1 Helmholtz Decomposition

We know in 2D the equation for the conserved quantity $\vec{G}$ in terms of the conserved quantity h and the primitive variables $\vec{u}$ is

$$\vec{G} = h\vec{u} - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right) \tag{1}$$

To look at this we consider the Helmholtz decomposition of $\vec{G}$ (assuming sufficient smoothness), in principle we can calculate these two parts since we know $\vec{G}$. Thus we have

$$\vec{G} = \nabla\phi + \nabla \times \vec{A}$$

Taking the curl of (1)

$$\nabla \times \vec{G} = \nabla \times (h\vec{u}) - \nabla \times \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

since the curl of a grad is 0

$$\nabla \times \vec{G} = \nabla \times (h\vec{u})$$

so the curl of $\vec{G}$ and $h\vec{u}$ are the same. Since the other term is a gradient, its helmholtz decomposition has no curl part and so it must be that for the helmholtz decomposition of $h\vec{u}$ (which we do not know) the curl part is the same as for $\vec{G}$ so that

$$h\vec{u} = \nabla\psi + \nabla \times \vec{A}$$

So we have that:

$$\nabla\phi + \nabla \times \vec{A} = \nabla\psi + \nabla \times \vec{A} - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

$$\nabla\phi = \nabla\psi - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

Assuming we can just integrate without a problem (probably need to impose some boundary conditions here)

$$\phi = \psi - \left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

From above by diving the Helmholtz decomposition of $\vec{u}$ by h we get (asssume $h > 0$)

$$\vec{u} = \frac{\nabla \psi}{h} + \frac{\nabla \times \vec{A}}{h}$$

Subbing this in gives

$$\phi = \psi - \left(\frac{h^3}{3}\left(\nabla \cdot \left(\frac{\nabla \psi}{h} + \frac{\nabla \times \vec{A}}{h}\right)\right)\right)$$

Getting ready for a integration by parts we divide out the $h^3/3$ factor to give

$$\frac{3}{h^3}\phi = \frac{3}{h^3}\psi - \nabla \cdot \left(\frac{\nabla \psi}{h} + \frac{\nabla \times \vec{A}}{h}\right)$$

So integrating over the domain and multiplying by a test function $v \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{3}{h^3}\phi v \; dx = \int_\Omega \frac{3}{h^3}\psi v \; dx - \int_\Omega \nabla \cdot \left(\frac{\nabla \psi}{h} + \frac{\nabla \times \vec{A}}{h}\right) v \; dx$$

By integrating by parts and remembering that v has is supported inside Ω we have

$$\int_\Omega \frac{3}{h^3}\phi v \; dx = \int_\Omega \frac{3}{h^3}\psi v \; dx - \int_\Omega \left(\frac{\nabla \psi}{h} + \frac{\nabla \times \vec{A}}{h}\right) \nabla \cdot v \; dx$$

$$\int_\Omega \frac{3}{h^3}\phi v \; dx = \int_\Omega \frac{3}{h^3}\psi v \; dx - \int_\Omega \frac{\nabla \psi}{h}\left(\nabla \cdot v\right) + \frac{\nabla \times \vec{A}}{h}\left(\nabla \cdot v\right) \; dx$$

$$\int_\Omega \frac{3}{h^3}\phi v - \frac{\nabla \times \vec{A}}{h}\left(\nabla \cdot v\right) \; dx = \int_\Omega \frac{3}{h^3}\psi v \; dx - \int_\Omega \frac{\nabla \psi}{h}\left(\nabla \cdot v\right) \; dx$$

In principle we can calculate the LHS for any v in the following way. We know $\vec{G}$ and h, by definition ϕ and $\vec{A}$ are given by

$$\vec{G} = \nabla\phi + \nabla \times \vec{A}$$

Note that we need ϕ and only $\nabla \times \vec{A}$, since

$$\nabla \cdot \vec{G} = \nabla \cdot (\nabla\phi) + \nabla \cdot (\nabla \times \vec{A})$$

$$\nabla \cdot \vec{G} = \nabla \cdot (\nabla\phi)$$

multiplying by $v \in C_0^\infty(\Omega)$ and integrating over Ω

$$\int_\Omega \nabla \cdot \vec{G} v \; dx = \int_\Omega \nabla \cdot (\nabla\phi) v \; dx$$

Integrating by parts gives:

$$\int_\Omega \vec{G}(\nabla \cdot v) \; dx = \int_\Omega (\nabla\phi)(\nabla \cdot v) \; dx$$