1 Serre Equations

The elliptic equation we want to solve in strong form is:

$$\vec{G} = h\vec{u} - \nabla \left[\frac{h^3}{3} \left(\nabla \cdot \vec{u} \right) \right]$$

First step is to take this equations dot product with a vector of smooth test functions $\vec{v}$.

$$\vec{G} \cdot \vec{v} = h\vec{u} \cdot \vec{v} - \nabla \left[\frac{h^3}{3} \left(\nabla \cdot \vec{u} \right) \right] \cdot \vec{v}$$

Integrate over the domain Ω

$$\int_\Omega \vec{G} \cdot \vec{v} \, dx = \int_\Omega h\vec{u} \cdot \vec{v} \, dx - \int_\Omega \nabla \left[\frac{h^3}{3} \left(\nabla \cdot \vec{u} \right) \right] \cdot \vec{v} \, dx$$

Now we can apply integration by parts to the bottom equation, we will use Dirichlet boundaries (with zero velocities), so we lose the surface integral.

$$\int_\Omega \vec{G} \cdot \vec{v} \, dx = \int_\Omega h\vec{u} \cdot \vec{v} \, dx + \int_\Omega \frac{h^3}{3} \left(\nabla \cdot \vec{u} \right) \left(\nabla \cdot \vec{v} \right) \, dx$$

So our problem becomes.

Find $\vec{u} \in V$ such that

$$\vec{u} = 0 \quad on \, \partial\Omega$$

$$\int_\Omega \vec{G} \cdot \vec{v} \, dx = \int_\Omega h\vec{u} \cdot \vec{v} \, dx + \int_\Omega \frac{h^3}{3} \left(\nabla \cdot \vec{u} \right) \left(\nabla \cdot \vec{v} \right) \, dx$$

$\forall \vec{v}$ who are smooth $\Re^2$ valued functions.

In particular this analysis gives that $\vec{u}$ is in $H(div)$ as long as $\vec{G}$ and h are square integrable, (proof?).

2 Is this sufficient?

Our equations for conservation are:

$$h_t + (uh)_x + (vh)_y = 0 \tag{1}$$

(1) we can expand this to get

$$h_t + h(u_x + v_y) + h_x u + h_y v = 0$$

$$h_t + \nabla \cdot (h\vec{u}) = 0$$

$$F^x = \left[\begin{array}{c} G^x u + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - v \frac{\partial}{\partial y} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \\ uvh \end{array} \right] \quad (2)$$

$$F^y = \left[\begin{array}{c} uvh \\ G^y v + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - u \frac{\partial}{\partial x} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \end{array} \right] \quad (3)$$

Such that

$$\frac{\partial \vec{G}}{\partial t} + \nabla \cdot \vec{F} = 0$$

Expanding

$$\frac{\partial G^x}{\partial t} + \frac{\partial}{\partial x} \left(G^x u + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - v \frac{\partial}{\partial y} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) + \frac{\partial}{\partial y} (uvh) = 0$$

$$\frac{\partial G^y}{\partial t} + \frac{\partial}{\partial x} (uvh) + \frac{\partial}{\partial y} \left(G^y v + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - u \frac{\partial}{\partial x} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) = 0$$

We can group things together a little differently. For instance

$$\left[\begin{array}{c} \frac{\partial}{\partial x} \left(G^x u + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - v \frac{\partial}{\partial y} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \\ \frac{\partial}{\partial y} \left(G^y v + \frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 - u \frac{\partial}{\partial x} \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \end{array} \right] =$$
$$\nabla \left(\vec{G} \cdot \vec{u} + \left(\frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 \right) \left[\begin{array}{c} 1 \\ 1 \end{array} \right] + \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left(\vec{u} \cdot \nabla \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \right) \quad (4)$$

And

$$\left[\begin{array}{c} \frac{\partial}{\partial y} (uvh) \\ \frac{\partial}{\partial x} (uvh) \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \nabla (uvh) = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \nabla \left(h\vec{u} \cdot \left[\begin{array}{cc} 0 & 0.5 \\ 0.5 & 0 \end{array} \right] \vec{u} \right) \quad (5)$$

So we have

$$\frac{\partial \vec{G}}{\partial t} + \nabla \left(\vec{G} \cdot \vec{u} + \left(\frac{gh^2}{2} - \frac{2h^3}{3} (\nabla \cdot \vec{u})^2 \right) \left[\begin{array}{c} 1 \\ 1 \end{array} \right] + \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left(\vec{u} \cdot \nabla \left(\frac{h^3}{3} (\nabla \cdot \vec{u}) \right) \right) \right)$$
$$+ \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \nabla (uvh) = 0 \quad (6)$$

Taking the dot product of this with a text function $\vec{w}$

$$\frac{\partial \vec{G}}{\partial t}+\nabla\left(\vec{G}\cdot\vec{u}+\left(\frac{gh^2}{2}-\frac{2h^3}{3}\left(\nabla\cdot\vec{u}\right)^2\right)\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}\left(\vec{u}\cdot\nabla\left(\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right)\right)\right)\right)$$
$$+\begin{bmatrix}0&1\\1&0\end{bmatrix}\nabla(uvh)=0 \quad (7)$$