Dispersive Shock Waves of the Serre equations

Jordan Pitt, Stephen Roberts and Christopher Zoppou
Australian National University

January 24, 2018

Introduction

Summary of our paper “Behaviour of the Serre equations in the presence of steep gradients revisited” (Wave Motion Volume 76, January 2018)

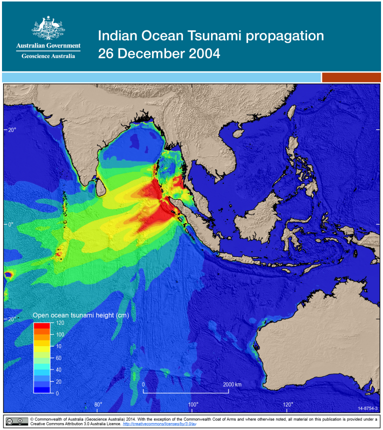
Outline of Presentation:

- Motivation
- Serre Equations
- Dispersive Shock Waves
- Investigation
- Results

Our Background

- Interest : Numerical methods for water waves. Focusing on ocean hazards.

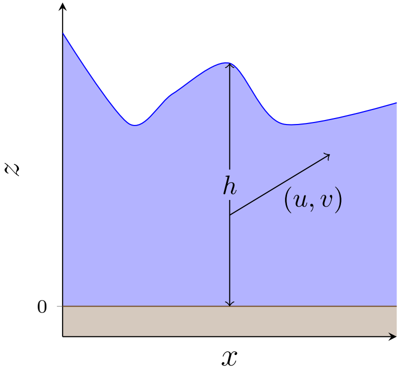
Indian ocean tsunami



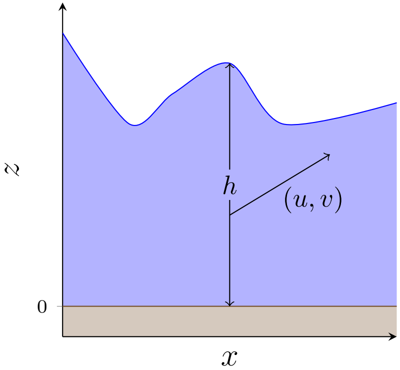
Our Background

- Interest : Numerical methods for water waves. Focusing on ocean hazards.
- Resulted In : Robust numerical method for the Shallow Water Wave Equations (ANUGA)

Shallow Water Wave Equations



Shallow Water Wave Equations



Conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2}\right) = 0$$

Serre equations

Conservation of mass

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Conservation of momentum

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Phi\right) = 0$$

$$\Phi = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}$$

Our Background

- Interest : Numerical methods for water waves.
  Focusing on ocean hazards.
- Resulted In : Robust numerical method for the
  Shallow Water Wave Equations (ANUGA)
- Current Goal : Robust numerical method for the
  Serre equations

Our Background

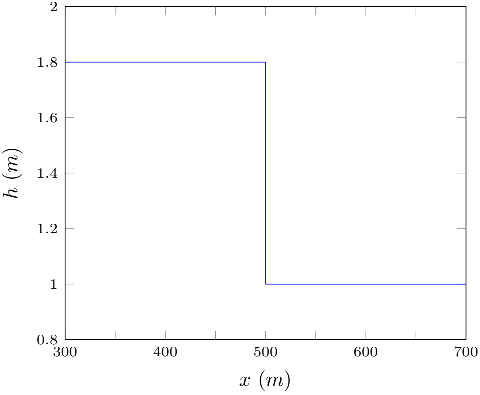
- Interest : Numerical methods for water waves. Focusing on ocean hazards.
- Resulted In : Robust numerical method for the Shallow Water Wave Equations (ANUGA)
- Current Goal : Robust numerical method for the Serre equations
- Problem : Handling discontinuous initial conditions (dam-break problem)

Model Problem : Dam Break Problem

$h_0 = 1m$, $h_1 = 1.8m$ and $x_0 = 500m$

$$h(x,0) = \begin{cases} h_1 & x \leq x_0 \\ h_0 & x > x_0 \end{cases}$$

$$u(x,0) = 0$$



Dispersive Shock Waves

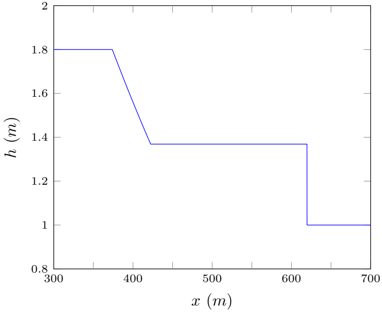


Figure: Shock Wave (analytical solution of the shallow water wave equations).

Dispersive Shock Waves

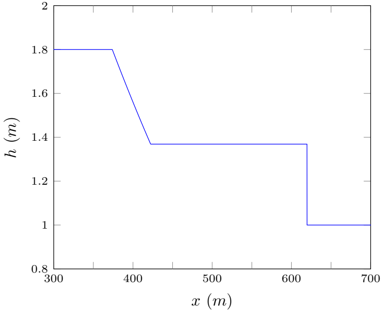


Figure: Shock Wave (analytical solution of the shallow water wave equations).

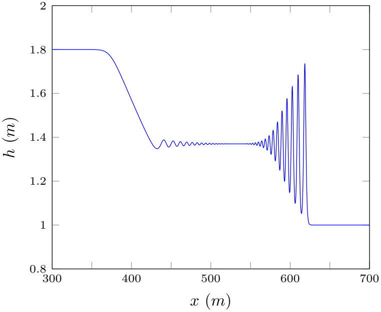


Figure: Dispersive Shock Wave (numerical solution of the Serre equations).

New Observed Behaviour

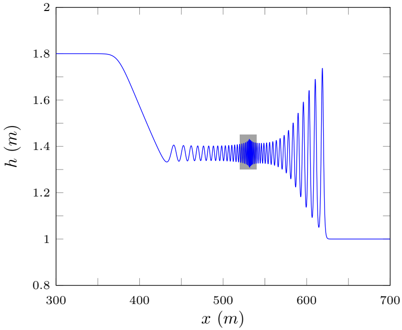


Figure: New observed structure.

New Observed Behaviour

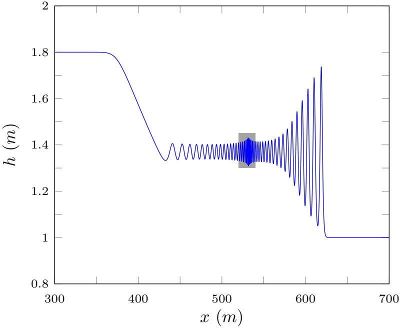


Figure: New observed structure.

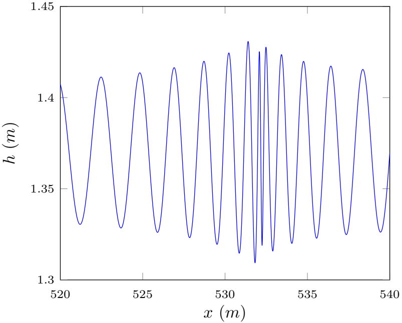


Figure: Zoomed in.

Properties of DSW for the Serre Equations

Asymptotic results for long times

- Whitham modulation results for leading wave amplitude and location

Whitham Modulation Results

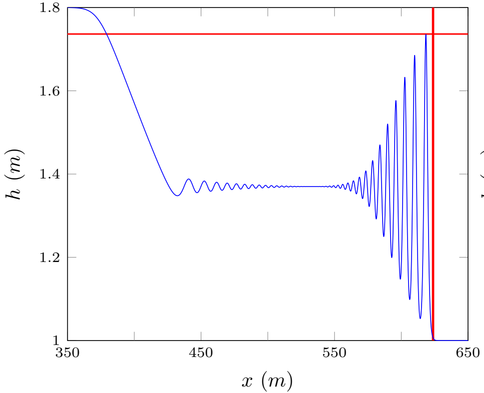


Figure: Common structure.

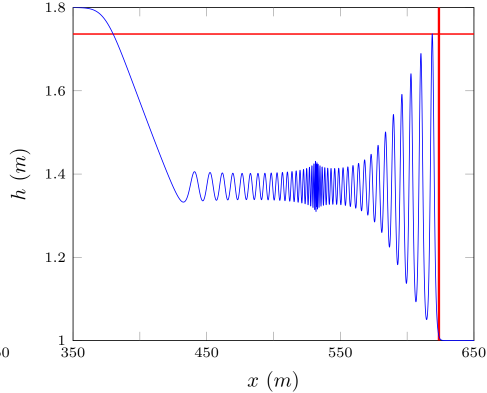


Figure: New structure.

Properties of DSW for the Serre Equations

Asymptotic results for long times

- Whitham modulation results for leading wave amplitude and location
- Oscillations of the DSW for the Serre equations oscillate around the SW of the SWWE

DSW comparison to SW

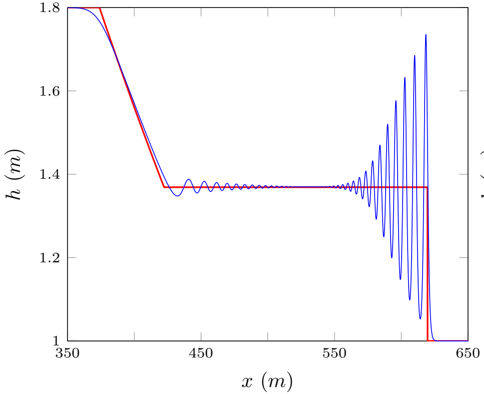


Figure: Common structure.

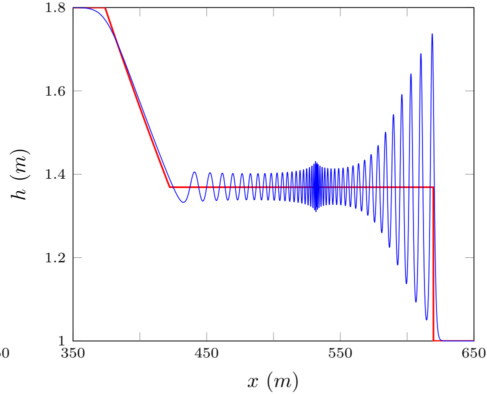


Figure: New structure.

Properties of DSW for the Serre Equations

Asymptotic results for long times

- Whitham modulation results for leading wave amplitude and location
- Oscillations of the DSW for the Serre equations oscillate around the SW of the SWWE

Linear results

- Separate dispersive wave trains

Separation of Dispersive Wave Trains

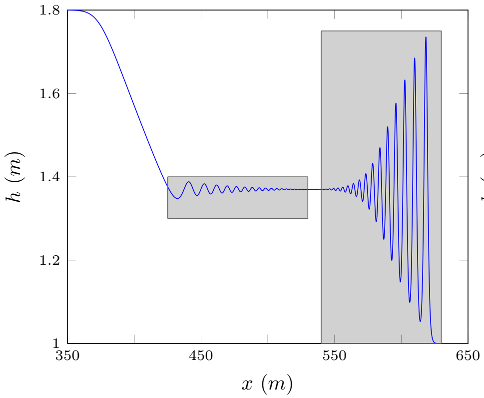


Figure: Common structure.

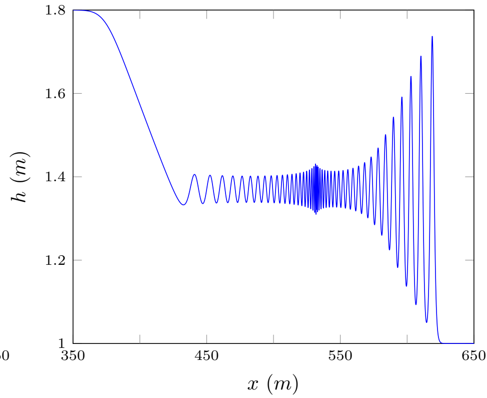


Figure: New structure.

Problem

- No analytic solution of the Serre equations for DSW
- Lack transient properties of DSW

Solution

- Use numerical solvers for the Serre equations on a problem with a smooth approximation to the initial conditions of the dam-break problem.

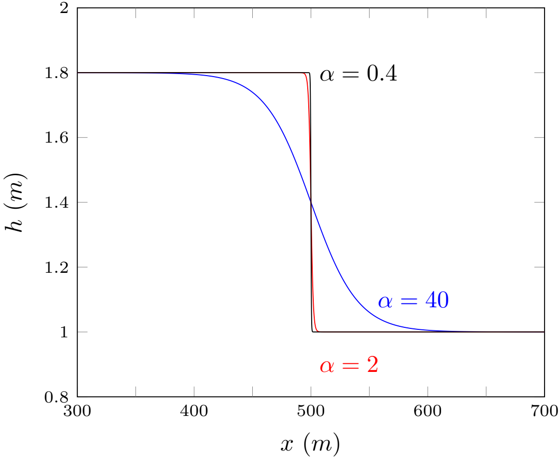
Smooth Dam Break Problem

$$h(x,0) = h_0 + \frac{h_1 - h_0}{2}\left(1 + \tanh\left(\frac{x_0 - x}{\alpha}\right)\right)$$

$$u(x,0) = 0$$

$$h_0 = 1m \ , \ h_1 = 1.8m \text{ and } x_0 = 500m$$

Smooth Dam Break Problem Examples



Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values

Non Oscillatory Structure $\alpha = 40$

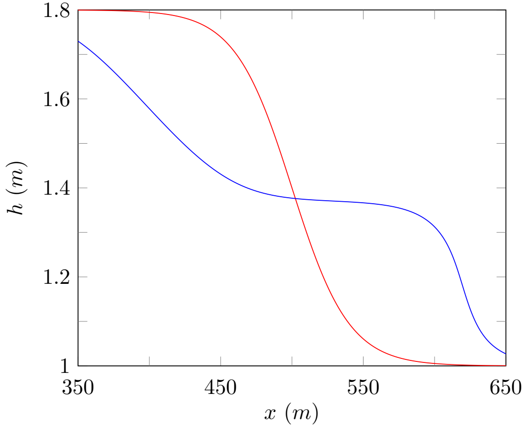


Figure: Highest resolution numerical solution at $t = 30s$

Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values
- Flat Structure (common one in the literature)

Flat Structure $\alpha = 2$

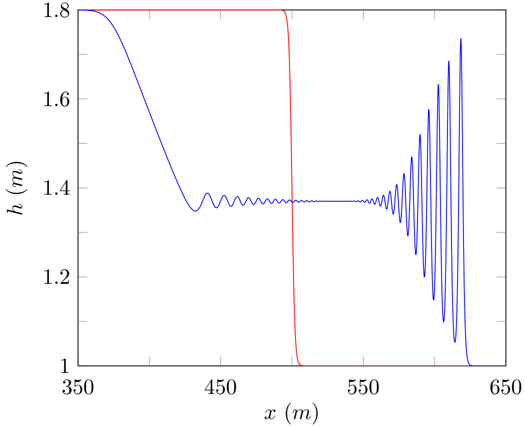


Figure: Highest resolution numerical solution at $t = 30s$

Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values
- Flat Structure (common one in the literature)
- Node Structure (also present in literature)

Node Structure $\alpha = 0.4$

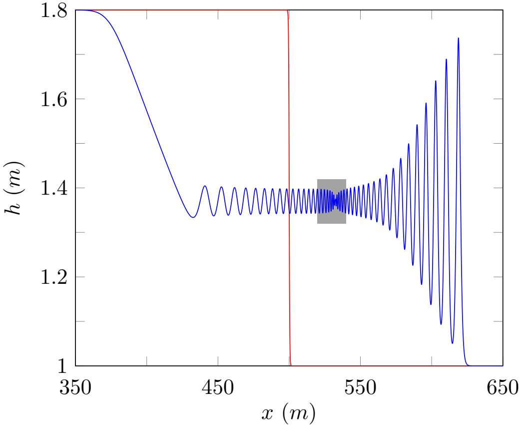
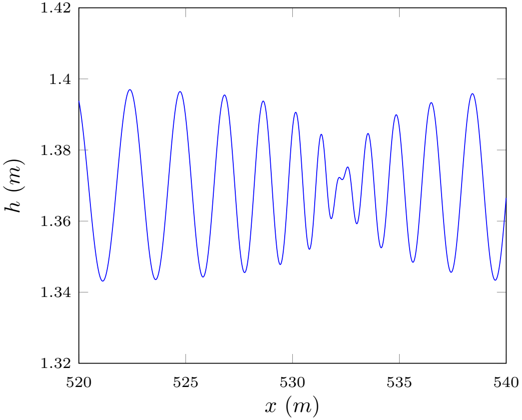


Figure: Highest resolution numerical solution at $t = 30s$

Node Structure $\alpha = 0.4$



Observed Behaviours

We observed four different behaviours of the numerical solution as $\alpha \to 0$

- Non Oscillatory Structure for large α values
- Flat Structure (common one in the literature)
- Node Structure (also present in literature)
- Growth Structure for small α values and the dam-break problem (New behaviour)

Growth Structure $\alpha = 0.1$

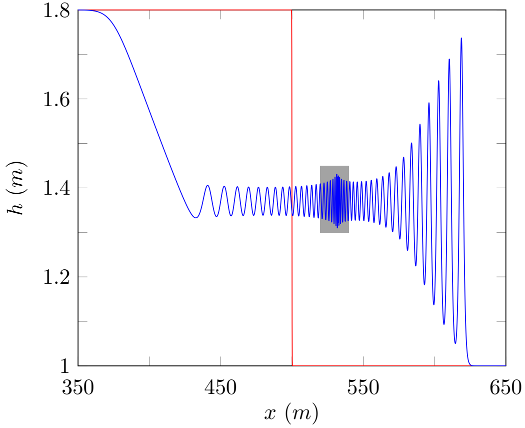
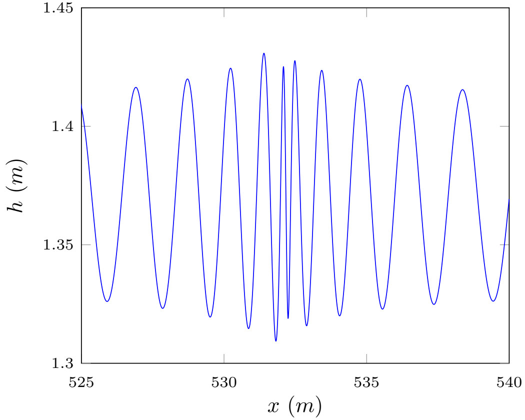
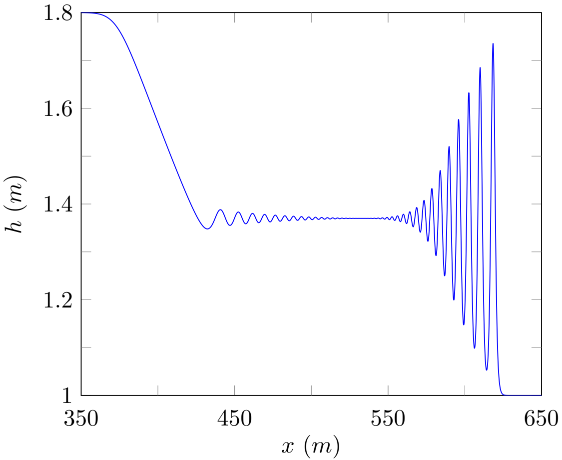


Figure: Highest resolution numerical solution at $t = 30s$

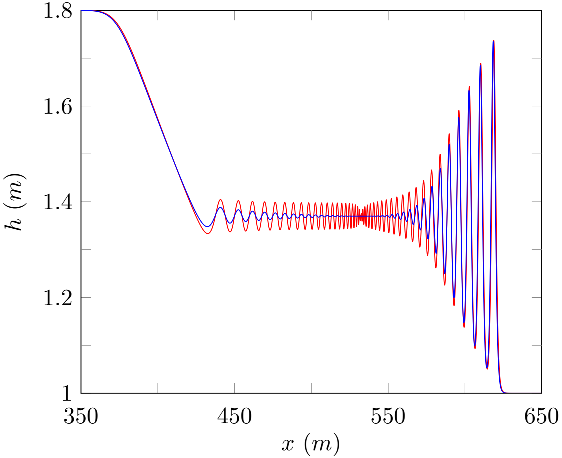
Growth Structure $\alpha = 0.1$



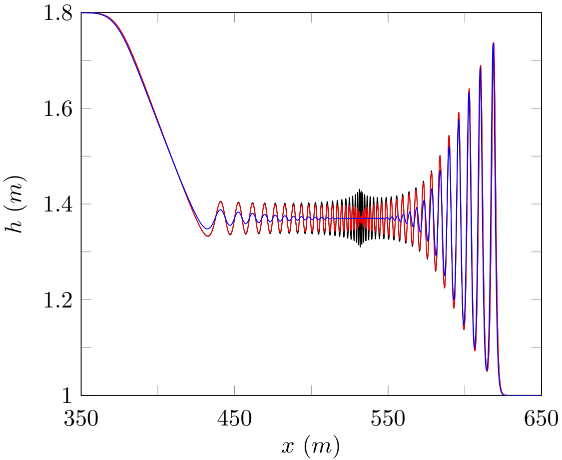
Structure Comparison: Flat



Structure Comparison: Flat & Node



Structure Comparison: Flat & Node & Growth



Justifying These Numerical Solutions

For a particular α value:

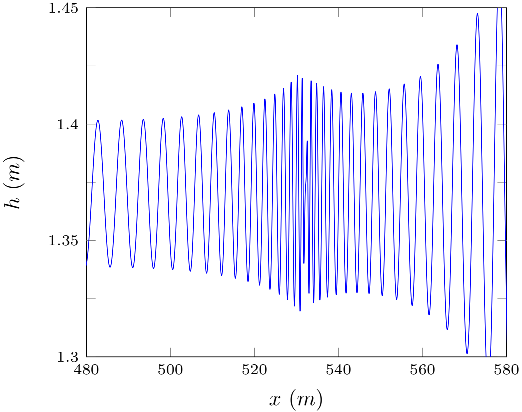
- Demonstrated convergence as the resolution of the method increases
- Demonstrated numerical solutions conserve mass, momentum and the Hamiltonian

These results demonstrated that the Growth Structure is correct structure of DSW for short time spans

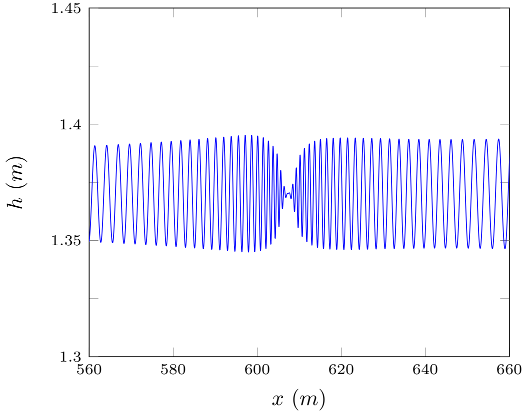
Long term solutions

- Growth structure agrees well with asymptotic results for short and long times
- Growth structure decays to node structure which decays to the flat structure
- Separation of dispersive wave trains over long times

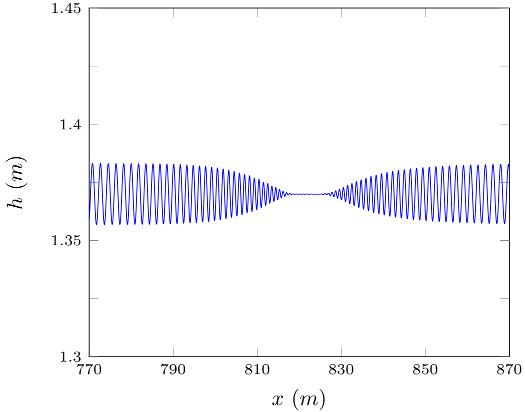
Dam break solution at $t = 30s$



Dam break solution at $t = 100s$



Dam break solution at $t = 300s$



Conclusion

Presentation

- Found new behaviour of DSW for short time spans not previously published in the literature
- Good agreement between numerical solutions and known properties of DSW for long time periods

Paper

- Explained why different behaviour published in the literature
- Justified the robustness of our numerical methods