Water and Dispersive Shocks

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Outline

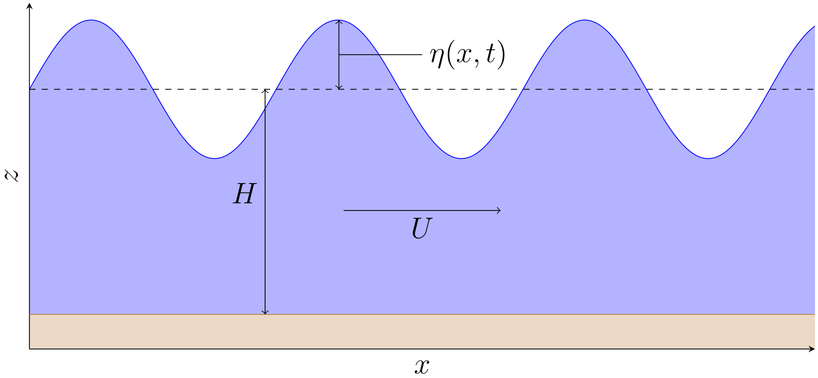
- Motivation: Water
- Nonlinear Shocks: Shallow Water Wave Equations
- Dispersive Shocks: Serre Equations

Water

(a) Sydney Harbour

(b) Lake Burley Griffin

Waves

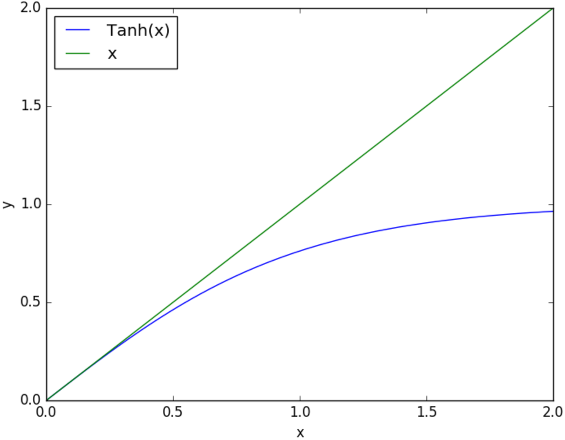


Linear Wave Speed

$$v_p = U \pm \sqrt{\frac{g}{k}\tanh\left(kH\right)}$$

- g is acceleration due to gravity $\left(m/s^2\right)$
- k is the wavenumber $\left(rad/m\right)$

Since $\tanh(x) \leq x$



Wave Speed Bounds

$$v_p = U \pm \sqrt{\frac{g}{k}\tanh\left(kH\right)}$$

$$U - \sqrt{gH} \leq v_p \leq U + \sqrt{gH}$$

Limits how fast information can propagate in water

Shocks

Due to bounds on information propagation we observe shocks in water.

Two main examples:

- Bores
- Hydraulic Jumps

Bores

Figure: Tidal Bore (Nova Scotia)

Hydraulic Jump

Figure: Hydraulic Jump (Mississippi)

Shallow Water Wave Equations

Mass: $$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Momentum: $$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) + \frac{\partial b}{\partial x}(gh) = 0.$$

Characteristics (Wave Speed)

$$v_p = U \pm \sqrt{gH}$$

So waves in SWWE travel at the fastest possible wave speed in water

Recall for water

$$v_p = U \pm \sqrt{\frac{g}{k}\tanh\left(kH\right)}$$

Replacing tanh with its Taylor series we get

$$v_p = U \pm \sqrt{gH - \frac{g}{3}H^3k^2 + \mathcal{O}\left(k^4H^5\right)}$$

Bore (Dam Break)

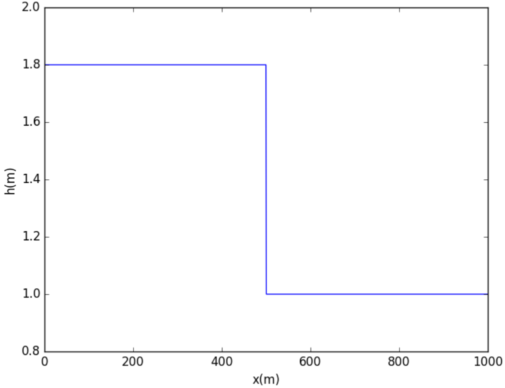


Figure: Dam-break problem

Bore (Dam Break)

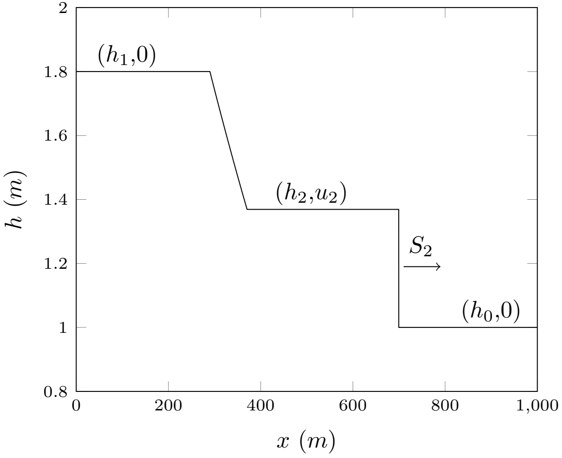


Figure: Dam-break problem Solution at 30s

Hydraulic Jump

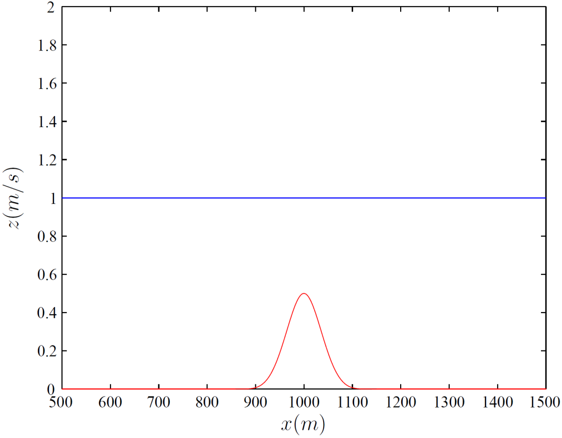


Figure: Flow over bump $u = 2m/s$ at $t = 0s$

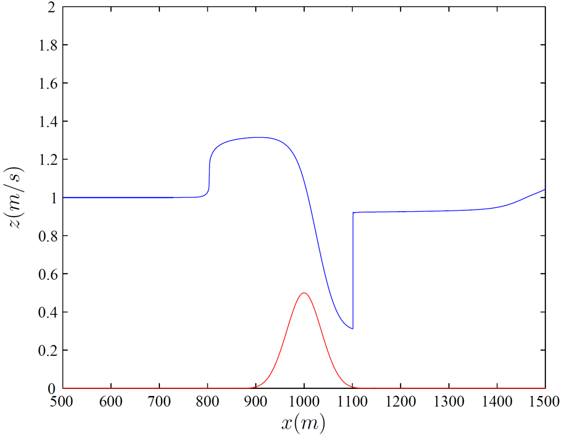


Figure: Flow over bump at $t = 100s$

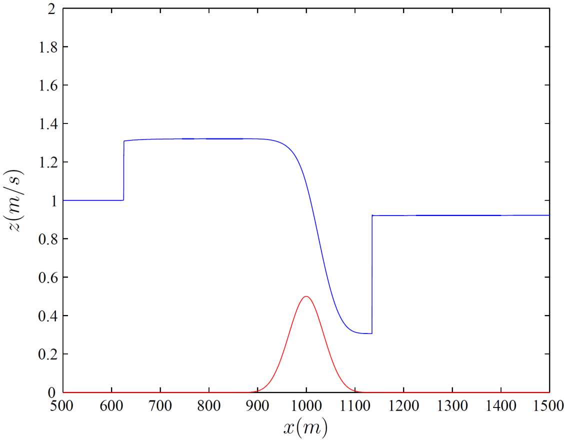


Figure: Flow over bump at $t = 200s$

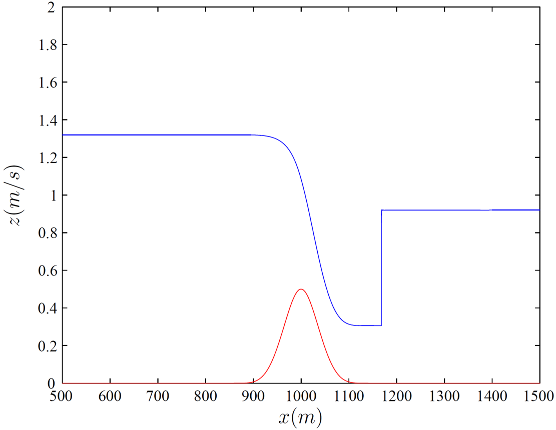


Figure: Flow over bump at $t = 300s$

Serre Equations

Mass:
$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

Momentum:
$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2}\Psi + \frac{h^3}{3}\Phi\right)$$

$$+ \frac{\partial b}{\partial x}\left(gh + h\Psi + \frac{h^2}{2}\Phi\right) = 0.$$

$$\Psi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u^2\frac{\partial^2 b}{\partial x^2}, \quad \Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Wave Speeds

$$v_p = U \pm \sqrt{gH}\sqrt{\frac{3}{k^2H^2+3}}$$

Taylor series of this is

$$v_p = U \pm \sqrt{gH - \frac{g}{3}H^3k^2 + \mathcal{O}\left(k^4H^5\right)}$$

Recall for water the Taylor series was

$$v_p = U \pm \sqrt{gH - \frac{g}{3}H^3k^2 + \mathcal{O}\left(k^4H^5\right)}$$

Result

Not only do we get the shocks due to the bounds on the wave speeds but we also get some of that internal structure we see in the real world

Dam Break

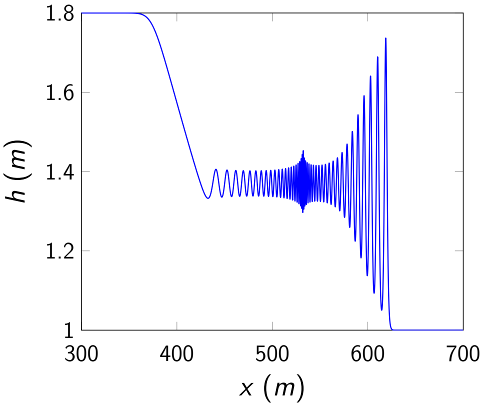


Figure: Dam-break problem solution at 30s

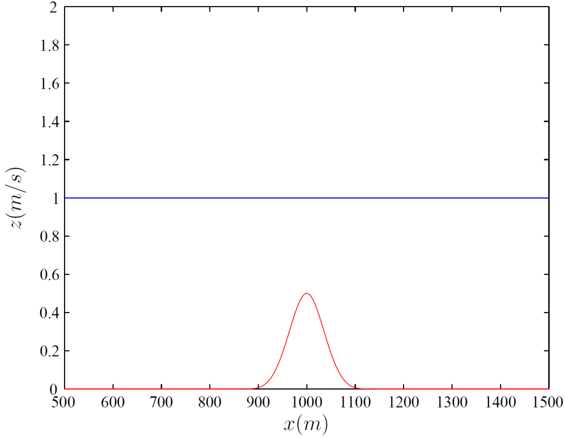
Wave Speed Effects

$$v_p = U \pm \sqrt{gH}\sqrt{\frac{3}{k^2H^2+3}}$$

$$v_p^- = U - \sqrt{gH}\sqrt{\frac{3}{k^2H^2+3}} \leq U \leq U + \sqrt{gH}\sqrt{\frac{3}{k^2H^2+3}} = v_p^+$$

Evolution

Hydraulic Jump



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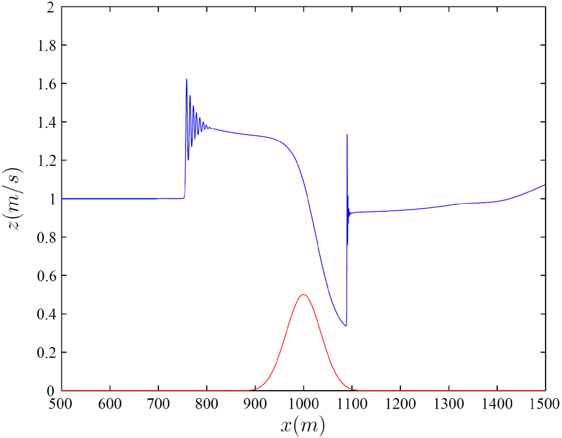

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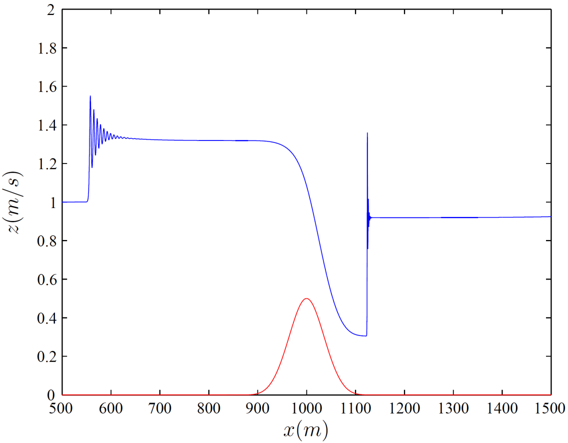

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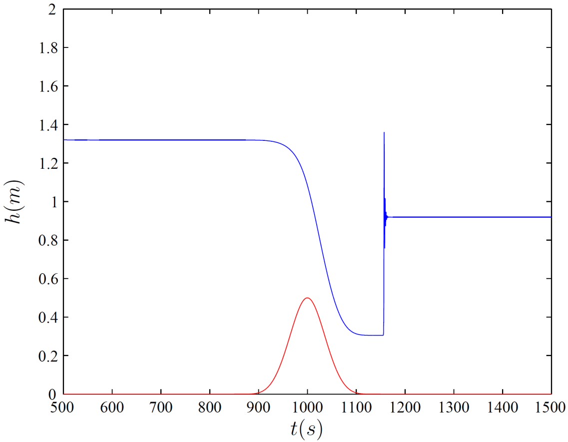

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