MATH 5210, HW III
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1) Let (Y, d) be a complete metric space and X a dense subset of Y. The set X is a also a metric space with respect to the same metric. Let X^* be the completion of X. Recall that X^* is the set of equivalence classes of Cauchy sequences (x_n) in X. Since Y is complete, $\lim_n x_n$ exists in Y. Equivalent Cauchy sequences have the same limit, hence $f((x_n)) = \lim_n x_n$ is a well defined map $f : X^* \to Y$. Show that f is an isomorphism of metric spaces.

Proof. To show $f : X^* \to Y$ is an isomorphism we need to show that it is a surjective isometry. Let $\{x_n\}$ be a Cauchy sequence such that $x_n \in X$ for all n. Let $\{\tilde{x}\}$ denote an equivalence class of Cauchy sequences in X.

X is dense in Y means for all $y \in Y$ there exists a sequence $\{x_n\}$ with elements in X such that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, y) < \epsilon/2$ if $n > N$, i.e. $\lim_n x_n = y \in Y$. By the triangle inequality we have $d(x_n, x_m) \leq d(x_n, y) + d(y, x_m) = \epsilon$ if $n, m > N$. Therefore $\{x_n\}$ is a Cauchy sequence and hence belongs to an equivalence class $\{\tilde{x}\} \in X^*$. This shows that f is surjective.

For f to be an isometry $d^*(\{\tilde{x}\}, \{\tilde{y}\}) = d(x, y)$ for all $\{\tilde{x}\}, \{\tilde{y}\} \in X^*$. The distance d^* is defined as the distance between any two representative elements from each equivalence class. Pick a Cauchy sequence out of each equivalence class, say $\{x_n\}$ from $\{\tilde{x}\}$ and $\{y_n\}$ from $\{\tilde{y}\}$. Since Y is complete the limits of each of these sequences exist in Y. Therefore $d(\{x_n\}, \{y_n\}) = d(\lim_n x_n, \lim_n y_n) = d(x, y)$. Thus f is isometric since distance is preserved. □

2) Let $V = C([0, 1])$ be the space of continuous functions on $[0, 1]$. Prove that the set of piece-wise linear function (i.e. whose graphs are obtained by connecting the dots in the plane) is dense in V, with respect to the sup norm, that is, for every $f \in V$ and every $\epsilon > 0$, there exists a piece-wise linear function g such that $|f(x) - g(x)| < \epsilon$ for all $x \in [0, 1]$. Hint: use uniform continuity of f.

Proof. Since f is continuous on a compact set this implies that f is uniformly continuous which means that for all $\epsilon > 0$ and $x \in [0, 1]$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Now take a partition $P : 0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = 1$ (n is finite) such that $x_{i+1} - x_i < \delta$. Let $g(x)$ be a piece-wise linear function passing through points $(x_i, f(x_i))$. The distance from $f(x)$ and $g(x)$ on each subinterval is bounded by ϵ by the uniform continuity of f. This means that even the supremum of the distances on each subinterval is bounded by ϵ, that is $\sup_{x\in[0,1]} |f(x) - g(x)| < \epsilon$. Therefore the set of all piece-wise linear functions is dense in V. □

3) Fix $K(x, y)$, a continuous function on $[0, 1]^2$. Let $f(x)$ be a continuous function on $[0, 1]$. Let

$$g(x) = \int_0^1 K(x, y)f(y)\, dy.$$

Prove that $g(x)$ is a continuous function on $[0,1]$. Hint: K is uniformly continuous, why? Let $V = C([0,1])$ be the space of continuous functions on $[0,1]$. Consider V as a normed space with the sup norm. Let $T : V \to V$, $T(f) = g$ for every $f \in V$, as above. Prove that T is bounded.

Proof. First note the function $K(x,y)$ is continuous on a compact set, therefore it is uniformly continuous and thus attains a maximum value M_k on $[0,1]^2$. $K(x,y)$ uniformly continuous means for all $\epsilon > 0$ and $(x,y) \in [0,1]^2$ there exists a $\delta > 0$ such that $|k(x,y) - k(a,b)| < \epsilon$ if $max\{|x-a|, |y-b|\} < \delta$.

Also note that every f is continuous on $[0,1]$, which is also a compact set, and therefore every f is uniformly continuous and thus attains a maximum value M_f on $[0,1]$. For $f(x)$ to be uniformly continuous means for all $\epsilon > 0$ and $x \in [0,1]$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ if $|x-y| < \delta$.

To prove that $g(x)$ is continuous assume $\sup_{x\in[0,1]} |f(x)| = ||f|| \leq 1$, then

$$\begin{aligned} |g(x) - g(z)| &= \left| \int_0^1 K(x,y)f(y)\ dy - \int_0^1 K(z,y)f(y)\ dy \right| \\ &= \left| \int_0^1 (K(x,y) - K(z,y))f(y)\ dy \right| \end{aligned}$$

Since K is uniformly continuous $|k(x,y) - k(z,y)| < \epsilon$.

$$\left| \int_0^1 (K(x,y) - K(z,y))f(y)\ dy \right| \leq \left| \int_0^1 \epsilon \right| = \epsilon$$

To show that T is bounded, which means $||T(f)|| \leq C||f||$ for all $f \in V$ assume that $f \neq 0$ then this inequality can be rewritten as

$$\frac{1}{||f||}||T(f)|| = ||T\left(\frac{f}{||f||}\right)|| \leq C$$

Since $f/||f||$ is an element of norm 1 T bounded is equivalent to demanding that

$$\sup_{||f||=1} ||T(f)|| = \sup_{||f||\leq 1} ||T(f)|| = \sup_{||f||<1} ||T(f)|| < \infty$$

So we have

$$||T(f)|| = \sup_{||f||=1} |T(f)| = \sup_{||f||=1} \left| \int_0^1 K(x,y)f(y)dy \right| < M_k \cdot M_f \cdot (1-0)$$

□

4) Let U be a dense subspace of a normed space V. Let $g : U \to \mathbb{R}$ be a bounded linear functional i.e. there exists $C \geq 0$ such that

$$|g(x)| \leq C||x||$$

for all $x \in U$. Then g can be extended (uniquely) to a linear functional $f : V \to \mathbb{R}$ satisfying the same bound. Hint: any $x \in V$ is a limit of a Cauchy sequence (x_n) in U.

Proof. U is dense in V means for all $v \in V$ there exists a sequence $\{u_n\}$ with elements in U such that for all $\epsilon > 0$ there exists N such that $||u_n - v|| < \epsilon$ if $n > N$. Since g is a bounded linear functional this means $|g(u_n)| \leq C||u_n||$ for all $u_n \in U$.

$$|g(u_n - u_m)| = |g(u_n) - g(u_m)| \leq C||u_n - u_m|| \leq C\left[||u_n - v|| + ||u_m - v||\right]$$

Since U is dense in V this means if $n, m > N$

$$|g(u_n) - g(u_m)| \leq C2\epsilon$$

This shows that the sequence $\{g_n\} = \{g(u_n)\}$ is Cauchy and since $\mathbb{R}$ is complete this sequence converges. Therefore

$$f(v) = \lim_{n\to\infty} g(u_n) \leq \lim_{n\to\infty} C||u_n|| = C||v||$$

To show that f is also linear consider the Cauchy sequences $\{u_n\}$ and $\{y_n\}$ with elements in U where $u_n \to v$ and $y_n \to x$ as $n \to \infty$ with $v, x \in V$. Then we have

$$f(v) + f(x) = \lim_{n\to\infty} g(u_n) + \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(u_n) + g(y_n) = \lim_{n\to\infty} g(u_n + y_n) = f(v + x)$$

□

5) Recall the normed space $\ell^2(\mathbb{N})$, the set of all infinite tuples of real numbers $x = (x_1, x_2, \ldots)$ such that $||x||^2 = \sum_{i=1}^{\infty} x_i^2 < \infty$, with the norm $||x||$ so defined. Let $S \subset \ell^2(\mathbb{N})$ be the subset of all x with $x_i \in \mathbb{Q}$ and almost all $x_i = 0$. This is a countable set. Prove that S is dense.

Proof. S dense in $\ell^2(\mathbb{N})$ means for all $x \in \ell^2(\mathbb{N})$ there exists $y \in S$ such that

$$||x - y|| = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2} < \epsilon$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can choose y such that $(x_i - y_i)^2 < \epsilon^2/2N$ for all $i \leq N$.

$$\begin{aligned} ||x - y|| &= \left[(x_1 - y_1)^2 + \cdots + (x_N - y_N)^2 + \sum_{i=N}^{\infty}(x_i - 0)^2\right]^{1/2} \\ &\leq \left[\frac{\epsilon^2}{2} + \sum_{i=N}^{\infty} x_i^2\right]^{1/2} \end{aligned}$$

Since every x is such that $\sum_{i=1}^{\infty} x_i^2 < \infty$ this means that if N is chosen sufficiently large $\sum_{i=N}^{\infty} x_i^2 < \epsilon^2/2$. Therefore $||x - y|| < \epsilon$. □

6) Let V be a normed space, and $A, B \subset V$ two open sets. Prove that $A + B = \{x + y \mid x \in A, y \in B\}$ is open.

Proof. Fix $y \in B$. The map $f : V \to V$ defined by $f(x) = x + y$ is continuous. The inverse map $g(x) = x - y$ is also continuous which means that the map f is a homeomorphism and therefore maps open sets to open sets. $A + y$ is open and $\cup_{y\in B} A + y = A + B$ is open since a union of open sets is open. □

7) Perhaps you have seen the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Where does this come from? The purpose of this exercise is to derive this formula as a special case of the Parseval's identity. Let $X = (-1/2, 1/2]$. Let $f(x) = x$ on X. Compute $||f||^2$, the square of $L^2(X)$ norm of f. Then Fourier expand f and then compute $||f||^2$ using the Parseval's identity. (Be careful, the norm of $\sin(2\pi n x)$ is not 1). Deduce the identity.

Proof. First we calculate $||f||^2$:

$$||f||^2 = \int_{-1/2}^{1/2} t^2 dt = \frac{1}{12}$$

Now calculating $||f||^2$ using Parseval's Identity:

$$||f||^2 = \sum_{m=1}^{\infty} \langle f, e_m \rangle^2 + \sum_{n=1}^{\infty} \langle f, e_n \rangle^2$$

The elements of the orthonormal basis, e_m and e_n, are defined by $u_m/||u_m||$ and $u_n/||u_n||$ respectively.

$$e_m = \frac{u_m}{||u_m||} = \frac{\cos(2\pi mt)}{||\cos(2\pi mt)||} = \frac{\cos(2\pi mt)}{\left(\int_{-1/2}^{1/2} \cos^2(2\pi mt)\right)^{1/2}} = \sqrt{2}\cos(2\pi mt)$$

$$e_n = \frac{u_n}{||u_n||} = \frac{\sin(2\pi nt)}{||\sin(2\pi nt)||} = \frac{\sin(2\pi nt)}{\left(\int_{-1/2}^{1/2} \sin^2(2\pi nt)\right)^{1/2}} = \sqrt{2}\sin(2\pi nt)$$

Parseval's Identity becomes

$$||f||^2 = \sum_{m=1}^{\infty} \left(\int_{-1/2}^{1/2} \sqrt{2} t \cos(2\pi mt) dt \right)^2 + \sum_{n=1}^{\infty} \left(\int_{-1/2}^{1/2} \sqrt{2} t \sin(2\pi nt) dt \right)^2$$

The first term on the right hand side integrates to zero, so after integrating the second term the identity simplifies to

$$||f||^2 = \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}\sin(\pi n) - \sqrt{2}\pi n \cos(\pi n)}{2\pi^2 n^2} \right)^2$$

Notice that $\sin(\pi n) = 0$ for all $n = 1, 2, \dots$ and $\cos(\pi n) = -1$ or 1 for all $n = 1, 2, \dots$. Therefore the numerator of this expression is simply equal to $\sqrt{2}\pi n$. Simplifying and setting this equal to $1/12$ we have

$$\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}\pi n}{2\pi^2 n^2} \right)^2 = \sum_{n=1}^{\infty} \frac{1}{2\pi^2 n^2} = \frac{1}{12} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\square$

8) Let $M \geq 0$. Let c_n be a sequence of real numbers such that $|c_n| \leq M/n^2$ for all n. Then the series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(2\pi nt)$$

converges uniformly, for all $t \in \mathbb{R}$. Hence f is a periodic and continuous function f. Prove that the series converges to f in $L^2((-1/2, 1/2])$ that is

$$\lim_n ||f - f_n|| = 0$$

where f_n is the sequence of partial sums, and $||\cdot||$ the L^2-norm. Hint: use Lebesgue dominated convergence theorem.

Proof. By assumption $\{f_N(t)\}$ converges uniformly to $f(t)$. Also $|f_N(t)|$ is bounded above for all $t \in (-1/2, 1/2]$ since

$$\begin{aligned} |f_N(t)| = \left|\sum_{n=1}^{N} c_n \sin(2\pi nt)\right| \ &\le \ \sum_{n=1}^{N} |c_n \sin(2\pi nt)| \ = \ \sum_{n=1}^{N} |c_n||\sin(2\pi nt)| \\ &\le \ \sum_{n=1}^{N} \frac{M}{n^2}|\sin(2\pi nt)| \ \le \ \sum_{n=1}^{N} \frac{M}{n^2} \ \le \ \frac{M\pi^2}{6} \end{aligned}$$

The function $\frac{M\pi^2}{6}$ is integrable on $(-1/2, 1/2]$ and by the Lebesgue Dominated Convergence theorem

$$\begin{aligned} &\lim_{n\to\infty} \int_{-1/2}^{1/2} f_n(t)dt = \int_{-1/2}^{1/2} f(t)dt \\ \Longrightarrow &\lim_{n\to\infty} \int_{-1/2}^{1/2} f(t)dt - \int_{-1/2}^{1/2} f_n(t)dt = 0 \\ \Longrightarrow &\lim_{n\to\infty} \int_{-1/2}^{1/2} f(t) - f_n(t)dt = 0 \end{aligned}$$

If the integral of $f(t) - f_n(t)$ is 0 then certainly the square root of the integral of $(f(t)-f_n(t))^2$ is 0. Hence $\lim_{n\to\infty} ||f - f_n|| = 0$ □

9) Let V be a Hilbert space. Let $W \subset V$ be a closed subspace. Prove that W contains a dense countable set, so it is also a Hilbert space. Hint: consider the projection $P : V \to W$.

Proof. Let $M = \{m_1, m_2, \cdots\}$ be a dense countable subset of V and let

$$a_n = \inf_{w\in W} ||m_n - w||$$

Then given any positive integer n and p there exists $w_{np} \in W$ such that

$$||m_n - w_{np}|| < a_n + \frac{1}{p}$$

Given any $\epsilon > 0$ and any $w \in W$ choose p such that $\frac{1}{p} < \frac{\epsilon}{3}$ and n such that $||m_n - w|| < \frac{\epsilon}{3}$ then

$$\begin{aligned} ||w - w_{np}|| &\le ||w - m_n|| + ||m_n - w_{np}|| \\ &\le \frac{\epsilon}{3} + a_n + \frac{1}{p} \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Therefore W has a dense countable set $\{w_{np}\}$ where $n, p = 1, 2, \cdots$ and hence W is also a Hilbert space. □