

A Finite Element-Volume Method for the Serre Equations

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Outline

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► Motivation

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- ▶ Motivation
- ▶ Method

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- ▶ Method
- ▶ Results

Ocean Wave Hazards

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► Tsunamis

Sulawesi 2018 Tsunami



Figure: Sulawesi Tsunami (Indonesia, 2018).

Ocean Wave Hazards

- ▶ Tsunamis
- ▶ Storm Surges

Storm Surge of Hurricane Florence and Michael

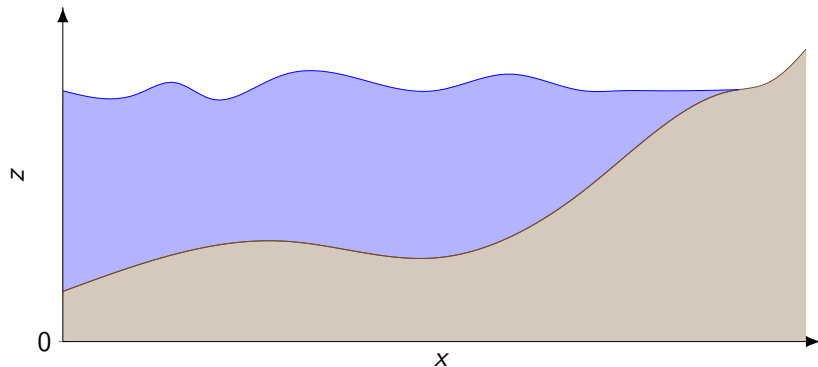


(a) Florence (U.S.A, 2018)

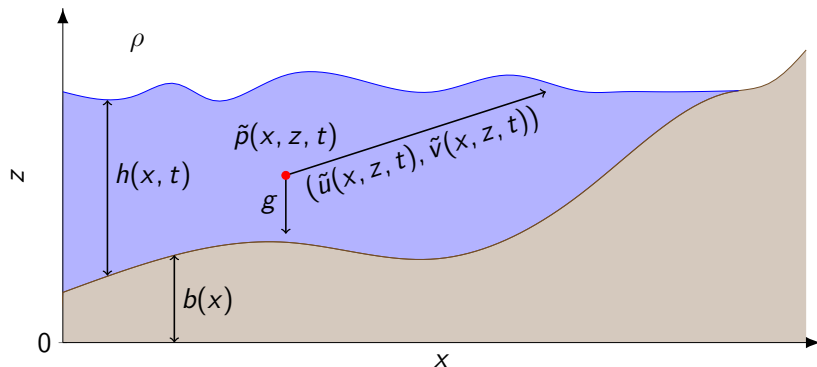


(b) Michael (U.S.A, 2018)

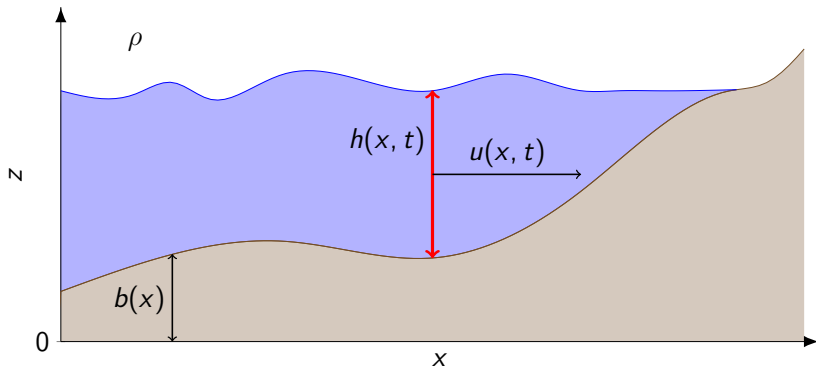
Two Dimensional Scenario



Navier-Stokes



Model Simplification: Serre Equations



Assumptions

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Quantity	Serre Equations
Particle: $\tilde{u}(x, z, t)$	$u(x, t)$

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Particle: $\tilde{u}(x, z, t)$	$u(x, t)$
Particle: $\tilde{v}(x, z, t)$	$u \frac{\partial b}{\partial x} - (z - b) \frac{\partial b}{\partial x}$

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Particle: $\tilde{v}(x, z, t)$	$u \frac{\partial b}{\partial x} - (z - b) \frac{\partial b}{\partial x}$
Particle: $\tilde{p}(x, z, t)$	$g\rho[h + b - z] + \rho[h + b - z] \psi$ $+ \frac{1}{2}\rho \left(h^2 - [z - b]^2 \right) \phi$

where

$$\psi = \frac{\partial b}{\partial x} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + u^2 \frac{\partial^2 b}{\partial x^2}, \quad \phi = \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - u \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

Equations

Mass:
$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0,$$

Momentum:
$$\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} + \frac{h^2}{2} \psi + \frac{h^3}{3} \phi \right)$$

$$+ \frac{\partial b}{\partial x} \left(gh + h\psi + \frac{h^2}{2} \phi \right) = 0.$$

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Method

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Goal: Adapt Finite Volume Methods for the Serre Equations

Finite Volume Method

- Conservation law form

Finite Volume Method

- ▶ Conservation law form
- ▶ Finite volume update

Finite Volume Method

- Conservation law form

Conservation Law Form

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left[f \left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n} \right) \right] + s \left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^m q}{\partial x^m} \right) = 0$$

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Conservation Law Form

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0,$$

$$\begin{aligned} \frac{\partial G}{\partial t} + \frac{\partial}{\partial x} \left(uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[\frac{\partial u}{\partial x} \right]^2 + h^2 u \frac{\partial u}{\partial x} \frac{\partial b}{\partial x} \right) \\ + \frac{1}{2}h^2 u \frac{\partial u}{\partial x} \frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x} \frac{\partial^2 b}{\partial x^2} + gh \frac{\partial b}{\partial x} = 0. \end{aligned}$$

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with

$$\mathbf{G} = hu \left(1 + \frac{\partial h}{\partial x} \frac{\partial b}{\partial x} + \frac{1}{2}h \frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x} \right]^2 \right) - \frac{\partial}{\partial x} \left(\frac{1}{3}h^3 \frac{\partial u}{\partial x} \right).$$

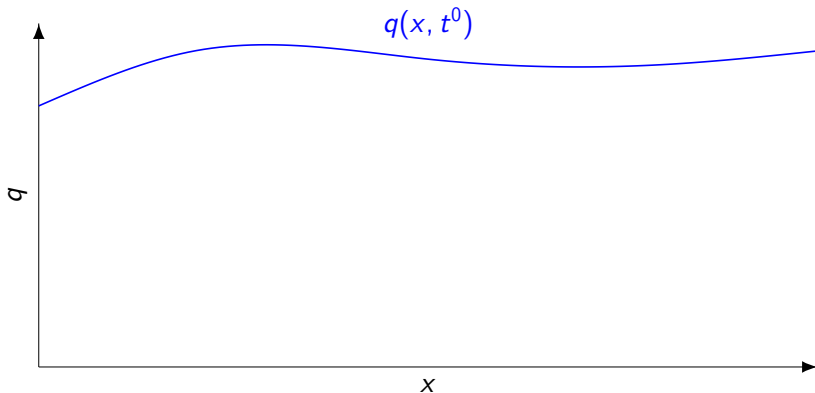
Finite Volume Method

- ▶ Conservation law form
- ▶ Finite volume update

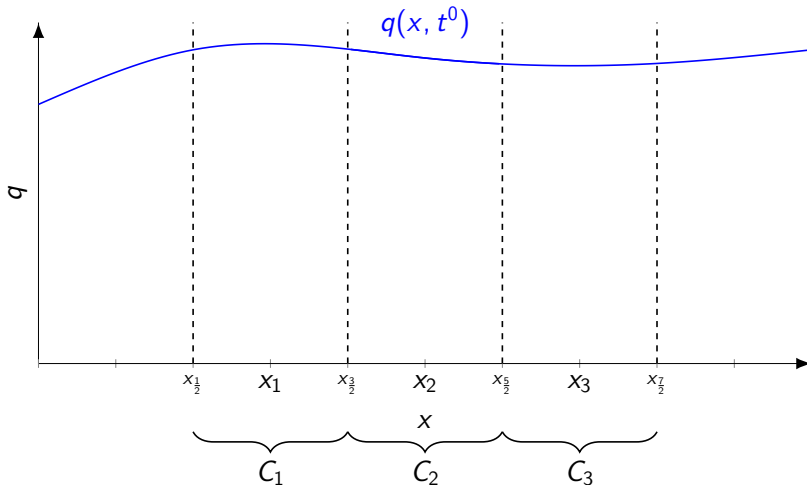
Conservation Law Form

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left[f \left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n} \right) \right] + s \left(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^m q}{\partial x^m} \right) = 0$$

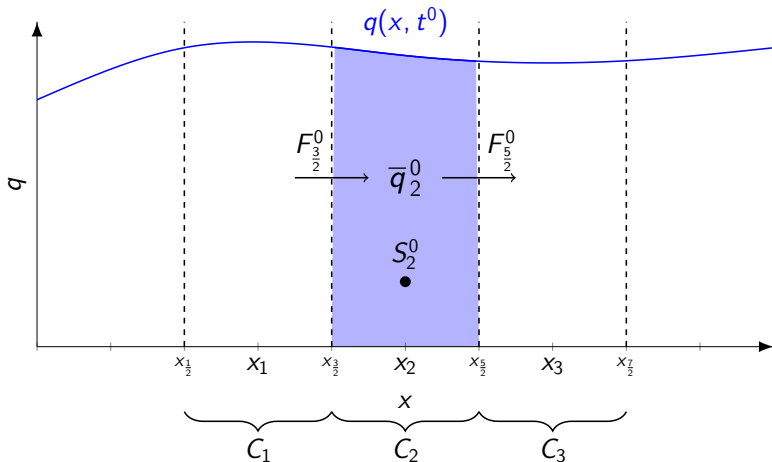
Finite Volume Method



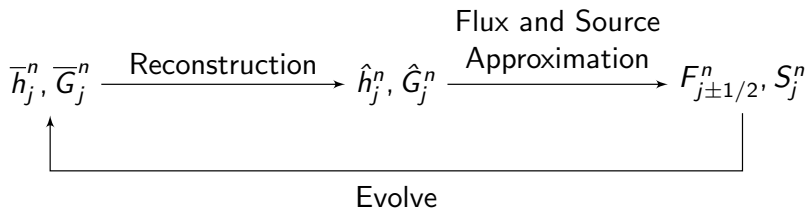
Discretisation



Update



Finite Volume Method



Require velocity to calculate flux and source

However to calculate the flux and source terms we require u

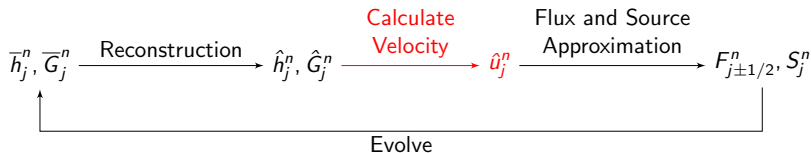
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Method



Reconstruction

- Determines spatial order of accuracy

Reconstruction

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Goal: Second-order accuracy

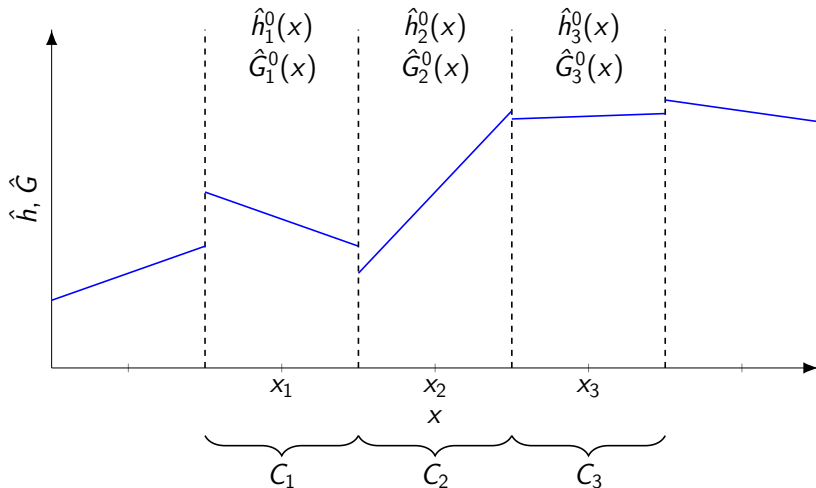
Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
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Reconstruction Spaces

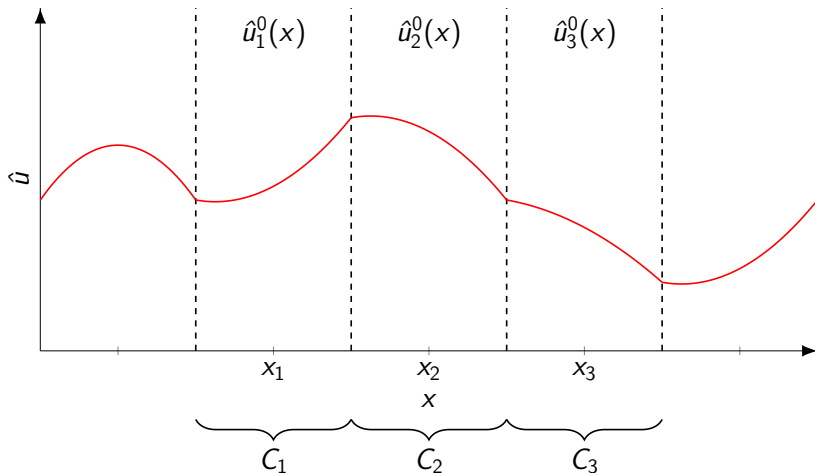
Quantity	Number of spatial derivatives	Reconstructed functions
h	zero	linear over cell, discontinuous at edges
G	zero	linear over cell, discontinuous at edges

Reconstruction

 \hat{h}, \hat{G} 

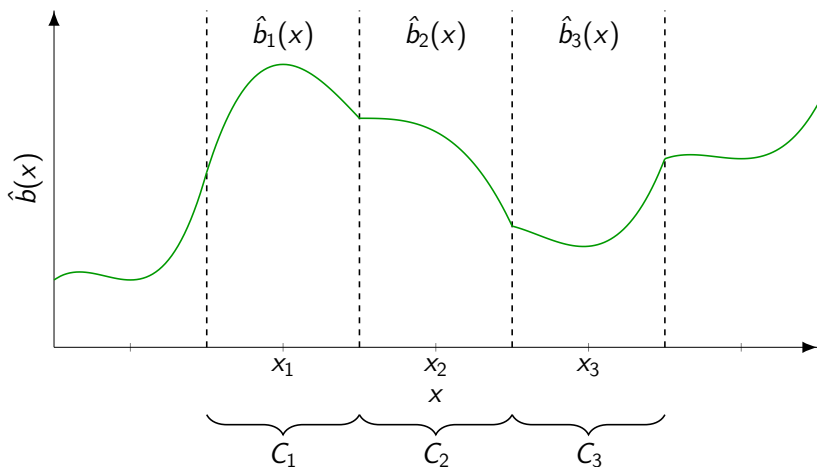
Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
h	zero	linear over cell, discontinuous at edges
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u	one	quadratic over cell, continuous at edges

\hat{u} 

Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
h	zero	linear over cell, discontinuous at edges
G	zero	linear over cell, discontinuous at edges
u	one	quadratic over cell, continuous at edges
b	two	cubic over cell, continuous at edges

\hat{b} 

Finite Element Calculation of Velocity

Finite Element Method to solve:

$$G = hu \left(1 + \frac{\partial h}{\partial x} \frac{\partial b}{\partial x} + \frac{1}{2} h \frac{\partial^2 b}{\partial x^2} + \left[\frac{\partial b}{\partial x} \right]^2 \right) - \frac{\partial}{\partial x} \left(\frac{1}{3} h^3 \frac{\partial u}{\partial x} \right).$$

for u given h , G and b

Finite Element Calculation of Velocity

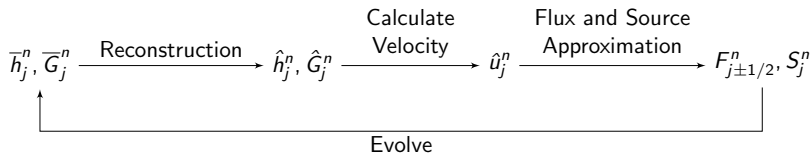
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for u given h , G and b

Solves the weak form replacing all quantities with their reconstructions \hat{h} , \hat{G} and \hat{b} to get \hat{u}

Method



Validation

► Analytic Solution

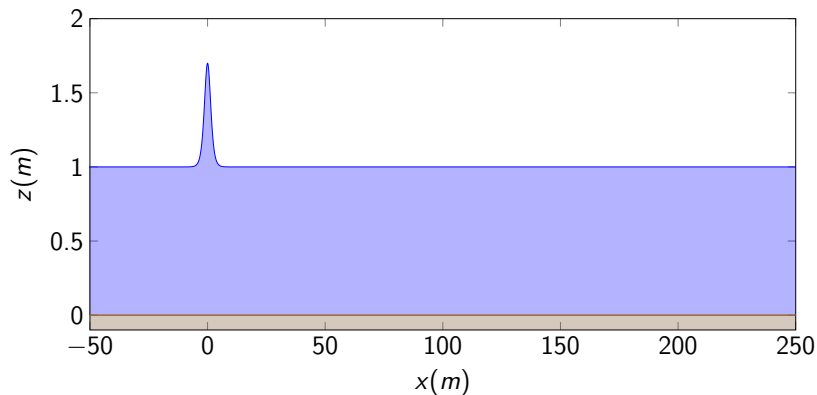
Validation

- ▶ Analytic Solution
- ▶ Experimental Results

Validation

► Analytic Solution

Soliton Example



Soliton Equations

$$h(x, t) = a_0 + a_1 \operatorname{sech}(\kappa(x - ct)),$$

$$u(x, t) = c \left(1 - \frac{a_0}{h(x, t)} \right),$$

$$b(x) = 0$$

Soliton Equations

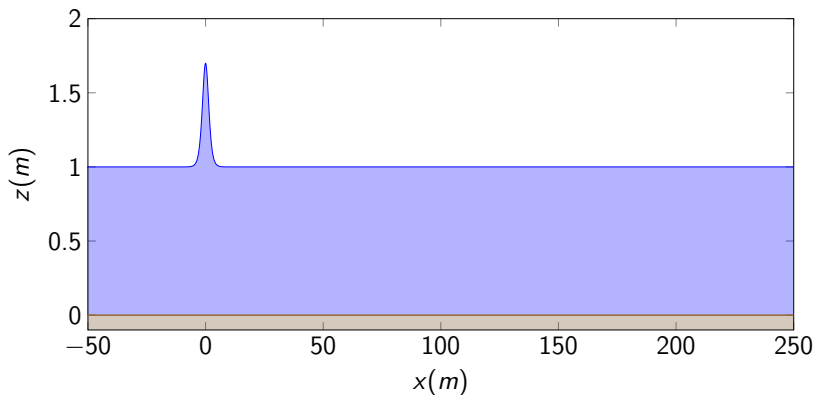
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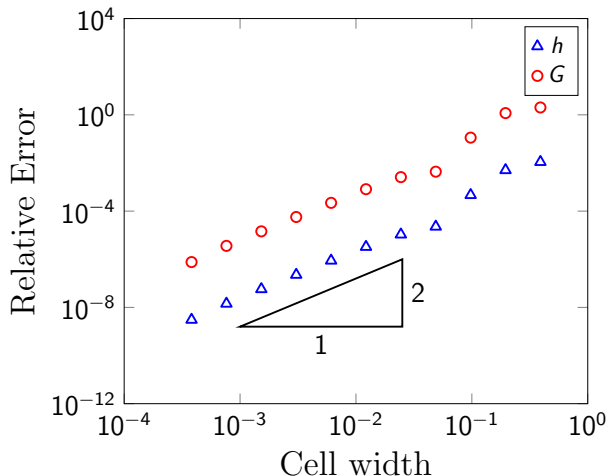
$$b(x) = 0$$

$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{(a_0 + a_1)}},$$

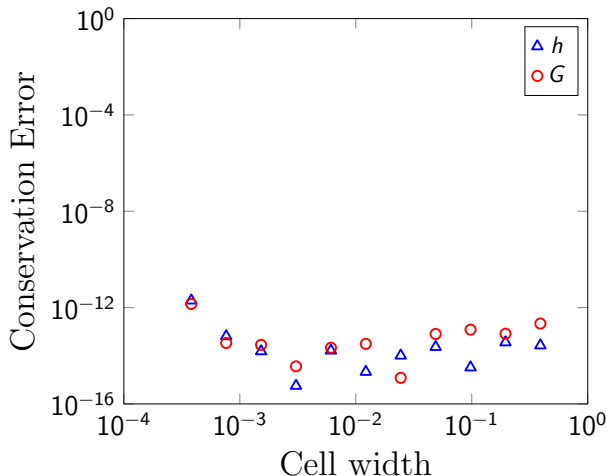
$$c = \sqrt{g(a_0 + a_1)}.$$

Numerical Solution $a_0 = 1m$, $a_1 = 0.7m$ 

Convergence



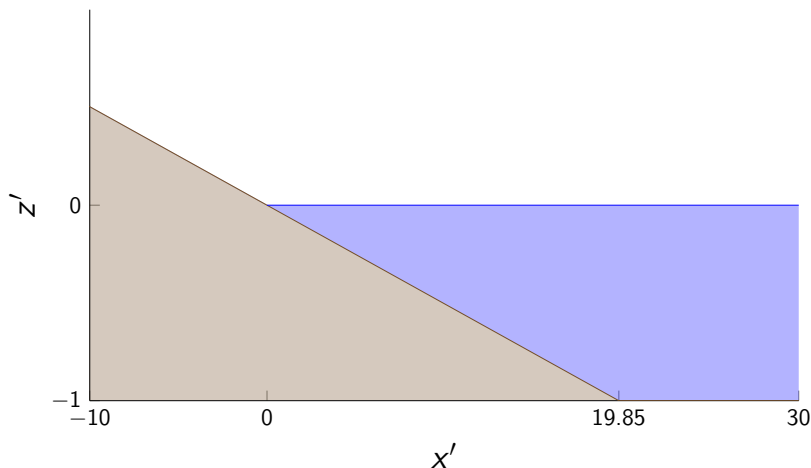
Conservation



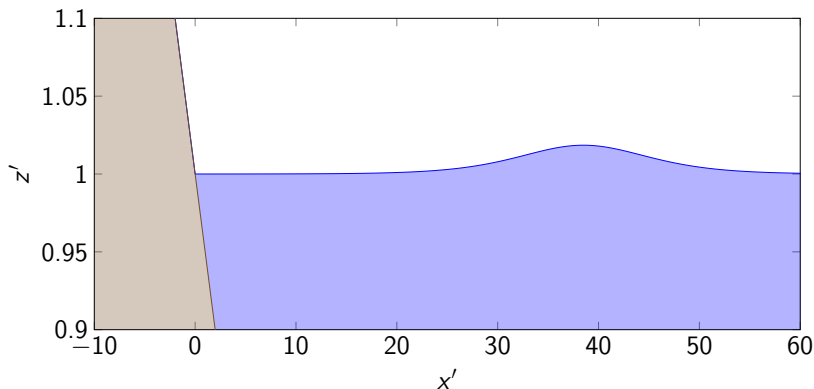
Validation

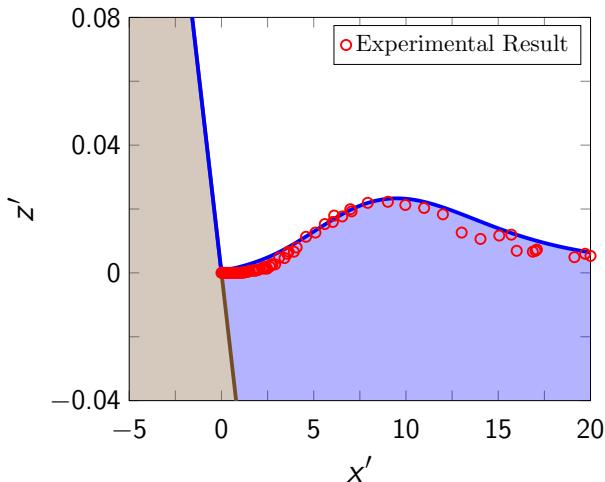
- ▶ Analytic Solution
- ▶ Experimental Results

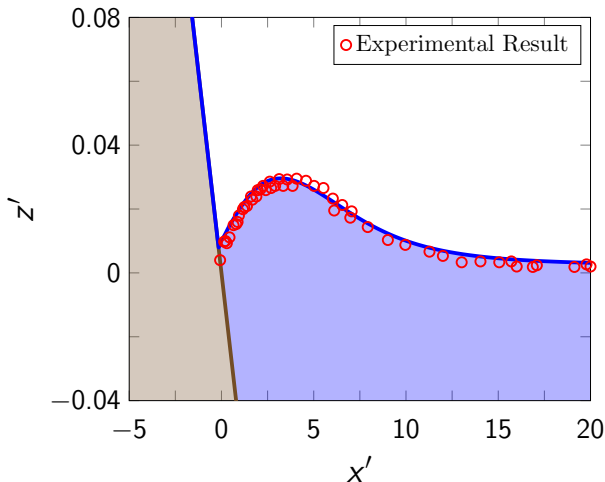
Synolakis Experiment

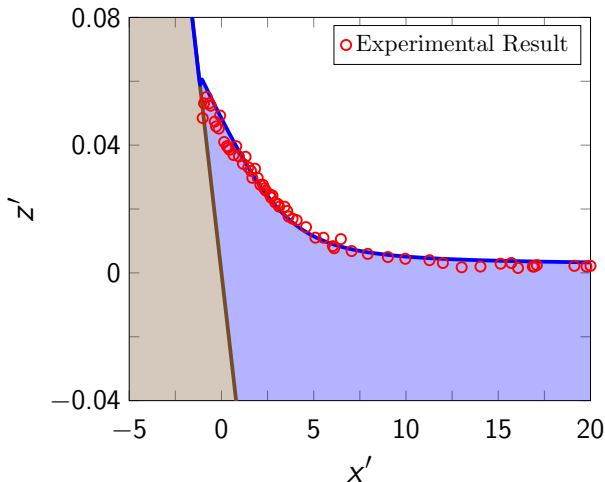


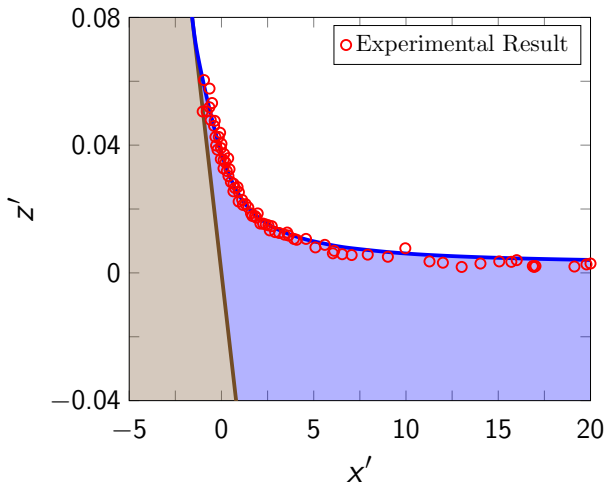
Numerical Solution

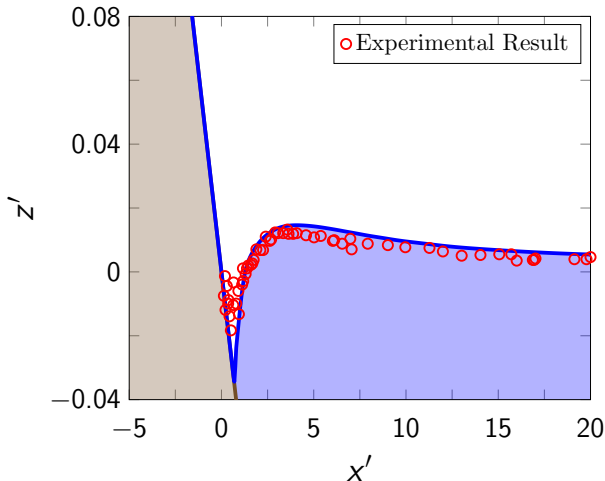


Comparison $t' = 30$ 

Comparison $t' = 40$ 

Comparison $t' = 50$ 

Comparison $t' = 60$ 

Comparison $t' = 70$ 

Conclusion

Finite Element Volume Method for The Serre Equations

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Thanks!