MATH 5210, HW II
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1) A metric space X is separable if it contains a dense countable set S. Prove that any open set V in X is a union of balls centered at points in S and with rational radii. (Since the set of such balls is countable, it follows that any open set is a countable union of balls).

Proof. X is a metric space with distance function d. Let $v \in V$ where V is any open set in X. There exists $r > 0$ such that $B_d(v, r) \subset V$. There is also a rational number $q \in (0, r/2)$ and an $s \in S$ such that $d(v, s) < q/2$ since S is dense in X. This implies $v \in B_d(s, q)$. Let $y \in B_d(s, q)$ also. This implies that $d(y, v) \leq d(y, s) + d(s, v) < q + q/2 < 2q < r$, which implies $y \in B_d(v, r) \subset V$ and $v \in B_d(s, q) \subset V$. □

2) Let $X = [0, 1]^2$. Choose the distance on X wisely, and use the previous exercise to prove that any open set in X is Lebesgue measurable.

Proof. Choose the distance on X to be $d(x, y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$. A ball in this space with this metric is a rectangle. Note that X is a separable metric space since it contains $\mathbb{Q} \cap [0, 1] \times \mathbb{Q} \cap [0, 1]$ which is a countable subset that is dense in X. By exercise (1) above any open set is the countable union of balls. A set is Lebesgue measurable if it is the countable union of rectangles. □

3) Let $P = [0, 1]^2$. If E and F are two elementary sets such that $E \cup F = P$ then $m(E \cap F) = m(E) + m(F) - 1$. Now assume $E = \cup_{i=1}^{\infty} E_i$ and $F = \cup_{i=1}^{\infty} F_i$, disjoint unions of elementary sets each, and $E \cup F = P$. Observe that $E \cap F$ is the disjoint union of $E_i \cap F_j$. Prove that

$$\sum_{i,j} m(E_i \cap F_j) = \sum_i m(E_i) + \sum_j m(F_j) - 1.$$

Proof. To show the equality above we first show that

$$\sum_{i,j} m(E_i \cap F_j) \leq \sum_i m(E_i) + \sum_j m(F_j) - 1.$$

It suffices to show that for all N

$$\sum_{i,j \leq N} m(E_i \cap F_j) \leq \sum_i m(E_i) + \sum_j m(F_j) - 1.$$

Let F_j's remain as they are and define the following new sets to remove some of the overlaps.

$$\begin{aligned}
E_1' &= E_1 \setminus ((E_1 \cap F_1) \cup (E_1 \cap F_2) \cup \cdots \cup (E_1 \cap F_N)) \\
E_2' &= E_2 \setminus ((E_2 \cap F_1) \cup (E_2 \cap F_2) \cup \cdots \cup (E_2 \cap F_N)) \\
&\vdots \\
E_N' &= E_N \setminus ((E_N \cap F_1) \cup (E_N \cap F_2) \cup \cdots \cup (E_N \cap F_N))
\end{aligned}$$

Now we have

$$\sum_{i,j \leq N} m(E_i \cap F_j) = \sum_i m(E_i) + \sum_j m(F_j) - \left[\sum_i m(E_i') + \sum_j m(F_j)\right]$$

Since not all overlap was removed $\sum_i m(E_i') + \sum_j m(F_j) \geq 1$ which implies that

$$\sum_{i,j \leq N} m(E_i \cap F_j) \leq \sum_i m(E_i) + \sum_j m(F_j) - 1.$$

To show the other direction, that is

$$\sum_{i,j} m(E_i \cap F_j) \geq \sum_i m(E_i) + \sum_j m(F_j) - 1$$

Set $E' = \cup_{i \leq N} E_i$ and $F' = \cup_{j \leq N} E_j$. We have that $E' \cup F' \subset P$. We know that

$$\begin{aligned}
&m(E') + m(F') - m(E' \cap F') = m(E' \cup F') \\
\implies &m(E') + m(F') - m(E' \cup F') = m(E' \cap F')
\end{aligned}$$

Now putting this all together we have

$$\sum_{i \leq N} m(E_i) + \sum_{j \leq N} m(F_j) - \sum_{i,j \leq N} m(E_i \cup F_j) = \sum_{i,j \leq N} m(E_i \cap F_j) \leq \sum_{i,j} m(E_i \cap F_j)$$

Using the fact that $m(E' \cup F') \leq m(P) = 1$ this implies that

$$\sum_{i \leq N} m(E_i) + \sum_{j \leq N} m(F_j) - 1 \leq \sum_{i \leq N} m(E_i) + \sum_{j \leq N} m(F_j) - \sum_{i,j \leq N} m(E_i \cup F_j) \leq \sum_{i,j} m(E_i \cap F_j)$$

This is true for all N. Pass to the limit and we have

$$\sum_i m(E_i) + \sum_j m(F_j) - 1 \leq \sum_{i,j} m(E_i \cap F_j)$$

$\square$

4) Let $\sum_{n=1}^{\infty} x_n$ be a series of non-negative real numbers. Show that its sum (which can be ∞) is equal to the supremum of the set of sums $\sum_{n\in S} x_n$ where S runs over all finite subsets of the set of natural numbers. Conclude that any sequence of non-negative numbers can be added in any order.

Proof. Certainly $\sum_{n\in S} x_n \leq \sum_{n=1}^{\infty} x_n$. We also have that $\sup\{\sum_{n\in S} x_n\} \leq \sum_{n=1}^{\infty} x_n$ this implies that there exists a $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} x_n \leq \sup\{\sum_{n\in S} x_n\}$. Now pass to the limit and we have

$$\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n \leq \sup\left(\sum_{n\in S} x_n\right)$$

Thus we have shown that

$$\sum_{n=1}^{\infty} x_n = \sup\left(\sum_{n\in S} x_n\right)$$

$\square$

5) In the following exercises, $\mathcal{M}$ is a σ-algebra of a non-empty set X, that is, a family of subsets of X closed under complements and countable unions, and μ is a σ-measure. Let $A_1 \supseteq A_2 \supseteq \dots$ be a sequence of sets in $\mathcal{M}$. Let $A = \cap_{i=1}^{\infty} A_i$. Prove that $\lim_{i\mapsto\infty} \mu(A_i) = \mu(A)$, assuming that $\mu(X) = 1$.

Proof. A_i's are not disjoint. Set $B_i = A_i \setminus A_{i+1} = A_i \cap (A_{i+1})^c$. B_i's are pairwise disjoint.

$$\begin{aligned}
\cup_{i=1}^{\infty} B_i &= \cup_{i=1}^{\infty} \left(A_i \cap (A_{i+1})^c\right) \\
&= \cup_{i=1}^{\infty} (A_i) \cap (\cup_{i=1}^{\infty} (A_{i+1})^c) \quad \text{(union is distributive)} \\
&= A_1 \cap (\cap_{i=1}^{\infty} A_{i+1})^c \quad \text{(since sets are nested and by Demorgan's Laws)} \\
&= A_1 \cap (A)^c \quad \text{(by definition of A)} \\
&\implies (\cup_{i=1}^{\infty} B_i) \cup A = A_1
\end{aligned}$$

$\cup_{i=1}^{\infty} B_i$ and A are disjoint therefore

$$\mu\left(\cup_{i=1}^{\infty} B_i\right) \cup A = \sum_{i=1}^{\infty} \mu(B_i) + \mu(A) = \mu(A_1)$$

Which implies

$$\begin{aligned}
\mu(A) &= \mu(A_1) - \sum_{i=1}^{\infty} \mu(B_i) \\
&= \mu(A_1) - \sum_{i=1}^{\infty} \left[\mu(A_i) - \mu(A_{i+1})\right] \\
&= \mu(A_1) - \left[(\mu(A_1) - \mu(A_2)) + (\mu(A_2) - \mu(A_3)) + \cdots\right] \\
&= \lim_{i\to\infty} \mu(A_i)
\end{aligned}$$

$\square$

6) A subset of X is called measurable if it belongs to $\mathcal{M}$. Let $f : X \to \mathbb{R}$ prove that $\{x|f(x) < c\}$ is measurable $\forall c \in \mathbb{R}$ if and only if $\{x|f(x) \leq c\}$ is measurable $\forall c \in \mathbb{R}$.

Proof. ($\implies$) First assume that $\{x|f(x) < c\}$ is measurable $\forall c \in \mathbb{R}$. Let $k \in \mathbb{N}$, then $\{x|f(x) < c + \frac{1}{k}\}$ is measurable. $\cap_k\{x|f(x) < c + \frac{1}{k}\}$ is also measurable since it is a union of measurable sets.

($\impliedby$) Now assume that $\{x|f(x) \leq c\}$ is measurable $\forall c \in \mathbb{R}$. Let $k \in \mathbb{N}$, then $\{x|f(x) \leq c - \frac{1}{k}\}$ is measurable. $\cap_k\{x|f(x) \leq c - \frac{1}{k}\}$ is measurable since it is a union of measurable sets.

Since both directions were proved we can conclude that $\{x|f(x) < c\}$ is measurable $\forall c \in \mathbb{R}$ if and only if $\{x|f(x) \leq c\}$ is measurable $\forall c \in \mathbb{R}$ □

7) Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions on X. Prove that

$$g(x) = \inf\{f_1(x), f_2(x), \dots\} \text{ and } G(x) = \sup\{f_1(x), f_2(x), \dots\}$$

are measurable functions.

Proof. **Supremum:**

First we will show that $G(x)$ is measurable. Let $c \in \mathbb{R}$. We want to show that $\{x|G(x) > c\} = \cup_{n\in\mathbb{N}}\{x|f_n(x) > c\}$, i.e. that $\{x|G(x) > c\}$ is a countable union of measurable sets.

Assume $y \in \{x|G(x) > c\}$, then $G(y) > c$. If $f_n(y) \leq c$ for all $n \in \mathbb{N}$ then c is an upper bound for $\{f_n(y)_{n\in\mathbb{N}}\}$. This would imply that $G(y) \leq c$, but $G(y) > c$ so $f_m(y) > c$ for some $m \in \mathbb{N}$, which means $y \in \{x|f_m(x) > c\}$ and $y \in \cup_{n\in\mathbb{N}}\{x|f_n(x) > c\}$. This shows that $\{x|G(x) > c\} \subseteq \cup_{n\in\mathbb{N}}\{x|f_n(x) > c\}$.

Now assume $z \in \cup_{n\in\mathbb{N}}\{x|f_n(x) > c\}$, then $z \in \{x|f_k(x) > c\}$ for some $k \in \mathbb{N}$ which implies $f_k(z) > c$. Since $G(z)$ is the least upper bound we have that $G(z) \geq f_k(z) > c$, hence $z \in \{x|G(x) > c\}$. This shows that $\cup_{n\in\mathbb{N}}\{x|f_n(x) > c\} \subseteq \{x|G(x) > c\}$.

Since both $\cup_{n\in\mathbb{N}}\{x|f_n(x) > c\} \subseteq \{x|G(x) > c\}$ and $\{x|G(x) > c\} \subseteq \cup_{n\in\mathbb{N}}\{x|f_n(x) > c\}$ are true we have that $\{x|G(x) > c\} = \cup_{n\in\mathbb{N}}\{x|f_n(x) > c\}$ and therefore since c is arbitrary $G(x)$ is measurable.

Infimum:

The proof that $g(x)$ is measurable is almost identical to the above argument. Let $c \in \mathbb{R}$. We want to show that $\{x|g(x) < c\} = \cup_{n\in\mathbb{N}}\{x|f_n(x) < c\}$, i.e. that $\{x|g(x) < c\}$ is a countable union of measurable sets.

Assume $y \in \{x|g(x) < c\}$, then $g(y) < c$. If $f_n(y) \geq c$ for all $n \in \mathbb{N}$ then c is a lower bound for $\{f_n(y)_{n\in\mathbb{N}}\}$. This would imply that $g(y) \geq c$, but $g(y) < c$ so $f_m(y) < c$ for some $m \in \mathbb{N}$, which means $y \in \{x|f_m(x) < c\}$ and $y \in \cup_{n\in\mathbb{N}}\{x|f_n(x) < c\}$. This shows that $\{x|g(x) < c\} \subseteq \cup_{n\in\mathbb{N}}\{x|f_n(x) < c\}$.

Now assume $z \in \cup_{n\in\mathbb{N}}\{x|f_n(x) < c\}$, then $z \in \{x|f_k(x) < c\}$ for some $k \in \mathbb{N}$ which implies $f_k(z) < c$. Since $g(z)$ is the greatest lower bound we have that $g(z) \leq f_k(z) < c$, hence $z \in \{x|g(x) < c\}$. This shows that $\cup_{n\in\mathbb{N}}\{x|f_n(x) < c\} \subseteq \{x|g(x) < c\}$.

Since both $\cup_{n\in\mathbb{N}}\{x|f_n(x) < c\} \subseteq \{x|g(x) < c\}$ and $\{x|g(x) < c\} \subseteq \cup_{n\in\mathbb{N}}\{x|f_n(x) < c\}$ are true we have that $\{x|g(x) < c\} = \cup_{n\in\mathbb{N}}\{x|f_n(x) < c\}$ and therefore since c is arbitrary $g(x)$ is measurable.

□

8) Let f be an integrable function on X, such that $f(x) \geq 0$ for all $x \in X$. Prove that $\int_X f = 0$ if and only if the measure of $A = \{x \in X|f(x) > 0\}$ is 0, that is $f = 0$ almost everywhere. Hint, consider the sets $A_n = \{x \in X|f(x) > 1/n\}$ for $n = 1, 2, \ldots$.

Proof. Assume $\int_X f = 0$. Then by the Chebychev inequality

$$0 \leq \mu(\{x \in X|f(x) \geq \frac{1}{n}\}) \leq \frac{1}{n}\int_X f = 0$$

which implies that $\mu(\{x \in X|f(x) \geq \frac{1}{n}\}) = 0$ for all $n \in \mathbb{N}$, that is $f = 0$ almost everywhere.

Now assume $f = 0$ almost everywhere. Then every nonnegative simple function f_n approximating f from below is zero almost everywhere and is of the form $f_n = \sum_{i=1}^n c_i\chi_{A_i}$ where either $c_i = 0$ or $\mu(A_i) = 0$. Then $\int_X f_n = \sum_{i=1}^n c_i\mu(A_i) = 0$.

Therefore $\int_X f = \sup\{\int_X f_n|f_n \textit{ is simple}, \ f_n \leq f\} = 0$. □

9) Let $X = (0, 1]$, with the usual measure, and let $f(x) = 1/\sqrt{x}$. Use the monotone convergence theorem to prove that f is integrable and compute it's integral.

Let $f_n(x) = f(x) \cdot \chi_{[\frac{1}{n},1]}$ so that $f_1(x) \leq f_2(x) \leq \cdots$, i.e. a monotone sequence.

$$\int_{[\frac{1}{n},1]} f_n(x) = \int_{\frac{1}{n}}^1 f = \int_{\frac{1}{n}}^1 \frac{1}{\sqrt{x}}\ dx = 2\sqrt{x}|_{x=\frac{1}{n}}^{x=1} = 2 - \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} 2 - \frac{1}{\sqrt{n}} = 2$$

Therefore by the Monotone Convergence Theorem

$$\int_{(0,1]} \frac{1}{\sqrt{x}} = 2$$