A Solution of the Conservation Law Form of the Serre Equations

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Abstract

The nonlinear and weakly dispersive Serre equations contain higher-order dispersive terms. This includes mixed spatial and temporal derivative flux terms which are difficult to handle numerically. These terms can be replaced by an alternative combination of equivalent temporal and spatial terms so that the Serre equations can be written in conservation law form. The water depth and new conserved quantities are evolved using a second-order finite volume scheme. The remaining primitive variable, the depth averaged horizontal velocity is obtained by solving a second-order elliptic equation using simple finite differences.

Using an analytical solution and by simulating the dam-break problem, the proposed scheme is shown to be accurate, simple to implement and stable for a range of problems, including flows with steep gradients. It is only slightly more computationally expensive than solving the shallow water wave equations.

Contents

1 Introduction

Dispersive waves can occur when the vertical fluid velocity has an influence on the behaviour of the flow and therefore cannot be ignored. This can occur when there is an abrupt change in fluid flow caused by an advancing front or when the fluid flows over changes in topography. For example, dispersive waves have been observed in the atmosphere with advancing cold fronts. Other examples include the trailing waves that accompany a tsunami or a tidal bore.

Two-dimensional systems of equations that include the influence of vertical velocity of the flow on the behaviour of the flow contain higher-order dispersive terms. The Serre equations belong to this class of equations. They are highly nonlinear and weakly dispersive and are based on the Euler equations describing the three-dimensional motion of fluid particles in incompressible flow with constant density. By integrating the Euler equations over the water depth and assuming that the variation of the horizontal velocity components remain constant with water depth, the Serre equations can be derived. The Serre equations contain second- and higher-order dispersive terms including a mixed spatial and temporal term. Ignoring these terms results in the shallow water wave equations. The mixed spatial and temporal term complicates the solution of the Serre equations.

There are numerous schemes for solving the Serre equations[8, 11]. These schemes only consider smooth problems. We have developed a scheme that can handle both smooth and steep gradient flows by employing very efficient schemes for solving conservation laws. If the Serre equations can be written in conservation law form, then these schemes can be used to solve the Serre equations. There is potentially a significant saving in computational effort if the equations can be written in conservation law form. In addition, these schemes are capable of handling steep gradients in a problem.

The Serre equations can be written in conservation law form by replacing the mixed derivative dispersive term by a combination of temporal and spatial terms. This is described in this paper. The water depth and new conserved quantity can be evolved using the finite volume method. The remaining primitive variable, the depth averaged velocity is obtained by solving a second-order elliptic equation using simple finite differences.

With this approach the conserved quantities can be discontinuous, which can be handled efficiently by the finite volume method and approximate Riemann solver and it is only slightly more computationally expensive than solving the shallow water wave equations.

In Section 2 the Serre equations are given, as well as expressions for the vertical velocity and pressure distribution predicted by these equations. We then derive the conservation law form of the Serre equations in Section 3 and explain why we want to write them in this form. How we solve the conservation law form of the Serre equations is described in Section 4. Convergence results for the simulation of a soliton are used to validate the proposed model in Section 5, and the results for the simulation of the dam-break problem are provided in Section 6.

2 One-dimensional Serre Equations

The Serre equations were first derived by Serre in 1953[13] and latter by Su and Gardner[15] and Seabra-Santos *et al.* [12] and are identical to the depth averaged Green-Naghdi equations[5].

The Serre equations assume that there is no variation in the horizontal velocity in the vertical direction. Other assumptions can be made such as a linear variation or parabolic variation[10, 18]. In these cases additional dispersive terms are introduced into the equations.

Using this assumption and integrating the Euler equations for incompressible fluid with constant density flowing over a fixed impermeable horizontal bed and satisfying certain kinematic and dynamic boundary conditions results in the Serre equation[12]

$$\frac{\partial h}{\partial t} + \frac{\partial (\bar{u}h)}{\partial x} = 0 \tag{1a}$$

and

$$\underbrace{\underbrace{\frac{\partial (\bar{u}h)}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}^2 h + \frac{gh^2}{2}\right)}_{\text{Shallow Water Wave Equations}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x} - \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial^2 \bar{u}}{\partial x \partial t}\right]\right)}_{\text{Dispersion Terms}} = 0,}_{\text{Serre Equations}} \tag{1b}$$

where, $\bar{u}(x,t)$ is the depth averaged velocity of the horizontal fluid velocity, $u(x,y,t)$, $h(x,t)$ is the water depth, g the acceleration due to gravity, t the time and (x, y) the horizontal and vertical coordinate system respectively. The kinematic boundary conditions simply state that a fluid particle on the water surface or on the bed remains on that surface and the only dynamic boundary condition is the atmospheric pressure at the water surface

For the Serre equations, the vertical fluid velocity is a linear function of water depth, zero at the bed and a maximum of

$$w(x, y, t) = \frac{\partial h}{\partial t} + \bar{u}\frac{\partial h}{\partial x}$$

at the water surface. Assuming that $w = 0$ results in the nonlinear shallow water wave equations.

The pressure distribution is the atmospheric pressure at the water surface and

$$p = p_a + \rho g h + \frac{\rho h^2}{2}\left(\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x} - \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial^2 \bar{u}}{\partial x \partial t}\right)$$

at the bed. It has a hydrostatic pressure term, $p_a + \rho g h$, the shallow water wave assumption, which is modified by a term that accounts for changes in the flow.

The continuity equation, (1a) is exact, no assumptions have been made about the vertical velocity distribution for u. However, in the derivation of the momentum equation, the assumption of constant u with water depth makes these equations weakly dispersive.

As highlighted in (1b), the Serre equations are the shallow water wave equations with additional higher-order terms. It includes a third-order spatial term as well as a mixed spatial and temporal term shown in red. The mixed spatial and temporal term complicates the solution of the Serre equations. The momentum equation can also be rewritten in terms of the depth averaged velocity. In this case the momentum equation resembles a dispersion equation with a third-order spatial derivative term. Ignoring third-order space derivatives and product derivatives, the Serre equations become the Boussinesq equations[1].

3 Conservation Law Form of the Serre Equations

We could solve the Serre equations using a variety of methods. However, if we can write the equations in conservation law form, then there are very efficient schemes for solving conservation laws which could be used to solve the Serre equations. These schemes are capable of handling steep gradients in a problem and there is potentially a significant saving in computational effort if the equations can be written in conservation law form. Smaller time steps would be necessary or a complicated implicit scheme would be required to solve the Serre equations in the form of (1).

For example, using a simple explicit finite difference schemes, stability analysis of the finite differences scheme for the advection equation (conservation law form) would show that the computational time step, Δt, is proportional to the computation distance step, Δx. For the diffusion equation the computational time step is proportional to Δx^2 and for the dispersion equation the time step is proportional to Δx^3. Potentially there is considerable savings to be made if the Serre equations can be written in conservation law form, where $\Delta t \leq \Delta x$ would be the requirement.

By making the following observation

$$\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial^2 \bar{u}}{\partial x \partial t}\right) = \frac{\partial}{\partial t}\left(h^2 \frac{\partial h}{\partial x}\frac{\partial \bar{u}}{\partial x} + \frac{h^3}{3}\frac{\partial^2 \bar{u}}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(h^2 \frac{\partial h}{\partial t}\frac{\partial \bar{u}}{\partial x}\right)$$

and using the continuity equation to eliminate $\partial h/\partial t$, then the momentum equation, (1b) can be re-written so that there are no temporal derivatives in the flux term as

$$\frac{\partial}{\partial t}\left(\bar{u}h - h^2 \frac{\partial h}{\partial x}\frac{\partial \bar{u}}{\partial x} - \frac{h^3}{3}\frac{\partial^2 \bar{u}}{\partial x^2}\right)$$
$$+ \frac{\partial}{\partial x}\left(\bar{u}^2 h + \frac{gh^2}{2} - \bar{u}h^2 \frac{\partial h}{\partial x}\frac{\partial \bar{u}}{\partial x} - \frac{\bar{u}h^3}{3}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{2h^3}{3}\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x}\right) = 0.$$

The temporal derivative terms have a corresponding flux term. These can be combined to produce a new non-physical conserved quantity

$$G = \bar{u}h - h^2 \frac{\partial h}{\partial x}\frac{\partial \bar{u}}{\partial x} - \frac{h^3}{3}\frac{\partial^2 \bar{u}}{\partial x^2}. \tag{2}$$

It consists of a flux of material, $\bar{u}h$, which is modified by some function of the water surface profile and velocity. The momentum equation, (1b), can be written in the desired conservation law form as

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}G + \frac{gh^2}{2} - \frac{2h^3}{3}\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial x}\right) = 0. \tag{3}$$

The continuity equation, (1a), is already in the appropriate form.

Given G and h, the remaining primitive variable, $\bar{u}$ is calculated through the operator $\mathcal{A}[h, G]$ which is the solution of the second-order elliptic equation

$$\bar{u}h - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial \bar{u}}{\partial x}\right) = G,$$

which is (2) written in divergent form and is easily solved using finite differences or the finite element method.

4 A Finite Volume Solution of the Conservation Law Form of the Serre Equations

We solve the Serre equations using a second-order finite volume method[17], where the cell interface values are obtained using linear interpolations which are limited using the *generalized minmod* limiter[16] to prevent unwanted oscillations. This ensures that the results are physical (bounded) and therefore stable. It includes a parameter, $\theta \in [1, 2]$, which controls the amount of diffusion introduced by the limiter.

Time integration of the semi-discrete system is performed using a second-order Strong Stability Preserving (SSP) Runge-Kutta scheme[14, 9].

The numerical approximation of the physical flux of a conserved quantity, $q^{\pm}_{j+1/2}$ across the boundary of a cell is given by the approximate Riemann solver proposed by Kurganov *et al.*[7] as

$$F_{j+1/2} = \frac{a^{+}_{j+1/2} f(q^{-}_{j+1/2}) - a^{-}_{j+1/2} f(q^{+}_{j+1/2})}{a^{+}_{j+1/2} - a^{-}_{j+1/2}} + \frac{a^{+}_{j+1/2}\, a^{-}_{j+1/2}}{a^{+}_{j+1/2} - a^{-}_{j+1/2}} \left[q^{+}_{j+1/2} - q^{-}_{j+1/2} \right].$$

This only requires an estimate of the wave speeds, $a^{\pm}_{j+1/2}$ and $f(q^{\pm}_{j+1/2})$ which is the corresponding finite difference analogue of the flux function in (1a) or (3) for $q^{\pm}_{j+1/2}$.

Performing a Fourier analysis of the linearized equations Serre equations, we find that the phase speed

$$v_p = u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{k^2h^2+3}}$$

differs from the group speed, v_g which is a property of dispersive waves. As the wave number, $k \to 0$, $v_p \to v_g \to u_0 \pm \sqrt{gh_0}$, they are equal to the phase speed of shallow water waves, where all wave components travel at the same speed. Indeed the phase speed for the Serre equations are bounded

$$u_0 - \sqrt{gh_0} \leq u_0 \pm \sqrt{gh_0}\sqrt{\frac{3}{k^2h^2+3}} \leq u_0 + \sqrt{gh_0}$$

by the phase speed of the shallow water wave equations. We now have an estimate of the maximum and minimum wave speed required by our chosen approximate Riemann solver.

4.1 Solution Process

The solution of the Serre equations involves the following steps

$$\underbrace{\begin{bmatrix} h \\ G \end{bmatrix}^{n} \xrightarrow{\mathcal{A}} \bar{u}^{n}}_{①} \rightarrow \underbrace{\begin{bmatrix} h \\ G \end{bmatrix}^{(1)} = \begin{bmatrix} h \\ G \end{bmatrix}^{n} - \Delta t \mathcal{L} \begin{bmatrix} h \\ G \end{bmatrix}^{n}}_{② \text{ First Euler Step}}$$

$$\underbrace{\begin{bmatrix} h \\ G \end{bmatrix}^{(1)} \xrightarrow{\mathcal{A}} \bar{u}^{(1)}}_{③} \rightarrow \underbrace{\begin{bmatrix} h \\ G \end{bmatrix}^{(2)} = \begin{bmatrix} h \\ G \end{bmatrix}^{(1)} - \Delta t \mathcal{L} \begin{bmatrix} h \\ G \end{bmatrix}^{(1)}}_{④ \text{ Second Euler Step}}$$

$$\underbrace{\begin{bmatrix} h \\ G \end{bmatrix}^{n+1} = \frac{1}{2} \begin{bmatrix} h \\ G \end{bmatrix}^{n} + \frac{1}{2} \begin{bmatrix} h \\ G \end{bmatrix}^{(2)}}_{⑤ \text{ Averaging Step}}$$

Step 1: Given the conserved quantities, h and G, the remaining primitive variable, $\bar{u}$, is obtained by solving the second-order elliptic equation, (2) using finite differences.
Step 2: Perform the reconstruction and solve a local Riemann problem to obtain the flux, $F_{j\pm1/2}$, of material across a cell interface. Evolve the solution using a first-order Euler time integration for the conserved quantities.
Steps 3 and 4: Repeat the process with the intermediate values and evolve using another first-order Euler step.
Step 5: The solution at the next time level is obtained by averaging the initial values and the values obtained from the second Euler step, which completes the second-order strong stability preserving Runge-Kutta time integration.

The second-order elliptic equation, (2) is solved for $\bar{u}$ using second-order central differences. Equation (2) is approximated by

$$G_j = a_j \bar{u}_{j+1} + b_j \bar{u}_j + c_j \bar{u}_{j-1}$$

where $a_j = -h_j^2(h_{j+1} - h_{j-1})/(4\Delta x^2) - h_j^3/(3\Delta x^2)$, $b_j = h_j + 2h_j^3/(3\Delta x^2)$ and $c_j = h_j^2(h_{j+1} - h_{j-1})/(4\Delta x^2) - h_j^3/(3\Delta x^2)$ for the m uniformly spaced computational nodes, $j = 1, \dots m$, which have been used to discretize the computational domain. This results in a tri-diagonal system of equations which can be solved efficiently using direct methods for $\bar{u}_j$ given G_j and h_j.

With this approach h and G can be discontinuous, which is handled by the finite volume method and approximate Riemann solver efficiently. An attractive feature of this approach is that even if G is discontinuous, $\bar{u}$ will always be smooth.

The resulting numerical scheme is theoretically $O(\Delta x^2, \Delta t^2)$ accurate. Theoretically, stability is satisfied when the time step, Δt, satisfies the *Courant-Friedrichs-Lewy*, (CFL) criteria[6]

$$\Delta t < \frac{\Delta x}{2\max(|\lambda_i|)} \quad \forall\ i.$$

5 Simulation of a Soliton

The only known analytical solution to the Serre equations is the Rayleigh solitary wave[2, 3], where

$$h(x,t) = a_0 + a_1 \text{sech}^2(\kappa(x - ct))$$

and

$$\bar{u}(x,t) = c\left(\frac{h(x,t) - a_0}{h(x,t)}\right)$$

with, $\kappa = \sqrt{3a_1}/(2a_0\sqrt{a_0 + a_1})$ and $c = \sqrt{g(a_0 + a_1)}$, which is a soliton advected over a horizontal bed. The soliton shown in Figure 1 has an amplitude of $a_1 = 1.0$ m and moves at a constant speed, $c = 10.387974$ m/s without changing shape in water that is $a_0 = 10$ m deep. It represents a balance between the dispersive and nonlinear terms. If there is an imbalance between the nonlinear and dispersive terms, we would observe trailing waves as well as attenuation of the simulated profile. The analytical solution is shown as the solid blue line and the red dots the simulated results using our model with $\Delta x = 2$ m, $\Delta t = 0.2\Delta x/\sqrt{ga_0}$ s, $\theta = 1.2$ and the solution is terminated at $t = 100$ s. There is very little phase error and attenuation in the simulated results.

We have established the convergence rate for the model using the soliton problem by calculating the non-dimensionalized L_1 norm

$$L_1 = \frac{\sum_{j=1}^{m} |h_j - h(x_j)|}{\sum_{j=1}^{m} |h(x_j)|}$$

shown here for h, where, h_j is the simulated values of $h(x,t)$ at x_j and $h(x_j)$ is the corresponding analytical solution. The L_1 norm is calculated using all the computational nodes. Figure 2 shows that the model is second-order accurate.

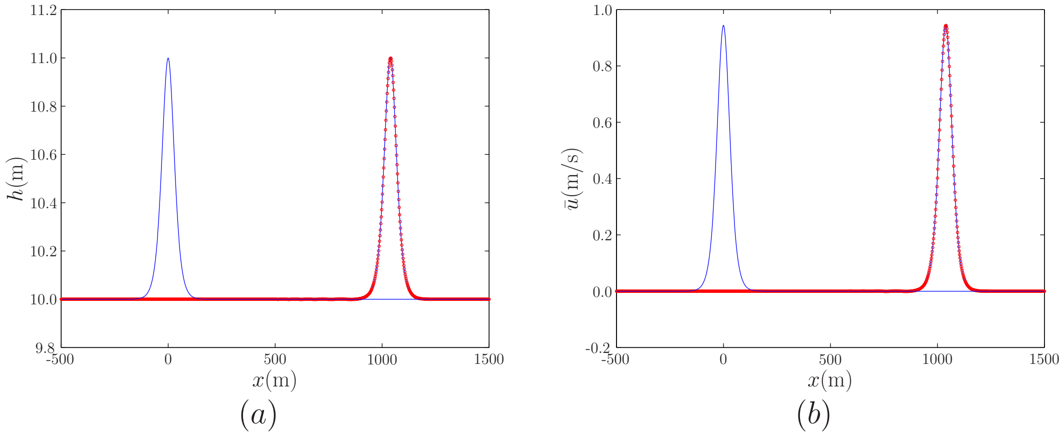

Figure 1: Initial and analytical solution of the Serre equations (blue line) plotted against the simulated solution of the Serre equations written in conservation law form (red circles) with h shown in (a) and $\bar{u}$ in (b).

6 Dam-Break Problem

Initially the water is still and $h_1 = 10$ m deep to the right and $h_0 = 2$ m deep to the left of a hypothetical dam on a frictionless horizontal channel. The solution is sought at $t = 30$ s after the dam is suddenly removed. In the simulations, $\Delta x = 0.02$ m, $m = 50000$, $\Delta t = 0.2\Delta x/\sqrt{gh_1}$ s and $\theta = 1.2$. The results shown in Figure 3(a) are for the shallow water wave equations, where dispersion is ignored and in Figure 3(b) for the solution of the Serre equations written in conservation law form. We have also plotted the analytical solution to this problem for the shallow water wave equations, which is shown by the solid blue line.

There are no dispersive waves predicted by the shallow water wave equations. The Serre equations has produced oscillations in the solution. It has also accurately captured the shock and rarefaction fan. The results shown in Figure 3 are similar to those obtained by El *et al.*[4] who used a finite difference scheme to solve the Serre equations.

Due to the additional solution of the second-order elliptic equation, the scheme is approximately 60% more computationally expensive than solving the shallow water wave equations.

7 Conclusions

The Serre equations contain dispersive terms which account for the effects of vertical velocity of the fluid particles on the behaviour of the flow. It contains

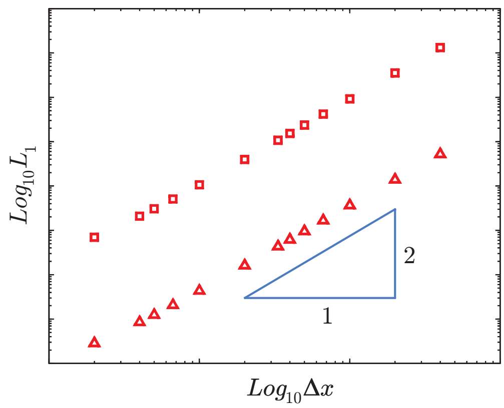

Figure 2: Convergence rate for the simulation of the soliton problem using the Serre equations written in conservation law form with, water depth (triangles) and velocity (squares).

higher-order spatial terms and a mixed spatial and temporal derivative term. The Serre equations have been written in conservation law form by replacing the mixed spatial and temporal derivative terms in the momentum equation by temporal terms and their corresponding fluxes. The temporal terms are combined to produce a new conservative quantity. The second-order finite volume method is used to evolve the conserved quantities. The remaining primitive variable is obtained by solving a second-order elliptic equation. The attractive features of this approach is that steep gradients in the flow can be handled, it is more accurate and stable than the use of finite difference scheme and is only slightly more computational expensive than solving the shallow water wave equation. This is demonstrated by simulating a hypothetical dam-break problem.

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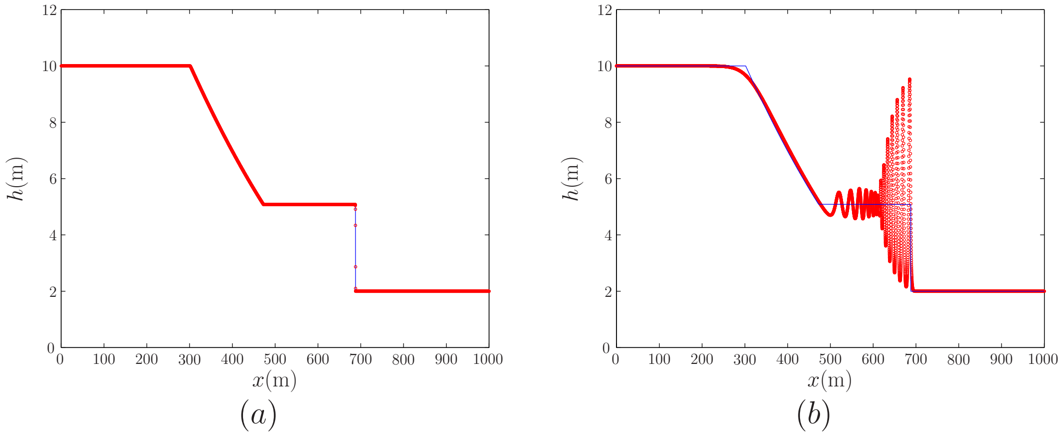

Figure 3: The simulated (○) water depth, h for the dam-break problem using the (a) shallow water wave equations and (b) Serre equations written in conservation law form shown along with the analytical solution (—) of the shallow water wave equations for this problem

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