

1 Serre Equations

The Serre Equations read (height/mass)

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0$$

Phi

$$\Phi = \frac{\partial b}{\partial x} \left(u \frac{\partial u}{\partial x} \right) + u^2 \frac{\partial^2 b}{\partial x^2} + \frac{\partial b}{\partial x} \frac{\partial u}{\partial t}$$

Gamma

$$\Gamma = \left(\frac{\partial u}{\partial x} \right)^2 - u \left(\frac{\partial^2 u}{\partial x^2} \right) - \left(\frac{\partial^2 u}{\partial x \partial t} \right)$$

Pressure

$$p|_{\xi} = p_a + \rho g \xi + \frac{\rho}{2} \xi (2h - \xi) \Gamma + \rho \xi \Phi$$

Momentum(velocity) x

$$\frac{\partial(uh)}{\partial t} + \frac{\partial}{\partial x} \left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3} \Gamma + \frac{h^2}{2} \Phi \right) + h \frac{\partial b}{\partial x} \left(g + \frac{h}{2} \Gamma + \Phi \right) = 0$$

z

$$w|_z = \frac{z - b}{h} \frac{\partial h}{\partial t} + u \frac{\partial b}{\partial x}$$

The total energy of the Euler system is $E = \frac{1}{2} |\vec{u}|^2 + \vec{g}z$:

$$\frac{\partial E}{\partial t} + \nabla \cdot (E \vec{u} + p \vec{u}) = 0$$

Integrating over depth $s = h + b$

$$\int_b^s \left(\frac{\partial E}{\partial t} + \nabla \cdot (E \vec{u} + p \vec{u}) \right) dz = 0$$

$$\int_b^s \frac{\partial E}{\partial t} dz + \int_b^s \frac{\partial}{\partial x} (Eu + pu) dz + \int_b^s \frac{\partial}{\partial z} (Ew + pw) dz = 0$$

$$\int_b^s \frac{\partial E}{\partial t} dz + \int_b^s \frac{\partial}{\partial x} (Eu + pu) dz + E(s)w(s) + p(s)w(s) - E(b)w(b) - p(b)w(b) = 0$$

at b $w = 0$ (no slip)

$$\int_b^s \frac{\partial E}{\partial t} dz + \int_b^s \frac{\partial}{\partial x} (Eu + pu) dz + E(s)w(s) + p(s)w(s) = 0$$

First term (Leibniz):

$$\int_b^s \frac{\partial E}{\partial t} dz = \frac{\partial}{\partial t} \int_b^s E dz - E(s) \frac{\partial(s)}{\partial t} + E(b) \frac{\partial(b)}{\partial t}$$

$$\int_b^s \frac{\partial E}{\partial t} dz = \frac{\partial}{\partial t} \int_b^s E dz - E(s) \frac{\partial h}{\partial t}$$

So need to calculate the integral of the energy over depth:

$$\int_b^s E dz = \int_b^s \frac{1}{2} (u^2 + w^2) + gz dz$$

$$\int_b^s \frac{1}{2} (u^2 + w^2) + gz dz = \frac{1}{2} \left(\bar{u}^2 h + \int_b^s w^2 dz \right) + \int_b^s gz dz$$

Calculating the P.E first (simplest)

$$\int_b^s gz dz = \left[\frac{1}{2} z^2 \right]_b^s = \frac{g}{2} (h^2 + 2hb)$$

Calculating the vertical velocity

$$\begin{aligned} \int_b^s w^2 dz &= \int_b^s \left(\frac{z-b}{h} \frac{\partial h}{\partial t} + u \frac{\partial b}{\partial x} \right)^2 dz \\ &= \int_b^s \left(\frac{z-b}{h} \frac{\partial h}{\partial t} \right)^2 + 2 \left(\frac{z-b}{h} \frac{\partial h}{\partial t} \right) \left(u \frac{\partial b}{\partial x} \right) + \left(u \frac{\partial b}{\partial x} \right)^2 dz \end{aligned}$$