1 Finite Element

$$Guh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)$$

To do so we begin by first multiplying by an arbitrary test function v so that

$$Gv = uhv - \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v$$

and then we integrate over the entire domain to get

$$\int_\Omega Gv dx = \int_\Omega uhv dx - \int_\Omega \frac{\partial}{\partial x}\left(\frac{h^3}{3}u_x\right)v dx$$

for all v

We then make use of integration by parts, with Dirchlet boundaries to get

$$\int_\Omega Gv dx = \int_\Omega uhv dx + \int_\Omega \frac{h^3}{3}u_x v_x dx$$

For u we are going to use $x_{j-1/2}$, x_j and $x_{j+1/2}$ as the nodes, which generate the basis functions $\phi_{j\pm1/2}$ and ϕ_j , which for us will be the space of continuous quadratic elements.

While for G and h we will choose basis functions w that are linear from $[x_{j-1/2}, x_{j+1/2}]$ but discontinuous at the edges.

We are going to look at the entire area where the basis functions are non-zero for $\phi_{j-1/2}$, ϕ_j and $\phi_{j+1/2}$. Which is the interval from $x_{j-3/2}$ to $x_{j+3/2}$. So we focus on the integrals on $[x_{j-3/2}, x_{j+3/2}]$ as

$$\begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}$$

$$\int_\Omega G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \int_\Omega uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \int_\Omega \frac{h^3}{3}u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

is

$$\sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx + \sum_j \int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx$$

$$x = \frac{3}{2}\xi\Delta x + x_j$$

Taking the derivatives we see
$dx = d\frac{3\xi}{2}\Delta x$, $\frac{dx}{d\xi} = \frac{3\Delta x}{2}$, $\frac{d\xi}{dx} = \frac{2}{3\Delta x}$.

We can describe the basis functions in the ξ space, where they are non-zero

2 Integrals

2.1 G

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx \qquad = \qquad \frac{3\Delta x}{2} \int_{-1}^{1} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi \quad (1)$$

We have

$$\int_{-1}^{1} G\phi_{j-1/2}d\xi = 0G^+_{j-3/2} + \frac{1}{9}G^-_{j-1/2} + \frac{1}{9}G^+_{j-1/2} + 0G^-_{j+1/2} = \frac{1}{9}G^-_{j-1/2} + \frac{1}{9}G^+_{j-1/2}$$

$$\int_{-1}^{1} G\phi_j d\xi = \frac{2}{9}G^+_{j-1/2} + \frac{2}{9}G^-_{j+1/2}$$

$$\int_{-1}^{1} G\phi_{j+1/2}d\xi = 0G^+_{j-1/2} + \frac{1}{9}G^-_{j+1/2} + \frac{1}{9}G^+_{j+1/2} + 0G^-_{j+3/2} = \frac{1}{9}G^-_{j+1/2} + \frac{1}{9}G^+_{j+1/2}$$

So

$$\int_{-1}^{1} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi \qquad = \qquad \begin{bmatrix} \frac{1}{9}G^-_{j-1/2} + \frac{1}{9}G^+_{j-1/2} \\ \frac{2}{9}G^+_{j-1/2} + \frac{2}{9}G^-_{j+1/2} \\ \frac{1}{9}G^-_{j+1/2} + \frac{1}{9}G^+_{j+1/2} \end{bmatrix}$$

So

$$\int_{x_{j-3/2}}^{x_{j+3/2}} G \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx = \frac{3\Delta x}{2} \begin{bmatrix} \frac{1}{9}G_{j-1/2}^- + \frac{1}{9}G_{j-1/2}^+ \\ \frac{2}{9}G_{j-1/2}^+ + \frac{2}{9}G_{j+1/2}^- \\ \frac{1}{9}G_{j+1/2}^- + \frac{1}{9}G_{j+1/2}^+ \end{bmatrix} = \frac{\Delta x}{6} \begin{bmatrix} G_{j-1/2}^- + G_{j-1/2}^+ \\ 2G_{j-1/2}^+ + 2G_{j+1/2}^- \\ G_{j+1/2}^- + G_{j+1/2}^+ \end{bmatrix}$$

$$(2)$$

2.2 uh

$$\int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx \qquad = \qquad \frac{3\Delta x}{2} \int_{-1}^{1} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi \quad (3)$$

We have

$$\int_{-1}^{1} uh\phi_{j-1/2}d\xi =$$

$$\left(-\frac{1}{90}h_{j-3/2}^+ - \frac{1}{90}h_{j-1/2}^- \right) u_{j-3/2}$$

$$+ \left(0h_{j-3/2}^+ + \frac{4}{90}h_{j-1/2}^- \right) u_{j-1}$$

$$+ \left(\frac{1}{90}h_{j-3/2}^+ + \frac{7}{90}h_{j-1/2}^- + \frac{7}{90}h_{j-1/2}^+ + \frac{1}{90}h_{j+1/2}^- \right) u_{j-1/2}$$

$$+ \left(\frac{4}{90}h_{j-1/2}^+ + 0h_{j+1/2}^- \right) u_j$$

$$+ \left(-\frac{1}{90}h_{j-1/2}^+ - \frac{1}{90}h_{j+1/2}^- \right) u_{j+1/2}$$

$$\int_{-1}^{1} u h \phi_j d\xi =$$

$$\left(\frac{4}{90} h_{j-1/2}^+ + 0 h_{j+1/2}^- \right) u_{j-1/2}$$

$$+ \left(\frac{16}{90} h_{j-1/2}^+ + \frac{16}{90} h_{j+1/2}^- \right) u_j$$

$$+ \left(0 h_{j-1/2}^+ + \frac{4}{90} h_{j+1/2}^- \right) u_{j+1/2}$$

$$=$$

$$\left(\frac{4}{90} h_{j-1/2}^+ \right) u_{j-1/2}$$

$$+ \left(\frac{16}{90} h_{j-1/2}^+ + \frac{16}{90} h_{j+1/2}^- \right) u_j$$

$$+ \left(\frac{4}{90} h_{j+1/2}^- \right) u_{j+1/2}$$

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$$\int_{-1}^{1} uh\phi_{j+1/2}d\xi =$$

$$\left(-\frac{1}{90}h_{j-1/2}^{+} - \frac{1}{90}h_{j+1/2}^{-}\right)u_{j-1/2}$$

$$+\left(0h_{j-1/2}^{+} + \frac{4}{90}h_{j+1/2}^{-}\right)u_{j}$$

$$+\left(\frac{1}{90}h_{j-1/2}^{+} + \frac{7}{90}h_{j+1/2}^{-} + \frac{7}{90}h_{j+1/2}^{+} + \frac{1}{90}h_{j+3/2}^{-}\right)u_{j+1/2}$$

$$+\left(\frac{4}{90}h_{j+1/2}^{+} + 0h_{j+3/2}^{-}\right)u_{j+1}$$

$$+\left(-\frac{1}{90}h_{j+1/2}^{+} - \frac{1}{90}h_{j+3/2}^{-}\right)u_{j+3/2} =$$

$$\left(-\frac{1}{90}h_{j-1/2}^{+} - \frac{1}{90}h_{j+1/2}^{-}\right)u_{j-1/2}$$

$$+\left(\frac{4}{90}h_{j+1/2}^{-}\right)u_{j}$$

$$+\left(\frac{1}{90}h_{j-1/2}^{+} + \frac{7}{90}h_{j+1/2}^{-} + \frac{7}{90}h_{j+1/2}^{+} + \frac{1}{90}h_{j+3/2}^{-}\right)u_{j+1/2}$$

$$+\left(\frac{4}{90}h_{j+1/2}^{+}\right)u_{j+1}$$

$$+\left(-\frac{1}{90}h_{j+1/2}^{+} - \frac{1}{90}h_{j+3/2}^{-}\right)u_{j+3/2}$$

$$\int_{-1}^{1} uh\phi_{j-1/2}d\xi =$$

$$\left(-\frac{1}{90}h_{j-3/2}^{+} - \frac{1}{90}h_{j-1/2}^{-}\right)u_{j-3/2}$$

$$+\left(\frac{4}{90}h_{j-1/2}^{-}\right)u_{j-1}$$

$$+\left(\frac{1}{90}h_{j-3/2}^{+} + \frac{7}{90}h_{j-1/2}^{-} + \frac{7}{90}h_{j-1/2}^{+} + \frac{1}{90}h_{j+1/2}^{-}\right)u_{j-1/2}$$

$$+\left(\frac{4}{90}h_{j-1/2}^{+}\right)u_{j}$$

$$+\left(-\frac{1}{90}h_{j-1/2}^{+} - \frac{1}{90}h_{j+1/2}^{-}\right)u_{j+1/2}$$

$$= \frac{1}{90}\begin{bmatrix} -h_{j-3/2}^{+} - h_{j-1/2}^{-} & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^{-} & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^{+} + 7h_{j-1/2}^{-} + 7h_{j-1/2}^{+} + h_{j+1/2}^{-} & 0 & 0 \\ 0 & 0 & 0 & 4h_{j-1/2}^{+} & 0 \\ 0 & 0 & 0 & 0 & -h_{j-1/2}^{+} - h_{j+1/2}^{-} \end{bmatrix}\begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_{j} \\ u_{j+1/2} \end{bmatrix}$$

$$\int_{-1}^{1} uh\phi_j d\xi =$$

$$\left(\frac{4}{90}h_{j-1/2}^+\right) u_{j-1/2}$$

$$+ \left(\frac{16}{90}h_{j-1/2}^+ + \frac{16}{90}h_{j+1/2}^-\right) u_j$$

$$+ \left(\frac{4}{90}h_{j+1/2}^-\right) u_{j+1/2}$$

$$= \frac{1}{90}\begin{bmatrix} 4h_{j-1/2}^+ & 0 & 0 \\ 0 & 16h_{j-1/2}^+ + 16h_{j+1/2}^- & 0 \\ 0 & 0 & 4h_{j+1/2}^- \end{bmatrix}\begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \end{bmatrix}$$

$$\int_{-1}^{1} uh\phi_{j+1/2}d\xi =$$

$$\left(-\frac{1}{90}h_{j-1/2}^+ - \frac{1}{90}h_{j+1/2}^-\right)u_{j-1/2}$$

$$+\left(\frac{4}{90}h_{j+1/2}^-\right)u_j$$

$$+\left(\frac{1}{90}h_{j-1/2}^+ + \frac{7}{90}h_{j+1/2}^- + \frac{7}{90}h_{j+1/2}^+ + \frac{1}{90}h_{j+3/2}^-\right)u_{j+1/2}$$

$$+\left(\frac{4}{90}h_{j+1/2}^+\right)u_{j+1}$$

$$+\left(-\frac{1}{90}h_{j+1/2}^+ - \frac{1}{90}h_{j+3/2}^-\right)u_{j+3/2}$$

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$$= \frac{1}{90}\begin{bmatrix} -h_{j-1/2}^+ - h_{j+1/2}^- & 0 & 0 & 0 & 0 \\ 0 & 4h_{j+1/2}^- & 0 & 0 & 0 \\ 0 & 0 & h_{j-1/2}^+ + 7h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 \\ 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 \\ 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - h_{j+3/2}^- \end{bmatrix}\begin{bmatrix} u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}$$

$$\int_{-1}^{1} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi =$$

$$\frac{1}{90} \begin{bmatrix} -h_{j-3/2}^+ - h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^+ + 7h_{j-1/2}^- + 10h_{j-1/2}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20h_{j-1/2}^+ + 20h_{j+1/2}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - \end{bmatrix}$$

$$\begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}$$

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$$\int_{-1}^{1} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} d\xi =$$

$$\frac{1}{90} \begin{bmatrix} -h_{j-3/2}^+ - h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4h_{j-1/2}^- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{j-3/2}^+ + 7h_{j-1/2}^- + 10h_{j-1/2}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20h_{j-1/2}^+ + 20h_{j+1/2}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10h_{j+1/2}^- + 7h_{j+1/2}^+ + h_{j+3/2}^- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4h_{j+1/2}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h_{j+1/2}^+ - h_{j+3/2}^- \end{bmatrix}$$

$$\begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}$$

$$\int_{x_{j-3/2}}^{x_{j+3/2}} uh \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix} dx =$$

$$\frac{\Delta x}{60} \left[\begin{array}{cccc} -h^+_{j-3/2} - h^-_{j-1/2} & 0 & 0 & 0 \\ 0 & 4h^-_{j-1/2} & 0 & 0 \\ 0 & 0 & h^+_{j-3/2} + 7h^-_{j-1/2} + 10h^+_{j-1/2} & 0 \\ 0 & 0 & 0 & 20h^+_{j-1/2} + 20h^-_{j+1/2} \end{array} \right.$$

$$\left. \begin{array}{ccccccc} & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10h^-_{j+1/2} + 7h^+_{j+1/2} + h^-_{j+3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4h^+_{j+1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h^+_{j+1/2} - h^-_{j+3/2} \end{array} \right]$$

$$\begin{bmatrix} u_{j-3/2} \\ u_{j-1} \\ u_{j-1/2} \\ u_j \\ u_{j+1/2} \\ u_{j+1} \\ u_{j+3/2} \end{bmatrix}$$

2.3 h3 u

First

$$(a + b + c + d)^3$$

where $ad = ac = bd = bc = 0$

is

$$(a + b + c + d)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + c^3 + 3c^2d + 3cd^2 + d^3$$

So that

$$\left(h^+_{j-3/2} + h^-_{j-1/2} + h^+_{j-1/2} + h^-_{j+1/2} \right)^3 =$$

$$\left(h^+_{j-3/2} \right)^3 + 3 \left(h^+_{j-3/2} \right)^2 \left(h^-_{j-1/2} \right) + 3 \left(h^+_{j-3/2} \right) \left(h^-_{j-1/2} \right)^2 + \left(h^-_{j-1/2} \right)^3$$

$$+ \left(h^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \left(h^-_{j+1/2} \right)^3$$

$$\int_{x_{j-3/2}}^{x_{j+3/2}} \frac{h^3}{3} u_x \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_x dx = \frac{2}{9\Delta x} \int_{-1}^{1} h^3 u_\xi \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_\xi d\xi$$

$$\int_{-1}^{1} h^3 u_\xi \phi_{j-1/2_\xi} d\xi =$$

$$\left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right.$$

$$\left. + \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-3/2}$$

$$+ \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right.$$

$$\left. + \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-1}$$

$$+ \left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right.$$

$$\left. + \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j-1/2}$$

$$\left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right.$$

$$\left. + \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_j$$

$$\left[\left(h_{j-3/2}^+ \right)^3 + 3 \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right.$$

$$\left. + \left(h_{j-1/2}^+ \right)^3 + 3 \left(h_{j-1/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + 3 \left(h_{j-1/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \left(h_{j-1/2}^- \right)^3 \right] u_{j+1/2}$$

Using where these values are not zero (messed up the subscripts for $h_{j+1/2}^-$)

$$
\begin{aligned}
&= \left[\left(h_{j-3/2}^+\right)^3 + 3\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + 3\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \left(h_{j-1/2}^-\right)^3 \right] u_{j-3/2} \\
&+ \left[\left(h_{j-3/2}^+\right)^3 + 3\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + 3\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \left(h_{j-1/2}^-\right)^3 \right] u_{j-1} \\
&+ \left[\left(h_{j-3/2}^+\right)^3 + 3\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + 3\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \left(h_{j-1/2}^-\right)^3 \right. \\
&+ \left. \left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right] u_{j-1/2} \\
&\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right] u_j \\
&\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right] u_{j+1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{17}{420}\left(h_{j-3/2}^+\right)^3 + \frac{3}{105}\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + \frac{3}{140}\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \frac{8}{105}\left(h_{j-1/2}^-\right)^3 \right] u_{j-3/2} \\
&+ \left[-\frac{1}{21}\left(h_{j-3/2}^+\right)^3 - \frac{6}{105}\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) - \frac{3}{21}\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 - \frac{44}{105}\left(h_{j-1/2}^-\right)^3 \right] u_{j-1} \\
&+ \left[\frac{1}{140}\left(h_{j-3/2}^+\right)^3 + 3\frac{1}{105}\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + 3\frac{17}{420}\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \frac{12}{35}\left(h_{j-1/2}^-\right)^3 \right. \\
&- \left. \frac{12}{35}\left(h_{j-1/2}^+\right)^3 - 3\frac{17}{420}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - 3\frac{1}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{1}{140}\left(h_{j+1/2}^-\right)^3 \right] u_{j-1/2} \\
&\left[\frac{44}{105}\left(h_{j-1/2}^+\right)^3 + 3\frac{1}{21}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + 3\frac{2}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \frac{1}{21}\left(h_{j+1/2}^-\right)^3 \right] u_j \\
&\left[-\frac{8}{105}\left(h_{j-1/2}^+\right)^3 - \frac{3}{140}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - \frac{3}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{17}{420}\left(h_{j+1/2}^-\right)^3 \right] u_{j+1/2}
\end{aligned}
$$

$$= \left[\frac{17}{420} \left(h^+_{j-3/2} \right)^3 + \frac{3}{105} \left(h^+_{j-3/2} \right)^2 \left(h^-_{j-1/2} \right) + \frac{3}{140} \left(h^+_{j-3/2} \right) \left(h^-_{j-1/2} \right)^2 + \frac{8}{105} \left(h^-_{j-1/2} \right)^3 \right] u_{j-3/2}$$

$$+ \left[-\frac{1}{21} \left(h^+_{j-3/2} \right)^3 - \frac{6}{105} \left(h^+_{j-3/2} \right)^2 \left(h^-_{j-1/2} \right) - \frac{3}{21} \left(h^+_{j-3/2} \right) \left(h^-_{j-1/2} \right)^2 - \frac{44}{105} \left(h^-_{j-1/2} \right)^3 \right] u_{j-1}$$

$$+ \left[\frac{1}{140} \left(h^+_{j-3/2} \right)^3 + \frac{3}{105} \left(h^+_{j-3/2} \right)^2 \left(h^-_{j-1/2} \right) + \frac{51}{420} \left(h^+_{j-3/2} \right) \left(h^-_{j-1/2} \right)^2 + \frac{12}{35} \left(h^-_{j-1/2} \right)^3 \right.$$

$$\left. -\frac{12}{35} \left(h^+_{j-1/2} \right)^3 - \frac{51}{420} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{3}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{1}{140} \left(h^-_{j+1/2} \right)^3 \right] u_{j-1/2}$$

$$\left[\frac{44}{105} \left(h^+_{j-1/2} \right)^3 + \frac{3}{21} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + \frac{6}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \frac{1}{21} \left(h^-_{j+1/2} \right)^3 \right] u_j$$

$$\left[-\frac{8}{105} \left(h^+_{j-1/2} \right)^3 - \frac{3}{140} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{3}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{17}{420} \left(h^-_{j+1/2} \right)^3 \right] u_{j+1/2}$$

$$\int_{-1}^{1} h^3 u_\xi \phi_{j\xi} d\xi =$$

$$\left(\left(h^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \left(h^-_{j+1/2} \right)^3 \right) u_{j-1/2}$$

$$+ \left(\left(h^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \left(h^-_{j+1/2} \right)^3 \right) u_j$$

$$\left(\left(h^+_{j-1/2} \right)^3 + 3 \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + 3 \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \left(h^-_{j+1/2} \right)^3 \right) u_{j+1/2}$$

$$\int_{-1}^{1} h^3 u_\xi \phi_{j\xi} d\xi =$$

$$\left(-\frac{26}{105} \left(h^+_{j-1/2} \right)^3 - \frac{9}{35} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{3}{21} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{2}{105} \left(h^-_{j+1/2} \right)^3 \right) u_{j-1/2}$$

$$+ \left(\frac{8}{35} \left(h^+_{j-1/2} \right)^3 + \frac{12}{105} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) - \frac{12}{105} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 - \frac{8}{35} \left(h^-_{j+1/2} \right)^3 \right) u_j$$

$$\left(\frac{2}{105} \left(h^+_{j-1/2} \right)^3 + \frac{3}{21} \left(h^+_{j-1/2} \right)^2 \left(h^-_{j+1/2} \right) + \frac{9}{35} \left(h^+_{j-1/2} \right) \left(h^-_{j+1/2} \right)^2 + \frac{26}{105} \left(h^-_{j+1/2} \right)^3 \right) u_{j+1/2}$$

$$\int_{-1}^{1} h^3 u_\xi \phi_{j+1/2_\xi} d\xi =$$

$$\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2 \left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right.$$

$$\left. +\left(h_{j+1/2}^+\right)^3 +3\left(h_{j+1/2}^+\right)^2 \left(h_{j+3/2}^-\right) +3\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 +\left(h_{j+3/2}^-\right)^3 \right] u_{j-1/2}$$

$$+\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2 \left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right.$$

$$\left. +\left(h_{j+1/2}^+\right)^3 + 3\left(h_{j+1/2}^+\right)^2 \left(h_{j+3/2}^-\right) + 3\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 + \left(h_{j+3/2}^-\right)^3 \right] u_j$$

$$+\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2 \left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right.$$

$$\left. +\left(h_{j+1/2}^+\right)^3 +3\left(h_{j+1/2}^+\right)^2 \left(h_{j+3/2}^-\right) +3\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 +\left(h_{j+3/2}^-\right)^3 \right] u_{j+1/2}$$

$$+\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2 \left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right.$$

$$\left. +\left(h_{j+1/2}^+\right)^3 + 3\left(h_{j+1/2}^+\right)^2 \left(h_{j+3/2}^-\right) + 3\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 + \left(h_{j+3/2}^-\right)^3 \right] u_{j+1}$$

$$+\left[\left(h_{j-1/2}^+\right)^3 + 3\left(h_{j-1/2}^+\right)^2 \left(h_{j+1/2}^-\right) + 3\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \left(h_{j+1/2}^-\right)^3 \right.$$

$$\left. +\left(h_{j+1/2}^+\right)^3 +3\left(h_{j+1/2}^+\right)^2 \left(h_{j+3/2}^-\right) +3\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 +\left(h_{j+3/2}^-\right)^3 \right] u_{j+3/2}$$

$$\int_{-1}^{1} h^3 u_\xi \phi_{j+1/2_\xi} d\xi =$$

$$\left[\frac{17}{420} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{3}{140} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{8}{105} \left(h_{j+1/2}^- \right)^3 \right] u_{j-1/2}$$

$$+ \left[-\frac{1}{21} \left(h_{j-1/2}^+ \right)^3 - \frac{6}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{21} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{44}{105} \left(h_{j+1/2}^- \right)^3 \right] u_j$$

$$+ \left[\frac{1}{140} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{51}{420} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{12}{35} \left(h_{j+1/2}^- \right)^3 \right.$$

$$\left. -\frac{12}{35} \left(h_{j+1/2}^+ \right)^3 - \frac{51}{420} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) - \frac{3}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 - \frac{1}{140} \left(h_{j+3/2}^- \right)^3 \right] u_{j+1/2}$$

$$+ \left[\frac{44}{105} \left(h_{j+1/2}^+ \right)^3 + \frac{3}{21} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + \frac{6}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \frac{1}{21} \left(h_{j+3/2}^- \right)^3 \right] u_{j+1}$$

$$+ \left[-\frac{8}{105} \left(h_{j+1/2}^+ \right)^3 - \frac{3}{140} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) - \frac{3}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 - \frac{17}{420} \left(h_{j+3/2}^- \right)^3 \right] u_{j+3/2}$$

So we have

$$\int_{-1}^{1} h^3 u_\xi \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_\xi d\xi =$$

$$\left[\frac{17}{420}\left(h_{j-3/2}^+\right)^3 + \frac{3}{105}\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + \frac{3}{140}\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \frac{8}{105}\left(h_{j-1/2}^-\right)^3 \right] u_{j-3/2}$$

$$+ \left[-\frac{1}{21}\left(h_{j-3/2}^+\right)^3 - \frac{6}{105}\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) - \frac{3}{21}\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 - \frac{44}{105}\left(h_{j-1/2}^-\right)^3 \right] u_{j-1}$$

$$+ \left[\frac{1}{140}\left(h_{j-3/2}^+\right)^3 + \frac{3}{105}\left(h_{j-3/2}^+\right)^2\left(h_{j-1/2}^-\right) + \frac{51}{420}\left(h_{j-3/2}^+\right)\left(h_{j-1/2}^-\right)^2 + \frac{12}{35}\left(h_{j-1/2}^-\right)^3 \right.$$

$$\left. -\frac{12}{35}\left(h_{j-1/2}^+\right)^3 - \frac{51}{420}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - \frac{3}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{1}{140}\left(h_{j+1/2}^-\right)^3 \right] u_{j-1/2}$$

$$\left[\frac{44}{105}\left(h_{j-1/2}^+\right)^3 + \frac{3}{21}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + \frac{6}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \frac{1}{21}\left(h_{j+1/2}^-\right)^3 \right] u_j$$

$$\left[-\frac{8}{105}\left(h_{j-1/2}^+\right)^3 - \frac{3}{140}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - \frac{3}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{17}{420}\left(h_{j+1/2}^-\right)^3 \right] u_{j+1/2}$$

$$+ \left(-\frac{26}{105}\left(h_{j-1/2}^+\right)^3 - \frac{9}{35}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - \frac{3}{21}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{2}{105}\left(h_{j+1/2}^-\right)^3 \right) u_{j-1/2}$$

$$+ \left(\frac{8}{35}\left(h_{j-1/2}^+\right)^3 + \frac{12}{105}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - \frac{12}{105}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{8}{35}\left(h_{j+1/2}^-\right)^3 \right) u_j$$

$$\left(\frac{2}{105}\left(h_{j-1/2}^+\right)^3 + \frac{3}{21}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + \frac{9}{35}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \frac{26}{105}\left(h_{j+1/2}^-\right)^3 \right) u_{j+1/2}$$

$$+ \left[\frac{17}{420}\left(h_{j-1/2}^+\right)^3 + \frac{3}{105}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + \frac{3}{140}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \frac{8}{105}\left(h_{j+1/2}^-\right)^3 \right] u_{j-1/2}$$

$$+ \left[-\frac{1}{21}\left(h_{j-1/2}^+\right)^3 - \frac{6}{105}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) - \frac{3}{21}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 - \frac{44}{105}\left(h_{j+1/2}^-\right)^3 \right] u_j$$

$$+ \left[\frac{1}{140}\left(h_{j-1/2}^+\right)^3 + \frac{3}{105}\left(h_{j-1/2}^+\right)^2\left(h_{j+1/2}^-\right) + \frac{51}{420}\left(h_{j-1/2}^+\right)\left(h_{j+1/2}^-\right)^2 + \frac{12}{35}\left(h_{j+1/2}^-\right)^3 \right.$$

$$\left. -\frac{12}{35}\left(h_{j+1/2}^+\right)^3 - \frac{51}{420}\left(h_{j+1/2}^+\right)^2\left(h_{j+3/2}^-\right) - \frac{3}{105}\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 - \frac{1}{140}\left(h_{j+3/2}^-\right)^3 \right] u_{j+1/2}$$

$$+ \left[\frac{44}{105}\left(h_{j+1/2}^+\right)^3 + \frac{3}{21}\left(h_{j+1/2}^+\right)^2\left(h_{j+3/2}^-\right) + \frac{6}{105}\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 + \frac{1}{21}\left(h_{j+3/2}^-\right)^3 \right] u_{j+1}$$

$$+ \left[-\frac{8}{105}\left(h_{j+1/2}^+\right)^3 - \frac{3}{140}\left(h_{j+1/2}^+\right)^2\left(h_{j+3/2}^{18}\right) - \frac{3}{105}\left(h_{j+1/2}^+\right)\left(h_{j+3/2}^-\right)^2 - \frac{17}{420}\left(h_{j+3/2}^-\right)^3 \right] u_{j+3/2}$$

$$\int_{-1}^{1} h^3 u_\xi \begin{bmatrix} \phi_{j-1/2} \\ \phi_j \\ \phi_{j+1/2} \end{bmatrix}_\xi d\xi =$$

$$\left[\frac{17}{420} \left(h_{j-3/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + \frac{3}{140} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \frac{8}{105} \left(h_{j-1/2}^- \right)^3 \right] u_{j-3/2}$$

$$+ \left[-\frac{1}{21} \left(h_{j-3/2}^+ \right)^3 - \frac{6}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) - \frac{3}{21} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 - \frac{44}{105} \left(h_{j-1/2}^- \right)^3 \right] u_{j-1}$$

$$+ \left[\frac{1}{140} \left(h_{j-3/2}^+ \right)^3 + \frac{3}{105} \left(h_{j-3/2}^+ \right)^2 \left(h_{j-1/2}^- \right) + \frac{51}{420} \left(h_{j-3/2}^+ \right) \left(h_{j-1/2}^- \right)^2 + \frac{12}{35} \left(h_{j-1/2}^- \right)^3 \right.$$

$$\left. - \frac{11}{20} \left(h_{j-1/2}^+ \right)^3 - \frac{7}{20} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{3}{20} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{1}{20} \left(h_{j+1/2}^- \right)^3 \right] u_{j-1/2}$$

$$\left[\frac{3}{5} \left(h_{j-1/2}^+ \right)^3 + \frac{1}{5} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) - \frac{1}{5} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 - \frac{3}{5} \left(h_{j+1/2}^- \right)^3 \right] u_j$$

$$+ \left[-\frac{1}{20} \left(h_{j-1/2}^+ \right)^3 + \frac{3}{20} \left(h_{j-1/2}^+ \right)^2 \left(h_{j+1/2}^- \right) + \frac{7}{20} \left(h_{j-1/2}^+ \right) \left(h_{j+1/2}^- \right)^2 + \frac{11}{20} \left(h_{j+1/2}^- \right)^3 \right.$$

$$\left. - \frac{12}{35} \left(h_{j+1/2}^+ \right)^3 - \frac{51}{420} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) - \frac{3}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 - \frac{1}{140} \left(h_{j+3/2}^- \right)^3 \right] u_{j+1/2}$$

$$+ \left[\frac{44}{105} \left(h_{j+1/2}^+ \right)^3 + \frac{3}{21} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) + \frac{6}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 + \frac{1}{21} \left(h_{j+3/2}^- \right)^3 \right] u_{j+1}$$

$$+ \left[-\frac{8}{105} \left(h_{j+1/2}^+ \right)^3 - \frac{3}{140} \left(h_{j+1/2}^+ \right)^2 \left(h_{j+3/2}^- \right) - \frac{3}{105} \left(h_{j+1/2}^+ \right) \left(h_{j+3/2}^- \right)^2 - \frac{17}{420} \left(h_{j+3/2}^- \right)^3 \right] u_{j+3/2}$$