Spreading Rumors

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April 15, 2020

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1 Rumors and the Logistic Equation

Many have modeled the spread of a rumor using the logistic model. The idea is that if you have a fixed population the change in the number of people who have heard a rumor (or just a piece of information) would be proportional to the product of those who have heard it and those who have not. Let $H(t)$ represent the number of people who have heard a rumor at a given time t. Then $H'(t) = kH(t)(P - H(t))$ where k is a proportionality constant and P is the total fixed population. Assume that at $t = 0$ only 1 person has heard the rumor, maybe the person who started it. This can be solved by separation of variables.

$$H(t) = \frac{PH_0}{H_0 + (P - H_0)e^{-kPt}}$$

Let $P = 1$ represent 100% of the population. The following depicts several solution curves with varying initial percentages of the population having heard the rumor at time $t = 0$ with growth constant $k = 0.5$.

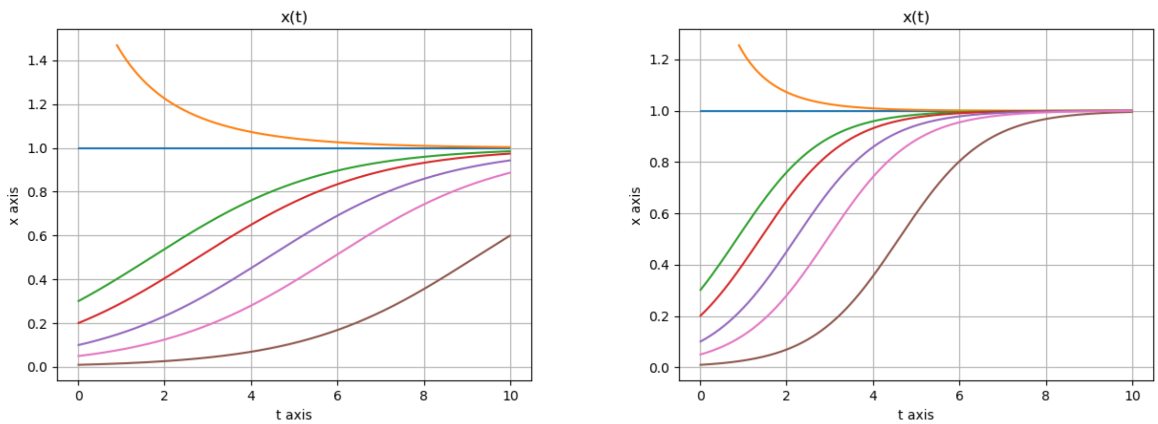


Figure 1: Solution curves of $H(t)$ for $k = 0.5$ and $k = 1$

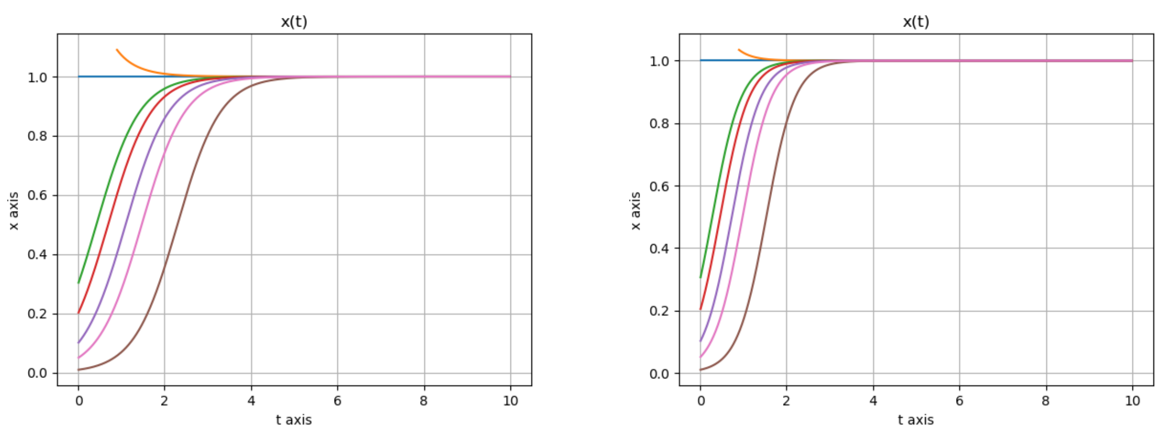


Figure 2: Solution curves of $H(t)$ for $k = 2$ and $k = 3$

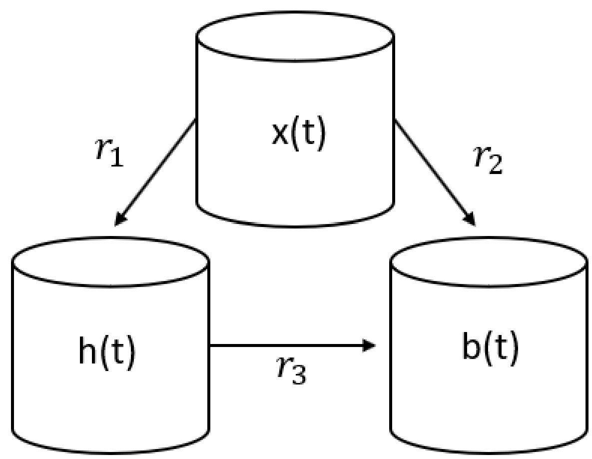
As k is increased the time it takes for the entire population to have heard the rumor decreases.

2 Rumors and Tanks

Before the internet rumors would spread based on direct contact with other people and through slow means of traditional forms of communication, i.e. letters, written articles in the local paper, etcetera. Today there are Twitter personalities that have obtained so many followers that they can make any remark they want and reach thousands (or even hundreds of thousands) of people at once. It's rather remarkable how the speed of communication has changed since the dawn of the internet. With this sort of power available our society is consequently subjected to much more information (and more importantly misinformation) than ever before. Now, more than ever, it would be useful to have a model that would describe how this information moves among a population.

2.1 Constructing the Model

Consider a population of people that become divided as a rumor spreads among the people. Initially everyone except for a few will not have heard the rumor. As the rumor spreads people will become categorized into groups, those who have not heard the rumor, those who have heard the rumor but didn't necessarily believe it, and those who heard and believed it. Let this categorization process be modeled by tanks filled with marbles. Each marble is a person and each tank it the category that each person identifies with. All marbles start in the "Haven't Heard it" tank. Let the number of people in this category be $x(t)$. Let $h(t)$ represent the number of people in the category "Heard, but didn't believe" and $b(t)$ be the number of people in the category "Heard and believed it". The variable t is an arbitrary unit of time. The rate at which people move from one category to another is represented by r_1, r_2, and r_3. Refer to the diagram above. The following differential equations can be constructed by considering what is coming in and going out of each tank.



$$x'(t) = -r_1x(t) - r_2x(t) \tag{1}$$
$$h'(t) = r_1x(t) - r_3h(t) \tag{2}$$
$$b'(t) = r_2x(t) + r_3h(t) \tag{3}$$

2.2 Solving the System

Equation 1 can be solved by separating variables.

$$x(t) = x_0e^{-(r_1+r_2)t}$$

The solution above can be used to obtain the solution to equation 2.

$$h(t) = \left(h_0 + \frac{r_1 x_0}{r_3 - r_2 - r_1}\right) e^{-r_3 t} - \left(\frac{r_1 x_0}{r_3 - r_2 - r_1}\right) e^{-(r_1+r_2)t}$$

Equation 3 is a linear combination of $x(t)$ and $h(t)$ and can be solved by direct integration.

$$b(t) = \left(h_0 + \frac{r_1 x_0}{r_3 - r_2 - r_1}\right)(1 - e^{-r_3 t}) + \left(\frac{r_2 x_0}{r_1 + r_2} + \frac{r_1 r_3 x_0}{(r_1+r_2)(r_3 - r_2 - r_1)}\right)(1 - e^{-(r_1+r_2)t}) + b_0$$

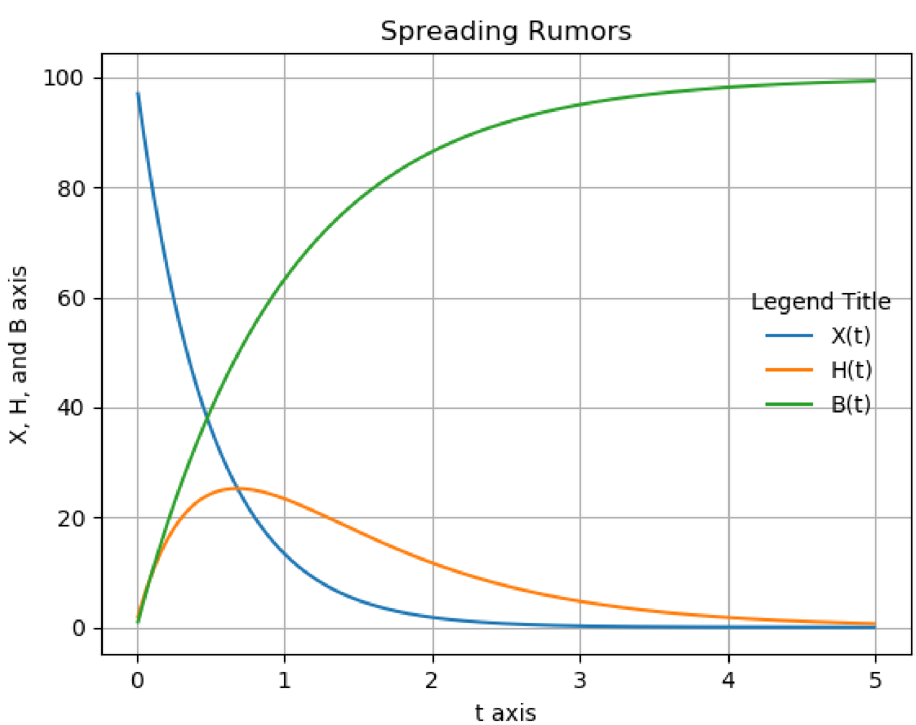
2.3 Analysis of the System

Assume that $r_1 = r_2 = r_3 = r$, that is the rates of people moving from tank to tank are all equal and constant. The system of differential equations becomes

$$\begin{aligned} x(t) =& x_0 e^{-2rt} \\ h(t) =& h_0 e^{-rt} + x_0 e^{-2rt} \\ b(t) =& (h_0 + x_0)\left(1 - e^{-rt}\right) + b_0 \end{aligned}$$

Assume the population consists of 100 people and at time $t = 0$ only one person has heard the rumor. This gives initial conditions $x(0) = 99$, $h(0) = 1$, $b(0) = 0$. Also assume that the rate r is one person per unit of time. Let time be measured by days.

$$\begin{aligned} x(t) &= 99e^{-2t} \\ h(t) &= 100e^{-t} - 99e^{-2t} \\ b(t) &= 100(1 - e^{-t}) \end{aligned}$$



2.4 Non-Constant Rates

Consider what happens when the rate between tanks is a function changing with time, that is $r = r(t)$.

3 Code for Graphs

```
import numpy as np
from matplotlib import pyplot as plt

x0 = 99
h0 = 1
r = 1
r1 = r
r2 = r
r3 = r
k1 = (r1*x0)/(r3-r2-r1)
k2 = (r2*x0)/(r1+r2) + (r1*r3*x0)/((r1+r2)*(r3-r2-r1))

t = np.arange(0.01,5,.01)
xt = x0*np.exp(-(r1+r2)*t)
ht = (h0 - k1)*np.exp(-r3*t) + k1*np.exp(-(r1+r2)*t)
bt = (h0 - k1)*(1 - np.exp(-r3*t))+ k2*(1 - np.exp(-(r1+r2)*t))

plt.title("Spreading Rumors")
plt.xlabel("t axis")
plt.ylabel("X, H, and B axis")
plt.plot(t,xt, label = "X(t)")
plt.plot(t,ht, label = "H(t)")
plt.plot(t,bt, label = "B(t)")
plt.grid(True)
plt.legend(loc="center right", title="Legend Title", frameon=False)
plt.show()
```