# 1 SWW soliton

The continuity equation is:

$$h_t + uh_x + u_x h = 0 \tag{1}$$

the momentum equation is

$$(uh)_t + \left(u^2 h + g\frac{h^2}{2}\right)_x = 0$$

expanding

$$u_t h + uh_t + \left(u^2 h\right)_x + ghh_x = 0$$

$$u_t h + uh_t + 2uu_x h + u^2 h_x + ghh_x = 0$$

using (1)

$$u_t h - u^2 h_x - uu_x h + 2uu_x h + u^2 h_x + ghh_x = 0$$

$$u_t h + uu_x h + ghh_x = 0$$

when $h > 0$

$$u_t + uu_x + gh_x = 0 \tag{2}$$

want solutions to (1) and (2) such that

$$h(x,t) = \phi(y), \quad y = x - ct$$
$$u(x,t) = \psi(y), \quad y = x - ct$$

Substituting these into (1) gives

$$-c\phi'(y) + \psi(y)\phi'(y) + \psi'(y)\phi(y) = 0$$

$$[\psi(y) - c]\,\phi'(y) + \psi'(y)\phi(y) = 0$$
$$[\psi(y) - c]\,\phi'(y) + \psi'(y)\phi(y) = 0 \tag{3}$$

Substituting these into (2) gives

$$-c\psi'(y) + \psi(y)\psi'(y) + g\phi'(y) = 0$$

$$[\psi(y) - c]\,\psi'(y) + g\phi'(y) = 0$$
$$[\psi(y) - c]\,\psi'(y) + g\phi'(y) = 0 \tag{4}$$

Rearranging (4) we get:

$$[\psi(y) - c]\,\psi'(y) = -g\phi'(y)$$
$$[\psi(y) - c] = \frac{-g\phi'(y)}{\psi'(y)}$$

subbing into (3):

$$\frac{-g\phi'(y)}{\psi'(y)}\phi'(y) + \psi'(y)\phi(y) = 0$$

$$(\psi'(y))^2\,\phi(y) = g\,(\phi'(y))^2$$

$$\psi'(y) = \sqrt{g}\frac{\phi'(y)}{\sqrt{\phi(y)}}$$

In particular since $(\sqrt{f(x)})' = f(x)f^{-1/2}(x)$ we must have

$$\psi(y) = \sqrt{g}\sqrt{\phi(y)} + D$$

$$u(x,t) = \sqrt{g}\sqrt{h(x,t)} + D$$