1 Elliptic Equation

The elliptic equation is

$$G = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

with bed terms it is

$$G = uh + u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right) - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

2 Finite Element

$$G = uh + u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right) - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv + uv\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right) - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gv\,dx = \int_\Omega uhv\,dx + \int_\Omega u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right)v\,dx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v\,dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx + \int_\Omega u\left(\frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right) + hb_x^2\right)v\,dx$$

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx + \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^2}{2}b_x\right)uv\,dx + \int_\Omega uhb_x^2 v\,dx$$

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx - \int_\Omega \frac{h^2}{2}b_x \frac{\partial}{\partial x}\left(uv\right)\, dx + \int_\Omega uhb_x^2 v\, dx$$

$$\int_\Omega Gvdx = \int_\Omega uhvdx + \int_\Omega \frac{h^3}{3}u_x v_x dx - \int_\Omega \frac{h^2}{2}b_x u_x v\, dx - \int_\Omega \frac{h^2}{2}b_x u v_x\, dx + \int_\Omega uhb_x^2 v\, dx$$

So importantly we just require that b_x is well behaved, and in particular it must be continuous

$$\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} Gvdx = \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhvdx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^3}{3}u_x v_x dx$$
$$-\sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2}b_x u_x v dx - \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} \frac{h^2}{2}b_x u v_x dx + \sum_j \int_{x_{j-1/2}}^{x_{j+3/2}} uhb_x^2 v dx$$

3 P1 FEM

Here are the terms for j for $v = \phi_{j+1/2}$

3.1 LHS

The LHS for each j is

$$\frac{\Delta x}{6}\left[G^+_{j-1/2} + 2G^-_{j+1/2} + 2G^+_{j+1/2} + G^-_{j+3/2}\right]$$

3.2 RHS

3.2.1 First Term

$$= \frac{\Delta x}{12}\Big[u_{j-1/2}h^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2} + u_{j+1/2}h^+_{j-1/2} + 3u_{j+1/2}h^-_{j+1/2}$$
$$+ 3u_{j+1/2}h^+_{j+1/2} + u_{j+1/2}h^-_{j-3/2} + u_{j+3/2}h^+_{j+1/2} + u_{j+3/2}h^-_{j-3/2}\Big] \quad (1)$$

Which breaking up into u terms gives

$$= \frac{\Delta x}{12}\Bigg[u_{j-1/2}\left(h^+_{j-1/2} + h^-_{j+1/2}\right) + u_{j+1/2}\left(h^+_{j-1/2} + 3h^-_{j+1/2} + 3h^+_{j+1/2} + h^-_{j-3/2}\right) + u_{j+3/2}\left(h^+_{j+1/2} + h^-_{j-3/2}\right)\Bigg] \quad (2)$$

3.2.2 Second Term

$$\frac{1}{12\Delta x}\Bigg[- u_{j-1/2}\Bigg[\left(h^+_{j-1/2}\right)^3 + \left(h^+_{j-1/2}\right)^2 h^-_{j+1/2} + h^+_{j-1/2}\left(h^-_{j+1/2}\right)^2 + \left(h^-_{j+1/2}\right)^3\Bigg] + u_{j+1/2}\Bigg[\left(h^+_{j-1/2}\right)^3 + \left(h^+_{j-1/2}\right)^2 h^-_{j+1/2} + h^+_{j-1/2}\left(h^-_{j+1/2}\right)^2 + \left(h^-_{j+1/2}\right)^3 + \left(h^+_{j+1/2}\right)^3 + 3\left(h^+_{j+1/2}\right)^2\left(h^-_{j-3/2}\right) + 3\left(h^+_{j+1/2}\right)\left(h^-_{j-3/2}\right)^2 + \left(h^-_{j-3/2}\right)^3\Bigg] - u_{j+3/2}\Bigg[\left(h^+_{j+1/2}\right)^3 + 3\left(h^+_{j+1/2}\right)^2\left(h^-_{j-3/2}\right) + 3\left(h^+_{j+1/2}\right)\left(h^-_{j-3/2}\right)^2 + \left(h^-_{j-3/2}\right)^3\Bigg]\Bigg] \quad (3)$$

3.2.3 Third Term

$$
\begin{aligned}
&\frac{1}{24\Delta x} \times \Bigg(\\
&b_{j-1/2}u_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b_{j-1/2}u_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b_{j-1/2}u_{j-1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&- b_{j-1/2}u_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} + 2b_{j-1/2}u_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} - 3b_{j-1/2}u_{j+1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&- b_{j+1/2}u_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} + 2b_{j+1/2}u_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} - 3b_{j+1/2}u_{j-1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&+ b_{j+1/2}u_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b_{j+1/2}u_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b_{j+1/2}u_{j+1/2}h^-_{j+1/2}h^-_{j+1/2} \\
&- 3b_{j+1/2}u_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b_{j+1/2}u_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b_{j+1/2}u_{j+1/2}h^-_{j+3/2}h^-_{j+3/2} \\
&+ 3b_{j+1/2}u_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} - 2b_{j+1/2}u_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} + b_{j+1/2}u_{j+3/2}h^-_{j+3/2}h^-_{j+3/2} \\
&+ 3b_{j+3/2}u_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} - 2b_{j+3/2}u_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} + b_{j+3/2}u_{j+1/2}h^-_{j+3/2}h^-_{j+3/2} \\
&- 3b_{j+3/2}u_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b_{j+3/2}u_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b_{j+3/2}u_{j+3/2}h^-_{j+3/2}h^-_{j+3/2} \Bigg)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&\frac{1}{24\Delta x} \times \Bigg(\\
&\quad u_{j-1/2}\left(b_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}\right) \\
&+ u_{j+1/2}\left(-b_{j-1/2}h^+_{j-1/2}h^+_{j-1/2} + 2b_{j-1/2}h^+_{j-1/2}h^-_{j+1/2} - 3b_{j-1/2}h^-_{j+1/2}h^-_{j+1/2}\right) \\
&+ u_{j-1/2}\left(-b_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} + 2b_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} - 3b_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}\right) \\
&\quad + u_{j+1/2}\left(b_{j+1/2}h^+_{j-1/2}h^+_{j-1/2} - 2b_{j+1/2}h^+_{j-1/2}h^-_{j+1/2} + 3b_{j+1/2}h^-_{j+1/2}h^-_{j+1/2}\right) \\
&+ u_{j+1/2}\left(-3b_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} + 2b_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} - b_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}\right) \\
&\quad + u_{j+3/2}\left(3b_{j+1/2}h^+_{j+1/2}h^+_{j+1/2} - 2b_{j+1/2}h^+_{j+1/2}h^-_{j+3/2} + b_{j+1/2}h^-_{j+3/2}h^-_{j+3/2}\right) \\
&\quad + u_{j+1/2}\left(3b_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} - 2b_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} + b_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}\right) \\
&+ u_{j+3/2}\left(-3b_{j+3/2}h^+_{j+1/2}h^+_{j+1/2} + 2b_{j+3/2}h^+_{j+1/2}h^-_{j+3/2} - b_{j+3/2}h^-_{j+3/2}h^-_{j+3/2}\right) \Bigg)
\end{aligned}
\tag{5}
$$

3.2.4 Fourth Term

$$\begin{aligned}
\frac{1}{24\Delta x}\Bigg(& - b_{j-1/2}u_{j-1/2}\Big[3h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + h^-_{j+1/2}h^-_{j+1/2}\Big] \\
& - b_{j-1/2}u_{j+1/2}\Big[h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + 3h^-_{j+1/2}h^-_{j+1/2}\Big] \\
& + b_{j+1/2}u_{j-1/2}\Big[3h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + h^-_{j+1/2}h^-_{j+1/2}\Big] \\
& + b_{j+1/2}u_{j+1/2}\Big[h^+_{j-1/2}h^+_{j-1/2} - 2h^+_{j-1/2}h^-_{j+1/2} + 3h^-_{j+1/2}h^-_{j+1/2}\Big] \\
& + b_{j+1/2}u_{j+1/2}\Big[-3h^+_{j+1/2}h^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2} - h^-_{j+3/2}h^-_{j+3/2}\Big] \\
& + b_{j+1/2}u_{j+3/2}\Big[h^+_{j+1/2}h^+_{j+1/2} - 2h^+_{j+1/2}h^-_{j+3/2} + 3h^-_{j+3/2}h^-_{j+3/2}\Big] \\
& - b_{j+3/2}u_{j+1/2}\Big[-3h^+_{j+1/2}h^+_{j+1/2} + 2h^+_{j+1/2}h^-_{j+3/2} - h^-_{j+3/2}h^-_{j+3/2}\Big] \\
& - b_{j+3/2}u_{j+3/2}\Big[h^+_{j+1/2}h^+_{j+1/2} - 2h^+_{j+1/2}h^-_{j+3/2} + 3h^-_{j+3/2}h^-_{j+3/2}\Big]\Bigg) \quad (6)
\end{aligned}$$

3.2.5 Fifth Term

$$\begin{aligned}
\frac{1}{12\Delta x}\Bigg\{ & \Big(b_{j-1/2}b_{j-1/2} - 2b_{j+1/2}b_{j-1/2} + b_{j+1/2}b_{j+1/2} - 2b_{j+3/2}b_{j+1/2} + b_{j+3/2}b_{j+3/2}\Big) \\
& \times \Big(u_{j-1/2}h^+_{j-1/2} + u_{j-1/2}h^-_{j+1/2} + u_{j+1/2}h^+_{j-1/2} + 3u_{j+1/2}h^-_{j+1/2} \\
& \quad + 3u_{j+1/2}h^+_{j+1/2} + u_{j+1/2}h^-_{j+3/2} + u_{j+3/2}h^+_{j+1/2} + u_{j+3/2}h^-_{j+3/2}\Big)\Bigg\}
\end{aligned}$$

$$\frac{1}{12\Delta x}\Bigg\{\Big(b_{j-1/2}b_{j-1/2}-2b_{j+1/2}b_{j-1/2}+b_{j+1/2}b_{j+1/2}-2b_{j+3/2}b_{j+1/2}+b_{j+3/2}b_{j+3/2}\Big)$$
$$\times\Big(u_{j-1/2}\left(h^+_{j-1/2}+h^-_{j+1/2}\right)$$
$$+u_{j+1/2}\left(h^+_{j-1/2}+3h^-_{j+1/2}+3h^+_{j+1/2}+h^-_{j+3/2}\right)$$
$$+u_{j+3/2}\left(h^+_{j+1/2}+h^-_{j+3/2}\right)\Big)\Bigg\}$$