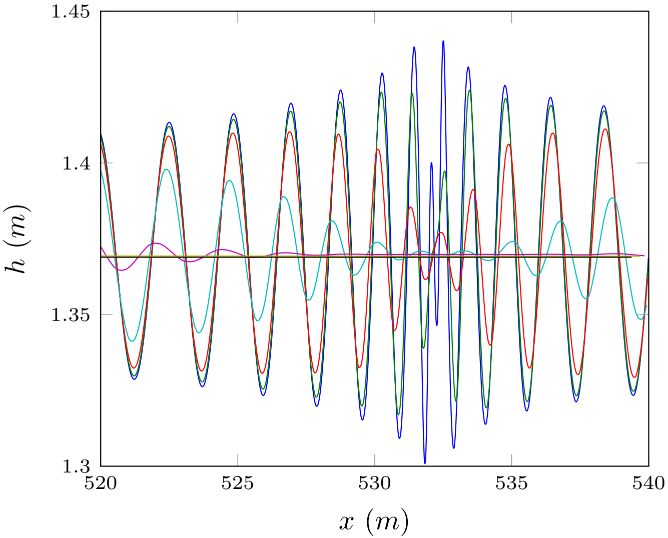


$\alpha = 1000$ and $\Delta x = \frac{10}{2^k}$m with $k = [3, 4, \cdots, 9]$

Varying Δx Scenarios

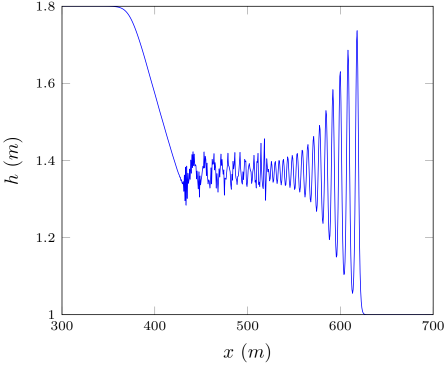


Varying Δx Scenarios



Finite Difference Schemes

Same results however very steep gradients cause problems



$\alpha = 5.0$ $\Delta x = \frac{10}{2^4}\text{m} = 0.625\text{m}$

Changing α

Can choose Δx such that as α gets larger our solutions converge to the same four scenarios as above.

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