1 PPM Method

Here I'm just going to rewrite the equations in Coolela and Woodwards landmark paper describing the PPM.

$$\Delta u_i = u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}$$

$$u_{6,i} = 6\left(\bar{u}_i - \frac{1}{2}\left(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}\right)\right)$$

$$u^{int}_{i+\frac{1}{2}} = \bar{u}_i + \frac{1\Delta x}{2\Delta x}\left(\bar{u}_{i+1} - \bar{u}_i\right) + \frac{1}{4\Delta x}\left\{\frac{2\Delta x^2}{2\Delta x}\left[\frac{2\Delta x}{3\Delta x} - \frac{2\Delta x}{3\Delta x}\right]\left(\bar{u}_{i+1} - \bar{u}_{i-1}\right) - \Delta x\frac{2\Delta x}{3\Delta x}\delta u_{i+1} + \Delta x\frac{2\Delta x}{3\Delta x}\delta u_i\right\}$$

$$u^{int}_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left(\bar{u}_{i+1} - \bar{u}_i\right) + \frac{1}{4\Delta x}\left\{-\Delta x\frac{2}{3}\delta u_{i+1} + \Delta x\frac{2}{3}\delta u_i\right\}$$

$$u^{int}_{i+\frac{1}{2}} = \bar{u}_i + \frac{1}{2}\left(\bar{u}_{i+1} - \bar{u}_i\right) + \frac{1}{6}\left\{\delta u_i - \delta u_{i+1}\right\}$$

Where δu_i is given by:

$$\delta u_i = \frac{1\Delta x}{3\Delta x}\left[\frac{3\Delta x}{2\Delta x}\left(\bar{u}_{i+1} - \bar{u}_i\right) + \frac{3\Delta x}{2\Delta x}\left(\bar{u}_i - \bar{u}_{i-1}\right)\right]$$

$$\delta u_i = \frac{1}{3}\left[\frac{3}{2}\left(\bar{u}_{i+1} - \bar{u}_i\right) + \frac{3}{2}\left(\bar{u}_i - \bar{u}_{i-1}\right)\right]$$

$$\delta u_i = \frac{1}{2}\left[\bar{u}_{i+1} - \bar{u}_i + \bar{u}_i - \bar{u}_{i-1}\right]$$

$$\delta u_i = \frac{1}{2}\left[\bar{u}_{i+1} - \bar{u}_{i-1}\right]$$

But actually we want to use $\delta_m u_i$ instead of δu_i which is given by

$$\delta_m u_i = \min\left\{|\delta u_i|, 2|\bar{u}_{i+1} - \bar{u}_i|, 2|\bar{u}_i - \bar{u}_{i-1}|\right\} sgn\left(\delta u_i\right)$$

if $(\bar{u}_{i+1} - \bar{u}_i)(\bar{u}_i - \bar{u}_{i-1}) > 0$ and 0 otherwise.

Moving onto calculating the edge values:

$$u_{i-\frac{1}{2}} \to \bar{u}_i \quad , \quad u_{i+\frac{1}{2}} \to \bar{u}_i$$

if $(u_{i+\frac{1}{2}} - \bar{u}_i)(\bar{u}_i - u_{i-\frac{1}{2}}) \geq 0$

$$u_{i-\frac{1}{2}} \to 3\bar{u}_i - 2u_{i+\frac{1}{2}}$$

if $\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right)\left(\bar{u}_i - \frac{1}{2}\left(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}\right)\right) > \frac{\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right)^2}{6}$

$$u_{i+\frac{1}{2}} \to 3\bar{u}_i - 2u_{i-\frac{1}{2}}$$

if $\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right)\left(\bar{u}_i - \frac{1}{2}\left(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}\right)\right) < -\frac{\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right)^2}{6}$

The parabola generated by the PPM is then the parabola that goes through $u_{i-\frac{1}{2}}$ and $u_{i+\frac{1}{2}}$ with the average of $\bar{u}_i$ over the ith cell.

Using Earlier text recon.tex we have:

$$P_i(x) = a(x - x_i)^2 + b(x - x_i) + c$$

The equation giving the correct cell average for i is:

$$\frac{2}{24}a(\Delta x)^2 + c = \bar{u}_i \tag{1}$$

for the edge values we get that:

$$u_{i-\frac{1}{2}} = P_i(x_{i-\frac{1}{2}}) = a\left(-\frac{1}{2}\Delta x\right)^2 + b\left(-\frac{1}{2}\Delta x\right) + c$$

$$u_{i+\frac{1}{2}} = P_i(x_{i+\frac{1}{2}}) = a\left(\frac{1}{2}\Delta x\right)^2 + b\left(\frac{1}{2}\Delta x\right) + c$$

For the first we get:

$$u_{i-\frac{1}{2}} = a\frac{1}{4}\Delta x^2 - b\frac{1}{2}\Delta x + c$$

For the second:

$$u_{i+\frac{1}{2}} = a\frac{1}{4}\Delta x^2 + b\frac{1}{2}\Delta x + c$$

Adding them together we get that:

$$u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} = a\frac{1}{2}\Delta x^2 + 2c$$

Taking away 2 times the average equation gives:

$$u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i = a\frac{1}{2}\Delta x^2 - \frac{4}{24}a(\Delta x)^2$$

$$u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i = a\frac{8}{24}\Delta x^2$$

$$u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i = a\frac{1}{3}\Delta x^2$$

$$3\frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{\Delta x^2} = a$$

Substituting this into the average equation gives:

$$\frac{2}{24}3\frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{\Delta x^2}(\Delta x)^2 + c = \bar{u}_i$$

$$\frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{4} + c = \bar{u}_i$$

$$c = \bar{u}_i - \frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{4}$$

Subbing this into the first equation we get that:

$$u_{i-\frac{1}{2}} = 3\frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{\Delta x^2}\frac{1}{4}\Delta x^2 - b\frac{1}{2}\Delta x + \bar{u}_i - \frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{4}$$

$$b\frac{1}{2}\Delta x = 3\frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{4} + \bar{u}_i - \frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{4} - u_{i-\frac{1}{2}}$$

$$b\frac{1}{2}\Delta x = \frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{2} + \bar{u}_i - u_{i-\frac{1}{2}}$$

$$b\frac{1}{2}\Delta x = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{2}$$

$$b = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\Delta x}$$

So we have:

$$a = 3\frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{\Delta x^2}$$

$$b = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\Delta x}$$

$$c = \bar{u}_i - \frac{u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 2\bar{u}_i}{4}$$

Also looking at the derivative we get that:

$$P_i'(x) = 2a(x - x_i) + b$$

Discontinuity detection can also be used to steepen curves (this is done before monotonicity corrections).

$$u_{i-\frac{1}{2}} \to (1-\eta_i)u_{i-\frac{1}{2}} + (\eta_i)\left(\bar{u}_{i-1} + \frac{1}{2}\delta_m u_{j-1}\right)$$

$$u_{i+\frac{1}{2}} \to (1-\eta_i)u_{i+\frac{1}{2}} + (\eta_i)\left(\bar{u}_{i+1} + \frac{1}{2}\delta_m u_{j+1}\right)$$

where:

$$\eta_i = \max\left\{0, \min\left\{\eta^{(1)}\left(\tilde{\eta}_i - \eta^{(2)}\right), 1\right\}\right\}$$

where:

$$\tilde{\eta}_i = -\left(\frac{\delta^2 u_{i+1} - \delta^2 u_{i-1}}{2\Delta x}\right)\left(\frac{2\Delta x^3}{\bar{u}_{i+1} - \bar{u}_{i-1}}\right)$$

$$\tilde{\eta}_i = -\Delta x^2 \frac{\delta^2 u_{i+1} - \delta^2 u_{i-1}}{\bar{u}_{i+1} - \bar{u}_{i-1}}$$

if $-\delta^2 u_{i+1}\delta^2 u_{i-1}$ and $|\bar{u}_{i+1} - \bar{u}_{i-1}| - \epsilon \min\{|\bar{u}_{i+1}|, |\bar{u}_{i-1}|\}$ are greater than 0. While being 0 otherwise.

Where:

$$\delta^2 u_i = \frac{1}{3\Delta x}\left[\frac{\bar{u}_{i+1} - \bar{u}_i}{2\Delta x} - \frac{\bar{u}_i - \bar{u}_{i-1}}{2\Delta x}\right]$$

$$\delta^2 u_i = \frac{1}{6\Delta x^2}\left[\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}\right]$$