Theorem 34 Suppose that each of (X, ρ) and (Y, η) is a metric space. (Y, η) is complete, X_0 is a ρ-dense subset of X and $f : X_0 \to Y$ is uniformly continuous on X_0, then there exists a unique continuous extension f_e of f to all of X, that is $f_e(x) = f(x)$ for all $x \in X_0$, f_e is continuous and $f_e : X \to Y$.

Proof. Suppose $x \in X$ and $x \notin X_0$. Since X_0 is dense in X there exists a sequence $\{x_n\} \subset X_0$ such that $\rho(x_n, x) \to 0$ and $n \to \infty$. This implies that $\{x_n\}$ is ρ-Cauchy (by Lemma 16, p. 159), that is $\rho(x_n, x_m) \leq \rho(x_m, x) + \rho(x, x_n) \to 0$ as $m, n \to \infty$.

Since f is uniformly continuous, given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(u, v) < \delta$ implies $\eta(f(u, f(v))) < \epsilon$. Apply this to the sequence $\{x_n\}$ and we have $\eta(f(x_m), f(x_n)) < \epsilon$ if $\rho(x_m, x_n) < \delta$. This implies that $\{f(x_n)\}$ is η-Cauchy in Y. Since Y is complete $\{f(x_n)\}$ converges to a point in Y which we shall call $f_e(x)$.

To see that $f_e(x)$ in Y depends on x but not the choice of the sequence $\{x_n\}$ which converges to x, suppose that $\{\tilde{x_n}\} \subset X_0$ and $\{\tilde{x_n}\} \xrightarrow{\rho} x$. Choose $N > 0$ so that $n > N$. This implies that $\rho(\tilde{x_n}, x_n) \leq \rho(\tilde{x_n}, x) + \rho(x, x_n) < \delta/2 + \delta/2 = \delta$ and by uniform continuity of f with $u = \tilde{x_n}$ and $v = x_n$ we have that for $n > N$, $\eta(f(\tilde{x_n}), f(x_n)) < \epsilon$. Therefore the two Cauchy sequences $\{f(\tilde{x_n})\}$ and $\{f(x_n)\}$ are equivalent Cauchy sequences and thus converge to the same point in Y. This shows that f_e is well-defined.

If there were a second continuous extension of f, say g, then $g(x) = \lim_{n\to\infty} f(x_n)$ must hold for all sequences $\{x_n\} \subset X_0$ such that $x_n \xrightarrow{\rho} x$, but for such sequences, the limit on the right is $f_e(x)$ by definition, thus $g(x) = f_e(x)$ on X.

Exercise 36

It remains to be proven that $f_e(x)$ is continuous. If $x, y \in X_0$ then $f_e(x) = f(x)$ which is uniformly continuous by assumption. If $x, y \in X$ and $x, y \notin X_0$ then there exists sequences $\{x_n\}$ and $\{y_n\}$ such that they converge to x and y, respectively. This follows from the fact that X_0 is dense in X. Now consider $\eta(f_e(x), f_e(y))$ and apply the triangle inequality twice,

$$\begin{aligned}\eta(f_e(x), f_e(y)) &\leq \eta(f_e(x), f(x_n)) + \eta(f(x_n), f_e(y)) \\ &\leq \eta(f_e(x), f(x_n)) + \eta(f(x_n), f(y_n)) + \eta(f(y_n), f_e(y))\end{aligned}$$

By the definition of $f_e(x)$ we have that $\eta(f_e(x), f(x_n)) < \epsilon/3$ if N is chosen large enough. The same argument applies for $\eta(f(y_n), f_e(y)) < \epsilon/3$. Lastly since f is uniformly continuous $\eta(f(x_n), f(y_n)) < \epsilon/3$ if $\rho(x_n, y_n) < \delta$. Therefore $f_e(x)$ is continuous. $\square$

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Suppose $\{f_n\}$ is a representative element of an equivalence class ξ in L_1, i.e. $\{f_n\}$ is L_1 Cauchy. Consider the function sequence $\{\int_a^x f_n(t)dt\} = \{F_n(x)\}$ in the space $C^{(1)}([a,b])$, and write

$$\begin{aligned}|F_n(x) - F_m(x)| &= \left|\int_a^x f_n(t)dt - \int_a^x f_m(t)dt\right| \\ &\leq \int_a^x |f_n(t) - f_m(t)|dt \\ &\leq \int_a^b |f_n(t) - f_m(t)|dt \\ &= ||f_n - f_m||_1\end{aligned}$$

Since $\{f_n\}$ is $||\cdot||_1$-Cauchy, this last expression converges to zero as $n, m \to \infty$ and we have that $\{F_n(x)\}$ is a $||\cdot||_\infty$-Cauchy sequence and thus converges uniformly to a continuous function on the interval $[a,b]$, say $F(x)$, i.e.,

$$F(x) = \lim_{n\to\infty} \int_a^x f_n(t)dt$$

where the convergence is uniform on $[a,b]$.

Exercise 37

If $\{g_n\}$ is also a representative element of the equivalence class ξ in L_1, then it is also true that

$$F(x) = \lim_{n\to\infty} \int_a^x g_n(t)dt$$

since if $\{\int_a^x g_n(t)dt\} = \{G_n(x)\}$ in the space $C^{(1)}([a,b])$ then

$$\begin{aligned}|F_n(x) - G_n(x)| &= \left|\int_a^x f_n(t)dt - \int_a^x g_n(t)dt\right| \\ &\leq \int_a^x |f_n(t) - g_n(t)|dt \\ &\leq \int_a^b |f_n(t) - g_n(t)|dt \\ &= ||f_n - g_n||_1\end{aligned}$$

Since $\{f_n\}$ and $\{g_n\}$ are both in ξ then $||f_n - g_n||_1 = 0$ which implies that

$$F(x) = \lim_{n\to\infty} \int_a^x g_n(t)dt$$