1 Cell Averaged G

Fit a parabola so that the averages over the neighbouring cells is correct. First the parabola is centred at the midpoint so that:

$$P_i(x) = a(x - x_i)^2 + b(x - x_i) + c$$

We have a list of cell averages $\bar{u}$ and we want this parabola to match the cell averages over the cell it is centred on and the two neighbouring cells. This gives 3 equations we can solve for for a,b and c. So we want:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} P_i(x)\,dx = \frac{1}{\Delta x}\bar{u}_i$$

$$\int_{x_{i-\frac{3}{2}}}^{x_{i-\frac{1}{2}}} P_i(x)\,dx = \frac{1}{\Delta x}\bar{u}_{i-1}$$

$$\int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}} P_i(x)\,dx = \frac{1}{\Delta x}\bar{u}_{i+1}$$

We know that:

$$\int_d^e a(x - x_i)^2 + b(x - x_i) + c\,dx = \left[\frac{1}{3}a(x - x_i)^3 + \frac{1}{2}b(x - x_i)^2 + c(x - x_i)\right]_d^e$$

So we get that

$$\left[\frac{1}{3}a(x - x_i)^3 + \frac{1}{2}b(x - x_i)^2 + c(x - x_i)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{\Delta x}\bar{u}_i$$

$$\left[\frac{1}{3}a(x - x_i)^3 + \frac{1}{2}b(x - x_i)^2 + c(x - x_i)\right]_{x_{i-\frac{3}{2}}}^{x_{i-\frac{1}{2}}} = \frac{1}{\Delta x}\bar{u}_{i-1}$$

$$\left[\frac{1}{3}a(x - x_i)^3 + \frac{1}{2}b(x - x_i)^2 + c(x - x_i)\right]_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}} = \frac{1}{\Delta x}\bar{u}_{i+1}$$

So:

$$\left[\frac{1}{3}a(x_{i+\frac{1}{2}} - x_i)^3 + \frac{1}{2}b(x_{i+\frac{1}{2}} - x_i)^2 + c(x_{i+\frac{1}{2}} - x_i)\right] - \left[\frac{1}{3}a(x_{i-\frac{1}{2}} - x_i)^3 + \frac{1}{2}b(x_{i-\frac{1}{2}} - x_i)^2 + c(x_{i-\frac{1}{2}} - x_i)\right] = \frac{1}{\Delta x}\bar{u}_i$$

$$\left[\frac{1}{3}a(\frac{1}{2}\Delta x)^3+\frac{1}{2}b(\frac{1}{2}\Delta x)^2+c(\frac{1}{2}\Delta x)\right]$$
$$-\left[\frac{1}{3}a(-\frac{1}{2}\Delta x)^3+\frac{1}{2}b(-\frac{1}{2}\Delta x)^2+c(-\frac{1}{2}\Delta x)\right]=\frac{1}{\Delta x}\bar{u}_i$$

$$\frac{2}{3}a(\frac{1}{2}\Delta x)^3+2c(\frac{1}{2}\Delta x)=\frac{1}{\Delta x}\bar{u}_i$$

$$\frac{2}{24}a(\Delta x)^2+c=\bar{u}_i \tag{1}$$

$$\left[\frac{1}{3}a(-\frac{1}{2}\Delta x)^3+\frac{1}{2}b(-\frac{1}{2}\Delta x)^2+c(-\frac{1}{2}\Delta x)\right]$$
$$-\left[\frac{1}{3}a(-\frac{3}{2}\Delta x)^3+\frac{1}{2}b(-\frac{3}{2}\Delta x)^2+c(-\frac{3}{2}\Delta x)\right]=\frac{1}{\Delta x}\bar{u}_{i-1}$$

$$\frac{1}{3}a\left(-\frac{1}{8}\Delta x^3+\frac{27}{8}\Delta x^3\right)+\frac{1}{2}b\left(\frac{1}{4}\Delta x^2-\frac{9}{4}\Delta x^2\right)+c\left(-\frac{1}{2}\Delta x+\frac{3}{2}\Delta x\right)=\frac{1}{\Delta x}\bar{u}_{i-1}$$

$$\frac{1}{3}a\frac{26}{8}\Delta x^2+\frac{1}{2}b\left(-\frac{8}{4}\Delta x\right)+c=\bar{u}_{i-1}$$

$$\frac{26}{24}a\Delta x^2-b\Delta x+c=\bar{u}_{i-1} \tag{2}$$

$$\left[\frac{1}{3}a(\frac{3}{2}\Delta x)^3+\frac{1}{2}b(\frac{3}{2}\Delta x)^2+c(\frac{3}{2}\Delta x)\right]-\left[\frac{1}{3}a(\frac{1}{2}\Delta x)^3+\frac{1}{2}b(\frac{1}{2}\Delta x)^2+c(\frac{1}{2}\Delta x)\right]$$
$$=\frac{1}{\Delta x}\bar{u}_{i+1}$$

$$\frac{1}{3}a\left(\frac{27}{8}\Delta x^3-\frac{1}{8}\Delta x^3\right)+\frac{1}{2}b\left(\frac{9}{4}\Delta x^2-\frac{1}{4}\Delta x^2\right)+c\left(\frac{3}{2}\Delta x-\frac{1}{2}\Delta x\right)=\frac{1}{\Delta x}\bar{u}_{i+1}$$

$$\frac{1}{3}a\frac{26}{8}\Delta x^2+\frac{1}{2}b\frac{8}{4}\Delta x+c=\bar{u}_{i+1}$$

$$\frac{26}{24}a\Delta x^2 + b\Delta x + c = \bar{u}_{i+1} \tag{3}$$

Adding (2) and (3) gives:

$$\frac{52}{24}a\Delta x^2 + 2c = \bar{u}_{i+1} + \bar{u}_{i-1}$$

Taking twice (1) from this gives:

$$\frac{48}{24}a\Delta x^2 = \bar{u}_{i+1} + \bar{u}_{i-1} - 2\bar{u}_i$$

$$a = \frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2} \tag{4}$$

Subbin in (4) into (1) gives:

$$\frac{2}{24}\left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}\right)(\Delta x)^2 + c = \bar{u}_i$$

$$\left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{24}\right) + c = \bar{u}_i$$

$$c = \bar{u}_i - \left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{24}\right) \tag{5}$$

Subbin in (4) and (5) into 3 gives:

$$\frac{26}{24}\left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}\right)\Delta x^2 + b\Delta x + \bar{u}_i - \left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{24}\right) = \bar{u}_{i+1}$$

$$\frac{26\bar{u}_{i+1} - 52\bar{u}_i + 26\bar{u}_{i-1}}{48} + b\Delta x - \left(\frac{\bar{u}_{i+1} - 26\bar{u}_i + \bar{u}_{i-1}}{24}\right) = \bar{u}_{i+1}$$

$$b\Delta x = \frac{\bar{u}_{i+1} - 26\bar{u}_i + \bar{u}_{i-1}}{24} + \bar{u}_{i+1} - \frac{26\bar{u}_{i+1} - 52\bar{u}_i + 26\bar{u}_{i-1}}{48}$$

$$b\Delta x = \frac{25\bar{u}_{i+1} - 26\bar{u}_i + \bar{u}_{i-1}}{24} - \frac{26\bar{u}_{i+1} - 52\bar{u}_i + 26\bar{u}_{i-1}}{48}$$

$$b\Delta x = \frac{50\bar{u}_{i+1} - 52\bar{u}_i + 2\bar{u}_{i-1} - 26\bar{u}_{i+1} + 52\bar{u}_i - 26\bar{u}_{i-1}}{48}$$

$$b\Delta x = \frac{24\bar{u}_{i+1} - 24\bar{u}_{i-1}}{48}$$

$$b = \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2\Delta x} \tag{6}$$

So we have $P_i(x)$ by these equations (4),(6), (5) Which are:

$$a = \frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}$$
$$b = \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2\Delta x}$$
$$c = \bar{u}_i - \left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{24}\right)$$

So we have :

$$P_i(x) = \frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}(x - x_i)^2 + \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2\Delta x}(x - x_i) + \bar{u}_i - \left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{24}\right)$$

This can be rewritten with another variable $-1 \leq \kappa \leq 1$ so that:

$$P_i(x) = \bar{u}_i + \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2\Delta x}(x - x_i) + 3\kappa\left[(x - x_i)^2 - \frac{\Delta x^2}{12}\right]\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}$$

Which is a second order reconstruction when $-1 \leq \kappa \leq 1$ and only third when $\kappa = \frac{1}{3}$ [?,proof]

Looking at the reconstruction at $\bar{u}^+_{i-\frac{1}{2}}$ which is the reconstruction based at i of its left boundary we get:

$$\bar{u}^+_{i-\frac{1}{2}} = P_i(x_{i-\frac{1}{2}}) = \bar{u}_i + \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2\Delta x}\left(x_{i-\frac{1}{2}} - x_i\right) + 3\kappa\left[(x_{i-\frac{1}{2}} - x_i)^2 - \frac{\Delta x^2}{12}\right]\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}$$

$$= \bar{u}_i + \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2\Delta x}\left(-\frac{1}{2}\Delta x\right) + 3\kappa\left[\left(-\frac{1}{2}\Delta x\right)^2 - \frac{\Delta x^2}{12}\right]\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}$$

$$= \bar{u}_i - \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{4} + 3\kappa\left[\frac{\Delta x^2}{4} - \frac{\Delta x^2}{12}\right]\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}$$

$$= \bar{u}_i - \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{4} + 3\kappa\left[\frac{3\Delta x^2}{12} - \frac{\Delta x^2}{12}\right]\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{2\Delta x^2}$$

$$= \bar{u}_i - \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{4} + 3\kappa\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{12}$$

$$= \bar{u}_i - \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{4} + \kappa\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{4}$$

$$= \bar{u}_i + \frac{1}{4}\left[-\bar{u}_{i+1} + \bar{u}_{i-1} + \kappa\bar{u}_{i+1} - 2\kappa\bar{u}_i + \kappa\bar{u}_{i-1}\right]$$

$$= \bar{u}_i + \frac{1}{4}\left[(\kappa - 1)\,\bar{u}_{i+1} - 2\kappa\bar{u}_i + (\kappa + 1)\,\bar{u}_{i-1}\right]$$

Since $2\kappa = (\kappa + 1) + (\kappa - 1)$ we have:

$$= \bar{u}_i + \frac{1}{4}\left[(\kappa - 1)\,\bar{u}_{i+1} - ((\kappa + 1) + (\kappa - 1))\,\bar{u}_i + (\kappa + 1)\,\bar{u}_{i-1}\right]$$

$$= \bar{u}_i + \frac{1}{4}\left[(\kappa - 1)\,(\bar{u}_{i+1} - \bar{u}_i) - (\kappa + 1)\,(\bar{u}_i - \bar{u}_{i-1})\right]$$

$$= \bar{u}_i + \frac{1}{4}(\kappa - 1)\,(\bar{u}_{i+1} - \bar{u}_i) - \frac{1}{4}(\kappa + 1)\,(\bar{u}_i - \bar{u}_{i-1})$$

$$= \bar{u}_i - \frac{1}{4}(1 - \kappa)\,(\bar{u}_{i+1} - \bar{u}_i) - \frac{1}{4}(1 + \kappa)\,(\bar{u}_i - \bar{u}_{i-1})$$

For the right side of cell i the sign is different on the first term only so we get

$$\bar{u}^-_{i+\frac{1}{2}} = P_i(x_{i-\frac{1}{2}}) = \bar{u}_i + \frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{4} + 3\kappa\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{12}$$

$$= \bar{u}_i + \frac{1}{4}\left[\bar{u}_{i+1} - \bar{u}_{i-1} + \kappa\bar{u}_{i+1} - 2\kappa\bar{u}_i + \kappa\bar{u}_{i-1}\right]$$

$$= \bar{u}_i + \frac{1}{4}\left[(\kappa+1)\,\bar{u}_{i+1} - 2\kappa\bar{u}_i + (\kappa-1)\,\bar{u}_{i-1}\right]$$

Using same trick as above we get:

$$= \bar{u}_i + \frac{1}{4}\left[(\kappa+1)\,\bar{u}_{i+1} - ((\kappa+1)+(\kappa-1))\,\bar{u}_i + (\kappa-1)\,\bar{u}_{i-1}\right]$$

$$= \bar{u}_i + \frac{1}{4}\left[(\kappa+1)\,(\bar{u}_{i+1}-\bar{u}_i) - (\kappa-1)\,(\bar{u}_i-\bar{u}_{i-1})\right]$$

$$= \bar{u}_i + \frac{1}{4}(\kappa+1)\,(\bar{u}_{i+1}-\bar{u}_i) - \frac{1}{4}(\kappa-1)\,(\bar{u}_i-\bar{u}_{i-1})$$

$$= \bar{u}_i + \frac{1}{4}(1+\kappa)\,(\bar{u}_{i+1}-\bar{u}_i) + \frac{1}{4}(1-\kappa)\,(\bar{u}_i-\bar{u}_{i-1})$$

[matches Chris]

Now lets focus on this last one for the moment doing some rearranging we get that:

$$\bar{u}^-_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left[\left(\frac{1}{2}+\frac{\kappa}{2}\right)(\bar{u}_{i+1}-\bar{u}_i) + \left(\frac{1}{2}-\frac{\kappa}{2}\right)(\bar{u}_i-\bar{u}_{i-1})\right]$$

Then:

$$\bar{u}^-_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left[\left(\frac{1}{2}+\frac{\kappa}{2}\right)\frac{\bar{u}_{i+1}-\bar{u}_i}{\bar{u}_i-\bar{u}_{i-1}} + \left(\frac{1}{2}-\frac{\kappa}{2}\right)\right](\bar{u}_i-\bar{u}_{i-1})$$

Define $r_i = \frac{\bar{u}_{i+1}-\bar{u}_i}{\bar{u}_i-\bar{u}_{i-1}}$ so:

$$\bar{u}^-_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left[\left(\frac{1}{2}+\frac{\kappa}{2}\right)r_i + \left(\frac{1}{2}-\frac{\kappa}{2}\right)\right](\bar{u}_i-\bar{u}_{i-1})$$

When $\kappa = \frac{1}{3}$ we get third order reconstruction and:

$$\bar{u}^-_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left[\left(\frac{1}{2}+\frac{1}{6}\right)r_i + \left(\frac{1}{2}-\frac{1}{6}\right)\right](\bar{u}_i-\bar{u}_{i-1})$$

$$\bar{u}^-_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left[\left(\frac{4}{6}\right)r_i + \left(\frac{2}{6}\right)\right](\bar{u}_i-\bar{u}_{i-1})$$

$$\bar{u}^-_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left[\left(\frac{2}{3}\right) r_i + \left(\frac{1}{3}\right)\right](\bar{u}_i - \bar{u}_{i-1})$$

So we want some nonlinear limiter $\phi^-(r_i)$ to reconstruct forwards such that in smooth situations:

$$\phi^-(r_i) = \frac{2}{3}r_i + \frac{1}{3}$$

2 quadratic fitting these points

Lets say we have a quadratic fitting the cell average (C_i) and the two edge values of a cell, and we want to calculate it.

$$P_i(x) = a(x - x_i)^2 + b(x - x_i) + c$$

We know from The cell average part that:

$$\frac{2}{24}a(\Delta x)^2 + c = \bar{u}_i \tag{7}$$

To fit the edge values we simply require that (poor notation above):

$$\begin{aligned} u_{i-\frac{1}{2}} &= P_i(x_{i-\frac{1}{2}}) \\ &= a(-\frac{1}{2}\Delta x)^2 + b(-\frac{1}{2}\Delta x) + c \\ &= \frac{a}{4}(\Delta x)^2 - \frac{b}{2}(\Delta x) + c \end{aligned} \tag{8}$$

$$\begin{aligned} u_{i+\frac{1}{2}} &= P_i(x_{i+\frac{1}{2}}) \\ &= a(\frac{1}{2}\Delta x)^2 + b(\frac{1}{2}\Delta x) + c \\ &= \frac{a}{4}(\Delta x)^2 + \frac{b}{2}(\Delta x) + c \end{aligned} \tag{9}$$

Adding (8) and (9) eliminates b

$$u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} = \frac{a}{4}(\Delta x)^2 - \frac{b}{2}(\Delta x) + c + \frac{a}{4}(\Delta x)^2 + \frac{b}{2}(\Delta x) + c$$

$$= \frac{a}{2}(\Delta x)^2 + 2c$$

taking 2×(7)

$$\frac{a}{2}(\Delta x)^2 + 2c - \frac{1}{6}a(\Delta x)^2 - 2c = u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2 * \bar{u}_i$$

$$\frac{a}{3}(\Delta x)^2 = u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i$$

$$a = \frac{3u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} - 6\bar{u}_i}{(\Delta x)^2}$$

So that now (7) says:

$$\frac{1}{12}\frac{3u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} - 6\bar{u}_i}{(\Delta x)^2}(\Delta x)^2 + c = \bar{u}_i$$

$$\frac{1}{4}\left(u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i\right) + c = \bar{u}_i$$

$$c = \bar{u}_i - \frac{1}{4}\left(u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i\right)$$

$$c = \frac{6\bar{u}_i - u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}}{4}$$

Now 9 reads:

$$u_{i+\frac{1}{2}} = \frac{3u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} - 6\bar{u}_i}{4(\Delta x)^2}(\Delta x)^2 + \frac{b}{2}(\Delta x) + \frac{6\bar{u}_i - u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}}{4}$$

$$u_{i+\frac{1}{2}} = \frac{3u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} - 6\bar{u}_i}{4} + \frac{6\bar{u}_i - u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}}{4} + \frac{b}{2}(\Delta x)$$

$$u_{i+\frac{1}{2}} = \frac{3u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} - 6\bar{u}_i + 6\bar{u}_i - u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}}{4} + \frac{b}{2}(\Delta x)$$

$$u_{i+\frac{1}{2}} = \frac{2u_{i-\frac{1}{2}} + 2u_{i+\frac{1}{2}}}{4} + \frac{b}{2}(\Delta x)$$

$$u_{i+\frac{1}{2}} = \frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}}{2} + \frac{b}{2}(\Delta x)$$

$$2u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} = b(\Delta x)$$

$$\frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\Delta x} = b$$

So we end up with the following equations:

$$a = \frac{3u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} - 6\bar{u}_i}{(\Delta x)^2}$$

$$b = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\Delta x}$$

$$c = \frac{6\bar{u}_i - u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}}{4}$$