

1 Cell Averaged G

We have:

$$G = uh - \frac{\partial}{\partial x} \left(\frac{h^3}{3} \frac{\partial u}{\partial x} \right)$$

For a soliton we have:

$$h(x, t) = a_0 + a_1 \operatorname{sech}^2(\kappa(x - ct))$$

$$u(x, t) = c \left(1 - \frac{a_0}{h(x, t)} \right)$$

$$\kappa = \frac{\sqrt{3a_1}}{2\sqrt{a_0(a_0 + a_1)}}$$

$$c = \sqrt{g(a_0 + a_1)}$$

Let's focus on $t = 0$

$$h(x) = a_0 + a_1 \operatorname{sech}^2(\kappa x)$$

$$u(x) = c \left(1 - \frac{a_0}{h(x)} \right)$$

$$\kappa = \frac{\sqrt{3a_1}}{2\sqrt{a_0(a_0 + a_1)}}$$

$$c = \sqrt{g(a_0 + a_1)}$$

So we have

$$\begin{aligned} \frac{\partial}{\partial x} u(x, t) &= -ca_0 \frac{\partial}{\partial x} \left(\frac{1}{h(x, t)} \right) \\ &= ca_0 \left[\frac{\partial h}{\partial x} \frac{1}{h^2} \right] \end{aligned}$$

Multiplying by $\frac{h^3}{3}$

$$\frac{h^3}{3} \frac{\partial}{\partial x} u(x, t) = \frac{ca_0}{3} h \left[\frac{\partial h}{\partial x} \right]$$

Taking the derivative:

$$\frac{\partial}{\partial x} \left(\frac{h^3}{3} \frac{\partial}{\partial x} u \right) = \frac{\partial}{\partial x} \left(\frac{ca_0}{3} h \frac{\partial h}{\partial x} \right)$$

$$\begin{aligned}
&= \frac{ca_0}{3} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) \\
&= \frac{ca_0}{3} \left(h \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x} \right)^2 \right)
\end{aligned}$$

So we have:

$$\frac{\partial}{\partial x} \left(\frac{h^3}{3} \frac{\partial u}{\partial x} \right) = \frac{ca_0}{3} \left(h \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x} \right)^2 \right)$$

Looking at momentum we have:

$$uh = c(h - a_0)$$

So G is:

$$G = c(h - a_0) - \frac{ca_0}{3} \left(h \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x} \right)^2 \right)$$

We will now calculate the derivatives of h:

$$h = a_0 + a_1 \operatorname{sech}^2(\kappa x)$$

$$\frac{\partial h}{\partial x} = a_1 \frac{\partial}{\partial x} \operatorname{sech}^2(\kappa x)$$

$$= a_1 \times 2 \times \kappa \times -\tanh(\kappa x) \operatorname{sech}(\kappa x) \times \operatorname{sech}(\kappa x)$$

$$\frac{\partial h}{\partial x} = -2a_1 \kappa \tanh(\kappa x) \operatorname{sech}^2(\kappa x)$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1 \kappa \frac{\partial}{\partial x} (\tanh(\kappa x) \operatorname{sech}^2(\kappa x))$$

$$\frac{\partial^2 h}{\partial x^2} = -2a_1 \kappa (-\kappa (\cosh(2\kappa x) - 2) \operatorname{sech}^4(\kappa x))$$

$$\frac{\partial^2 h}{\partial x^2} = 2a_1 \kappa^2 ((\cosh(2\kappa x) - 2) \operatorname{sech}^4(\kappa x))$$