1 Linearised Equations

$$G = uh - \frac{h^3}{3} u_{xx}$$

$$\eta_t + hu_x = 0$$

$$hu_t - \frac{h^3}{3} u_{xxt} + gh\eta_x = 0$$

$$(G)_t + gh\eta_x = 0$$

2 Numerical Approximation

We investigate our numerical technique by adding in a fourier mode so $W_j = W_0 e^{i(vt + kx_j)}$, and rewriting the equations using our spatial discretisation

2.1 G

Analytic:

$$G_j = u_j h_j - (\frac{h_j^3}{3} u_{xx})_j$$

Numerical approximation, we used second order central differences so we replace the second derivative of u with this approximation to it So we get

$$G_j = u_j h_j - \frac{h_j^3}{3} \left(\frac{-u_{j+2} + 16u_{j+1} - 30u_j + 16u_{j-1} - u_{j-2}}{12\Delta x^2} \right)$$

$$G_j = u_j h_j - \frac{h_j^3}{3} u_j \left(\frac{-e^{2k\Delta x} + 16e^{k\Delta x} - 30 + 16e^{-k\Delta x} - e^{-2k\Delta x}}{12\Delta x^2} \right)$$

$$G_j = u_j h_0 - \frac{h_0^3}{3} u_j \left(\frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2} \right)$$

$$G_j = u_j \left(h_0 - \frac{h_0^3}{3} \left(\frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2} \right) \right)$$

We are dealing with time continuous variables so, we first take the derivative in time exactly for the Fourier nodes so that:

So what we have is something that depends on the order used to approximate $u_x x$, lets call it $\mathcal{C}_3$ Thus:

$$\mathcal{C}_3 = \frac{-2\cos(2k\Delta x) + 32\cos(k\Delta x) - 30}{12\Delta x^2}$$

$$G_j = u_j\left(h_0 - \frac{h_0^3}{3}\mathcal{C}_3\right)$$

Furthermore we will call this whole thing $\mathcal{G}_2$ So we have

$$\mathcal{G}_3 = \left(h_0 - \frac{h_0^3}{3}\mathcal{C}_3\right)$$

then

$$G_j = u_j\mathcal{G}_3$$

Now we move on to

$$\eta_t + hu_x = 0$$

our equations are time continuous so that:

$$\eta_t + hu_x = 0$$

$$iv\eta + hu_x = 0$$

next we approximate

our conservation equations of the form

$$q_t + [f(q)]_x = 0$$

by

$$q_t + \frac{1}{\Delta x}\left[F_{j+1/2} - F_{j-1/2}\right] = 0$$

where $F_{j\pm1/2}$ given by Kurganovs method. In this equation h is constant so $f(\eta, u) = hu$. We start Kurganovs method by doing a reconstruction, we start by doing a central differencing approximation to obtain that

we note that the result is something like

$$q^-_{j+1/2} = \bar{q}_j + \frac{1}{6}\left(\bar{q}_j - \bar{q}_{j-1}\right) + \frac{1}{3}\left(\bar{q}_{j+1} - \bar{q}_j\right)$$

$$q^+_{j+1/2} = \bar{q}_{j+1} - \frac{1}{3}\left(\bar{q}_{j+1} - \bar{q}_j\right) - \frac{1}{6}\left(\bar{q}_{j+2} - \bar{q}_{j+1}\right)$$

we need to convert aberages to midpoints which we do by the following formula:

$$q_j = \frac{-\bar{q}_{j+1} + 26\bar{q}_j - \bar{q}_{j-1}}{24}$$

$$q_j = \bar{q}_j \frac{-e^{ik\Delta x} + 26 - e^{-ik\Delta x}}{24}$$

$$q_j = \bar{q}_j \frac{26 - 2\cos\left(k\Delta x\right)}{24}$$

Defining

$$\mathcal{M}_3 = \frac{26 - 2\cos\left(k\Delta x\right)}{24}$$

$$q_j = \bar{q}_j \mathcal{M}_3$$

So we have

$$q^-_{j+1/2} = \mathcal{M}_3\left[q_j + \frac{1}{6}\left(q_j - q_{j-1}\right) + \frac{1}{3}\left(q_{j+1} - q_j\right)\right]$$

$$q^-_{j+1/2} = \mathcal{M}_3 q_j\left[1 + \frac{1}{6}\left(1 - e^{-ik\Delta x}\right) + \frac{1}{3}\left(e^{ik\Delta x} - 1\right)\right]$$

$$q^-_{j+1/2} = \mathcal{M}_3 q_j\left[\frac{5}{6} + \frac{1}{6}\left(-e^{-ik\Delta x}\right) + \frac{1}{3}\left(e^{ik\Delta x}\right)\right]$$

$$q^-_{j+1/2} = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right] q_j$$

So deifning

$$R^-_3 = \frac{\mathcal{M}_3}{6}\left[5 + -e^{-ik\Delta x} + 2e^{ik\Delta x}\right]$$

$$q^-_{j+1/2} = R^-_3 q_j$$

for plus

$$q^+_{j+1/2} = \mathcal{M}_3\left[q_{j+1} - \frac{1}{3}\left(q_{j+1} - q_j\right) - \frac{1}{6}\left(q_{j+2} - q_{j+1}\right)\right]$$

$$q^+_{j+1/2} = \mathcal{M}_3 \left[1 - \frac{1}{3}\left(1 - e^{-k\Delta x}\right) - \frac{1}{6}\left(e^{k\Delta x} - 1\right)\right] q_{j+1}$$

$$q^+_{j+1/2} = \mathcal{M}_3 \left[\frac{5}{6} + \frac{1}{3}e^{-k\Delta x} - \frac{1}{6}e^{k\Delta x}\right] q_{j+1}$$

$$q^+_{j+1/2} = \frac{\mathcal{M}_3}{6}\left[5 + 2e^{-k\Delta x} - e^{k\Delta x}\right] q_{j+1}$$

$$q^+_{j+1/2} = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right] q_j$$

Defining

$$R^+_3 = \frac{\mathcal{M}_3 e^{ik\Delta x}}{6}\left[5 + 2e^{-ik\Delta x} - e^{ik\Delta x}\right]$$

$$q^+_{j+1/2} = R^+_3 q_j$$

THIS IS ALL THE SAME NOW?

Next we have to use the wavespeeds, up to our linearisation assuming still water the velocities are zero so

$$a^-_{j+1/2} = -\sqrt{gh^-_{j+1/2}}$$

$$a^+_{j+1/2} = +\sqrt{gh^-_{j+1/2}}$$

We have that

$$F_{i+\frac{1}{2}} = \frac{a^+_{i+\frac{1}{2}} f\left(q^-_{i+\frac{1}{2}}\right) - a^-_{i+\frac{1}{2}} f\left(q^+_{i+\frac{1}{2}}\right)}{a^+_{i+\frac{1}{2}} - a^-_{i+\frac{1}{2}}} + \frac{a^+_{i+\frac{1}{2}}\, a^-_{i+\frac{1}{2}}}{a^+_{i+\frac{1}{2}} - a^-_{i+\frac{1}{2}}}\left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \quad (1)$$

$$F_{i+\frac{1}{2}} = \frac{\left(\sqrt{gh^-_{j+1/2}}\right) f\left(q^-_{i+\frac{1}{2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right) f\left(q^+_{i+\frac{1}{2}}\right)}{\left(\sqrt{gh^-_{j+1/2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right)} + \frac{\left(\sqrt{gh^-_{j+1/2}}\right)\left(-\sqrt{gh^-_{j+1/2}}\right)}{\left(+\sqrt{gh^-_{j+1/2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right)}\left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \quad (2)$$

$$F_{i+\frac{1}{2}} = \frac{\left(+\sqrt{gh^-_{j+1/2}}\right) f\left(q^-_{i+\frac{1}{2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right) f\left(q^+_{i+\frac{1}{2}}\right)}{2\sqrt{gh^-_{j+1/2}}} + \frac{\left(+\sqrt{gh^-_{j+1/2}}\right)\left(-\sqrt{gh^-_{j+1/2}}\right)}{2\sqrt{gh^-_{j+1/2}}}\left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \tag{3}$$

$$F_{i+\frac{1}{2}} = \frac{\left(+\sqrt{gh^-_{j+1/2}}\right) f\left(q^-_{i+\frac{1}{2}}\right) - \left(-\sqrt{gh^-_{j+1/2}}\right) f\left(q^+_{i+\frac{1}{2}}\right)}{2\sqrt{gh^-_{j+1/2}}} + \frac{-gh^-_{j+1/2}}{2\sqrt{gh^-_{j+1/2}}}\left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \tag{4}$$

$$F_{i+\frac{1}{2}} = \frac{f\left(q^-_{i+\frac{1}{2}}\right) + f\left(q^+_{i+\frac{1}{2}}\right)}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}}\right] \tag{5}$$

for eta this becomes

$$F_{i+\frac{1}{2}}(\eta) = \frac{hu^-_{i+\frac{1}{2}} + hu^+_{i+\frac{1}{2}}}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[\eta^+_{i+\frac{1}{2}} - \eta^-_{i+\frac{1}{2}}\right]$$

up to order the last term becomes

$$F_{i+\frac{1}{2}}(\eta) = \frac{hu^-_{i+\frac{1}{2}} + hu^+_{i+\frac{1}{2}}}{2} - \frac{\sqrt{gh}}{2}\left[\eta^+_{i+\frac{1}{2}} - \eta^-_{i+\frac{1}{2}}\right]$$

$$F_{i+\frac{1}{2}}(\eta) = \frac{h\mathcal{R}^- u_j + h\mathcal{R}^+ u_j}{2} - \frac{\sqrt{gh}}{2}\left[\mathcal{R}^+\eta_j - \mathcal{R}^-\eta_j\right]$$

$$F_{i+\frac{1}{2}}(\eta) = \frac{h\mathcal{R}^- + h\mathcal{R}^+}{2} u_j - \frac{\sqrt{gh}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]\eta_j$$

$$F_{i+\frac{1}{2}}(\eta) = F_2^{\eta,u} u_j + F_2^{\eta,\eta}\eta_j$$

where

$$F_2^{\eta,u} = \frac{h\mathcal{R}^- + h\mathcal{R}^+}{2}$$

$$F_2^{\eta,\eta} = -\frac{\sqrt{gh}}{2}\left[\mathcal{R}^+ - \mathcal{R}^-\right]$$

For G this becomes

$$F_{i+\frac{1}{2}}(G) = \frac{gh\eta^-_{i+\frac{1}{2}} + gh\eta^+_{i+\frac{1}{2}}}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[\mathcal{G}u^+_{i+\frac{1}{2}} - \mathcal{G}u^-_{i+\frac{1}{2}}\right] \quad (6)$$

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^-\eta_j + gh\mathcal{R}^+\eta_j}{2} - \frac{\sqrt{gh^-_{j+1/2}}}{2}\left[\mathcal{G}\mathcal{R}^+u_j - \mathcal{G}\mathcal{R}^-u_j\right] \quad (7)$$

up to order the last term becomes

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^-\eta_j + gh\mathcal{R}^+\eta_j}{2} - \frac{\sqrt{gh}}{2}\left[\mathcal{G}\mathcal{R}^+u_j - \mathcal{G}\mathcal{R}^-u_j\right] \quad (8)$$

$$F_{i+\frac{1}{2}}(G) = \frac{gh\mathcal{R}^- + gh\mathcal{R}^+}{2}\eta_j - \frac{\sqrt{gh}}{2}\left[\mathcal{G}\mathcal{R}^+ - \mathcal{G}\mathcal{R}^-\right]u_j \quad (9)$$

$$F_{i+\frac{1}{2}}(G) = F_2^{G,u}u_j + F_2^{G,\eta}\eta_j$$

where

$$F_2^{G,u} = -\frac{\sqrt{gh}}{2}\left[\mathcal{G}\mathcal{R}^+ - \mathcal{G}\mathcal{R}^-\right]$$

$$F_2^{G,\eta} = \frac{gh\mathcal{R}^- + gh\mathcal{R}^+}{2}$$

So we have

$$ivG_j + gh\eta_x = 0$$

$$iv\eta + hu_x = 0$$

become

$$iv\eta + \frac{1}{\Delta x}\left[(1-e^{-ik\Delta x})F_2^{\eta,u}u_j + (1-e^{-ik\Delta x})F_2^{\eta,\eta}\eta_j\right] = 0$$

We let $\mathcal{D} = (1-e^{-ik\Delta x})$

$$iv\eta_j + \frac{1}{\Delta x}\left[\mathcal{D}F_2^{\eta,u}u_j + \mathcal{D}F_2^{\eta,\eta}\eta_j\right] = 0$$

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}F_2^{\eta,\eta}\right]\eta_j + \frac{1}{\Delta x}\left[\mathcal{D}F_2^{\eta,u}\right]u_j = 0$$

Similarly for G

$$\mathcal{G}\left[iv + \frac{1}{\Delta x}\mathcal{D}F_2^{G,u}\right]u_j + \frac{1}{\Delta x}\left[\mathcal{D}F_2^{G,\eta}\right]\eta_j = 0$$

$$\left[\begin{array}{cc} iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)} & \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,u)} \\ \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,\eta)} & iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} \end{array}\right]\left[\begin{array}{c} \eta_j \\ u_j \end{array}\right] = 0$$

for a nontrivial solution

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]\left[iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}\right] - \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,u)}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,\eta)}$$

$$\left[iv + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]\left[iv\mathcal{G} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)}\right] - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)}$$

$$-v^2\mathcal{G} + iv\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + iv\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)} + \frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2 + i\left[\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]v + \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{G}\mathcal{F}_2^{(\eta,\eta)}\mathcal{F}_2^{(G,u)} - \frac{1}{\Delta x^2}\mathcal{D}^2\mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)} = 0$$

$$-\mathcal{G}v^2 + i\left[\mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(G,u)} + \mathcal{G}\frac{1}{\Delta x}\mathcal{D}\mathcal{F}_2^{(\eta,\eta)}\right]v + \frac{1}{\Delta x^2}\mathcal{D}^2\left[\mathcal{G}\mathcal{F}_2^{(\eta,\eta)}\mathcal{F}_2^{(G,u)} - \mathcal{F}_2^{(\eta,u)}\mathcal{F}_2^{(G,\eta)}\right] = 0$$