Groundwater Contamination Metal Company is a mine for metals. Part of their process involves disposing of waste rock. The waste is deposited into ponds that have clean water flowing into them. A certain type of algae grows in the pond that degrades a harmful substance contained in the waste rock. The algae degrades the harmful substance at a rate proportional to the concentration of the harmful substance in the pond. Let $x(t)$ be the amount of the harmful substance in the pond at time t and $x(0) = x_0$ be the initial amount of the harmful substance in the pond at $t = 0$. Let r_{in} and r_{out} be the rate of water into and out of the pond. Let V be the volume of the pond and k be a proportionality constant for the degradation term.

(a) Assume for a moment that the pond does not have algae. Derive a differential equation to describe how the amount of the harmful substance in the pond changes with time.

Solution:

$$\frac{dx}{dt} = r_{in}c_{in} - r_{out}\frac{x(t)}{V} = -r_{out}\frac{x(t)}{V}$$

(b) Solve the differential equation derived in part (a).

Solution:

$$\int_{x(0)}^{x(t)} \frac{dx}{x(t)} = \int_0^t \frac{-r_{out}}{V} dt$$

$$\implies \ln(x(t)) - \ln(x(0)) = \frac{-r_{out}}{V} t$$

$$\implies \ln\left(\frac{x(t)}{x_0}\right) = \frac{-r_{out}}{V} t$$

$$\implies \frac{x(t)}{x_0} = e^{\frac{-r_{out}}{V} t}$$

$$\implies x(t) = x_0 e^{\frac{-r_{out}}{V} t}$$

(c) Now consider how the differential equation would be different if algae is in the pond. Derive the differential equation. What type of model is this?

Solution:

$$\frac{dx}{dt} = -r_{out}\frac{x(t)}{V} - f\left(\frac{x(t)}{V}\right)\frac{x(t)}{V}$$

$$\frac{dx}{dt} = -r_{out}\frac{x(t)}{V} - k\frac{x^2(t)}{V^2}$$

$$\frac{dx}{dt} = \frac{x(t)}{V}\left(-r_{out} - k\frac{x(t)}{V}\right)$$

Logistic Model

(d) Solve the differential equation derived in part (c). *Hint: Use partial fractions.*

Solution:

$$\int_{x(0)}^{x(t)} \frac{dx}{\frac{x(t)}{V}\left(-r_{out} - k\frac{x(t)}{V}\right)} = \int_0^t dt$$

Use partial fraction decomposition:

$$\frac{1}{\frac{x(t)}{V}\left(-r_{out} - k\frac{x(t)}{V}\right)} = \frac{A}{\frac{x(t)}{V}} + \frac{B}{\left(-r_{out} - k\frac{x(t)}{V}\right)}$$

$$\implies 1 = A\left(-r_{out} - k\frac{x(t)}{V}\right) + B\frac{x(t)}{V}$$

If $x(t) = 0$:

$$1 = A\left(-r_{out} - k\frac{0}{V}\right) + B\frac{0}{V} \implies A = \frac{-1}{r_{out}}$$

If $x(t) = \frac{Vr_{out}}{k}$:

$$1 = A\left(-r_{out} - k\frac{\frac{Vr_{out}}{k}}{V}\right) + B\frac{\frac{Vr_{out}}{k}}{V} \implies B = \frac{-k}{r_{out}}$$

Now solve

$$\int_{x(0)}^{x(t)} \frac{\frac{-1}{r_{out}}}{\frac{x(t)}{V}} + \frac{\frac{-k}{r_{out}}}{\left(-r_{out} - k\frac{x(t)}{V}\right)} = \int_0^t dt$$

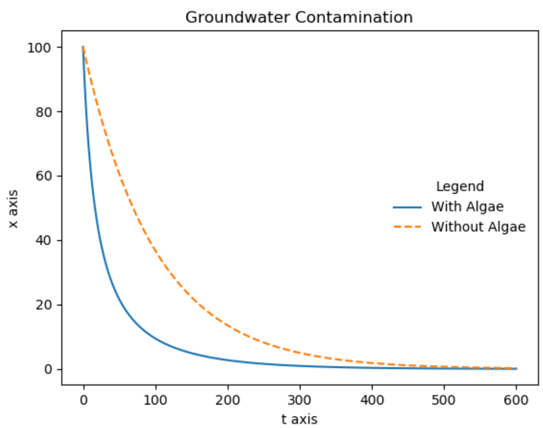
$$\implies \frac{-V}{r_{out}}\int_{x(0)}^{x(t)} \frac{1}{x(t)}dx - \frac{k}{r_{out}}\int_{x(0)}^{x(t)} \frac{1}{-r_{out} - k\frac{x(t)}{V}}dx = t$$

$$-\frac{V}{r_{out}}\left(\ln(x(t)) - \ln(x_0)\right) + \frac{V}{r_{out}}\left(\ln\left(-r_{out} - \frac{k}{V}x(t)\right) - \ln\left(-r_{out} - \frac{k}{V}x_0\right)\right) = t$$

Skipping some algebra...

$$x(t) = \frac{-r_{out}}{\left(\frac{-r_{out}}{x_0} - \frac{k}{V}\right) e^{\frac{r_{out}}{V} t} + \frac{k}{V}}$$

(e) The following graph shows how the amount of the harmful substance changes with time both with algae in the pond and with out based on the following conditions:



- t measured in days
- $r_{in} = r_{out} = 100\ gal/day$
- $x(0) = 100$ lbs
- $k = 45,638$
- $V = 10,000$ gallons

i How much of the harmful substance is being degraded by the algae and how much is flowing into the ground?

Solution:

ii If more than 10% of the initial amount of the harmful substance flows into the ground Metal Mines risks contaminating the groundwater of the neighboring town. Is the groundwater becoming contaminated?

Solution:

iii For what value of r ensures the groundwater will not be contaminated?

Solution: