

# A Finite Element-Volume Method for the Serre Equations

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# Outline

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## ► Motivation

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- ▶ Motivation
- ▶ Method

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- ▶ Motivation
- ▶ Method
- ▶ Results

# Ocean Wave Hazards

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## ► Tsunamis

# Sulawesi 2018 Tsunami



Figure: Sulawesi Tsunami (Indonesia, 2018).



# Ocean Wave Hazards

- ▶ Tsunamis
- ▶ Storm Surges

# Storm Surge of Hurricane Florence and Michael

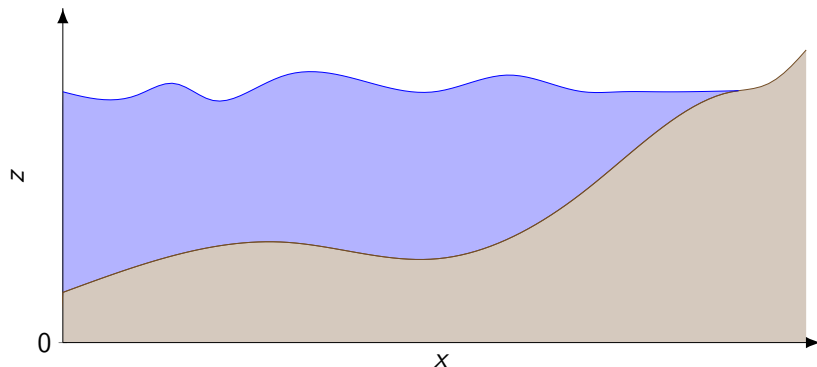


(a) Florence (U.S.A, 2018)

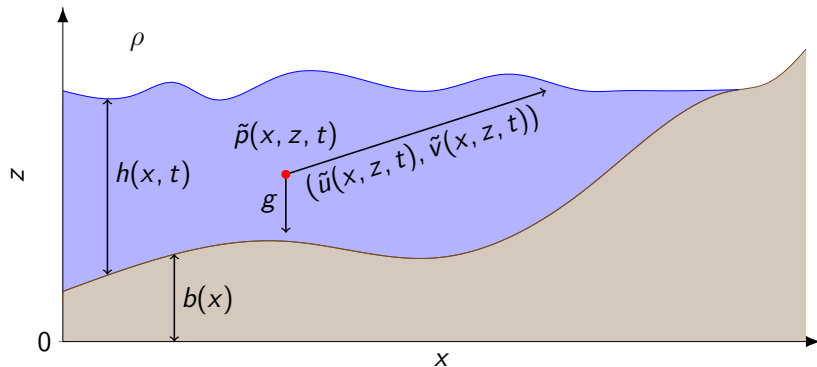


(b) Michael (U.S.A, 2018)

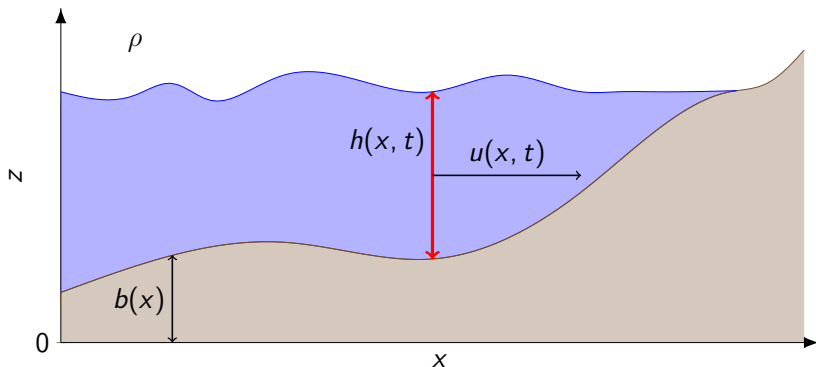
## Two Dimensional Scenario



# Navier-Stokes



# Model Simplification: Serre Equations



# Assumptions

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Quantity	Serre Equations
Particle: $\tilde{u}(x, z, t)$	$u(x, t)$

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Particle: $\tilde{u}(x, z, t)$	$u(x, t)$
Particle: $\tilde{v}(x, z, t)$	$u \frac{\partial b}{\partial x} - (z - b) \frac{\partial b}{\partial x}$



## Assumptions

Quantity	Serre Equations
Particle: $\tilde{u}(x, z, t)$	$u(x, t)$
Particle: $\tilde{v}(x, z, t)$	$u \frac{\partial b}{\partial x} - (z - b) \frac{\partial b}{\partial x}$
Particle: $\tilde{p}(x, z, t)$	$g\rho[h + b - z] + \rho[h + b - z] \psi$ $+ \frac{1}{2}\rho \left( h^2 - [z - b]^2 \right) \phi$

where

$$\psi = \frac{\partial b}{\partial x} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + u^2 \frac{\partial^2 b}{\partial x^2}, \quad \phi = \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - u \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t}.$$

# Equations

Mass: 
$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0,$$

Momentum: 
$$\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left( u^2 h + \frac{gh^2}{2} + \frac{h^2}{2} \psi + \frac{h^3}{3} \phi \right)$$

$$+ \frac{\partial b}{\partial x} \left( gh + h\psi + \frac{h^2}{2} \phi \right) = 0.$$

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**Goal:** Adapt Finite Volume Methods for the Serre Equations

# Finite Volume Method

- Conservation law form

# Finite Volume Method

- ▶ Conservation law form
- ▶ Finite volume update

# Finite Volume Method

- Conservation law form



# Conservation Law Form

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left[ f \left( q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n} \right) \right] + s \left( q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^m q}{\partial x^m} \right) = 0$$

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# Conservation Law Form

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0,$$

$$\begin{aligned} \frac{\partial G}{\partial t} + \frac{\partial}{\partial x} \left( uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[ \frac{\partial u}{\partial x} \right]^2 + h^2 u \frac{\partial u}{\partial x} \frac{\partial b}{\partial x} \right) \\ + \frac{1}{2}h^2 u \frac{\partial u}{\partial x} \frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x} \frac{\partial^2 b}{\partial x^2} + gh \frac{\partial b}{\partial x} = 0. \end{aligned}$$

## Conservation Law Form

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with

$$\mathbf{G} = hu \left( 1 + \frac{\partial h}{\partial x} \frac{\partial b}{\partial x} + \frac{1}{2}h \frac{\partial^2 b}{\partial x^2} + \left[ \frac{\partial b}{\partial x} \right]^2 \right) - \frac{\partial}{\partial x} \left( \frac{1}{3}h^3 \frac{\partial u}{\partial x} \right).$$

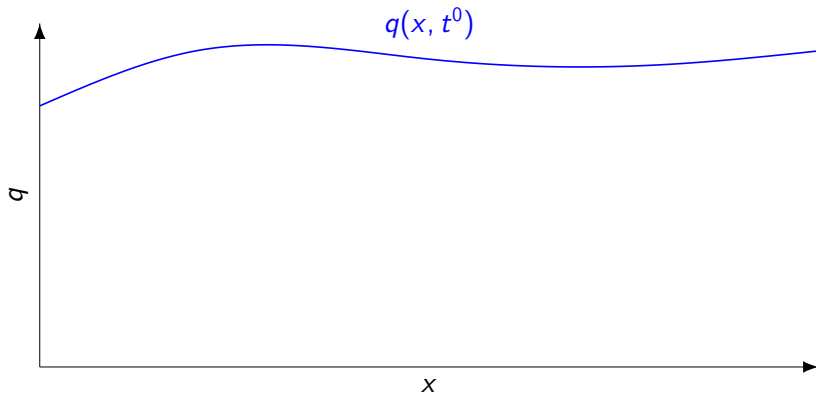
# Finite Volume Method

- ▶ Conservation law form
- ▶ Finite volume update

# Conservation Law Form

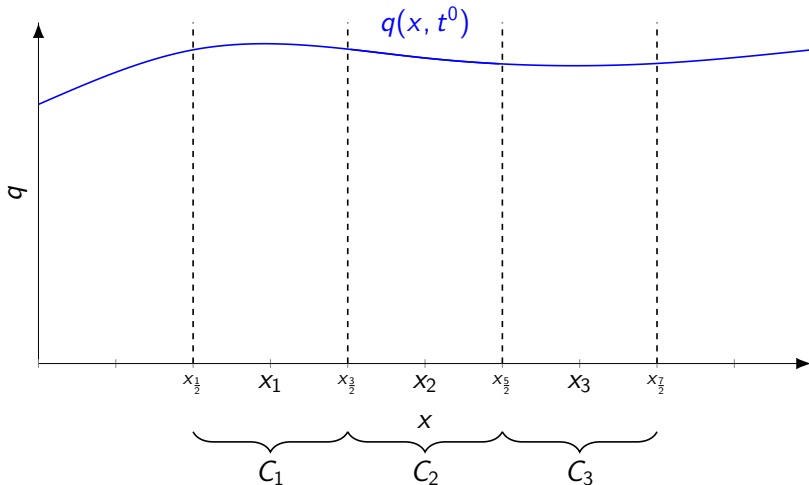
$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left[ f \left( q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^n q}{\partial x^n} \right) \right] + s \left( q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \dots, \frac{\partial^m q}{\partial x^m} \right) = 0$$

# Finite Volume Method

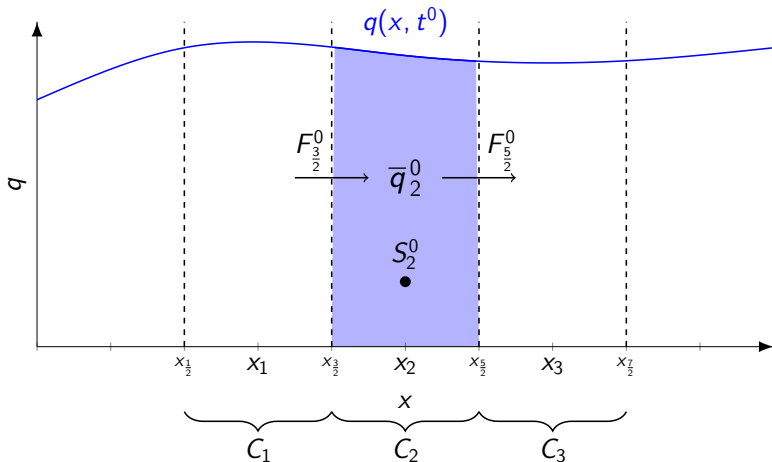




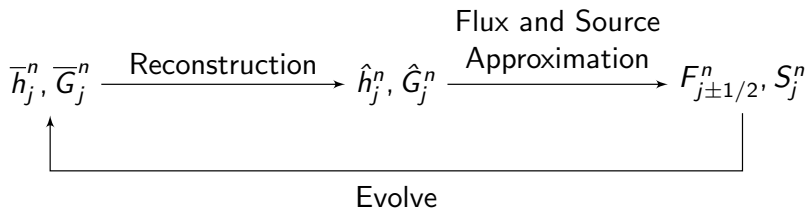
# Discretisation



## Update



# Finite Volume Method



## Require velocity to calculate flux and source

However to calculate the flux and source terms we require  $u$

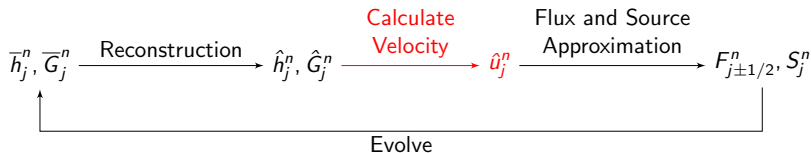
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However to calculate the flux and source terms we require  $u$

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$$\begin{aligned} \frac{\partial G}{\partial t} + \frac{\partial}{\partial x} \left( uG + \frac{gh^2}{2} - \frac{2}{3}h^3 \left[ \frac{\partial u}{\partial x} \right]^2 + h^2 u \frac{\partial u}{\partial x} \frac{\partial b}{\partial x} \right) \\ + \frac{1}{2}h^2 u \frac{\partial u}{\partial x} \frac{\partial^2 b}{\partial x^2} - hu^2 \frac{\partial b}{\partial x} \frac{\partial^2 b}{\partial x^2} + gh \frac{\partial b}{\partial x} = 0. \end{aligned}$$

# Method



# Reconstruction

- Determines spatial order of accuracy

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Goal: Second-order accuracy



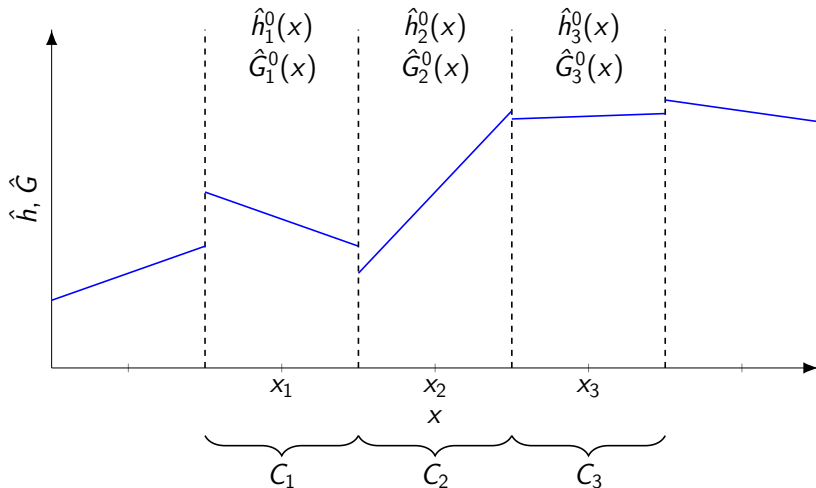
# Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
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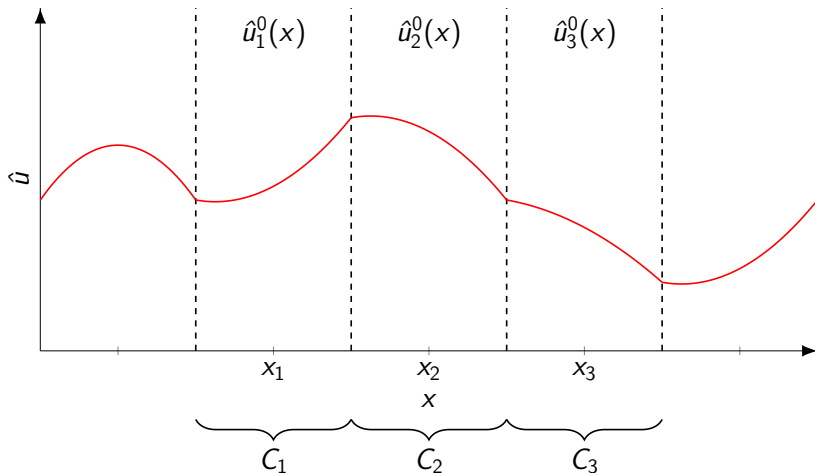
Quantity	Number of spatial derivatives	Reconstructed functions
$h$	zero	linear over cell, discontinuous at edges
$G$	zero	linear over cell, discontinuous at edges

## Reconstruction

 $\hat{h}, \hat{G}$ 

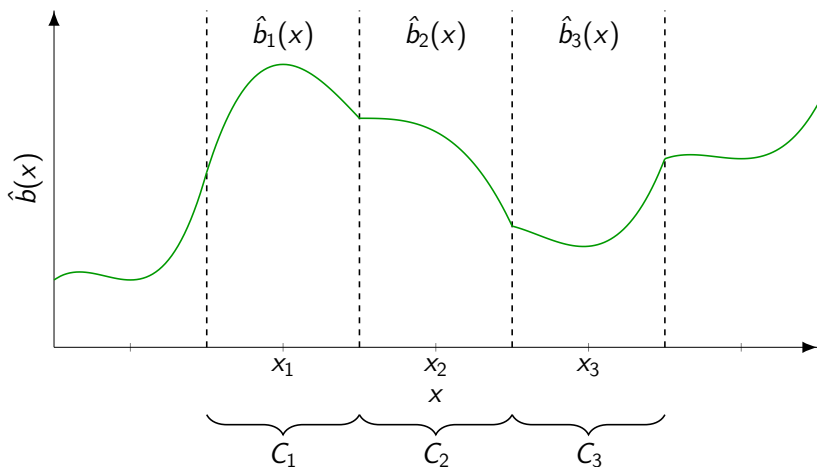
# Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
$h$	zero	linear over cell, discontinuous at edges
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$u$	one	quadratic over cell, continuous at edges

$\hat{u}$ 

# Reconstruction Spaces

Quantity	Number of spatial derivatives	Reconstructed functions
$h$	zero	linear over cell, discontinuous at edges
$G$	zero	linear over cell, discontinuous at edges
$u$	one	quadratic over cell, continuous at edges
$b$	two	cubic over cell, continuous at edges

$\hat{b}$ 

# Finite Element Calculation of Velocity

Finite Element Method to solve:

$$G = hu \left( 1 + \frac{\partial h}{\partial x} \frac{\partial b}{\partial x} + \frac{1}{2} h \frac{\partial^2 b}{\partial x^2} + \left[ \frac{\partial b}{\partial x} \right]^2 \right) - \frac{\partial}{\partial x} \left( \frac{1}{3} h^3 \frac{\partial u}{\partial x} \right).$$

for  $u$  given  $h$ ,  $G$  and  $b$



## Finite Element Calculation of Velocity

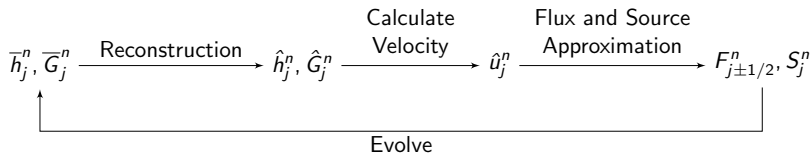
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Solves the weak form replacing all quantities with their reconstructions  $\hat{h}$ ,  $\hat{G}$  and  $\hat{b}$  to get  $\hat{u}$

# Method



# Validation

- Analytic Solution

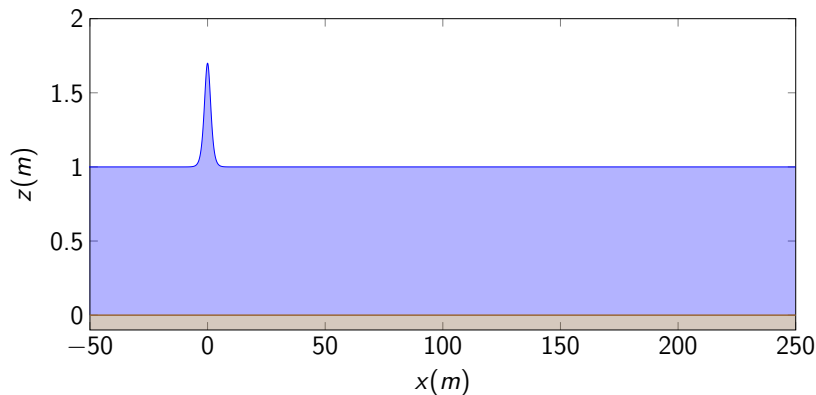
# Validation

- ▶ Analytic Solution
- ▶ Experimental Results

# Validation

## ► Analytic Solution

# Soliton Example





# Soliton Equations

$$h(x, t) = a_0 + a_1 \operatorname{sech}(\kappa(x - ct)),$$

$$u(x, t) = c \left( 1 - \frac{a_0}{h(x, t)} \right),$$

$$b(x) = 0$$



# Soliton Equations

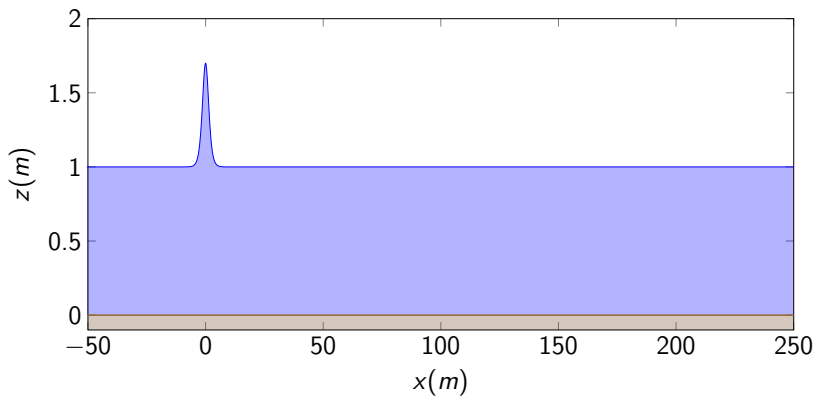
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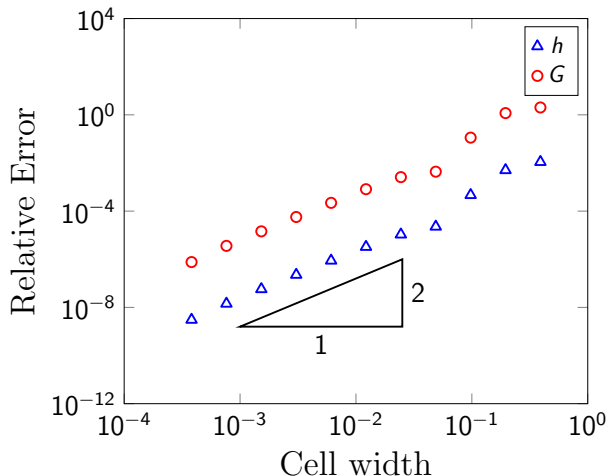
$$\kappa = \frac{\sqrt{3a_1}}{2a_0\sqrt{(a_0 + a_1)}},$$

$$c = \sqrt{g(a_0 + a_1)}.$$

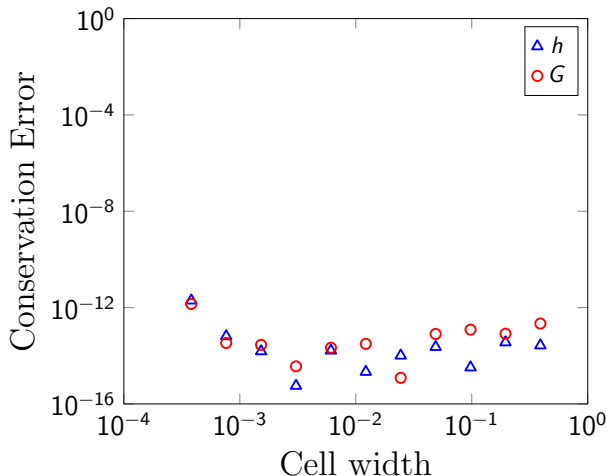
Numerical Solution  $a_0 = 1$ ,  $a_1 = 0.7$ 



# Convergence



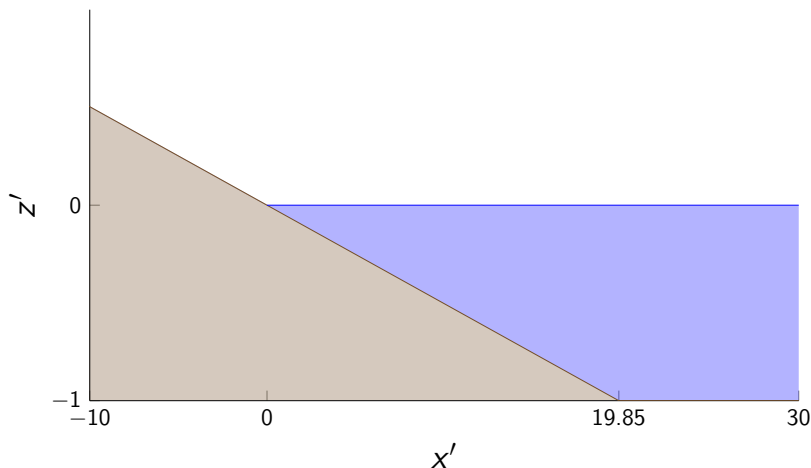
# Conservation



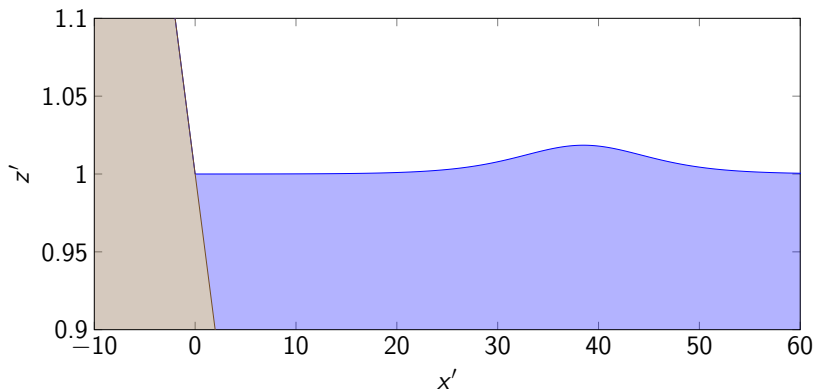
# Validation

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# Synolakis Experiment

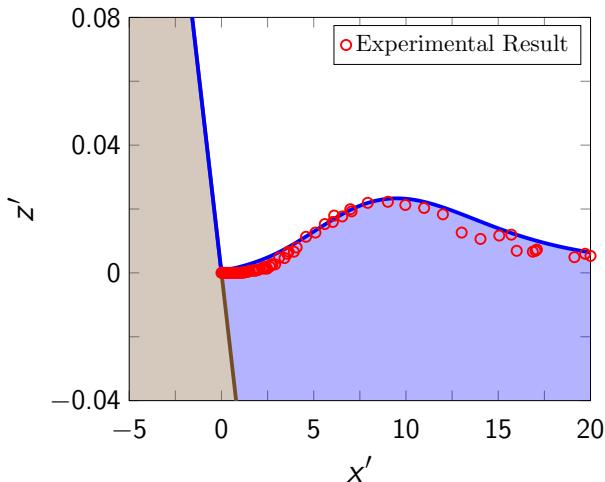


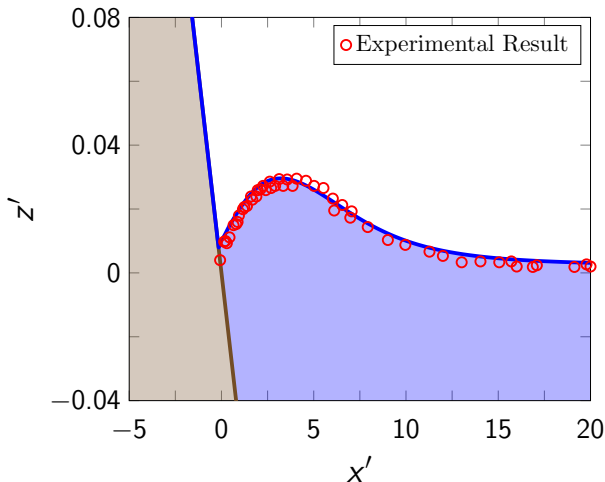
# Numerical Solution

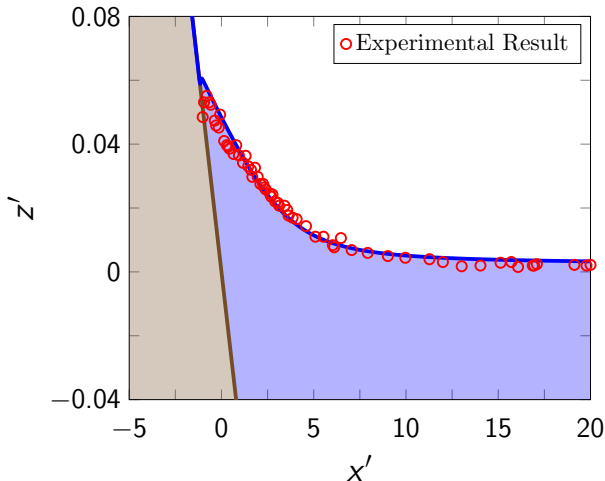


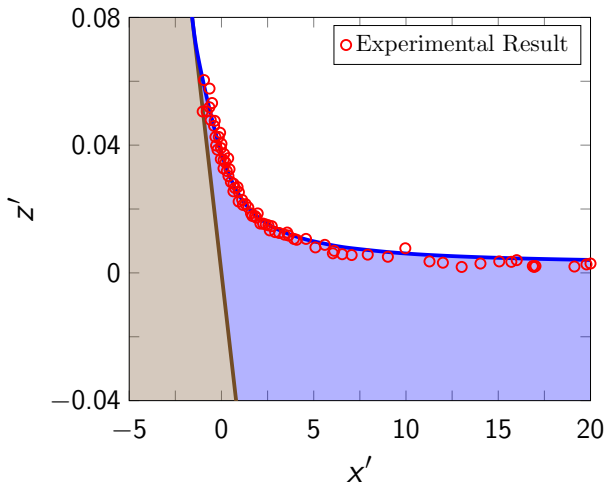


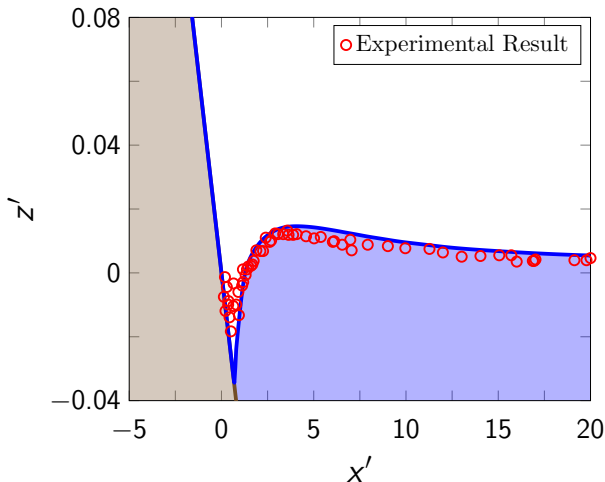


Comparison  $t' = 30$ 

Comparison  $t' = 40$ 

Comparison  $t' = 50$ 

Comparison  $t' = 60$ 

Comparison  $t' = 70$ 

# Conclusion

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- ▶ Reproduces experimental results

# Thanks!