Thesis report:
“Simulation of rapidly varying and dry bed flow using the Serre equations solved by a finite element volume method”
by Jordan Pitt

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The main results of this thesis are the development and validation of a numerical technique for the solution of the Serre equations which describe the long-wave limit of finite-amplitude surface waves. The particular method uses finite-volumes for the unsteady components and finite-elements for the solution of a boundary-value problem. This method is compared with a number of other methods, but primarily with a similar method that uses finite-differences for the boundary-value problem. These two methods are extended beyond the constant coefficients Serre equations to variable bottom topography and dry beds where the fluid depth approaches zero. The main conclusion of the research is that the proposed method is the best-performing method out a suite of methods for the Serre equations, in particular when the solution has extreme gradients. However, the final test case appears to be flawed, as the solutions are compared against experimental observations in which the Serre equations are not valid. In these scenarios I would anticipate that given sufficient resolution the solutions of the Serre equations should remain bounded. This may not be the case, as there are examples such as the fifth-order Korteweg–de Vries equation which develop singularities in finite time. This point is discussed in more detail below.

The thesis present a comprehensive analysis of various methods for the Serre equations, and Mr Pitt has shown a very good understanding of the numerical methods. His numerical analysis of the methods is sound, though in some places in Chapter 4 some further explanations could have been included. The choice of test cases for the two main numerical methods is very appropriate, and clearly shows the strengths and weaknesses of all the methods that are considered.

Although I would question which is the most useful method for one-dimensional calculations, I agree that that FEVM would be the preferred method for typical simulations in two-dimensions. Hopefully further work can be done to extend this to run on unstructured meshes in two-dimensions and to develop a parallel version of the code. I would foresee that one of the major difficulties would be coupling this to a shallow-water code that applies in the open ocean, but this should be surmountable.

The main aspect of the thesis which lets it down is the presentation. The problems related to this are: organization, reference to the work done by Mr Pitt, punctation and flow of text, and use of casual expressions.

The organization of the content makes the thesis quite confusing to follow. I understand the justification for this in that Mr Pitt wants to make it clear the work that he primarily undertook for this thesis. However, that then means that important results appear out of place. For example, the observations of the dam-break problem in §2 could have been expanded and put as a larger section in §5 as part of the validation and comparison of the various methods. Similarly, since the thesis focusses on the comparison of FEVM and FDVM_2, and it appears that Mr Pitt performed the analysis of FDVM_2, it would have made more sense to include this work in Chapter 4 or as a separate chapter.

Throughout the thesis there are many references to 'I' or 'my'. For example §1.2 commences with the sentence 'My research ...', which could be better expressed as 'The research undertaken for this thesis ...'. On p. 7 there is the sentence 'This paper was produced by me with the support of my coauthors based on my own work'. This could be better expressed as 'This paper was primarily prepared by me in collaboration with my coathors, based on research that I primarily undertook'. The are many other occurrences referrring to 'my research'. This detracts significantly from the contribution made by Mr Pitt's supervisors, in amongst other things, suggesting the original research topic, checking on calculations, providing contacts and suggesting various avenues of research.

The punctuation and flow of the writing is confusing and in places creates possible ambiguities. For example on p. 1 at the beginning of the second paragraph 'The physics of water can be described using Newton's second law. From which ...'. As a new sentence has commenced it is not clear what 'which' refers to. In this case the full-stop after 'law' should be replaced by a comma, which removes the ambiguity. On p. 8 in the first paragraph the sentence 'Followed ...' should a continuation of the previous sentence or written as 'This is followed ...'. On p. 9, line 4, the semi-colon should be removed. On p. 14 the sentence 'Additionally, the Green–Naghdi equations [21] which are equivalent to the Serre equations for one-dimensional flows were ...' should have commas after the reference and 'were'. On p. 15 in the first line a comma should be inserted rather than commencing a new sentence or 'Where' should be removed. This connection between sentences is a common problem throughout the thesis. However, a thorough proofreading of the thesis of the final version of the thesis should occur to correct all the punctation.

Two very common expressions that occur in this thesis are 'we get' and 'So that'. The first could be replaced by 'we obtain', 'gives' or 'gives rise to'. The second could be replaced by 'Consequently', 'Therefore' or 'Hence'.

In Chapter 3 the variable $w = h + b$ is introduced. This is a derived variable and does not occur in the Serre equations (2.6), hence I do not follow why it is carried through for all the calculations . For example, in point (iv) of p. 29 $\mathcal{S}_j$ only needs to be a function of h, b and u. The only point that w is explicitly written is in (3.19)–(3.20), and could be defined just for this calculation.

The justification for FEVM compared to FDVM_2 largely relies on §6.3. As mentioned earlier, I believe there are significant problems with this. Firstly, I think the figures for the FDVM_2, Figures 6.19–6.20 are incorrect. It looks as the experimental results have been plotted in blue rather than the numerical results. The main reason I think this is that the description in the text does not concur with the plots, there are dynamics due to reflected waves which should not be present in the numerical results and in all other simulations the dynamics of FEVM and FDVM_2 are very similar. Secondly, it is known that the Serre equations do not model breaking, therefore it is pointless to compare the solutions after breaking. I would except that close to breaking dispersion must act to smooth any discontinuities, and create solitary waves, dispersive wavetrains or undular bores. I would therefore suggest that the breakdown of FEVM or FDVM_2 is due to the solutions being under-resolved. FEVM may not become singular, but it would be diverging from the solution of the Serre equations in this limit. Hence to say from this test that FEVM is superior to FDVM_2 is unrealistic.

I think that the boundary conditions in §6.2 are incorrect, in particular the equation on p. 106. Since this is a periodic wave this should be

$$u(x,t) = \frac{\omega}{k}\frac{h(x,t)-h_0}{h(x,t)}.$$

For the low frequency case the wave speed should be 1.84 m/s and for the high frequency this should be 1.64, m/s, rather than $\sqrt{gh_0} = 1.98\,\mathrm{m/s}$. As the error for the high frequency case seems to be larger than for the low frequency case, this could explain some of the difference between the numerical and experimental results.

Chapter 4 and §2.2.2 would be much simplified by showing that the linearised Serre equations are just the Boussinesq equations:

$$\left(1 - \frac{H^2}{3}\frac{\partial^2}{\partial X^2}\right)\frac{\partial^2 \mu}{\partial t^2} - gH\frac{\partial^2 \mu}{\partial X^2} = 0,$$

or

$$\left(1 - \frac{H^2}{3}\frac{\partial^2}{\partial X^2}\right)\frac{\partial \mu}{\partial t} + g\frac{\partial \eta}{\partial X} = 0,$$
$$\frac{\partial \eta}{\partial t} + H\frac{\partial \mu}{\partial X} = 0,$$

where $X = x - Ut$.

Finally, there are two comments for consideration, that do not necessarily need to be addressed in the final version of the thesis.

Regarding Figure 2.2 and the related text. Does the behaviour in the middle of the domain oscillate between the node structure and the growth structure as the two wavetrains pass through each other? Since it is concluded that the dynamics in the vicinity of this region are linear, it possible that this structure is just superposition or annihilation of wavetrains.

On p. 79 the error in FDVM_3 was attributed to the fractional coefficients not being able to be represented exactly in finite-precision. Were other third-order Runge–Kutta schemes tested for which the coefficients can be represented exactly?

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Minor points

An incomplete list of minor points that should be addressed in the final version of the thesis are:

- Units for parameters are typically in roman font, e.g., 1.84 m/s.
- p. 4, line -7: '~~whose results~~ , and the analysis and results for this method'.
- p. 8: 'It's' should be 'Its'.
- p. 10, lines 11–13: This sentence is ambiguous and should be clarified. This could interpreted to mean that in a numerical method the Serre equations are solved without bottom friction and then the results are post-processed to include bottom friction.
- p. 10, line -8: 'average horizontal velocity'.
- p. 10, line -4: 'a fluid particle'.
- p. 10, line -1: 'vertical momentum equation of the Euler equations'.
- p. 11, line -4: 'dispersion of waves on the free surface'.
- p. 12, lines (-2)–(-1): 'By ~~rewriting the Serre equations and~~ introducing'.
- p. 13, lines (-2)–(-1): 'and a dispersion relation'.
- p. 17, line -12: 'by ~~summing~~ integrating'.
- p. 17, line -4: '~~of~~ for the Serre equations'.
- p. 19 line 9–10: 'initial ~~condition~~ value problems'.
- p. 25, line -1: This statement that the finite-volume method only requires values within the cell is not technically correct, as it is dependent through G on values outside the cell. This is clear from the top of p. 37 where it is stated that the matrix that must be inverted is penta-diagonal.
- p. 26, line -9: What equations are to be solved when it is stated 'solving the equations'?
- p. 32, line -9: 'Integrating this equation by parts'.
- p. 44, lines (-13)–(-12): 'small ~~chosen~~ positive parameter. The analytical error introduced by this transformation ~~is smallest~~ decreases with h_{base} ~~is smallest~~.'
- p. 44, line -2: '~~also~~ serves'.
- p. 45, line -10: Font for 'LU' should be consistent.
- p. 45, line -5: 'We ~~possess~~ obtain'.
- p. 46, line 2: New sentence.
- p. 48, line 7: 'We ~~analysed~~ analyse'.

- p. 48: Third paragraph should be the concluding sentence of the second paragraph.
- p. 48, line -5: 'and which is independent'.
- p. 48, line -3: '~~propagating Fourier modes through the numerical scheme~~ representing the numerical solution in terms of Fourier modes'.
- p. 52: For the second and third equations, general typesetting convention is to start each line with an operator, so second line in both should start with an '='.
- p. 52, line -1: '~~the~~ this'.
- p. 56, line 2: 'When the flow is ~~flowing~~ to'.
- p. 58: Last paragraph reads like dot points. Needs connections between sentences to improve the flow.
- p. 59, line 9: 'g' should be in math font.
- p. 59, line -8: 'This ~~is different~~ differs'.
- p. 59, line -3: '~~enough~~ sufficient'.
- p. 63, line 4: 'determines the error in the speed'.
- p. 72: In §5.1.2 the initial quantity that should be used is the numerical representation of the initial condition, rather than the actual initial condition. This is acknowledged at the end of the section, and borne out in §5.3.
- p. 73, line 17: 'will be useful'.
- p. 74, line -5: '~~over~~ as a function of'.
- p. 77, line 1: '~~h's evolution equation~~ the evolution equation for h'.
- p. 77, lines 2–3: '~~; with the error in u being dominated by the error in G~~ . The error in the calculation of u is then dominated by the error in G.'
- p. 77, line -11: '~~the~~ uh'.
- p. 80, para. 4: In the third sentence I do not follow why the second-order increase in the error follows as $\Delta x \to 0$.
- p. 83, line -3: 'numerical ~~solutions~~ reproduction'.
- p. 86, lines 5–6: '~~an a_1 high Gaussian bump~~ a Gaussian bump of amplitude a_1'.
- p. 87, line 17: 'in Figure 5.12 for'.
- p. 95, line -2: '~~in~~ on the water surface'.
- p. 97, line 15: '~~observe~~ generate oscillations'.
- p. 103, lines 8–9: '~~except mass; which is conserved exactly~~ . The exception is the mass, which is conserved exactly'.

- p. 103, line 9: ‘machine ~~epsilon~~ precision’.
- p. 116, line -1: ‘and these’.
- p. 123, line 8: ‘~~methods~~ robustness of the numerical methods’.
- p. 124, lines 11–12: ‘~~method~~ simulation during the run-up process due to the ~~methods~~ handling’.
- p. 129, line -12: Font for D and W should be calligraphic.
- p. 130, line -2: Including wave breaking is not a natural extension of this model, as the Serre equations do not apply. The formulation of the model would have to change completely.
- p. 143, equation (C.5): Subsequent to this equation $\mathcal{F}^{\eta,G}$ is not mentioned. Should this term be zero following from (C.2a)?