1 Serre Equations

The Serre Equations read (height/mass)

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} + \frac{\partial vh}{\partial y} = 0$$

Phi

$$\Phi = \frac{\partial b}{\partial x}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial b}{\partial y}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + u^2\frac{\partial^2 b}{\partial x^2} + 2uv\frac{\partial^2 b}{\partial x \partial y} + v^2\frac{\partial^2 b}{\partial y^2} + \frac{\partial b}{\partial x}\frac{\partial u}{\partial t} + \frac{\partial b}{\partial y}\frac{\partial v}{\partial t}$$

Gamma

$$\Gamma = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 - u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) - v\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t}\right)$$

Pressure

$$p|_\xi = p_a + \rho g \xi + \frac{\rho}{2}\xi\left(2h - \xi\right)\Gamma + \rho\xi\Phi$$

Momentum(velocity) x

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Gamma + \frac{h^2}{2}\Phi\right) + \frac{\partial uvh}{\partial y} + h\frac{\partial b}{\partial x}\left(g + \frac{h}{2}\Gamma + \Phi\right) = 0$$

y

$$\frac{\partial (vh)}{\partial t} + \frac{\partial}{\partial y}\left(v^2 h + \frac{gh^2}{2} + \frac{h^3}{3}\Gamma + \frac{h^2}{2}\Phi\right) + \frac{\partial uvh}{\partial x} + h\frac{\partial b}{\partial y}\left(g + \frac{h}{2}\Gamma + \Phi\right) = 0$$

z

$$w|_z = \frac{z-b}{h}\frac{\partial h}{\partial t} + u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}$$

The total energy of the Euler system is $E = \frac{1}{2}|\vec{u}|^2 + \vec{g}z$:

$$\frac{\partial E}{\partial t} + \nabla \cdot (E\vec{u} + p\vec{u}) = 0$$

Integrating over depth $s = h + b$

$$\int_b^s \left(\frac{\partial E}{\partial t} + \nabla \cdot (E\vec{u} + p\vec{u})\right) dz = 0$$

$$\int_b^s \frac{\partial E}{\partial t}\,dz+\int_b^s \frac{\partial}{\partial x}\left(Eu+pu\right)\,dz+\int_b^s \frac{\partial}{\partial y}\left(Ev+pv\right)\,dz+\int_b^s \frac{\partial}{\partial z}\left(Ew+pw\right)\,dz=0$$

$\int_b^s \frac{\partial E}{\partial t}\,dz+\int_b^s \frac{\partial}{\partial x}\left(Eu+pu\right)\,dz+\int_b^s \frac{\partial}{\partial y}\left(Ev+pv\right)\,dz+E(s)w(s)+p(s)w(s)-E(b)w(b)-p(b)w(b)=0$

at b $w=0$ (no slip)

$$\int_b^s \frac{\partial E}{\partial t}\,dz+\int_b^s \frac{\partial}{\partial x}\left(Eu+pu\right)\,dz+\int_b^s \frac{\partial}{\partial y}\left(Ev+pv\right)\,dz+E(s)w(s)+p(s)w(s)=0$$

First term (Leibeniz):

$$\int_b^s \frac{\partial E}{\partial t}\,dz=\frac{\partial}{\partial t}\int_b^s E\,dz-E(s)\frac{\partial(s)}{\partial t}+E(b)\frac{\partial(b)}{\partial t}$$

$$\int_b^s \frac{\partial E}{\partial t}\,dz=\frac{\partial}{\partial t}\int_b^s E\,dz-E(s)\frac{\partial h}{\partial t}$$

So need to calculate the integral of the energy over depth:

$$\int_b^s E\,dz=\int_b^s \frac{1}{2}\left(u^2+v^2+w^2\right)+gz\,dz$$

$$\int_b^s \frac{1}{2}\left(u^2+v^2+w^2\right)+gz\,dz=\frac{1}{2}\left(\bar{u}^2h+\bar{v}^2h+\int_b^s w^2\,dz\right)+\int_b^s gz\,dz$$

Calculating the P.E first (simplest)

$$\int_b^s gz\,dz=\left[\frac{1}{2}z^2\right]_b^s=\frac{g}{2}\left(h^2+2hb\right)$$

Calculating the vertical velocity

$$\int_b^s w^2\,dz=\int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t}+u\frac{\partial b}{\partial x}+v\frac{\partial b}{\partial y}\right)^2\,dz$$

$$= \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)^2 + 2\left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)\left(u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\right) + \left(u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\right)^2 \, dz$$

$$= \int_b^s \left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)^2 + 2\left(\frac{z-b}{h}\frac{\partial h}{\partial t}\right)\left(u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\right) + \left(u\frac{\partial b}{\partial x}\right)^2 + 2\left(u\frac{\partial b}{\partial x}v\frac{\partial b}{\partial y}\right) + \left(v\frac{\partial b}{\partial y}\right)^2 \, dz$$