

Dispersive Shock Waves of the Serre equations

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Introduction

Summary of our paper “Behaviour of the Serre equations in the presence of steep gradients revisited” (Wave Motion Volume 76, January 2018).

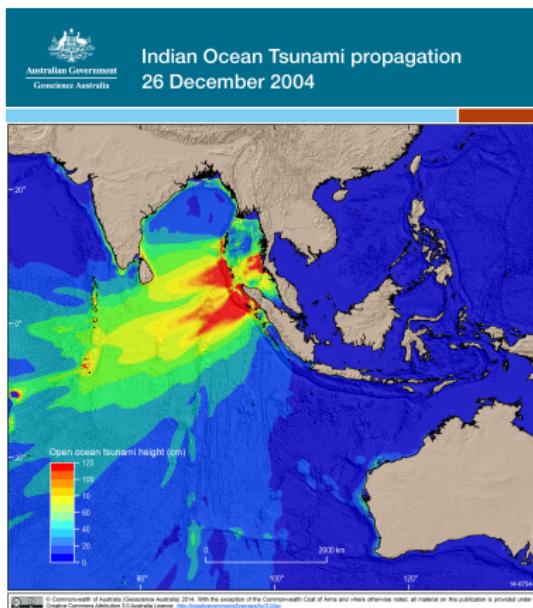
Outline of Presentation:

- ▶ Motivation.
- ▶ Serre Equations.
- ▶ Dispersive Shock Waves.
- ▶ Investigation.
- ▶ Results.

Our Background

- ▶ Interest : Numerical methods for water waves focusing on ocean hazards.

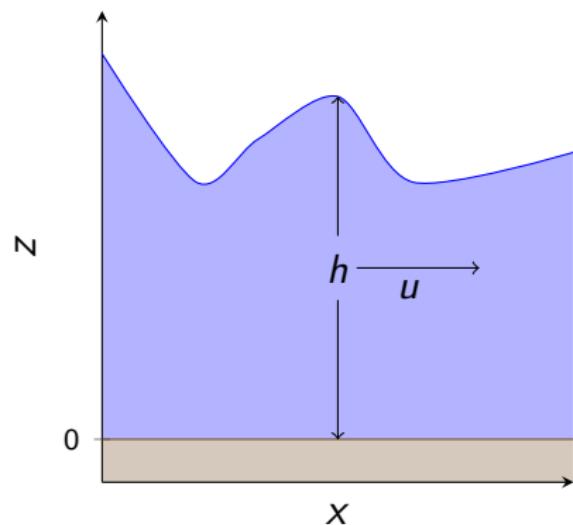
Indian ocean tsunami



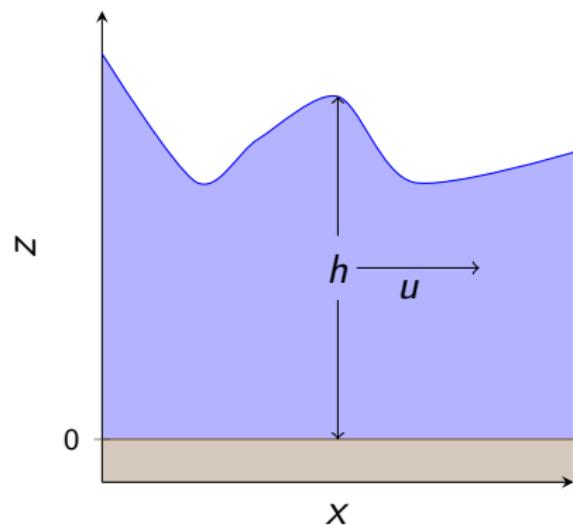
Our Background

- ▶ Interest : Numerical methods for water waves focusing on ocean hazards.
- ▶ Resulted In : Robust numerical method for the Shallow Water Wave Equations (ANUGA).

Shallow Water Wave Equations



Shallow Water Wave Equations



Conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0$$

Conservation of momentum

$$\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} \right) = 0$$

Serre equations

Conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0$$

Conservation of momentum

$$\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3} \Phi \right) = 0$$

$$\Phi = \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - u \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}$$

Our Background

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- ▶ Current Goal : Robust numerical method for the Serre equations.

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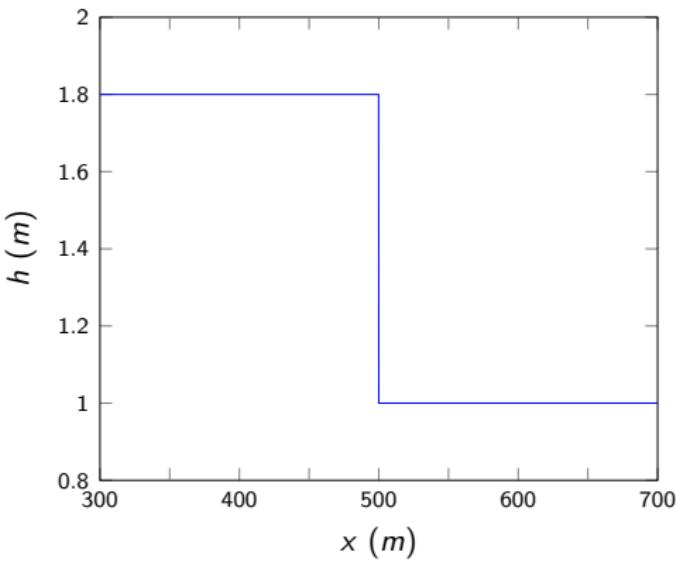
- ▶ Interest : Numerical methods for water waves focusing on ocean hazards.
- ▶ Resulted In : Robust numerical method for the Shallow Water Wave Equations (ANUGA).
- ▶ Current Goal : Robust numerical method for the Serre equations.
- ▶ Problem : Handling discontinuous initial conditions (dam-break problem).

Model Problem : Dam Break Problem

$$h_0 = 1m, h_1 = 1.8m \text{ and } x_0 = 500m$$

$$h(x, 0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

$$u(x, 0) = 0$$



Dambreak Problem Solutions

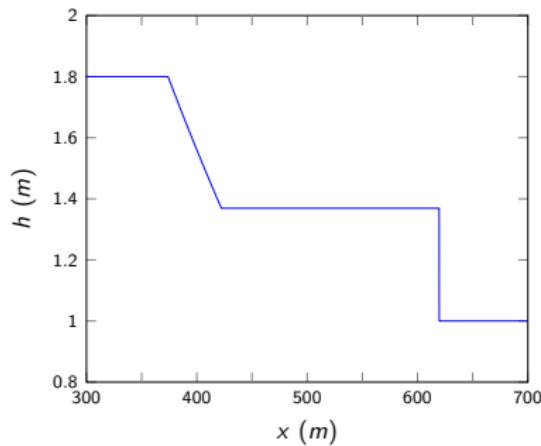


Figure: Shock wave (analytical solution of the shallow water wave equations).

Dambreak Problem Solutions

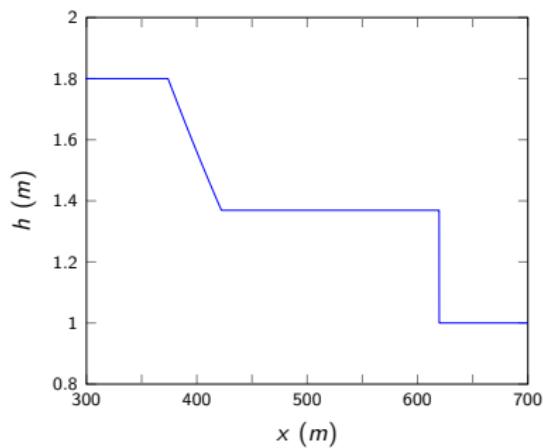


Figure: Shock wave (analytical solution of the shallow water wave equations).

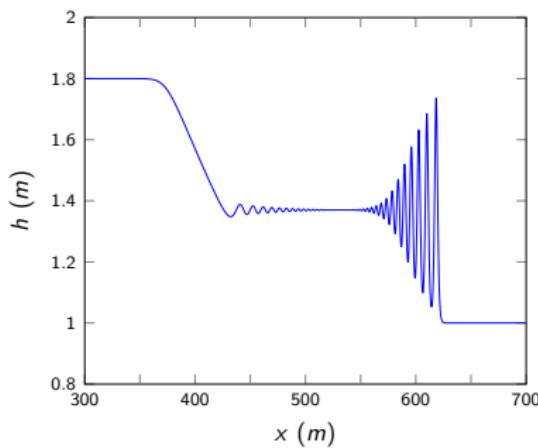


Figure: Dispersive shock wave (common numerical solution of the Serre equations in the literature).

New Observed Behaviour for Dispersive Shock Waves

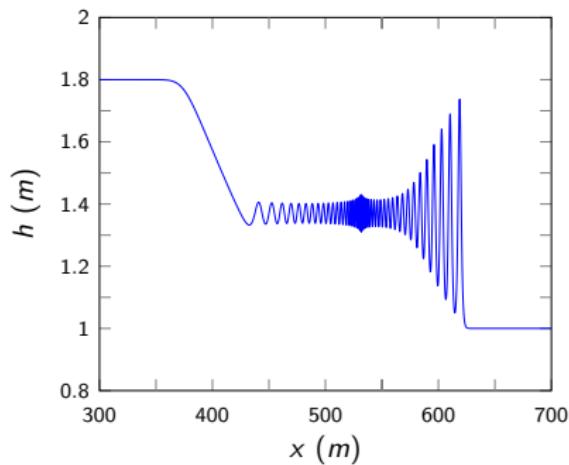


Figure: New observed structure.

New Observed Behaviour for Dispersive Shock Waves

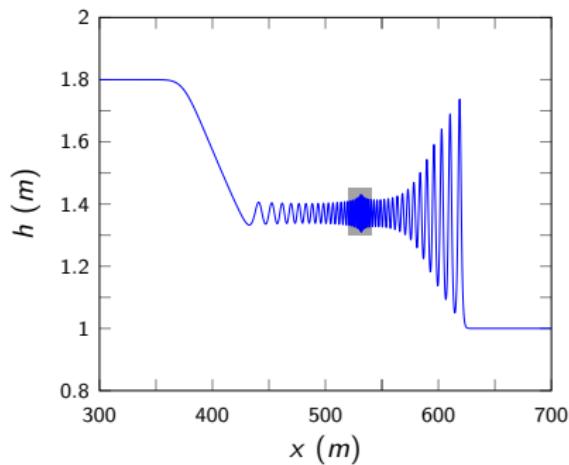


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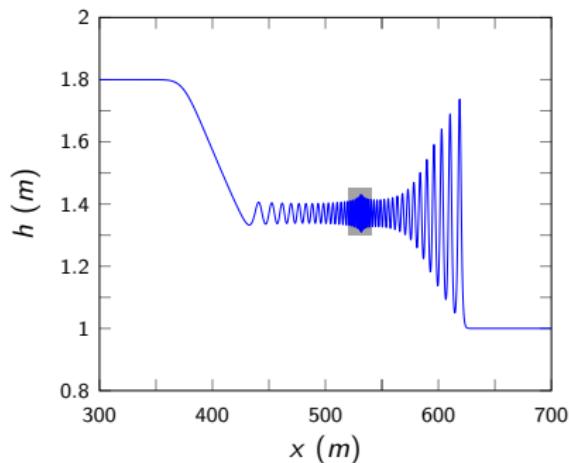


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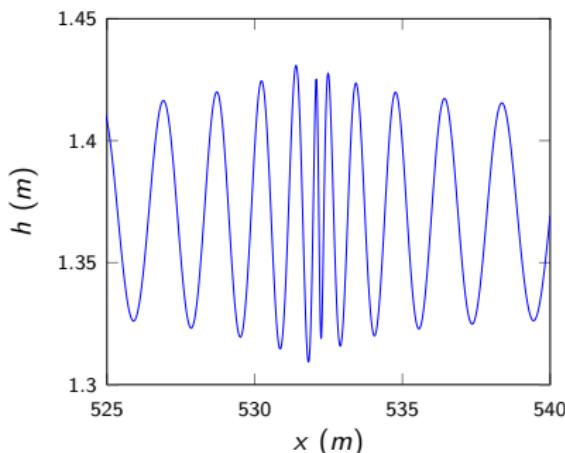


Figure: Zoomed in.

Properties of Dispersive Shock Waves of the Serre Equations

Asymptotic results for long times:

- ▶ Whitham modulation results for leading wave amplitude and location.

Dispersive Shock Waves

Whitham Modulation Results

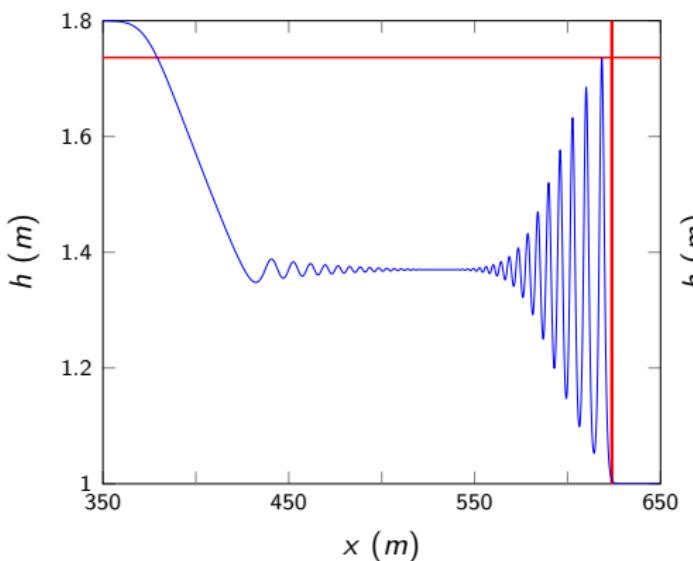


Figure: Common structure.

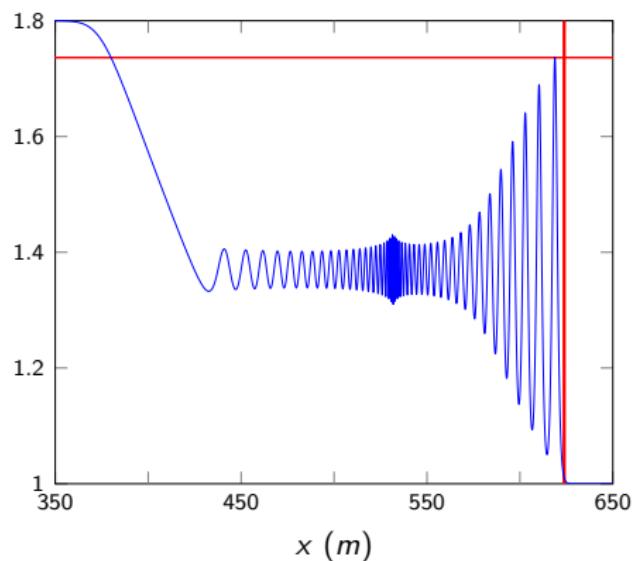


Figure: New structure.

Properties of Dispersive Shock Waves of the Serre Equations

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- ▶ Oscillations of the dispersive shock waves for the Serre equations oscillate around the shock waves of the shallow water wave equations.

Dispersive Shock Waves

Dispersive Shock Waves compared to Shock Waves

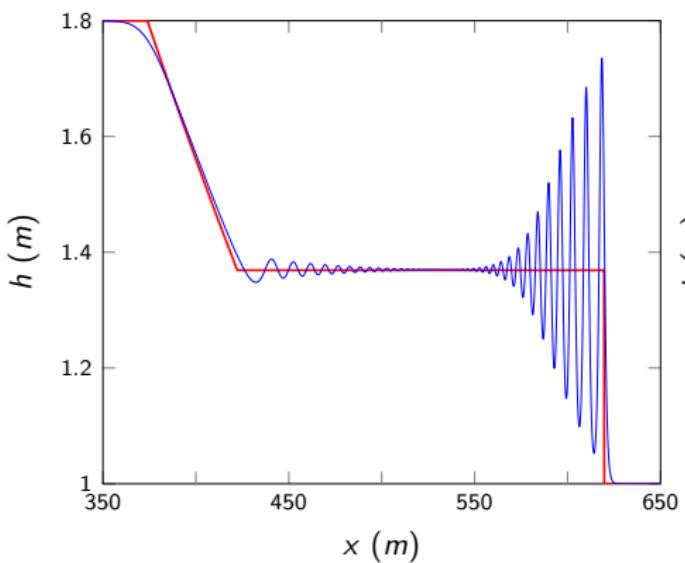


Figure: Common structure.

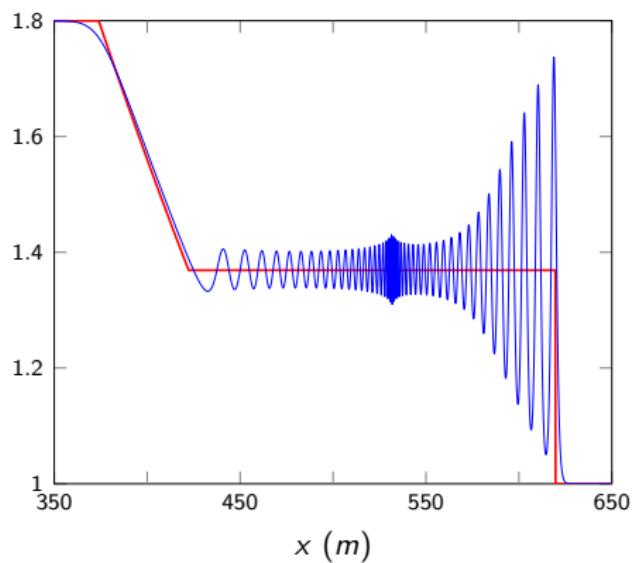


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Properties of Dispersive Shock Waves of the Serre Equations

Asymptotic results for long times:

- ▶ Whitham modulation results for leading wave amplitude and location.
- ▶ Oscillations of the dispersive shock waves for the Serre equations oscillate around the shock waves of the shallow water wave equations.

Linear results:

- ▶ Separate dispersive wave trains.

Dispersive Shock Waves

Separation of Dispersive Wave Trains

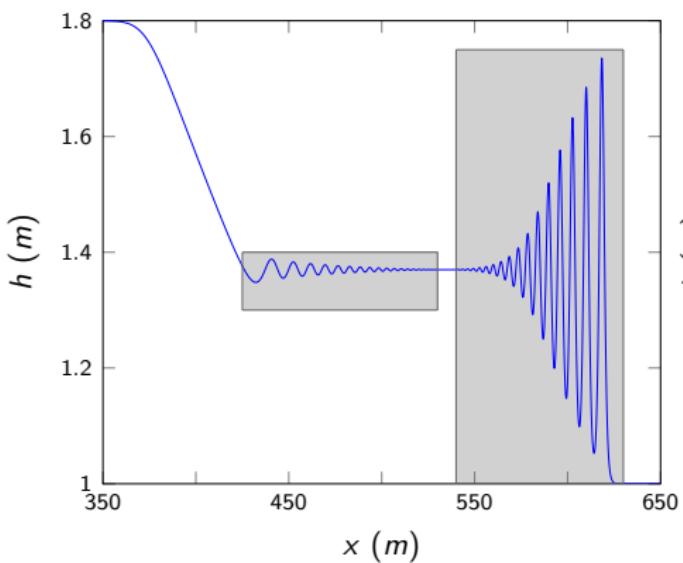


Figure: Common structure.

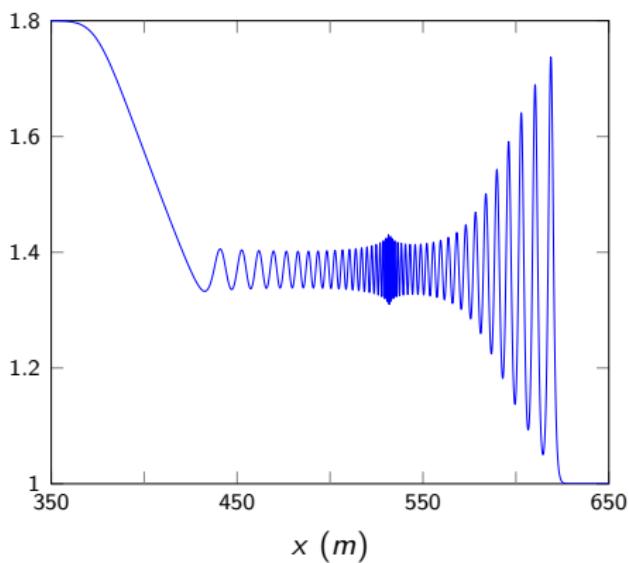


Figure: New structure.

Problem

- ▶ No analytic solution of the Serre equations for dispersive shock waves.
- ▶ Lack transient properties of dispersive shock waves.

Solution

Use numerical solvers for the Serre equations on a problem with a smooth approximation to the initial conditions of the dam-break problem.

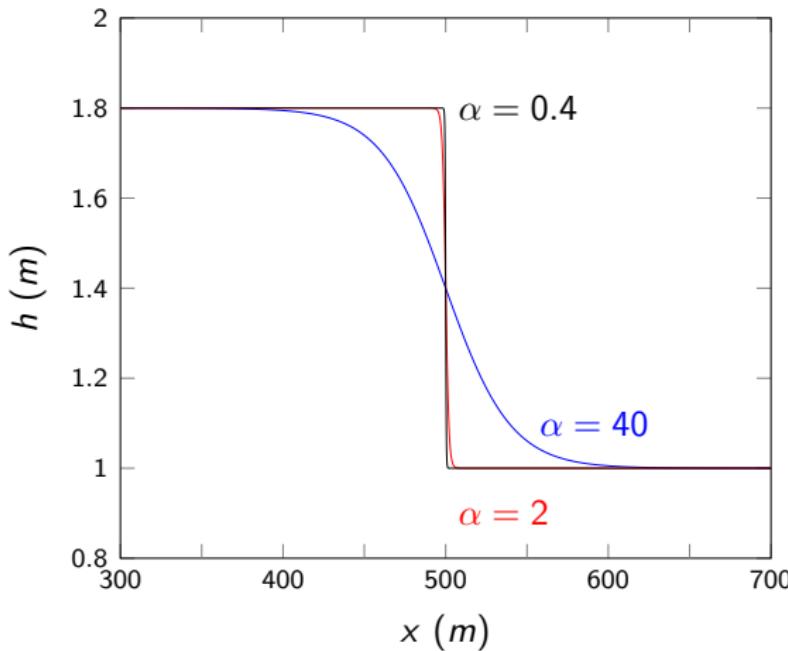
Smooth Dam Break Problem

$$h(x, 0) = h_0 + \frac{h_1 - h_0}{2} \left(1 + \tanh \left(\frac{x_0 - x}{\alpha} \right) \right)$$

$$u(x, 0) = 0$$

$$h_0 = 1m, h_1 = 1.8m \text{ and } x_0 = 500m$$

Smooth Dam Break Problem Examples



Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \rightarrow 0$

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- ▶ Non Oscillatory Structure for large α values.

Non Oscillatory Structure $\alpha = 40$

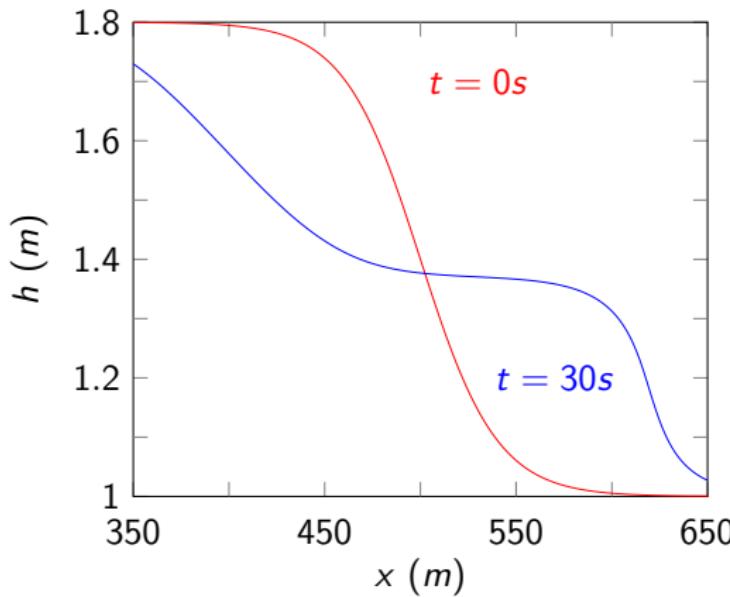


Figure: Highest resolution numerical solution at $t = 30s$

Observed Behaviours

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- ▶ Non Oscillatory Structure for large α values.
- ▶ Flat Structure (common one in the literature).

Flat Structure $\alpha = 2$

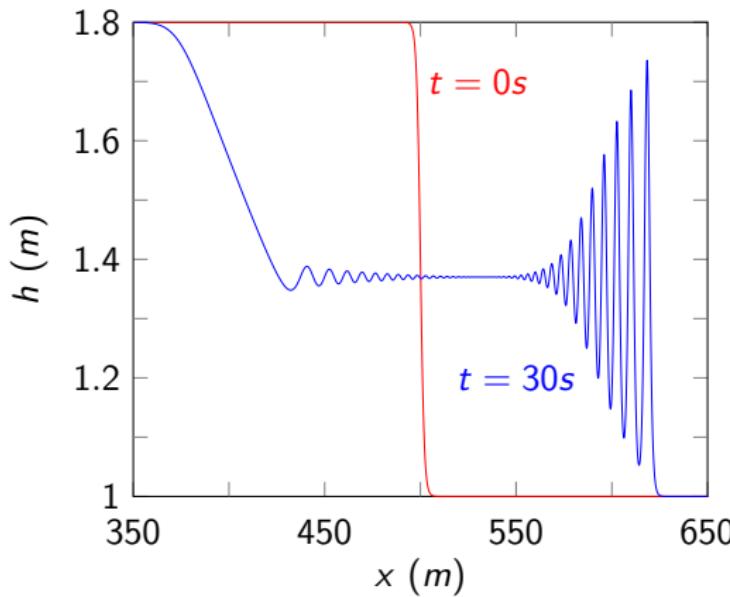


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Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \rightarrow 0$

- ▶ Non Oscillatory Structure for large α values.
- ▶ Flat Structure (common one in the literature).
- ▶ Node Structure (also present in literature).

Node Structure $\alpha = 0.4$

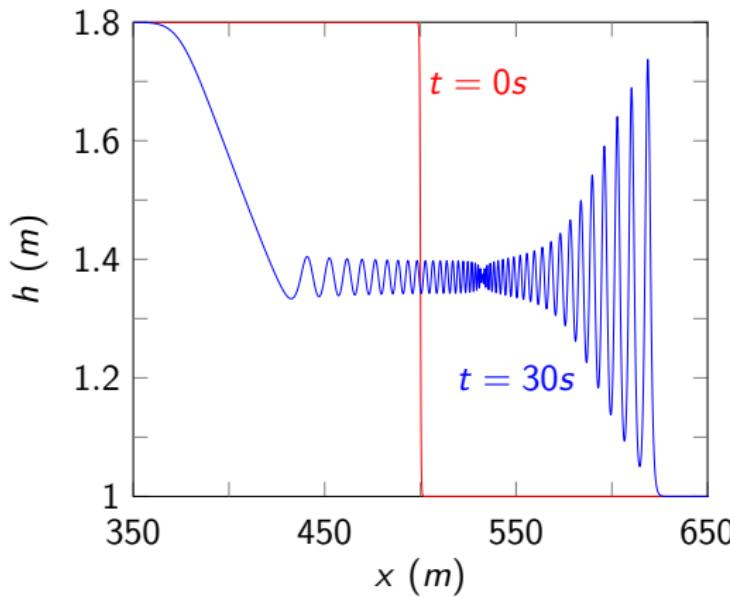


Figure: Highest resolution numerical solution at $t = 30s$

Node Structure $\alpha = 0.4$

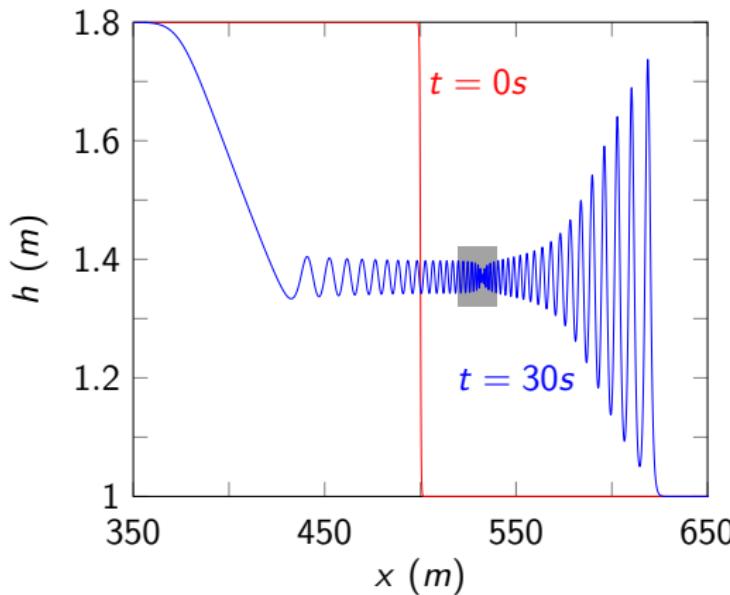
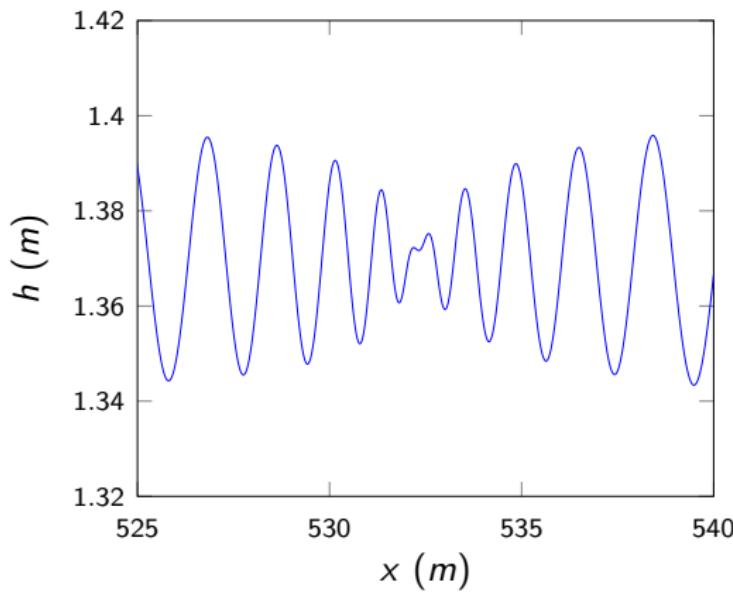


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Node Structure $\alpha = 0.4$



Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \rightarrow 0$

- ▶ Non Oscillatory Structure for large α values.
- ▶ Flat Structure (common one in the literature).
- ▶ Node Structure (also present in literature).
- ▶ Growth Structure for small α values and the dam-break problem (New behaviour).

Growth Structure $\alpha = 0.1$

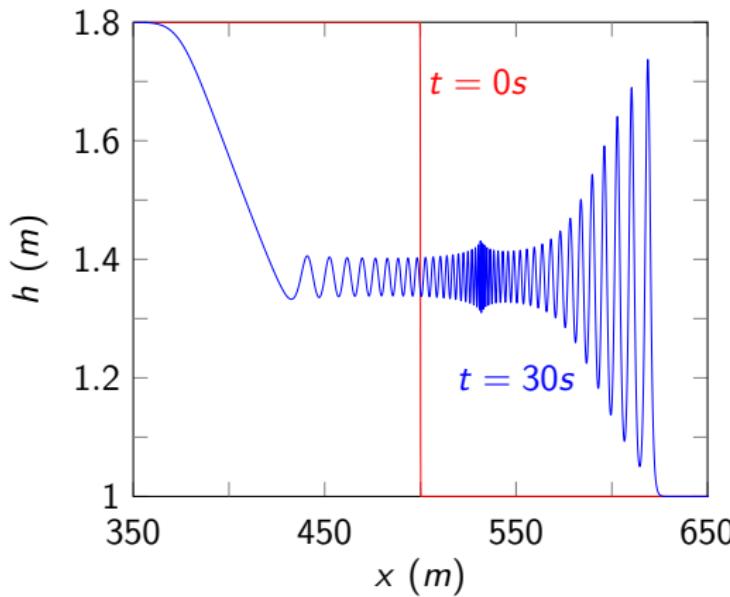


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Growth Structure $\alpha = 0.1$

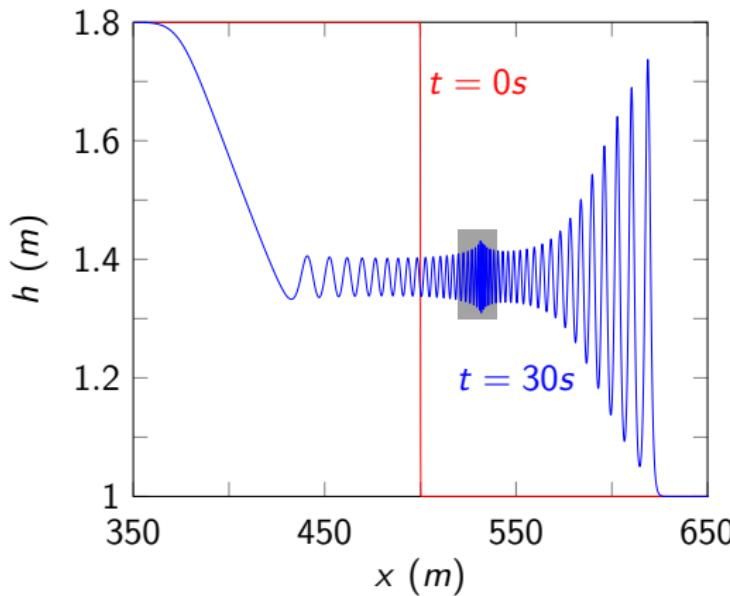
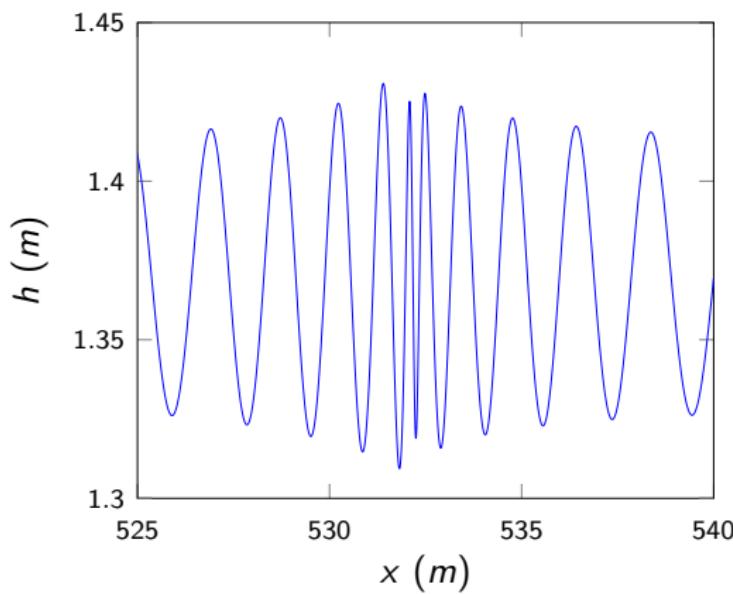
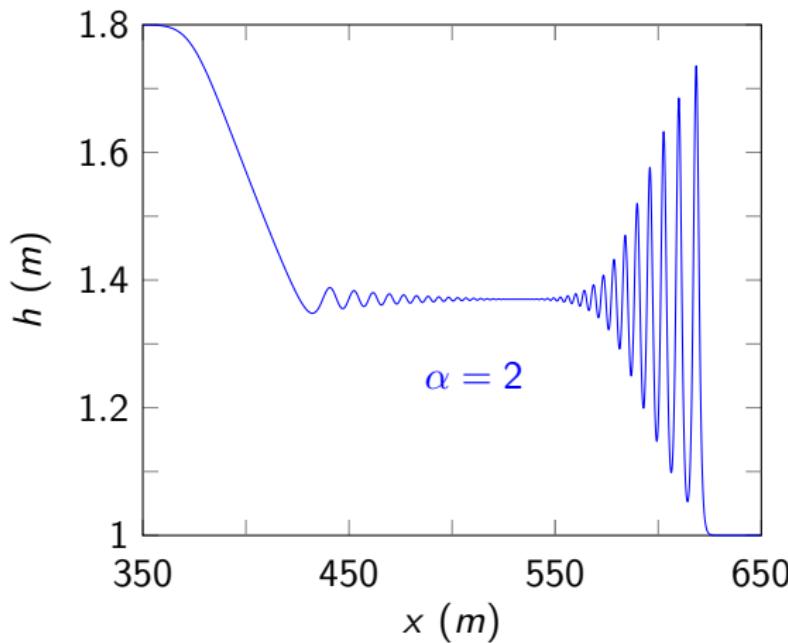


Figure: Highest resolution numerical solution at $t = 30s$

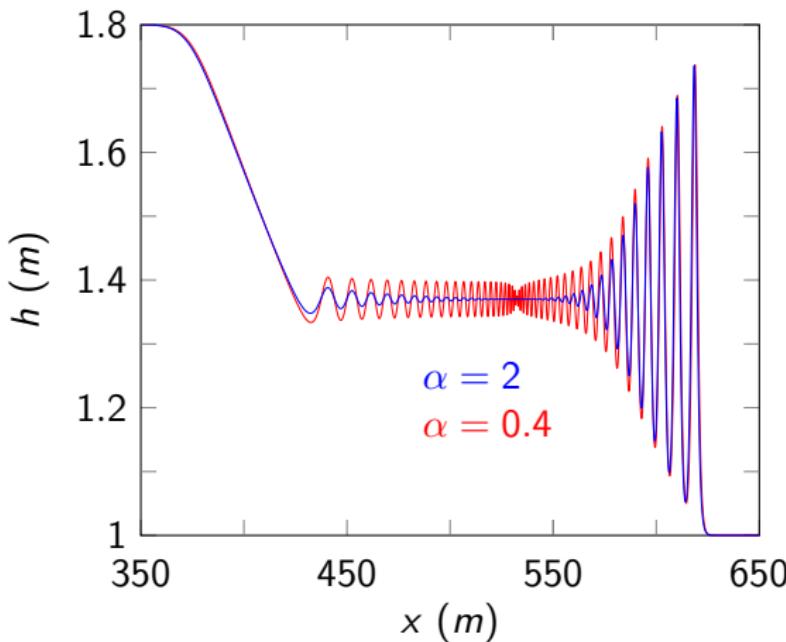
Growth Structure $\alpha = 0.1$



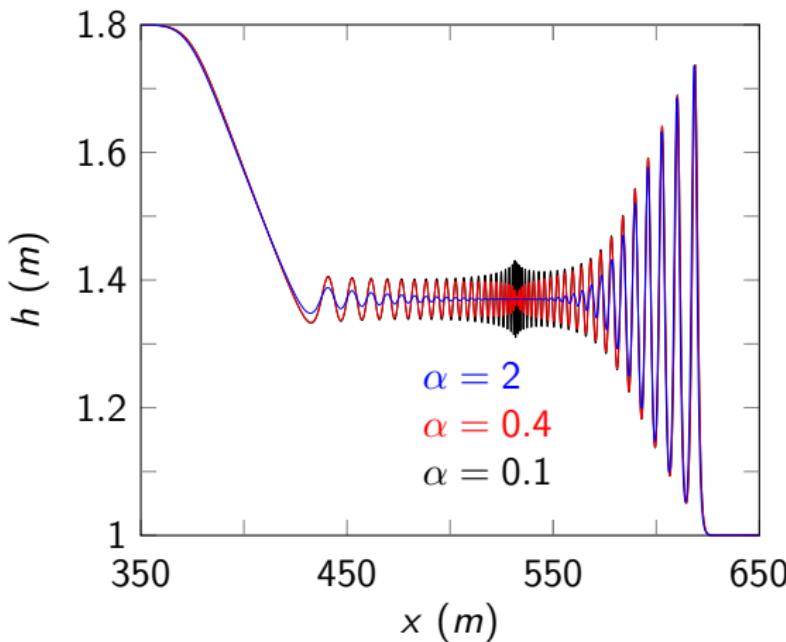
Structure Comparison: Flat



Structure Comparison: Flat & Node



Structure Comparison: Flat & Node & Growth



Structure Comparison Energy of Initial Conditions

Hamiltonian (Energy):

$$\mathcal{H}(x, t) = \frac{1}{2} \left(hu^2 + \frac{h^3}{3} \left(\frac{\partial u}{\partial x} \right)^2 + gh^2 \right)$$

Comparison of Energy of Initial Conditions:

Structure	α	\mathcal{H}
Non-Oscillatory	40	10335.8
Flat	2	10395.5
Node	0.4	10398.0
Growth	0.1	10398.4

Justifying These Numerical Solutions

For a particular α value:

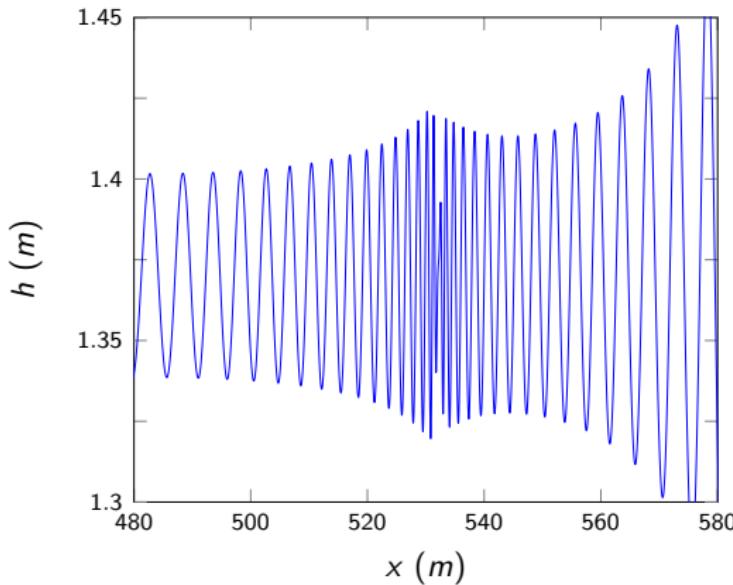
- ▶ Demonstrated convergence as the resolution of the method increases.
- ▶ Demonstrated numerical solutions conserve mass, momentum and energy.

These results demonstrated that the Growth Structure is correct structure of DSW for short time spans.

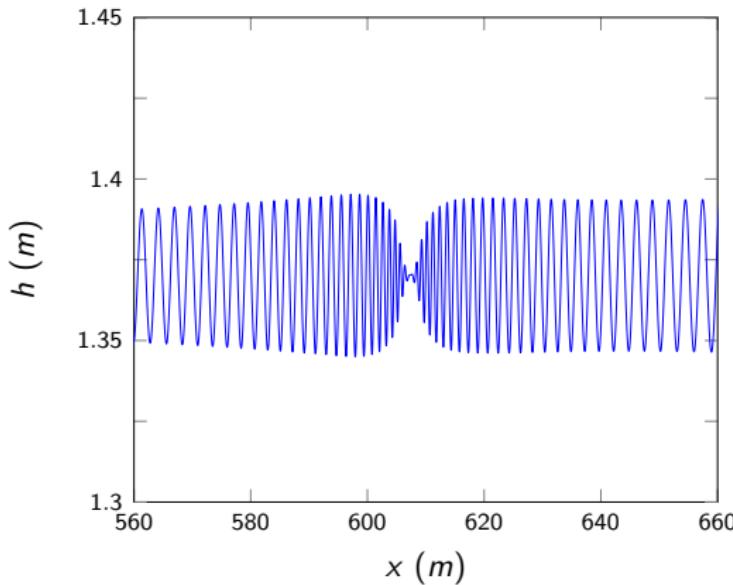
Long term solutions

- ▶ Growth structure agrees well with asymptotic results for short and long times.
- ▶ Growth structure decays to node structure which decays to the flat structure.
- ▶ Separation of dispersive wave trains over long times.

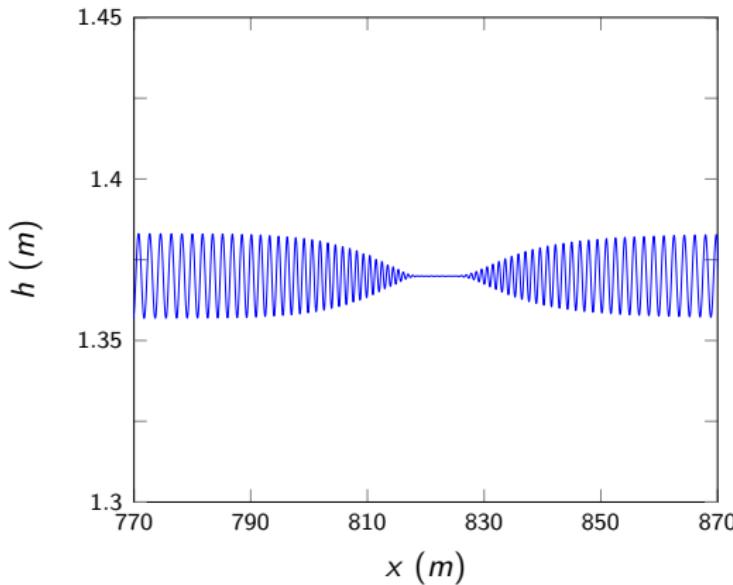
Dam break solution at $t = 30s$ ($\alpha = 0$)



Dam break solution at $t = 100s$ ($\alpha = 0$)



Dam break solution at $t = 300s$ ($\alpha = 0$)



Conclusion

Presentation

- ▶ Found new behaviour of dispersive shock waves for short time spans not previously published in the literature.
- ▶ Good agreement between numerical solutions and known properties of dispersive shock waves for long time periods.

Paper

- ▶ Explained why different behaviour published in the literature.
- ▶ Justified the robustness of our numerical methods.

Comparison of Node and Growth Structures

