1 Numerical Method Break Down

Our conservative update is, for our equations is

$$\bar{q}_j^{n+1} = \bar{q}_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right]$$

This converts to (both analytical and numerical)

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}v_{j-1} - \mathcal{F}^{q,q}q_{j-1}\right]$$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j - \mathcal{F}^{q,v}e^{-ik\Delta x}v_j - \mathcal{F}^{q,q}e^{-ik\Delta x}q_j\right]$$

Defining $\mathcal{D}_x = 1 - e^{-ik\Delta x}$

$$\mathcal{M}q_j^{n+1} = \mathcal{M}q_j^n - \frac{\Delta t}{\Delta x}\left[\mathcal{D}_x\mathcal{F}^{q,v}v_j + \mathcal{D}_x\mathcal{F}^{q,q}q_j\right]$$

So we have

$$q_j^{n+1} = q_j^n - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\left[\mathcal{F}^{q,v}v_j + \mathcal{F}^{q,q}q_j\right]$$

Thus we have

$$\begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ \mathcal{G}u \end{bmatrix}_j^{n} - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \mathcal{F}^{u,h} & \mathcal{F}^{u,u} \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} - \frac{\mathcal{D}_x\Delta t}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{bmatrix}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

Lets define

$$\boldsymbol{F} = \frac{\mathcal{D}_x}{\mathcal{M}\Delta x}\begin{bmatrix} \mathcal{F}^{h,h} & \mathcal{F}^{h,u} \\ \frac{1}{\mathcal{G}}\mathcal{F}^{u,h} & \frac{1}{\mathcal{G}}\mathcal{F}^{u,u} \end{bmatrix}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \begin{bmatrix} h \\ u \end{bmatrix}_j^{n} - \Delta t\boldsymbol{F}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = (\boldsymbol{I} - \Delta t \boldsymbol{F}) \begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

Thats our Euler Step, the difference between this and the previous version is we didn't divide that bottom Row by $\mathcal{G}$ So we have to change our approximation stuff. Also we would like the know what the analytic value of $\boldsymbol{F}$ is and approximations to it.

1.1 Analytic

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,u} = ikH$$

$$\frac{\mathcal{D}_a}{\Delta x \mathcal{M}_a} \mathcal{F}_a^{h,h} = 0$$

$$\frac{\mathcal{D}_a}{\mathcal{G}_a \Delta x \mathcal{M}_a} \mathcal{F}_a^{u,h} = \frac{ikgH}{H + \frac{H^3}{3} k^2} = i \frac{k}{H} gH \frac{3}{3 + H^2 k^2}$$

using $\omega = \pm k \sqrt{gH} \sqrt{\frac{3}{H^2 k^2 + 3}}$, $\omega^2 = k^2 gH \frac{3}{H^2 k^2 + 3}$

$$\frac{\mathcal{D}_a}{\mathcal{G} \Delta x \mathcal{M}_a} \mathcal{F}_a^{u,h} = i \frac{k}{H} \frac{\omega^2}{k^2} = -i \frac{\omega^2}{kH}$$

$$\frac{\mathcal{D}_a}{\mathcal{G} \Delta x \mathcal{M}_a} \mathcal{F}_a^{u,u} = 0$$

So we have

$$\boldsymbol{F} = \begin{bmatrix} 0 & ikH \\ \frac{\omega^2}{ikH} & 0 \end{bmatrix} = \frac{1}{ikH} \begin{bmatrix} 0 & -k^2 H^2 \\ \omega^2 & 0 \end{bmatrix}$$

We can diagonalise this ($A = PDP^-1$) with the follwing matricers

$$\boldsymbol{F} = \frac{1}{ikH} \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -ikH\omega & 0 \\ 0 & ikH\omega \end{bmatrix} \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\boldsymbol{F} = \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix}^{-1}$$

We will use the following notation

$$\boldsymbol{W}_a = \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix}$$

$$\boldsymbol{S}_a = \begin{bmatrix} -\frac{ikH}{\omega} & \frac{ikH}{\omega} \\ 1 & 1 \end{bmatrix}$$

So we have

$$\boldsymbol{F}_a = \boldsymbol{S}_a \boldsymbol{W}_a \boldsymbol{S}_a^{-1}$$

1.2 First Order

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,u} = iHk - \frac{iHk^3}{6}(\Delta x)^2 + O(\Delta x^3)$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1} \mathcal{F}_1^{h,h} = \frac{k^2\sqrt{gH}}{2}\Delta x + O(\Delta x^2)$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,h} = \frac{gH}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left(1+e^{ik\Delta x}\right)\left[H - \frac{H^3}{3}\left(\frac{2\cos\left(k\Delta x\right)-2}{\Delta x^2}\right)\right]^{-1}$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,h} = \frac{3igk}{H^2k^2+3} - \frac{igk^3(H^2k^2+6)}{4(H^2k^2+3)^2}\Delta x^2 + O(\Delta x^3)$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}-1\right]$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1} \mathcal{F}_1^{u,u} = \frac{1}{2}k^2\sqrt{gH}\Delta x + O(\Delta x^2)$$

So

$$\boldsymbol{F} = \begin{bmatrix} \frac{k^2\sqrt{gH}}{2}\Delta x + O(\Delta x^2) & iHk - \frac{iHk^3}{6}(\Delta x)^2 + O(\Delta x^3) \\ \dfrac{3igk}{H^2k^2+3} - \dfrac{igk^3(H^2k^2+6)}{4(H^2k^2+3)^2}\Delta x^2 + O(\Delta x^3) & \frac{1}{2}k^2\sqrt{gH}\Delta x + O(\Delta x^2) \end{bmatrix}$$

$$\boldsymbol{F} = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x + O(\Delta x^2) & iHk - \frac{iHk^3}{6}(\Delta x)^2 + O(\Delta x^3) \\ i\frac{\omega^2}{Hk} - i\frac{\omega^2}{4kH}\dfrac{(H^2k^2+6)}{(H^2k^2+3)}\Delta x^2 + O(\Delta x^3) & \frac{1}{2}k^2\sqrt{gH}\Delta x + O(\Delta x^2) \end{array} \right]$$

$$\boldsymbol{F} = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x + O(\Delta x^2) & iHk - \frac{iHk^3}{6}(\Delta x)^2 + O(\Delta x^3) \\ i\frac{\omega^2}{Hk} - i\frac{\omega^2}{4kH}\left[1 + \frac{3}{H^2k^2+3}\right]\Delta x^2 + O(\Delta x^3) & \frac{1}{2}k^2\sqrt{gH}\Delta x + O(\Delta x^2) \end{array} \right]$$

$$\boldsymbol{F} = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x + O(\Delta x^2) & iHk - \frac{iHk^3}{6}(\Delta x)^2 + O(\Delta x^3) \\ i\frac{\omega^2}{Hk} - i\frac{\omega^2}{4kH}\left[1 + \frac{\omega^2}{k^2gH}\right]\Delta x^2 + O(\Delta x^3) & \frac{1}{2}k^2\sqrt{gH}\Delta x + O(\Delta x^2) \end{array} \right]$$

$$\boldsymbol{F} = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x & iHk - \frac{iHk^3}{6}(\Delta x)^2 \\ i\frac{\omega^2}{Hk} - i\frac{\omega^2}{4kH}\left[1 + \frac{\omega^2}{k^2gH}\right]\Delta x^2 & \frac{k^2\sqrt{gH}}{2}\Delta x \end{array} \right] + O(\Delta x^2)$$

$$\boldsymbol{F} = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x & iHk - \frac{iHk^3}{6}(\Delta x)^2 \\ i\frac{\omega^2}{Hk} - i\frac{\omega^2}{4kH}\left[1 + \frac{\omega^2}{k^2gH}\right]\Delta x^2 & \frac{k^2\sqrt{gH}}{2}\Delta x \end{array} \right] + O(\Delta x^2)$$

$$\boldsymbol{F} = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x & iHk \\ i\frac{\omega^2}{Hk} & \frac{k^2\sqrt{gH}}{2}\Delta x \end{array} \right] + O(\Delta x^2)$$

We neglect the $O(\Delta x^2)$ terms and get

$$\boldsymbol{F}_1 = \left[\begin{array}{cc} \frac{k^2\sqrt{gH}}{2}\Delta x & iHk \\ i\frac{\omega^2}{Hk} & \frac{k^2\sqrt{gH}}{2}\Delta x \end{array} \right]$$

which has the following eigenvalues

$$\lambda_{1,\pm} = \frac{\pm 2ikH\omega + Hk^3\sqrt{gH}\Delta x}{2Hk}$$

So

$$\boldsymbol{F}_1 = \boldsymbol{S}_1 \left[\begin{array}{cc} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{array} \right] \boldsymbol{S}_1^{-1}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{S}_1 \left[\begin{array}{cc} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{array}\right] \boldsymbol{S}_1^{-1}\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\boldsymbol{S}_1^{-1}\left(\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1}\right) = \left(\boldsymbol{S}_1^{-1} - \Delta t \left[\begin{array}{cc} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{array}\right] \boldsymbol{S}_1^{-1}\right) \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\boldsymbol{S}_1^{-1}\mathcal{D}^t\left(\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right) = -\Delta t \left[\begin{array}{cc} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{array}\right] \boldsymbol{S}_1^{-1} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\frac{\mathcal{D}^t}{\Delta t}\left(\boldsymbol{S}_1^{-1}\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n}\right) = - \left[\begin{array}{cc} \lambda_{1,-} & 0 \\ 0 & \lambda_{1,+} \end{array}\right] \left(\boldsymbol{S}_1^{-1} \left[\begin{array}{c} h \\ u \end{array}\right]_j^n\right)$$

so
$\frac{\mathcal{D}^t}{\Delta t} = -\lambda_{1,\pm}$
So

$$\frac{\mathcal{D}^t}{\Delta t} = -\frac{\pm 2ikH\omega + Hk^3\sqrt{gH}\Delta x}{2Hk}$$

$$\frac{\mathcal{D}^t}{\Delta t} = -(\pm i\omega + \frac{1}{2}k^2\sqrt{gH}\Delta x)$$

$$\frac{\mathcal{D}^t}{\Delta t} = \mp i\omega - \frac{1}{2}k^2\sqrt{gH}\Delta x$$

$$\frac{\mathcal{D}^t}{\Delta t} + \frac{1}{2}k^2\sqrt{gH}\Delta x = \mp i\omega$$

$$\pm i\left(\frac{\mathcal{D}^t}{\Delta t} + \frac{1}{2}k^2\sqrt{gH}\Delta x\right) = \omega$$

Can we do it with full analytic values? Ok our matrix is the following

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{h,u} = \frac{1 - e^{-ik\Delta x}}{\Delta x} H \frac{e^{ik\Delta x} + 1}{2}$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{M}_1}\mathcal{F}_1^{h,h} = -\frac{\sqrt{gH}}{2}\frac{1 - e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x} - 1\right]$$

$$\frac{\mathcal{D}}{\Delta x \mathcal{G}_1 \mathcal{M}_1}\mathcal{F}_1^{u,h} = \frac{gH}{2}\frac{1 - e^{-ik\Delta x}}{\Delta x}\left(1 + e^{ik\Delta x}\right)\left[H - \frac{H^3}{3}\left(\frac{2\cos(k\Delta x) - 2}{\Delta x^2}\right)\right]^{-1}$$

$$\frac{\mathcal{D}}{\mathcal{G}_1 \Delta x \mathcal{M}_1}\mathcal{F}_1^{u,u} = -\frac{\sqrt{gH}}{2}\frac{1-e^{-ik\Delta x}}{\Delta x}\left[e^{ik\Delta x}-1\right]$$

$$\boldsymbol{F} = \frac{1-e^{-ik\Delta x}}{\Delta x}\left[\begin{array}{cc} -\frac{\sqrt{gH}}{2}\left[e^{ik\Delta x}-1\right] & H\frac{e^{ik\Delta x}+1}{2} \\ \frac{g}{2}\left(1+e^{ik\Delta x}\right)\left[1-\frac{H^2}{3}\left(\frac{2\cos(k\Delta x)-2}{\Delta x^2}\right)\right]^{-1} & -\frac{\sqrt{gH}}{2}\left[e^{ik\Delta x}-1\right] \end{array}\right]$$

1.3 Second Order

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^1 = (\boldsymbol{I}-\Delta t\boldsymbol{F})\left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^2 = (\boldsymbol{I}-\Delta t\boldsymbol{F})\left[\begin{array}{c} h \\ u \end{array}\right]_j^1$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\left[\begin{array}{c} h \\ u \end{array}\right]_j^n + \left[\begin{array}{c} h \\ u \end{array}\right]_j^2\right)$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\left[\begin{array}{c} h \\ u \end{array}\right]_j^n + (\boldsymbol{I}-\Delta t\boldsymbol{F})^2\left[\begin{array}{c} h \\ u \end{array}\right]_j^n\right)$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\boldsymbol{I}+(\boldsymbol{I}-\Delta t\boldsymbol{F})^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(\boldsymbol{I}+\boldsymbol{I}-2\Delta t\boldsymbol{F}+\Delta t^2\boldsymbol{F}^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I}-2\Delta t\boldsymbol{F}+\Delta t^2\boldsymbol{F}^2\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

lets say we have

$$\boldsymbol{F}_2 = \boldsymbol{S}_2\left[\begin{array}{cc} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{array}\right]\boldsymbol{S}_2^{-1}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{I}-2\Delta t\boldsymbol{S}_2\left[\begin{array}{cc} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{array}\right]\boldsymbol{S}_2^{-1}+\Delta t^2\boldsymbol{S}_2\left[\begin{array}{cc} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{array}\right]\boldsymbol{S}_2^{-1}\right)\left[\begin{array}{c} h \\ u \end{array}\right]_j^n$$

$$\boldsymbol{S}_2^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{S}_2^{-1} - 2\Delta t\begin{bmatrix} \lambda_{2,-} & 0 \\ 0 & \lambda_{2,+} \end{bmatrix}\boldsymbol{S}_2^{-1} + \Delta t^2\begin{bmatrix} \lambda_{2,-}^2 & 0 \\ 0 & \lambda_{2,+}^2 \end{bmatrix}\boldsymbol{S}_2^{-1}\right)\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$\boldsymbol{S}_2^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n+1} = \frac{1}{2}\left(2\boldsymbol{S}_2^{-1} + \begin{bmatrix} \Delta t^2\lambda_{2,-}^2 - 2\Delta t\lambda_{2,-} & 0 \\ 0 & \Delta t^2\lambda_{2,+}^2 - 2\Delta t\lambda_{2,-} \end{bmatrix}\boldsymbol{S}_2^{-1}\right)\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}$$

$$e^{i\omega\Delta t}\left(\boldsymbol{S}_2^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}\right) = \frac{1}{2}\left(2 + \begin{bmatrix} \Delta t^2\lambda_{2,-}^2 - 2\Delta t\lambda_{2,-} & 0 \\ 0 & \Delta t^2\lambda_{2,+}^2 - 2\Delta t\lambda_{2,-} \end{bmatrix}\right)\left(\boldsymbol{S}_2^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}\right)$$

$$e^{i\omega\Delta t}\left(\boldsymbol{S}_2^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}\right) = \frac{1}{2}\begin{bmatrix} 2 + \Delta t^2\lambda_{2,-}^2 - 2\Delta t\lambda_{2,-} & 0 \\ 0 & 2 + \Delta t^2\lambda_{2,+}^2 - 2\Delta t\lambda_{2,-} \end{bmatrix}\left(\boldsymbol{S}_2^{-1}\begin{bmatrix} h \\ u \end{bmatrix}_j^{n}\right)$$

So we have

$$e^{i\omega\Delta t} = 1 + \frac{1}{2}\Delta t^2\lambda_{2,\pm}^2 - \Delta t\lambda_{2,\pm}$$

1.4 Third Order

$$\begin{bmatrix} h \\ u \end{bmatrix}^{1} = (\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\begin{bmatrix} h \\ u \end{bmatrix}^{n}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{2} = (\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\begin{bmatrix} h \\ u \end{bmatrix}^{1}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{3} = \frac{3}{4}\begin{bmatrix} h \\ u \end{bmatrix}^{n} + \frac{1}{4}\begin{bmatrix} h \\ u \end{bmatrix}^{2}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{4} = (\boldsymbol{I} - \Delta t\boldsymbol{F}_3)\begin{bmatrix} h \\ u \end{bmatrix}^{3}$$

$$\begin{bmatrix} h \\ u \end{bmatrix}^{n+1} = \frac{1}{3}\begin{bmatrix} h \\ u \end{bmatrix}^{n} + \frac{2}{3}\begin{bmatrix} h \\ u \end{bmatrix}^{4}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{3} = \frac{3}{4}\left[\begin{array}{c} h \\ u \end{array}\right]^{n} + \frac{1}{4}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)^2 \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{3} = \left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)^2\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \frac{1}{3}\left[\begin{array}{c} h \\ u \end{array}\right]^{n} + \frac{2}{3}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}\left(\boldsymbol{I} - \Delta t \boldsymbol{F}_3\right)^2\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + \frac{2}{3}(\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{3}{4}\boldsymbol{I} + \frac{1}{4}\left(\boldsymbol{I} - \Delta t \boldsymbol{F}_3\right)^2\right)\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + (\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{1}{2}\boldsymbol{I} + \frac{1}{6}\left(\boldsymbol{I} - 2\Delta t \boldsymbol{F}_3 + \Delta t^2 \boldsymbol{F}_3^2\right)\right)\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + (\boldsymbol{I} - \Delta t \boldsymbol{F}_3)\left(\frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2\right)\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\frac{1}{3}\boldsymbol{I} + \frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 + (-\Delta t \boldsymbol{F}_3)\left(\frac{2}{3}\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2\right)\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \frac{1}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 - \frac{2}{3}\Delta t \boldsymbol{F}_3 + \frac{1}{3}\Delta t \boldsymbol{F}_3 \Delta t \boldsymbol{F}_3 - \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 \Delta t \boldsymbol{F}_3\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \boldsymbol{F}_3 + \frac{1}{6}\Delta t^2 \boldsymbol{F}_3^2 + \frac{1}{3}\Delta t^2 \boldsymbol{F}_3^2 - \frac{1}{6}\Delta t^3 \boldsymbol{F}_3^3\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

$$\left[\begin{array}{c} h \\ u \end{array}\right]^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{F}_3 + \frac{1}{2}\Delta t^2 \boldsymbol{F}_3^2 - \frac{1}{6}\Delta t^3 \boldsymbol{F}_3^3\right) \left[\begin{array}{c} h \\ u \end{array}\right]^{n}$$

lets say we have

$$\boldsymbol{F}_3 = \boldsymbol{S}_3 \boldsymbol{D}_3 \boldsymbol{S}_3^{-1}$$

$$\left[\begin{array}{c} h \\ u \end{array} \right]^{n+1} = \left(\boldsymbol{I} - \Delta t \boldsymbol{S}_3 \boldsymbol{D}_3 \boldsymbol{S}_3^{-1} + \frac{1}{2} \Delta t^2 \boldsymbol{S}_3 \boldsymbol{D}_3^2 \boldsymbol{S}_3^{-1} - \frac{1}{6} \Delta t^3 \boldsymbol{S}_3 \boldsymbol{D}_3^3 \boldsymbol{S}_3^{-1} \right) \left[\begin{array}{c} h \\ u \end{array} \right]^{n}$$

$$\boldsymbol{S}_3^{-1} \left[\begin{array}{c} h \\ u \end{array} \right]^{n+1} = \left(\boldsymbol{S}_3^{-1} - \Delta t \boldsymbol{D}_3 \boldsymbol{S}_3^{-1} + \frac{1}{2} \Delta t^2 \boldsymbol{D}_3^2 \boldsymbol{S}_3^{-1} - \frac{1}{6} \Delta t^3 \boldsymbol{D}_3^3 \boldsymbol{S}_3^{-1} \right) \left[\begin{array}{c} h \\ u \end{array} \right]^{n}$$

$$e^{i\omega\Delta t} \boldsymbol{S}_3^{-1} \left[\begin{array}{c} h \\ u \end{array} \right]^{n} = \left(\boldsymbol{I} - \Delta t \boldsymbol{D}_3 + \frac{1}{2} \Delta t^2 \boldsymbol{D}_3^2 - \frac{1}{6} \Delta t^3 \boldsymbol{D}_3^3 \right) \boldsymbol{S}_3^{-1} \left[\begin{array}{c} h \\ u \end{array} \right]^{n}$$

$$e^{i\omega\Delta t} \boldsymbol{S}_3^{-1} \left[\begin{array}{c} h \\ u \end{array} \right]^{n} = \left[\begin{array}{cc} 1 - \Delta t \lambda_{3,-} + \frac{\Delta t^2}{2} \lambda_{3,-}^2 - \frac{\Delta t^3}{6} \lambda_{3,-}^3 & 0 \\ 0 & 1 - \Delta t \lambda_{3,+} + \frac{\Delta t^2}{2} \lambda_{3,+}^2 - \frac{\Delta t^3}{6} \lambda_{3,+}^3 \end{array} \right] \boldsymbol{S}_3^{-1} \left[\begin{array}{c} h \\ u \end{array} \right.$$

$$e^{i\omega\Delta t} = 1 - \Delta t \lambda_{3,\pm} + \frac{\Delta t^2}{2} \lambda_{3,\pm}^2 - \frac{\Delta t^3}{6} \lambda_{3,\pm}^3$$