$$\frac{\partial h}{\partial t} + \frac{\partial(\bar{u}h)}{\partial x} = 0 \tag{1a}$$

$$\underbrace{\underbrace{\frac{\partial(\bar{u}h)}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x} - \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial^2 \bar{u}}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}} = 0.}_{\text{Serre Equations}} \tag{1b}$$

Want FD approximation to (1b):

$$\frac{\partial(\bar{u}h)}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}^2 h + \frac{gh^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x} - \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial^2 \bar{u}}{\partial x \partial t}\right]\right) = 0$$

Expanding gives:

$$\frac{\partial(uh)}{\partial t} + 2uh\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x} + gh\frac{\partial h}{\partial x} + \frac{\partial}{\partial x}\left[\frac{h^3}{3}\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x} - \frac{h^3}{3}\bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{h^3}{3}\frac{\partial^2 \bar{u}}{\partial x \partial t}\right]$$

Let

$$D = 2uh\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x} + gh\frac{\partial h}{\partial x}$$

$$\frac{\partial(uh)}{\partial t} + D + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3} - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} = 0$$

Let

$$F = D + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}$$

$$\frac{\partial(uh)}{\partial t} + F - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} = 0$$

$$\frac{\partial(uh)}{\partial t} - h^2\frac{\partial^2 u}{\partial x \partial t} - \frac{h^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} + F = 0$$

$$\frac{u^{n+1}h^{n+1} - u^{n-1}h^{n-1}}{2\Delta t} - h^2\frac{\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}}{2\Delta t} - \frac{h^3}{3}\frac{\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}}{2\Delta t} + F^n = 0$$

$$u^{n+1}h^{n+1} - u^{n-1}h^{n-1} - h^2\left[\left(\frac{\partial u}{\partial x}\right)^{n+1} - \left(\frac{\partial u}{\partial x}\right)^{n-1}\right] - \frac{h^3}{3}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}\right] + 2\Delta t F^n = 0$$

$$u^{n+1}h^{n+1} - h^2\left(\frac{\partial u}{\partial x}\right)^{n+1} - \frac{h^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} + 2\Delta t F^n - u^{n-1}h^{n-1} + h^2\left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{h^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} = 0$$

$$u^{n+1}h^{n+1} - h^2\left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right) - \frac{h^3}{3}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right) + 2\Delta t F^n - u^{n-1}h^{n-1} + h^2\left(\frac{\partial u}{\partial x}\right)^{n-1} + \frac{h^3}{3}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1} = 0$$

So for a tri-diagonal matrix

$$a_i = \frac{h^2}{2\Delta x} - \frac{h^3}{3\Delta x^2}$$

$$b_i = h^{n+1} + \frac{2h^3}{3\Delta x^2}$$

$$c_i = -\frac{h^2}{2\Delta x} - \frac{h^3}{3\Delta x^2}$$

$$F = 2uh\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x} + gh\frac{\partial h}{\partial x} + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}$$

$$F = 2uh\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x} + gh\frac{\partial h}{\partial x} + h^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{h^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - h^2 u\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3}u\frac{\partial^3 u}{\partial x^3}$$