# 1 Serre Equations

Dimensional variables have astreicks as superscript Scale length: $x^* = \lambda x$, typical depth $h^* = dh$, typical speed $u^* = u\epsilon c$ (Linear theory)

Results: $t^* = t\frac{\lambda}{c}$, $g^* = g\frac{c^2}{\lambda}$, $b^* = bd$

Serre Equations Mass

$$\frac{\partial h^*}{\partial t^*} + u^*\frac{\partial h^*}{\partial x^*} + h^*\frac{\partial u^*}{\partial x^*} = 0$$

$$\frac{\partial dh}{\partial t\frac{\lambda}{c}} + u\epsilon c\frac{\partial dh}{\partial \lambda x} + dh\frac{\partial u\epsilon c}{\partial \lambda x} = 0$$

$$\frac{dc}{\lambda}\frac{\partial h}{\partial t} + \frac{dc}{\lambda}\epsilon u\frac{\partial h}{\partial x} + \frac{dc}{\lambda}\epsilon h\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \epsilon u\frac{\partial h}{\partial x} + \epsilon h\frac{\partial u}{\partial x} = 0$$

Serre Equations Momentum

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+g^*\frac{\partial h^*}{\partial x^*}+h^*\frac{\partial h^*}{\partial x^*}\Gamma^*+\frac{(h^*)^2}{3}\frac{\partial \Gamma^*}{\partial x^*}+\frac{\partial h^*}{\partial x^*}\Phi^*+\frac{h^*}{2}\frac{\partial \Phi^*}{\partial x^*}+h^*\frac{\partial b^*}{\partial x^*}\left(g^* + \frac{h^*}{2}\Gamma^* + \Phi^*\right) = 0$$

$$\Gamma^* = \frac{\partial u^*}{\partial x^*}\frac{\partial u^*}{\partial x^*} - u^*\frac{\partial^2 u^*}{\partial (x^*)^2} - \frac{\partial^2 u^*}{\partial x^* \partial t^*}$$

$$\Phi^* = \frac{\partial b^*}{\partial x^*}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*}\right) + (u^*)^2\frac{\partial^2 b^*}{\partial (x^*)^2}$$

$$\Gamma^* = \frac{\partial u\epsilon c}{\partial \lambda x}\frac{\partial u\epsilon c}{\partial \lambda x} - u\epsilon c\frac{\partial^2 u\epsilon c}{\partial (\lambda x)^2} - \frac{\partial^2 u\epsilon c}{\partial \lambda x \partial t\frac{\lambda}{c}}$$

$$\Gamma^* = \frac{\epsilon^2 c^2}{\lambda^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\epsilon^3 c^3}{\lambda^2}u\frac{\partial^2 u}{\partial x^2} - \frac{\epsilon^2 c^3}{\lambda^2}\frac{\partial^2 u}{\partial x \partial t}$$

$$\Gamma^* = \frac{\epsilon^2 c^2}{\lambda^2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \epsilon c u\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^2 u}{\partial x \partial t}\right)$$

$$\Gamma^* = \frac{\epsilon^2 c^2}{\lambda^2}\Gamma$$

where

$$\Gamma = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \epsilon c u \frac{\partial^2 u}{\partial x^2} - c\frac{\partial^2 u}{\partial x \partial t}$$

$$\Phi^* = \frac{\partial b^*}{\partial x^*}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*}\right) + (u^*)^2\frac{\partial^2 b^*}{\partial (x^*)^2}$$

$$\Phi^* = \frac{\partial db}{\partial \lambda x}\left(\frac{\partial u\epsilon c}{\partial t\frac{\lambda}{c}} + u\epsilon c\frac{\partial u\epsilon c}{\partial x\lambda}\right) + (u\epsilon c)^2\frac{\partial^2 db}{\partial (\lambda x)^2}$$

$$\Phi^* = \frac{d}{\lambda}\frac{\partial b}{\partial x}\left(\frac{\epsilon c^2}{\lambda}\frac{\partial u}{\partial t} + \frac{\epsilon^2 c^2}{\lambda}u\frac{\partial u}{\partial x}\right) + \frac{d^2\epsilon^2 c^2}{\lambda^2}u^2\frac{\partial^2 b}{\partial x^2}$$

$$\Phi^* = \frac{dc^2\epsilon}{\lambda^2}\frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + \frac{d^2\epsilon^2 c^2}{\lambda^2}u^2\frac{\partial^2 b}{\partial x^2}$$

$$\Phi^* = \frac{dc^2\epsilon}{\lambda^2}\left(\frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + du^2\frac{\partial^2 b}{\partial x^2}\right)$$

$$\Phi^* = \frac{dc^2\epsilon}{\lambda^2}\Phi$$

where

$$\Phi = \frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + du^2\frac{\partial^2 b}{\partial x^2}$$

$$\frac{\partial u^*}{\partial t^*} = \frac{\partial u\epsilon c}{\partial t\frac{\lambda}{c}} = \frac{\epsilon c^2}{\lambda}\frac{\partial u}{\partial t}$$

$$u^*\frac{\partial u^*}{\partial x^*} = u\epsilon c\frac{\partial u\epsilon c}{\partial \lambda x} = \frac{\epsilon^2 c^2}{\lambda}u\frac{\partial u}{\partial x}$$

$$g^*\frac{\partial h^*}{\partial x^*} = g\frac{c^2}{\lambda}\frac{\partial dh}{\partial \lambda x} = g\frac{dc^2}{\lambda^2}\frac{\partial h}{\partial x}$$

$$h^*\frac{\partial h^*}{\partial x^*}\Gamma^* = dh\frac{\partial dh}{\partial \lambda x}\frac{\epsilon^2 c^2}{\lambda^2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \epsilon c u\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^2 u}{\partial x\partial t}\right) = \frac{d^2\epsilon^2 c^2}{\lambda^3}h\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \epsilon c u\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^2 u}{\partial x\partial t}\right)$$

$$\frac{(h^*)^2}{3}\frac{\partial \Gamma^*}{\partial x^*} = \frac{h^2 d^2}{3}\frac{\partial}{\partial \lambda x}\left(\frac{\epsilon^2 c^2}{\lambda^2}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \epsilon c u\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^2 u}{\partial x\partial t}\right)\right)$$

$$= \frac{d^2\epsilon^2c^2}{\lambda^3}\frac{h^2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \epsilon cu\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^2 u}{\partial x\partial t}\right)$$

$$\frac{\partial h^*}{\partial x^*}\Phi^* = \frac{\partial dh}{\partial \lambda x}\left(\frac{dc^2\epsilon}{\lambda^2}\left(\frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + du^2\frac{\partial^2 b}{\partial x^2}\right)\right)$$

$$= \frac{d^2c^2\epsilon}{\lambda^3}\frac{\partial h}{\partial x}\left(\frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + du^2\frac{\partial^2 b}{\partial x^2}\right)$$

$$\frac{h^*}{2}\frac{\partial \Phi^*}{\partial x^*} = \frac{dh}{2}\frac{\partial}{\partial x^*}\left(\frac{dc^2\epsilon}{\lambda^2}\left(\frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + du^2\frac{\partial^2 b}{\partial x^2}\right)\right)$$

$$= \frac{d^2c^2\epsilon}{\lambda^2}\frac{h}{2}\frac{\partial}{\partial x^*}\left(\left(\frac{\partial b}{\partial x}\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x}\right) + du^2\frac{\partial^2 b}{\partial x^2}\right)\right)$$

Thus

$$\begin{aligned}\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + g^*\frac{\partial h^*}{\partial x^*} + h^*\frac{\partial h^*}{\partial x^*}\Gamma^* + \frac{(h^*)^2}{3}\frac{\partial \Gamma^*}{\partial x^*} + \frac{\partial h^*}{\partial x^*}\Phi^* + \frac{h^*}{2}\frac{\partial \Phi^*}{\partial x^*} \\ = \frac{\epsilon c^2}{\lambda}\frac{\partial u}{\partial t} + \frac{\epsilon^2c^2}{\lambda}u\frac{\partial u}{\partial x} + g\frac{dc^2}{\lambda^2}\frac{\partial h}{\partial x} \\ + \frac{d^2\epsilon^2c^2}{\lambda^3}h\frac{\partial h}{\partial x}\Gamma + \frac{d^2\epsilon^2c^2}{\lambda^3}\frac{h^2}{3}\frac{\partial \Gamma}{\partial x} + \frac{d^2c^2\epsilon}{\lambda^3}\frac{\partial h}{\partial x}\Phi + \frac{d^2c^2\epsilon}{\lambda^2}\frac{h}{2}\frac{\partial \Phi}{\partial x^*}\end{aligned} \tag{1}$$

$$= \frac{1}{\lambda}\left(\epsilon c^2\frac{\partial u}{\partial t} + \epsilon^2c^2u\frac{\partial u}{\partial x} + g\frac{dc^2}{\lambda}\frac{\partial h}{\partial x} + \frac{d^2\epsilon^2c^2}{\lambda^2}h\frac{\partial h}{\partial x}\Gamma + \frac{d^2\epsilon^2c^2}{\lambda^2}\frac{h^2}{3}\frac{\partial \Gamma}{\partial x} + \frac{d^2c^2\epsilon}{\lambda^2}\frac{\partial h}{\partial x}\Phi + \frac{d^2c^2\epsilon}{\lambda}\frac{h}{2}\frac{\partial \Phi}{\partial x}\right) \tag{2}$$

$$= \frac{1}{\lambda}\left(g\frac{dc^2}{\lambda}\frac{\partial h}{\partial x} + \epsilon c^2\left(\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x} + \frac{d^2\epsilon}{\lambda^2}h\frac{\partial h}{\partial x}\Gamma + \frac{d^2\epsilon}{\lambda^2}\frac{h^2}{3}\frac{\partial \Gamma}{\partial x} + \frac{d^2}{\lambda^2}\frac{\partial h}{\partial x}\Phi + \frac{d^2}{\lambda}\frac{h}{2}\frac{\partial \Phi}{\partial x}\right)\right) \tag{3}$$

$$h^*\frac{\partial b^*}{\partial x^*}\left(g^* + \frac{h^*}{2}\Gamma^* + \Phi^*\right) = d^2h\frac{\partial b}{\partial x}\left(g\frac{c^2}{\lambda} + \frac{dh}{2}\frac{\epsilon^2c^2}{\lambda^2}\Gamma + \frac{dc^2\epsilon}{\lambda^2}\Phi\right)$$

$$= \frac{d^2c^2}{\lambda}h\frac{\partial b}{\partial x}\left(g + \frac{dh}{2}\frac{\epsilon^2}{\lambda}\Gamma + \frac{d\epsilon}{\lambda}\Phi\right)$$

$$
\frac{1}{\lambda}\left(g\frac{dc^2}{\lambda}\frac{\partial h}{\partial x}+\epsilon c^2\left(\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{d^2\epsilon}{\lambda^2}h\frac{\partial h}{\partial x}\Gamma+\frac{d^2\epsilon}{\lambda^2}\frac{h^2}{3}\frac{\partial \Gamma}{\partial x}+\frac{d^2}{\lambda^2}\frac{\partial h}{\partial x}\Phi+\frac{d^2}{\lambda}\frac{h}{2}\frac{\partial \Phi}{\partial x}\right)\right) \\ +\frac{d^2c^2}{\lambda}h\frac{\partial b}{\partial x}\left(g+\frac{dh}{2}\frac{\epsilon^2}{\lambda}\Gamma+\frac{d\epsilon}{\lambda}\Phi\right)=0 \tag{4}
$$

$$
\frac{c^2}{\lambda}\left(g\frac{d}{\lambda}\frac{\partial h}{\partial x}+\epsilon\left(\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{d^2\epsilon}{\lambda^2}h\frac{\partial h}{\partial x}\Gamma+\frac{d^2\epsilon}{\lambda^2}\frac{h^2}{3}\frac{\partial \Gamma}{\partial x}+\frac{d^2}{\lambda^2}\frac{\partial h}{\partial x}\Phi+\frac{d^2}{\lambda}\frac{h}{2}\frac{\partial \Phi}{\partial x}\right)\right) \\ +\frac{d^2c^2}{\lambda}h\frac{\partial b}{\partial x}\left(g+\frac{dh}{2}\frac{\epsilon^2}{\lambda}\Gamma+\frac{d\epsilon}{\lambda}\Phi\right)=0 \tag{5}
$$

$\sigma = d/\lambda$

$$
g\sigma\frac{\partial h}{\partial x}+\epsilon\left(\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\sigma^2\epsilon h\frac{\partial h}{\partial x}\Gamma+\sigma\epsilon\frac{h^2}{3}\frac{\partial \Gamma}{\partial x}+\sigma^2\frac{\partial h}{\partial x}\Phi+\sigma d\frac{h}{2}\frac{\partial \Phi}{\partial x}\right) \\ +d^2h\frac{\partial b}{\partial x}\left(g+\sigma\epsilon^2\frac{h}{2}\Gamma+\sigma\epsilon\Phi\right)=0 \tag{6}
$$

$$
g\frac{\sigma}{\epsilon}\frac{\partial h}{\partial x}+\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\sigma^2\epsilon h\frac{\partial h}{\partial x}\Gamma+\sigma\epsilon\frac{h^2}{3}\frac{\partial \Gamma}{\partial x}+\sigma^2\frac{\partial h}{\partial x}\Phi+\sigma d\frac{h}{2}\frac{\partial \Phi}{\partial x} \\ +\frac{d^2}{\epsilon}h\frac{\partial b}{\partial x}\left(g+\sigma\epsilon^2\frac{h}{2}\Gamma+\sigma\epsilon\Phi\right)=0 \tag{7}
$$