1 3rd order edges

$$C_j(x) = a_j(x-x_j)^3 + b_j(x-x_j)^2 + c_j(x-x_j) + d_j$$

From the values at the 4 cells around the edge we have

$$C_j(x_j) = u_j = d_j$$
$$C_j(x_{j+1}) = u_{j+1} = a_j(\Delta x)^3 + b_j(\Delta x)^2 + c_j(\Delta x) + d_j$$
$$C_j(x_{j+2}) = u_{j+2} = a_j(2\Delta x)^3 + b_j(2\Delta x)^2 + c_j(2\Delta x) + d_j$$
$$C_j(x_{j-1}) = u_{j-1} = a_j(-\Delta x)^3 + b_j(-\Delta x)^2 + c_j(-\Delta x) + d_j$$

$$u_{j-1} = -a_j(\Delta x)^3 + b_j(\Delta x)^2 - c_j(\Delta x) + d_j$$

$$u_{j+2} = 8a_j\Delta x^3 + 4b_j\Delta x^2 + 2c_j\Delta x + d_j$$

Add the u_{j-1} and u_{j+1} equations

$$u_{j+1} + u_{j-1} = 2b_j(\Delta x)^2 + 2u_j$$

$$u_{j+1} - 2u_j + u_{j-1} = 2b_j(\Delta x)^2$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2} = b_j$$

Add the u_{j+2} and 2 times the u_{j-1} equations

$$u_{j+2} + 2u_{j-1} = 6a_j\Delta x^3 + 6b_j\Delta x^2 + 3d_j$$

$$u_{j+2} + 2u_{j-1} = 6a_j\Delta x^3 + 6\frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2}\Delta x^2 + 3u_j$$

$$u_{j+2} + 2u_{j-1} = 6a_j\Delta x^3 + 3\left(u_{j+1} - 2u_j + u_{j-1}\right) + 3u_j$$
$$u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1} = 6a_j\Delta x^3$$

$$\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6\Delta x^3} = a_j$$

Subbing it in

$$u_{j+1} = \left(\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6\Delta x^3}\right)(\Delta x)^3 + \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2}\right)(\Delta x)^2 + c_j(\Delta x) + d_j$$

$$u_{j+1} = \left(\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6}\right) + \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{2}\right) + c_j(\Delta x) + u_j$$

$$u_{j+1} - \left(\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6}\right) - \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{2}\right) - u_j = +c_j(\Delta x)$$

$$u_{j+1} + \left(\frac{-u_{j+2} + 3u_{j+1} - 3u_j + u_{j-1}}{6}\right) + \left(\frac{-u_{j+1} + 2u_j - u_{j-1}}{2}\right) - u_j = +c_j(\Delta x)$$

$$\frac{6u_{j+1} - u_{j+2} + 3u_{j+1} - 3u_j + u_{j-1} - 3u_{j+1} + 6u_j - 3u_{j-1} - 6u_j}{6} = c_j(\Delta x)$$

$$\frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{6\Delta x} = c_j$$

So to be clear

$$a_j = \frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6\Delta x^3}$$

$$b_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2}$$

$$c_j = \frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{6\Delta x}$$

$$d_j = u_j$$

Thus

$$C_j(x_{j+1/2}) = a_j(\frac{\Delta x}{2})^3 + b_j(\frac{\Delta x}{2})^2 + c_j(\frac{\Delta x}{2}) + d_j$$

$$u_{j+1/2} = \frac{1}{8}a_j\Delta x^3 + \frac{1}{4}b_j\Delta x^2 + \frac{1}{2}c_j\Delta x + d_j$$

$$u_{j+1/2} = \frac{1}{8}\left(\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6\Delta x^3}\right)\Delta x^3 + \frac{1}{4}\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2}\right)\Delta x^2 + \frac{1}{2}\left(\frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{6\Delta x}\right)\Delta x + u_j \tag{1}$$

$$u_{j+1/2} = \left(\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{48}\right) + \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{8}\right) + \left(\frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{12}\right) + u_j \tag{2}$$

$$u_{j+1/2} = \left(\frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{48}\right) + \left(\frac{6u_{j+1} - 12u_j + 6u_{j-1}}{48}\right) + \left(\frac{-4u_{j+2} + 24u_{j+1} - 12u_j - 8u_{j-1}}{48}\right) + u_j \tag{3}$$

$$u_{j+1/2} = \left(\frac{-3u_{j+2} + 27u_{j+1} - 21u_j - 3u_{j-1}}{48}\right) + u_j \tag{4}$$

$$u_{j+1/2} = \left(\frac{-3u_{j+2} + 27u_{j+1} + 27u_j - 3u_{j-1}}{48}\right) \tag{5}$$

Now for the derivative at this point,

$$\frac{\partial}{\partial x}C_j(x_{j+1/2}) = 3a_j(\frac{\Delta x}{2})^2 + 2b_j(\frac{\Delta x}{2}) + c_j$$

$$\frac{\partial}{\partial x}C_j(x_{j+1/2}) = a_j\frac{3\Delta x^2}{4} + b_j\Delta x + c_j$$

$$\frac{\partial u}{\partial x}_{j+1/2} = \frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{6\Delta x^3}\frac{3\Delta x^2}{4} + \frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2}\Delta x + \frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{6\Delta x} \tag{6}$$

$$
\begin{aligned}
\frac{\partial u}{\partial x}_{j+1/2} &= \frac{u_{j+2} - 3u_{j+1} + 3u_j - u_{j-1}}{8\Delta x} \\
+\frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x} &+ \frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{6\Delta x}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial x}_{j+1/2} &= \frac{3u_{j+2} - 9u_{j+1} + 9u_j - 3u_{j-1}}{24\Delta x} \\
+\frac{12u_{j+1} - 24u_j + 12u_{j-1}}{24\Delta x} &+ \frac{-4u_{j+2} + 24u_{j+1} - 12u_j - 8u_{j-1}}{24\Delta x}
\end{aligned} \tag{8}
$$

$$
\frac{\partial u}{\partial x}_{j+1/2} = \frac{-u_{j+2} + 27u_{j+1} - 27u_j + u_{j-1}}{24\Delta x} \tag{9}
$$