Definition 28 Suppose X is a (real or complex) vector space. An *inner product* on X is a mapping $\langle \cdot, \cdot \rangle$ from $X \times X$ into $\mathbb{C}$ such that

i. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

ii. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

iii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

iv. $\langle x, x \rangle > 0$ if $x \neq \theta$

It should be observed that properties (i) - (iv) imply also that $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ and $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$.

(Check) By property (iii): $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle}$. By property (i): $\overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle}$. By properties (iii) again: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and $\langle x, z \rangle = \overline{\langle z, x \rangle} \implies \langle x, y \rangle + \langle x, z \rangle$. For the second part, by property (iii) we have $\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle}$. Using property (ii) $\overline{\alpha}\overline{\langle y, x \rangle}$ and by property (iii) again we have $\overline{\alpha} \langle x, y \rangle$.

Note also that if $\langle x, x \rangle = 0$, then x must be θ, and if $\langle x, y \rangle = 0$ for all y in X, then $x = \theta$ since $\langle x, x \rangle = 0$.

For our case in question we will be considering X the space of continuous functions on $[a, b]$ and for our inner product we will use

$$\langle u, v \rangle = \int_a^b u(t)\overline{v(t)}w(t)dt, \quad w(t) \geq 0$$

(Check)
Property (i):

$$\begin{aligned}
\langle x+y, z \rangle &= \int_a^b \left(x(t) + y(t)\right) \overline{z(t)} w(t)dt \\
&= \int_a^b x(t)\overline{z(t)}w(t) + y(t)\overline{z(t)}w(t)dt \\
&= \int_a^b x(t)\overline{z(t)}w(t)dt + \int_a^b y(t)\overline{z(t)}w(t)dt \\
&= \langle x, z \rangle + \langle y, z \rangle
\end{aligned}$$

Property (ii)

$$\begin{aligned}
\langle \alpha x, y \rangle &= \int_a^b \alpha x(t)\overline{y(t)}w(t)dt \\
&= \alpha \int_a^b x(t)\overline{y(t)}w(t)dt \\
&= \alpha \langle x, y \rangle
\end{aligned}$$

Property (iii)

$$\begin{aligned}\langle x, y\rangle &= \int_a^b x(t)\overline{y(t)}w(t)dt \\ &= \int_a^b \overline{y(t)}x(t)w(t)dt \\ &= \overline{\langle y, x\rangle}\end{aligned}$$

Property (iv)

$$\begin{aligned}\langle x, x\rangle &= \int_a^b x(t)\overline{x(t)}w(t)dt \\ &= \int_a^b |x(t)|^2 w(t)dt > 0 \text{ if } x \neq \theta\end{aligned}$$

We shall say that u and v are *orthogonal* if and only if $\langle u, v\rangle = 0$ and that u is *normal* if $\langle u, u\rangle = 1$. Now define $||u||^2 = \langle u, u\rangle$.

Theorem 36 (Cauchy-Schwarz-Bunyakowski)

$$|\langle u, v\rangle|^2 \leq ||u||^2 \cdot ||v||^2 \leq \langle u, u\rangle\langle v, v\rangle$$

Proof. Notice that $0 \leq \langle v - \alpha u, v - \alpha u\rangle$ holds for all $\alpha \in \mathbb{C}$. Set $\alpha = \frac{\langle v,u\rangle}{\langle u,u\rangle}$ and we get

$$\begin{aligned}0 \leq \langle v - \alpha u, v - \alpha u\rangle &= \langle v, v\rangle \\ &= \langle v, v\rangle + \langle v, -\alpha u\rangle + \langle -\alpha u, v\rangle + \langle -\alpha u, -\alpha u\rangle \\ &= \langle v, v\rangle - \langle v, \alpha u\rangle - \langle \alpha u, v\rangle + \langle \alpha u, \alpha u\rangle \\ &= \langle v, v\rangle - \overline{\alpha}\langle v, u\rangle - \alpha\langle u, v\rangle + \alpha\overline{\alpha}\langle u, u\rangle \\ &= \langle v, v\rangle - \overline{\frac{\langle v, u\rangle}{\langle u, u\rangle}}\langle v, u\rangle - \frac{\langle v, u\rangle}{\langle u, u\rangle}\langle u, v\rangle + \left|\frac{\langle v, u\rangle}{\langle u, u\rangle}\right|^2 \langle u, u\rangle \\ &= \langle v, v\rangle - \overline{\frac{\langle v, u\rangle}{\langle u, u\rangle}}\langle v, u\rangle - \frac{\overline{\langle u, v\rangle}}{\langle u, u\rangle}\langle u, v\rangle + \left|\frac{\langle v, u\rangle}{\langle u, u\rangle}\right|^2 \langle u, u\rangle \\ &= \langle v, v\rangle - \overline{\frac{\langle v, u\rangle}{\langle u, u\rangle}}\langle v, u\rangle - \frac{\overline{\langle v, u\rangle}}{\langle u, u\rangle}\langle v, u\rangle + \left|\frac{\langle v, u\rangle}{\langle u, u\rangle}\right|^2 \langle u, u\rangle \\ &= \langle v, v\rangle - 2\frac{|\langle v, u\rangle|^2}{\langle u, u\rangle} + \frac{|\langle v, u\rangle|^2}{\langle u, u\rangle} \\ &= \langle v, v\rangle - \frac{|\langle v, u\rangle|^2}{\langle u, u\rangle}\end{aligned}$$

Now we have

$$|\langle v, u\rangle|^2 \leq \langle v, v\rangle\langle u, u\rangle = ||u||^2||v||^2$$

$\square$

In our setting this says

$$\left|\int_a^b v(x)\overline{u(x)}w(x)\right| \leq \left[\int_a^b |v(x)|^2 w(x)\right]\left[\int_a^b |u(x)|^2 w(x)\right]$$

Theorem 37 $||v||^2 = \langle v, v\rangle$ defines a norm $||v||$ on $[C[a,b]]X$.

Proof. To be a norm $||\cdot||$ needs to be a function $||\cdot|| : [C[a,b]]X \to \mathbb{R}$ that satisfies three properties.

i. $||v|| > 0$ and $||v|| = 0$ if and only if $v = 0 \ \forall \ v \in C[a,b]]X$

ii. $||\alpha v|| = |\alpha| ||v|| \ \forall \ \lambda \in \mathbb{R} \ \ and \ \ \forall \ \ v \in C[a,b]]X$

iii. $||u + v|| \leq ||u|| + ||v|| \ \forall u, v \in C[a,b]]X$

Property (i) and (ii) are true by the definition of the inner product. We need only check property (iii) the triangle inequality.

$$\begin{aligned}
0 \leq ||u+v||^2 &= \langle u+v, u+v\rangle \\
&= \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} + \langle v,v\rangle \\
&= \langle u,u\rangle + 2Re\langle u,v\rangle + \langle v,v\rangle \\
&\leq \langle u,u\rangle + 2|\langle u,v\rangle| + \langle v,v\rangle \\
&\leq [||u|| + ||v||]^2
\end{aligned}$$

□

Our space of functions is now $X, \mu \equiv (C[a,b], \mu)$ as a metric space and a normed space with norm being the L_2 norm with weight function $w(x)$:

$$||f-g||^2 = \int_a^b w(x)|f(x) - g(x)|^2 dx$$

and $\mu(f,g) = ||f-g||$ being the metric. This space of course has a completion which is unique up to isometric isomorphisms and we denote this space by $L_{2_w}([a,b])$. As we saw earlier the Riemann integral is uniformly continuous on this original space and thus has a unique extension to the entire space L_{2_w}.

Definition 29 A *Hilbert Space* is a complete inner product space.

Exercise 39

1. Show that if u and v are points in an inner product space, then

$$||u+v||^2 + ||u-v||^2 = 2\left[||u||^2 + ||v||^2\right]$$

   If the inner product is real valued, then this is called the Parallelogram Law.

Proof. Expanding the first term on the left hand side we have

$$\begin{aligned}||u+v||^2 &= \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= ||u||^2 + \langle u,v\rangle + \langle v,u\rangle + ||v||^2\end{aligned}$$

Similarly expand the second term on the left hand side

$$\begin{aligned}||u+v||^2 &= \langle u-v, u-v\rangle = \langle u, u-v\rangle - \langle v, u-v\rangle \\ &= \langle u,u\rangle - \langle u,v\rangle - [\langle v,u\rangle - \langle v,v\rangle] \\ &= ||u||^2 - \langle u,v\rangle - \langle v,u\rangle + ||v||^2\end{aligned}$$

Add these two together and terms cancel to arrive at

$$||u+v||^2 + ||u-v||^2 = 2\left[||u||^2 + ||v||^2\right]$$

□

2. Show that $||u+v||^2 = ||u||^2 + ||v||^2$ if $u \perp v$, that is $\langle u,v\rangle = 0$. This is called the Pythagorean Theorem.

   Proof.

$$\begin{aligned}||u+v||^2 &= \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= ||u||^2 + ||v||^2\end{aligned}$$

□

3. Find an example of a complex inner product space in which (2) above fails. Also give an example where the converse fails.

   If $\langle u,v\rangle = 0$ then $||u+v||^2 = ||u||^2 + ||v||^2$. An example of an complex inner product space where this fails is:... If $||u+v||^2 = ||u||^2 + ||v||^2$ then $\langle u,v\rangle = 0$. An example of an complex inner product space where this fails is:...

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p. 177

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Definition 30

1. If $(X, \langle\cdot,\cdot\rangle)$ is a given inner product space and $x \in X$, then $x^\perp = \{y \in X | \langle x,y\rangle = 0\}$ and is called the *orthogonal complement of* x.

   Note: Clearly $x^\perp$ is a vector subspace of X since if $y_1 \in x^\perp$ and $y_2 \in x^\perp$, then so is $\alpha y_1 + \beta y_2$. Moreover, $x^\perp$ is closed, that is if $y_n \in x^\perp$, and $y_n \to y$, then $|\langle x, y_n - y\rangle| \leq ||x|| \cdot ||y_n - y|| \to 0$ and since $\langle x, y_n\rangle = 0$, we have that $\langle x,y\rangle = 0$ also.

2. Suppose M is a subset of X, then we define $M^{\perp} \equiv \cap_{x \in M^{\perp}} x^{\perp}$. Since the intersection of vector spaces is again a vector space; moreover, since the intersection of closed sets is closed, $M^{\perp}$ is a closed vector subspace of X for all subsets M of X.

3. A subset $N \in M$ is said to be convex if $x \in N$ and $y \in N$ and also $0 \leq \lambda \leq 1$, then $z = \lambda x + (1 - \lambda) y \in N$; this says that the segment joining x and y lies in N.

Exercise 40

1. Show that the intersection of a vector space is again a vector space.

   Proof. Suppose X and Y are both vector spaces. To show that $X \cap Y$ is a vector space we need to verify that it contains θ (the zero vector), and that it is closed under vector addition and scalar multiplication.

   Since X is a vector space we know that it contains θ. Since Y is also a vector space it also contains θ which implies that $\theta \in \{x | \; x \in X \textit{ and } x \in Y\} = X \cap Y$. First property verified.

   Suppose $x, y \in X \cap Y$. If $x \in X$ and $y \in X$ since X is a vector space $x + y \in X$. If $x \in Y$ and $y \in Y$ since Y is a vector space $x + y \in Y$. Which implies that $x + y \in \{x | \; x \in X \textit{ and } x \in Y\} = X \cap Y$, thus the second property is verified.

   Finally, suppose $x \in X \cap Y$ and $\lambda \in F$, where F is any scalar field. Since $x \in X$ and X is a vector space this means that $\lambda x \in X$. Similarly since $x \in Y$ and Y is a vector space this means that $\lambda y \in Y$ which implies that $\lambda x \in \{x | \; x \in X \textit{ and } x \in Y\} = X \cap Y$. Third property verified. $\square$

2. Show that the intersection of convex space is convex.

   Proof. Suppose X, Y are two convex spaces and that $x, y \in X \cap Y$. Since X is convex and since $x, y \in X$ this means $z = \lambda x + (1 - \lambda) y \in X$. Similarly since Y is convex and since $x, y \in Y$ this means $z = \lambda x + (1 - \lambda) y \in Y$. Thus $z = \lambda x + (1 - \lambda) y \in X \cap Y$ which shows that $X \cap Y$ is convex. $\square$

3. Show that the intersection of closed spaces is closed.

   Proof. Suppose X, Y are closed spaces. This means that X^c and Y^c are open. This implies that $X^c \cup Y^c$ is open. The complement of $X^c \cup Y^c$ is therefore closed which by DeMorgan's Law is $X \cap Y$. $\square$