1 Serre Equations for horizontal beds

The elliptic equation:

$$G_x = uh - h^2 \frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{h^3}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right)$$

$$G_y = vh - h^2 \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{h^3}{3}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right)$$

Then

$$G_x = uh - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\mathrm{div}\left(\vec{u}\right)\right)$$

$$G_y = vh - \frac{\partial}{\partial y}\left(\frac{h^3}{3}\mathrm{div}\left(\vec{u}\right)\right)$$

$$\vec{G} = \vec{u}h - \nabla\left(\frac{h^3}{3}\left(\nabla \cdot \vec{u}\right)\right)$$

The continuity equation

$$\frac{\partial h}{\partial t} + \nabla \cdot \left(h\vec{u}\right) = 0$$

The flux terms we want to rewrite the flux terms into vector notation:
What we have is that

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{F} = 0$$

Where

$$\vec{U} = \begin{bmatrix} h \\ G_x \\ G_y \end{bmatrix}$$

$$\vec{F} = [F_x, F_y]$$

With F given by

$$F_x = \begin{bmatrix} uh \\ G_x u + \frac{gh^2}{2} - \frac{2h^3}{3}\left(\nabla \cdot \vec{u}\right)^2 - \frac{vh^3}{3}\frac{\partial}{\partial y}\left(\nabla \cdot \vec{u}\right) - vh^2 \frac{\partial h}{\partial y}\left(\nabla \cdot \vec{u}\right) \\ uvh \end{bmatrix}$$

$$F_y = \begin{bmatrix} vh \\ uvh \\ G_y v + \frac{gh^2}{2} - \frac{2h^3}{3}\left(\nabla\cdot\vec{u}\right)^2 - \frac{uh^3}{3}\frac{\partial}{\partial x}\left(\nabla\cdot\vec{u}\right) - uh^2\frac{\partial h}{\partial x}\left(\nabla\cdot\vec{u}\right) \end{bmatrix}$$

Let's just look at the G related components:

$$FG_x = \begin{bmatrix} G_x u + \frac{gh^2}{2} - \frac{2h^3}{3}\left(\nabla\cdot\vec{u}\right)^2 - \frac{vh^3}{3}\frac{\partial}{\partial y}\left(\nabla\cdot\vec{u}\right) - vh^2\frac{\partial h}{\partial y}\left(\nabla\cdot\vec{u}\right) \\ uvh \end{bmatrix}$$

$$FG_y = \begin{bmatrix} uvh \\ G_y v + \frac{gh^2}{2} - \frac{2h^3}{3}\left(\nabla\cdot\vec{u}\right)^2 - \frac{uh^3}{3}\frac{\partial}{\partial x}\left(\nabla\cdot\vec{u}\right) - uh^2\frac{\partial h}{\partial x}\left(\nabla\cdot\vec{u}\right) \end{bmatrix}$$

Note: We have that

$$\frac{\partial}{\partial y}\left(\frac{vh^3}{3}\left(\nabla\cdot\vec{u}\right)\right) = \frac{\partial v}{\partial y}\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right) + vh^2\frac{\partial h}{\partial y}\left(\nabla\cdot\vec{u}\right) + \frac{vh^3}{3}\frac{\partial}{\partial y}\left(\nabla\cdot\vec{u}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{vh^3}{3}\left(\nabla\cdot\vec{u}\right)\right) - \frac{\partial v}{\partial y}\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right) = vh^2\frac{\partial h}{\partial y}\left(\nabla\cdot\vec{u}\right) + \frac{vh^3}{3}\frac{\partial}{\partial y}\left(\nabla\cdot\vec{u}\right)$$

$$\frac{\partial v}{\partial y}\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right) - \frac{\partial}{\partial y}\left(\frac{vh^3}{3}\left(\nabla\cdot\vec{u}\right)\right) = -vh^2\frac{\partial h}{\partial y}\left(\nabla\cdot\vec{u}\right) - \frac{vh^3}{3}\frac{\partial}{\partial y}\left(\nabla\cdot\vec{u}\right)$$

Similarly:

$$\frac{\partial u}{\partial x}\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right) - \frac{\partial}{\partial x}\left(\frac{uh^3}{3}\left(\nabla\cdot\vec{u}\right)\right) = -uh^2\frac{\partial h}{\partial x}\left(\nabla\cdot\vec{u}\right) - \frac{uh^3}{3}\frac{\partial}{\partial x}\left(\nabla\cdot\vec{u}\right)$$

So we can rewrite it as :

$$FG_x = \begin{bmatrix} G_x u + \frac{gh^2}{2} - \frac{2h^3}{3}\left(\nabla\cdot\vec{u}\right)^2 + \frac{\partial v}{\partial y}\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right) - \frac{\partial}{\partial y}\left(\frac{vh^3}{3}\left(\nabla\cdot\vec{u}\right)\right) \\ uvh \end{bmatrix}$$

$$FG_y = \begin{bmatrix} uvh \\ G_y v + \frac{gh^2}{2} - \frac{2h^3}{3}\left(\nabla\cdot\vec{u}\right)^2 + \frac{\partial u}{\partial x}\frac{h^3}{3}\left(\nabla\cdot\vec{u}\right) - \frac{\partial}{\partial x}\left(\frac{uh^3}{3}\left(\nabla\cdot\vec{u}\right)\right) \end{bmatrix}$$