1 Helmholtz Decomposition

We know in 2D the equation for the conserved quantity $\vec{G}$ in terms of the conserved quantity h and the primitive variables $\vec{u}$ is

$$\vec{G} = h\vec{u} - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right) \tag{1}$$

To look at this we consider the Helmholtz decomposition of $\vec{G}$ (assuming sufficient smoothness), in principle we can calculate these two parts since we know $\vec{G}$. Thus we have

$$\vec{G} = \nabla\phi + \nabla \times \vec{A}$$

Taking the curl of (1)

$$\nabla \times \vec{G} = \nabla \times \left(h\vec{u}\right) - \nabla \times \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

since the curl of a grad is 0

$$\nabla \times \vec{G} = \nabla \times \left(h\vec{u}\right)$$

so the curl of $\vec{G}$ and $h\vec{u}$ are the same. Since the other term is a gradient, its helmholtz decomposition has no curl part and so it must be that for the helmholtz decomposition of $h\vec{u}$ (which we do not know) the curl part is the same as for $\vec{G}$ so that

$$h\vec{u} = \nabla\psi + \nabla \times \vec{A}$$

So we have that:

$$\nabla\phi + \nabla \times \vec{A} = \nabla\psi + \nabla \times \vec{A} - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

$$\nabla\phi = \nabla\psi - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

Assuming we can integrate(seems reasonable) we get that (and dropping the constant)

$$\phi = \psi - \left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

It would be nice to rewrite the divergence of $\vec{u}$ in terms of ψ since

$$h\vec{u} = \nabla\psi + \nabla \times \vec{A}$$

$$\vec{u} = \frac{\nabla\psi}{h} + \frac{\nabla \times \vec{A}}{h}$$

substituting this in gives

$$\phi = \psi - \left(\frac{h^3}{3}\left(\nabla \cdot \left[h^{-1}\nabla\psi + h^{-1}\nabla \times \vec{A}\right]\right)\right)$$

$$\phi = \psi - \left(\frac{h^3}{3}\left(h^{-1}\left(\nabla \cdot \nabla\psi\right) + \nabla\psi \cdot \nabla h^{-1} + \left(\nabla \times A\right) \cdot \nabla h^{-1}\right)\right)$$

Since

$$\nabla h^{-1} = -h^{-2}\nabla h$$

$$\phi = \psi - \left(\frac{1}{3}\left(h^2\nabla^2\psi - h\nabla\psi \cdot \nabla h - h\left(\nabla \times A\right) \cdot \nabla h\right)\right)$$

$$\phi = \psi - \frac{h^2}{3}\nabla^2\psi - \frac{h}{3}\nabla\psi \cdot \nabla h - \frac{h}{3}\left(\nabla \times A\right) \cdot \nabla h$$

$$\psi - \frac{h^2}{3}\nabla^2\psi - \frac{h}{3}\nabla\psi \cdot \nabla h = \phi + \frac{h}{3}\left(\nabla \times A\right) \cdot \nabla h$$

Where only the LHS has unknowns. So this is solvable (meaningful?) if we are allowed to firstly take the helmholtz decomposition of both $\vec{G}$ and $h\vec{u}$. We also integrate all the gradient terms. We must also assume that $h > 0$ to do the divisions and finally take the gradient of h. Remember that we have dropped a constant as well during integration.

To do such a thing we must assume that at least

- Ω is the boundary of the problem to do a helmholtz decomposition we must have that Ω is a bounded, simply-connected, Lipschitz domain
- $\vec{G}$ and $h\vec{u}$ must be $L^2\left(\Omega\right)^3$ function to do a helmholtz decomposition. The resultant decomposition as the base divergence free part is in $H(\text{curl}, \Omega)$ while the base of curl free part is in $H^1\left(\Omega\right)$.

By these equations we start with:

$G \in L^2$ and $h \in H^1$ then $\phi \in H^1$ and $\nabla \times A \in L^2$. Also $\nabla h \in L^2$, so we have that $\psi \in H^2$ thus $\vec{u} \in L^2$