# A Finite Element-Volume Method for the Serre Equations

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# Introduction

- Motivation

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- Method

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- Method
- Results

# Ocean Wave Hazards

- Tsunamis

# Sulawesi 2018 Tsunami

Figure: Sulawesi Tsunami (Indonesia, 2018).

# Ocean Wave Hazards

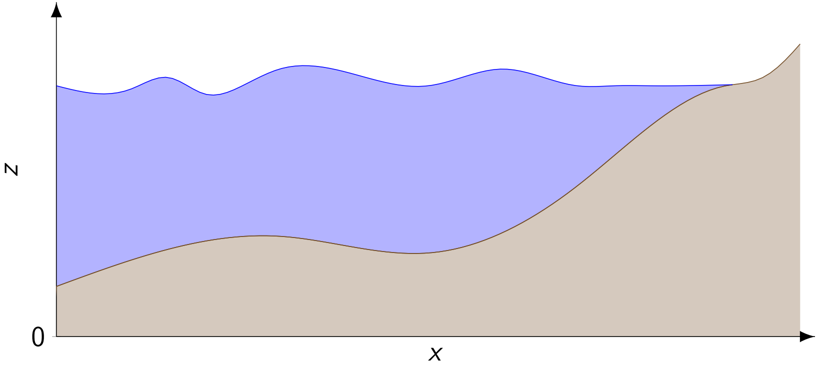
- Tsunamis
- Storm Surges

# Storm Surge of Hurricane Florence and Michael

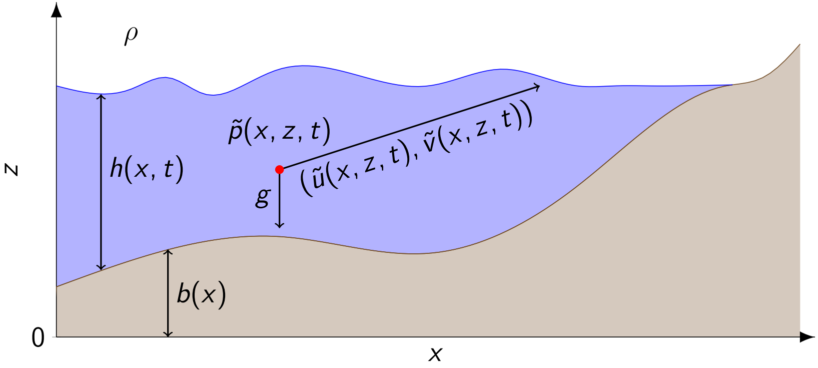
(a) Florence (U.S.A, 2018)


(b) Michael (U.S.A, 2018)

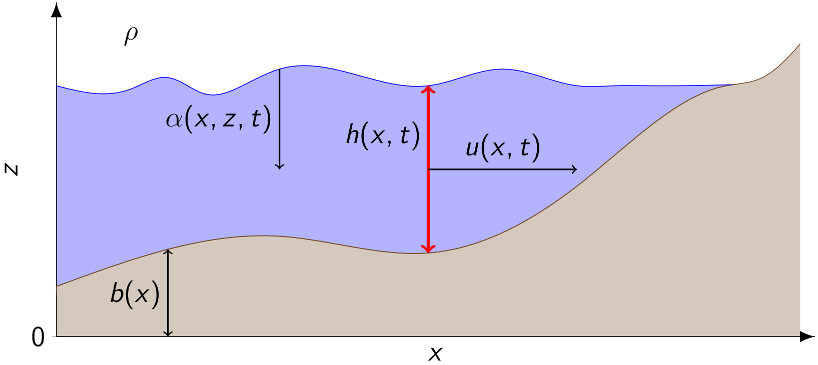
# Two Dimensional Scenario



# Navier-Stokes



# Model Simplification: Serre Equations



# Assumptions

| Quantity | Serre Equations |
|---|---|
| Particle: $\tilde{v}(x,z,t)$ | $u\frac{\partial b}{\partial x} - (h-\alpha)\frac{\partial b}{\partial x}$ |
| Particle: $\tilde{p}(x,z,t)$ | $g\rho\alpha + \rho\alpha\Psi + \frac{1}{2}\rho\alpha\left(2h-\alpha\right)\Phi$ |

where

$$\alpha(x,z,t) = (h(x,t) + b(x)) - z$$

and

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

# Equations

Mass:
$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Momentum:
$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right)$$

$$+ \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0.$$

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# Method

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Goal: Adapt Finite Volume Methods for the Serre Equations

# Finite Volume Method

- Conservation law form

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- Finite volume update

# Finite Volume Method

- Conservation law form

# Conservation Law Form

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left[f\left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n}\right)\right] + s\left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n}\right) = 0$$

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# Conservation Law Form

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3\left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2\frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.$$

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with

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

# Finite Volume Method

- Conservation law form
- Finite volume update

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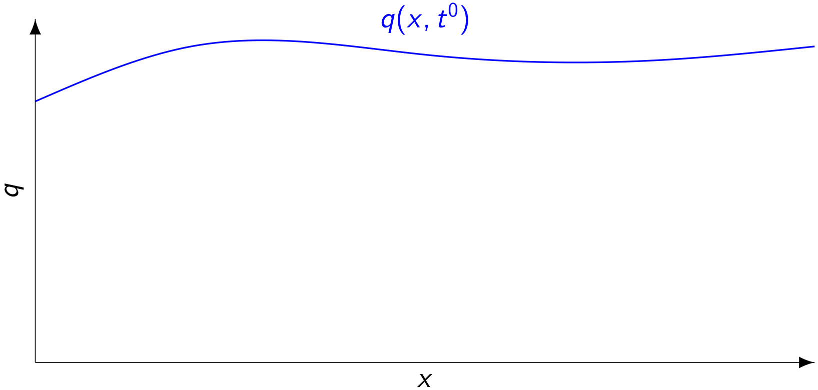
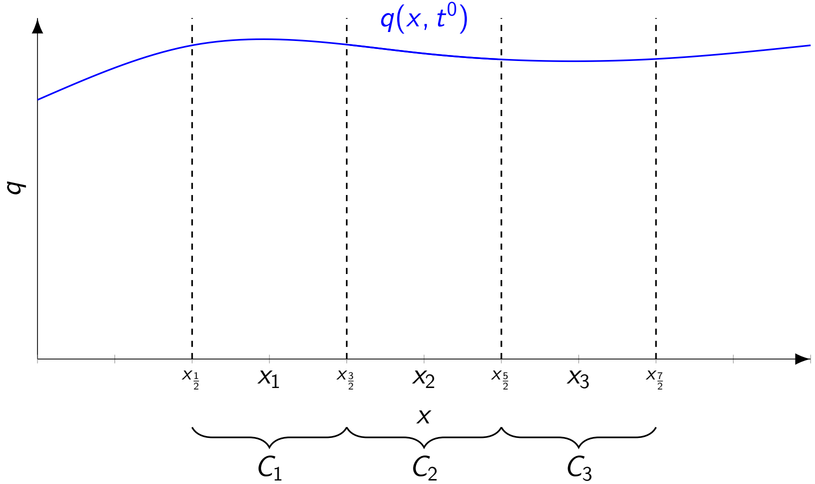
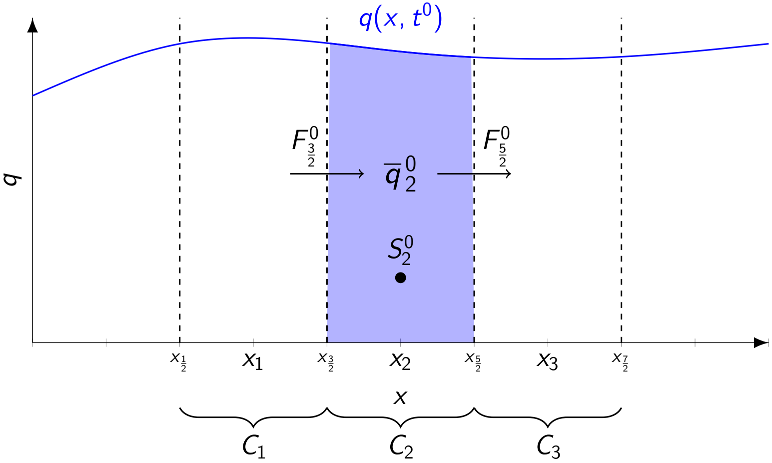
# Finite Volume Method



# Discretisation



# Update



$q(x, t^0)$
$q$
$F^0_{\frac{3}{2}}$
$\overline{q}^{\,0}_2$
$F^0_{\frac{5}{2}}$
$S^0_2$
$x_{\frac{1}{2}}$
$x_1$
$x_{\frac{3}{2}}$
$x_2$
$x_{\frac{5}{2}}$
$x_3$
$x_{\frac{7}{2}}$
$x$
$C_1$
$C_2$
$C_3$

# Finite Volume Method

$$\begin{tikzcd}[column sep=huge]
\overline{h}_j^n, \overline{G}_j^n \arrow[r, "\text{Reconstruction}"] & \hat{h}_j^n, \hat{G}_j^n \arrow[r, "\text{Flux Approximation}"] & F^n_{j\pm 1/2}, S_j^n \arrow[ll, bend left, "\text{Evolve}"]
\end{tikzcd}$$

# Require velocity to calculate flux

However to calculate the flux and source terms we require $u$

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$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left( uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[\frac{\partial u}{\partial x}\right]^2 + h^2 u \frac{\partial u}{\partial x}\frac{\partial b}{\partial x}\right) + \frac{1}{2}h^2 u \frac{\partial u}{\partial x}\frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial x^2} + gh\frac{\partial b}{\partial x} = 0.$$

# Method

$$
\begin{tikzcd}
\bar{h}^n_j, \bar{G}^n_j \arrow[r, "\text{Reconstruction}"] & \hat{h}^n_j, \hat{G}^n_j \arrow[r, "\text{Calculate Velocity}"] & \hat{u}^n_j \arrow[r, "\text{Flux Approximation}"] & F^n_{j\pm 1/2}, S^n_j \arrow[lll, bend left, "\text{Evolve}"]
\end{tikzcd}
$$

# Reconstruction

- Determines spatial order of accuracy

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Goal: Second-order accuracy

# Reconstruction Spaces

| Quantity | Number of spatial derivatives | Reconstructed functions |
|---|---|---|

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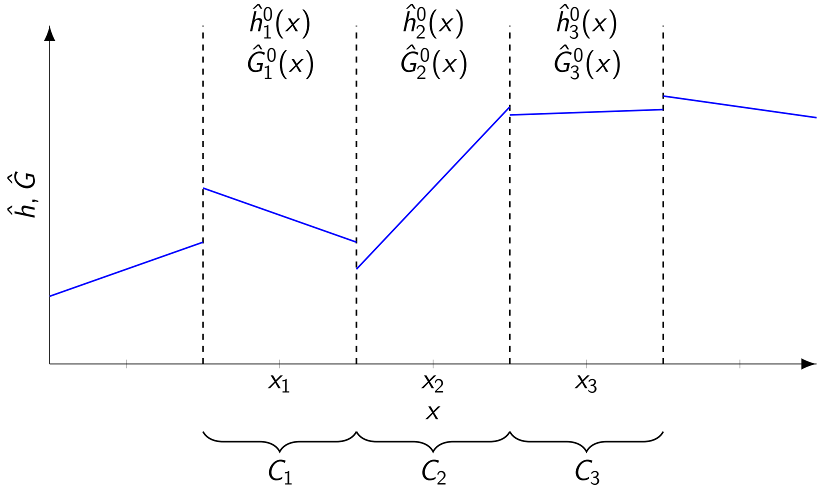
| Quantity | Number of spatial derivatives | Reconstructed functions |
|---|---|---|
| $h$ | zero | linear over cell, discontinuous at edges |
| $G$ | zero | linear over cell, discontinuous at edges |

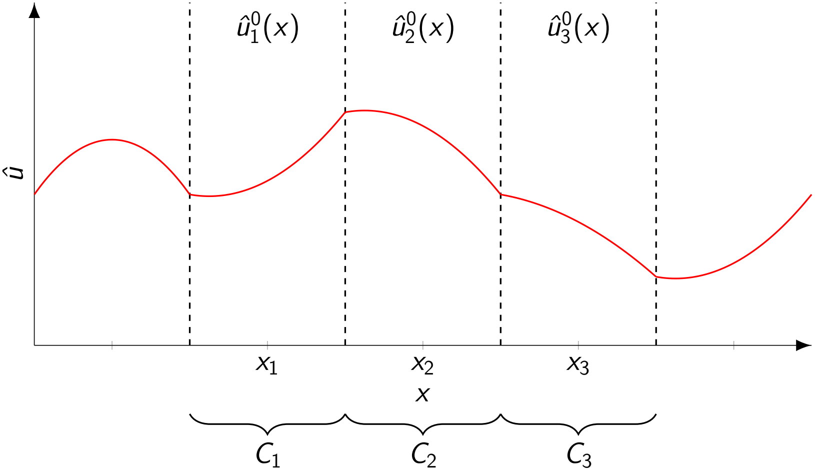
# $\hat{h}$, $\hat{G}$


$\hat{h}^0_1(x)$
$\hat{G}^0_1(x)$
$\hat{h}^0_2(x)$
$\hat{G}^0_2(x)$
$\hat{h}^0_3(x)$
$\hat{G}^0_3(x)$
$\hat{h}$, $\hat{G}$
$x_1$
$x_2$
$x_3$
$x$
$C_1$
$C_2$
$C_3$

# Reconstruction Spaces

| Quantity | Number of spatial derivatives | Reconstructed functions |
|---|---|---|
| $h$ | zero | linear over cell, discontinuous at edges |
| $G$ | zero | linear over cell, discontinuous at edges |
| $u$ | one | quadratic over cell, continuous at edges |

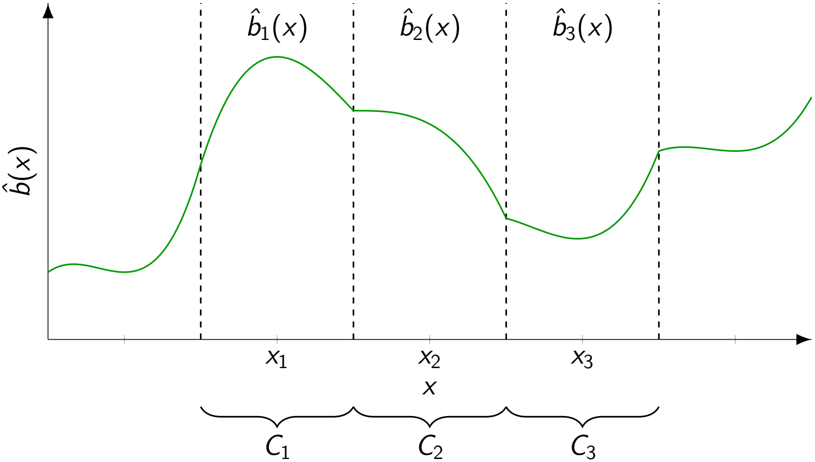
## $\hat{u}$



# Reconstruction Spaces

| Quantity | Number of spatial derivatives | Reconstructed functions |
|---|---|---|
| $h$ | zero | linear over cell, discontinuous at edges |
| $G$ | zero | linear over cell, discontinuous at edges |
| $u$ | one | quadratic over cell, continuous at edges |
| $b$ | two | cubic over cell, continuous at edges |

## $\hat{b}$



# Finite Element Calculation of Velocity

Finite Element Method to solve:

$$G = hu\left(1 + \frac{\partial h}{\partial x}\frac{\partial b}{\partial x} + \frac{1}{2}h\frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}h^3\frac{\partial u}{\partial x}\right).$$

for $u$ given $h$, $G$ and $b$

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for $u$ given $h$, $G$ and $b$

Solves the weak form replacing all quantities with their reconstructions $\hat{h}$, $\hat{G}$ and $\hat{b}$ to get $\hat{u}$

# Method

$$
\begin{tikzcd}[column sep=huge]
\overline{h}_j^n, \overline{G}_j^n \arrow[r, "\text{Reconstruction}"] & \hat{h}_j^n, \hat{G}_j^n \arrow[r, "\text{Calculate Velocity}"] & \hat{u}_j^n \arrow[r, "\text{Flux Approximation}"] & F_{j\pm1/2}^n, S_j^n \arrow[lll, bend left, "\text{Evolve}"]
\end{tikzcd}
$$

# Validation

- Analytic Solution

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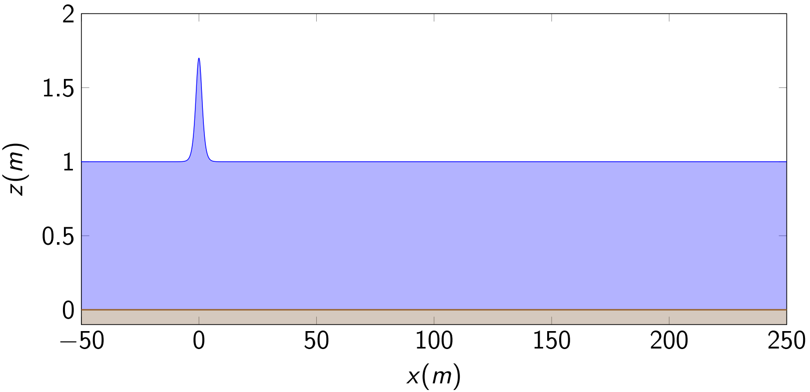
- Experimental Results

# Validation

- Analytic Solution

# Soliton Example



# Soliton Equations

$$h(x,t) = a_0 + a_1 \text{sech}\left(\kappa\left(x - ct\right)\right),$$

$$u(x,t) = c\left(1 - \frac{a_0}{h(x,t)}\right),$$
$$b(x) = 0$$

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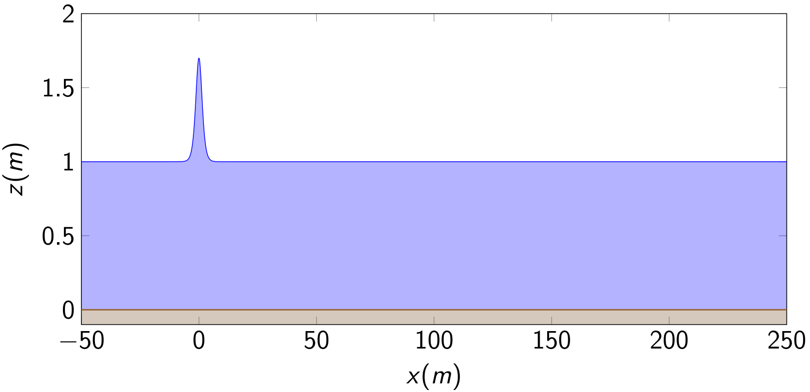
$$b(x) = 0$$

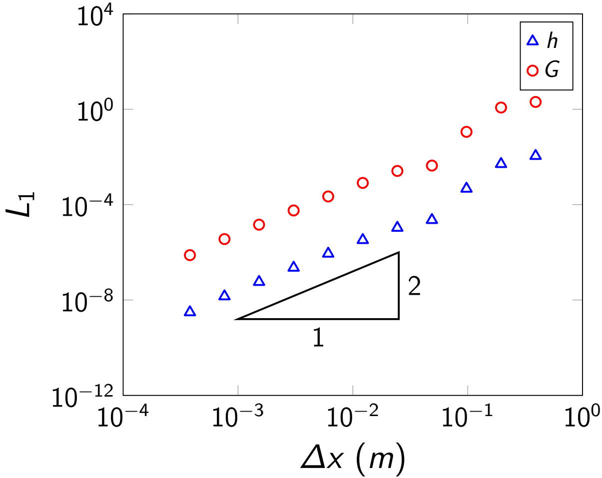
$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{(a_0 + a_1)}},$$

$$c = \sqrt{g\left(a_0 + a_1\right)}.$$

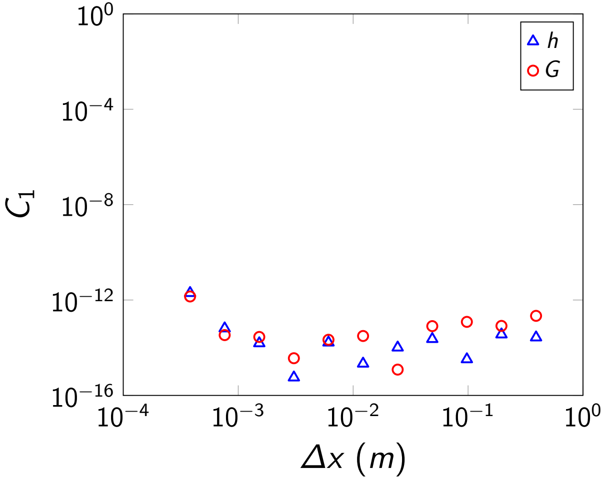
# Numerical Solution $a_0 = 1$, $a_1 = 0.7$



# Convergence



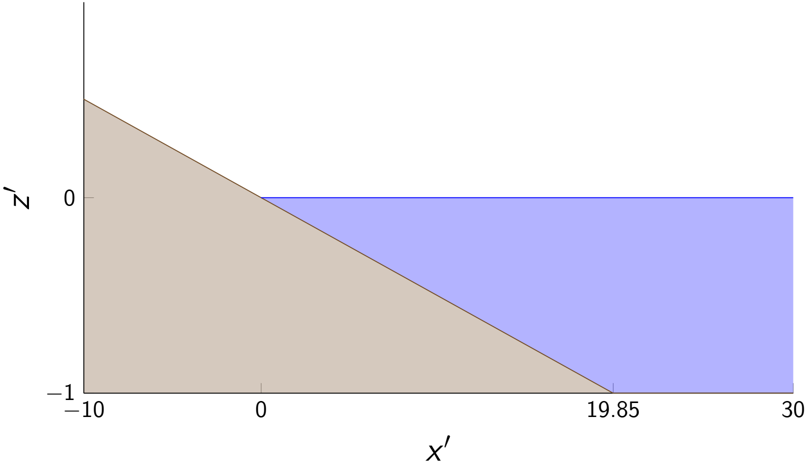
# Conservation



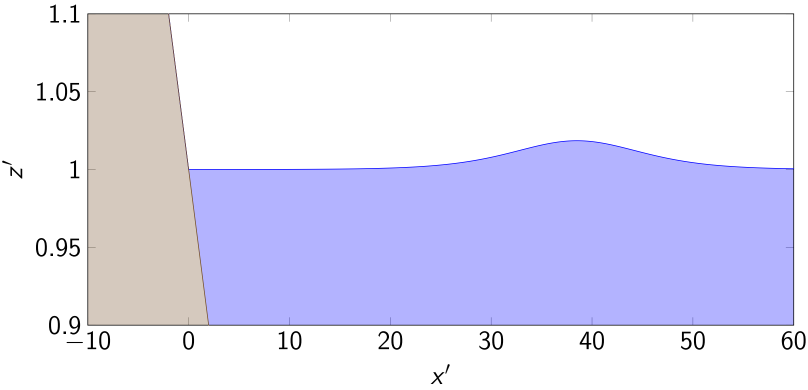
# Validation

- Analytic Solution
- Experimental Results

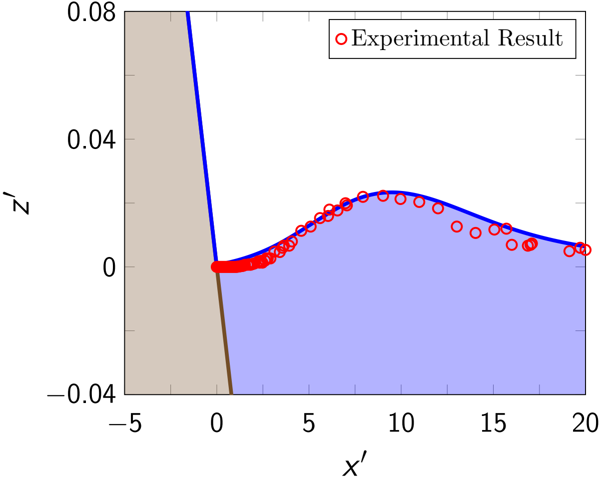
# Synolakis Experiment



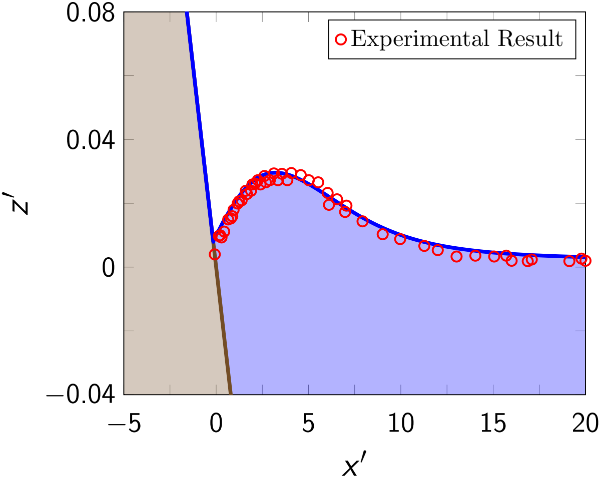
# Numerical Solution



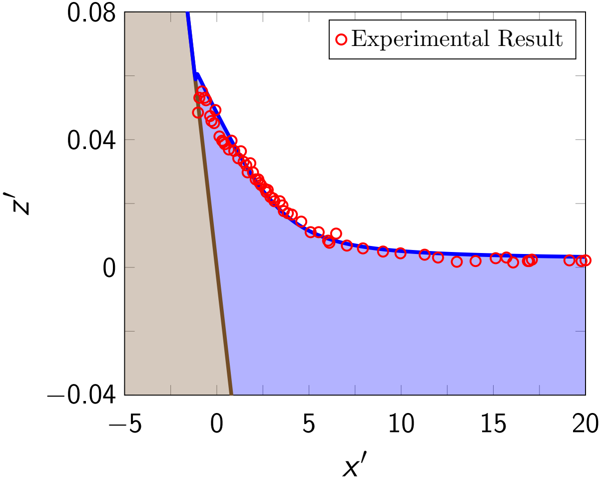
# Comparison $t = 30s$



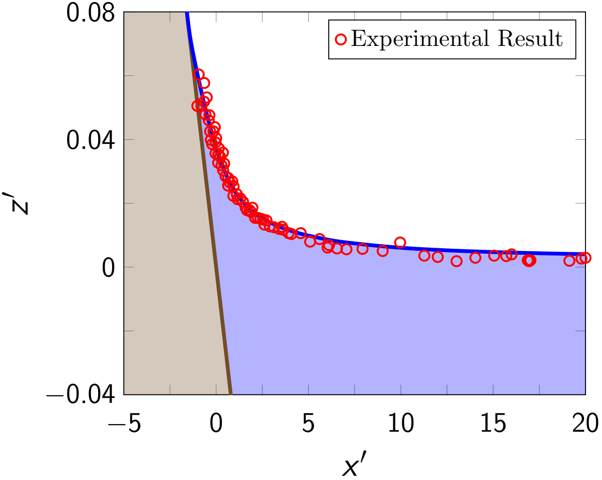
# Comparison $t = 40s$



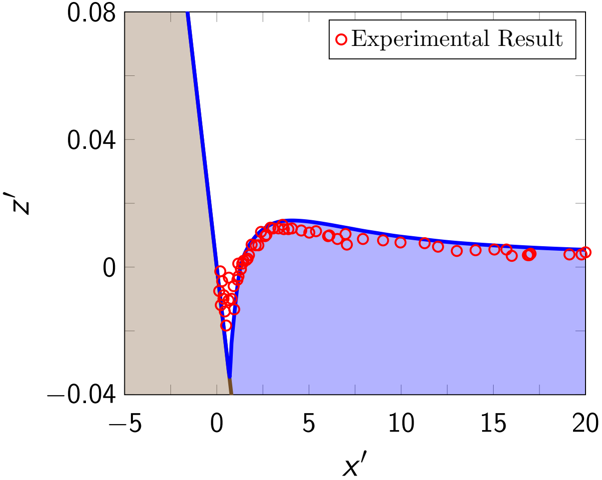
# Comparison $t = 50s$



# Comparison $t = 60s$



# Comparison $t = 70s$



# Thanks!