1 SWW soliton

The continuity equation is:

$$h_t + uh_x + u_x h = 0 \tag{1}$$

the momentum equation is

$$(uh)_t + \left(u^2 h + g\frac{h^2}{2}\right)_x = 0$$

expanding

$$u_t h + uh_t + \left(u^2 h\right)_x + ghh_x = 0$$

$$u_t h + uh_t + 2uu_x h + u^2 h_x + ghh_x = 0$$

using (1)

$$u_t h - u^2 h_x - uu_x h + 2uu_x h + u^2 h_x + ghh_x = 0$$

$$u_t h + uu_x h + ghh_x = 0$$

when $h > 0$

$$u_t + uu_x + gh_x = 0 \tag{2}$$

want solutions to (1) and (2) such that

$$h(x,t) = \phi(x-ct)$$
$$u(x,t) = \psi(x-ct)$$

Substituting these into (1) gives

$$-c\phi'(x-ct) + \psi(x-ct)\phi'(x-ct) + \psi'(x-ct)\phi(x-ct) = 0$$

$$\left(\psi(x-ct) - c\right)\phi'(x-ct) + \psi'(x-ct)\phi(x-ct) = 0$$

$$\left(\psi(x-ct) - c\right)\phi'(x-ct) + \psi'(x-ct)\phi(x-ct) = 0 \tag{3}$$

Substituting these into (2) gives

$$-c\psi'(x-ct)+\psi(x-ct)\psi'(x-ct)+g\phi'(x-ct)=0$$

$$(\psi(x-ct)-c)\,\psi'(x-ct)+g\phi'(x-ct)=0$$

$$(\psi(x-ct)-c)\,\psi'(x-ct)+g\phi'(x-ct)=0 \tag{4}$$

So we find that by (4)

$$(\psi(x-ct)-c)=\frac{-g\phi'(x-ct)}{\psi'(x-ct)}$$

subbing this into (3)

$$\frac{-g\phi'(x-ct)}{\psi'(x-ct)}\phi'(x-ct)+\psi'(x-ct)\phi(x-ct)=0$$

$$g\left(\phi'(x-ct)\right)^2=\left(\psi'(x-ct)\right)^2\phi(x-ct)$$

$$g\frac{\left(\phi'(x-ct)\right)^2}{\phi(x-ct)}=\left(\psi'(x-ct)\right)^2$$

$$\sqrt{g}\frac{\phi'(x-ct)}{\sqrt{\phi(x-ct)}}=\psi'(x-ct)$$

After we differentiated with respect to x and t it will be easier to see what we have with a change of variables so $y=x-ct$:

$$\sqrt{g}\frac{\phi'(y)}{\sqrt{\phi(y)}}=\psi'(y)$$

and

$$(\psi(y)-c)\,\phi'(y)+\psi'(y)\phi(y)=0 \tag{5}$$

$$(\psi(y)-c)\,\psi'(y)+g\phi'(y)=0 \tag{6}$$

We know by the fundamental theorem of calculus that

$$\psi(y)=\int_0^y\psi'(s)ds-\psi(0)$$

$$\psi(y) = \sqrt{g}\int_0^y \frac{\phi'(s)}{\sqrt{\phi(s)}}ds + \psi(0)$$

It is quite easy to see that the antiderivative of $\frac{\phi'(s)}{\sqrt{\phi(s)}}$ is $2\sqrt{\phi(s)}$

$$\psi(y) = 2\sqrt{g}\left[\sqrt{\phi(y)} - \sqrt{\phi(0)}\right] + \psi(0)$$

$$\psi(y) = 2\sqrt{g}\sqrt{\phi(y)} - 2\sqrt{g}\sqrt{\phi(0)} + \psi(0)$$

defining $d = -2\sqrt{g}\sqrt{\phi(0)} + \psi(0)$ which is a constant

$$\psi(y) = 2\sqrt{g}\sqrt{\phi(y)} + d$$

So we have that (assume $c \neq 0$)

$$u(x,t) = 2\sqrt{g}\sqrt{h(x,t)} + d \tag{7}$$

Assume that $d = 0$? $\implies \psi(0) = 2\sqrt{g}\sqrt{\phi(0)}$ which isn't too far fetched.

$$h(x,t) = \frac{u(x,t)^2}{4g} \tag{8}$$

so (2) gives

$$u_t + uu_x + g\left(\frac{u^2}{4g}\right)_x = 0$$

$$u_t + uu_x + \frac{2uu_x}{4} = 0$$

$$u_t + \frac{3uu_x}{2} = 0$$

$$-c\psi'(x-ct) + \frac{3\psi(x-ct)\psi'(x-ct)}{2} = 0$$

$$\left(\frac{3\psi(x-ct)}{2} - c\right)\psi'(x-ct) = 0$$

So either $\psi'(x-ct) = 0$ or $\psi(x-ct) = 2c/3$