Dynamical Systems

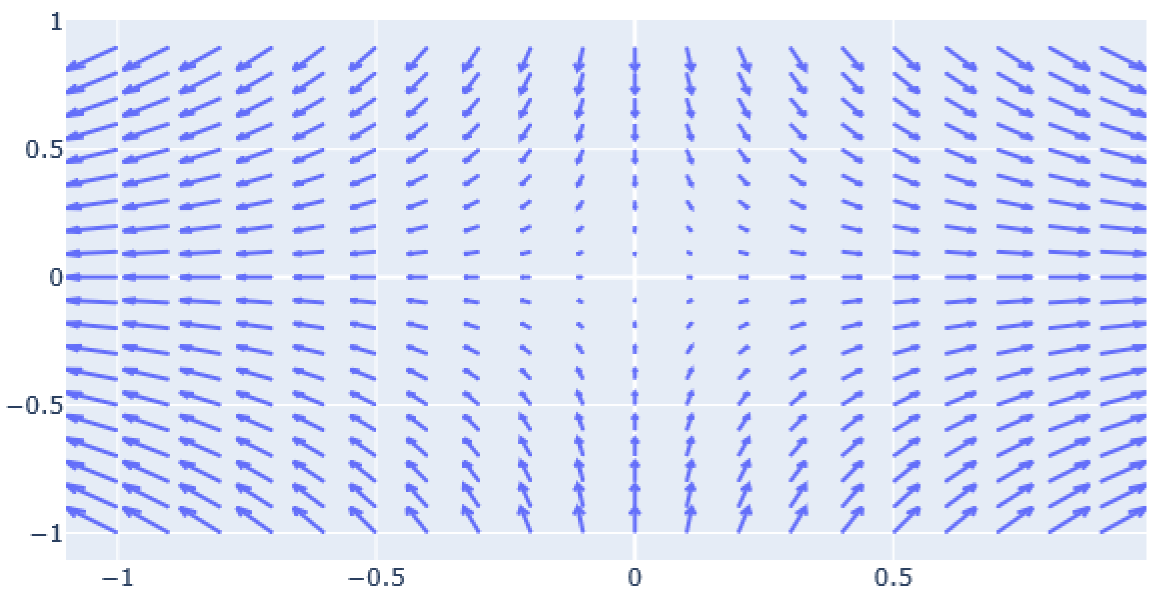
Jordan Saethre

November 12, 2019

Exercise 25 Part I: For each of the following constant matrices of A, Compute the eigenvalues and discuss the behavior of solution about the equilibrium point θ. If you have access to appropriate computing facilities, sketch the vector field and several trajectories for each case.

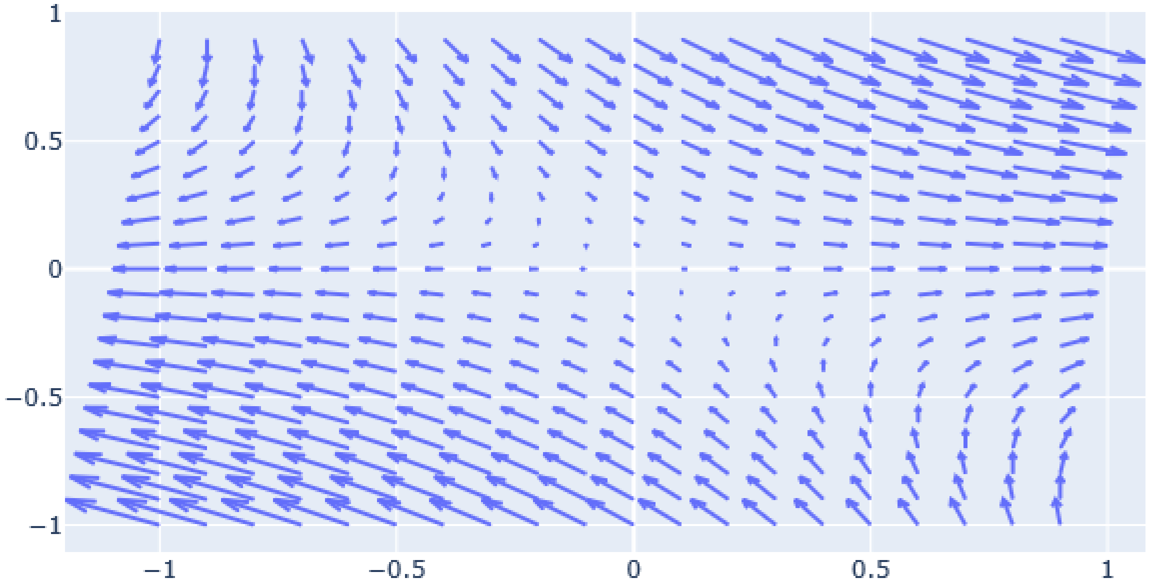
1. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 0 = \lambda^2 - 1 = 0 \implies \lambda = \pm 1$

   $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x \ and \ y' = y$



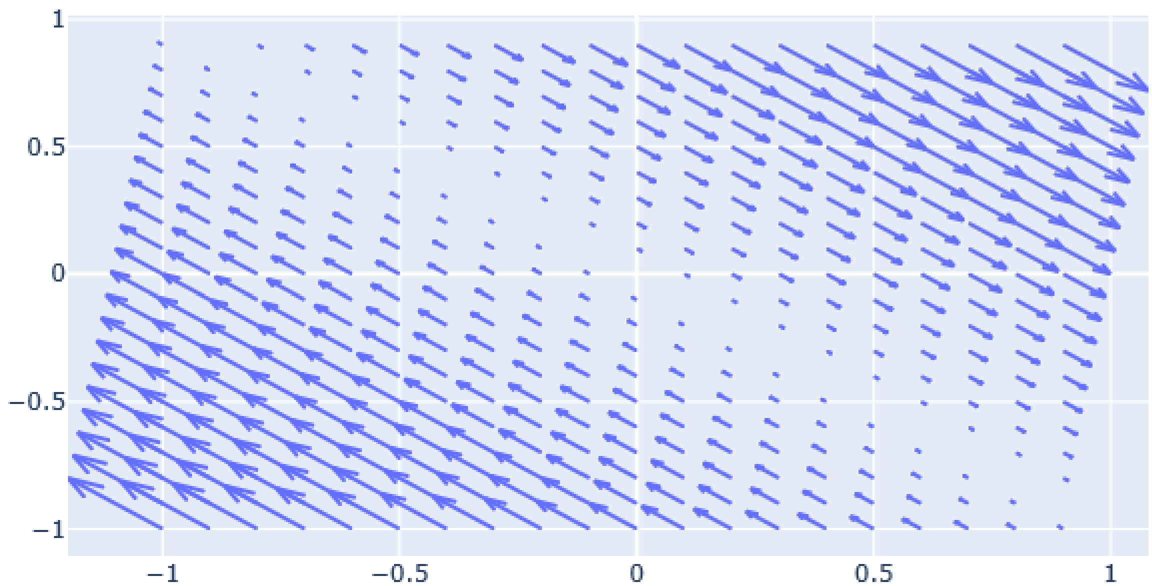
2. $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 0 = \lambda^2 - 1 = 0 \implies \lambda = \pm 1$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x + y \; and \; y' = -y$$



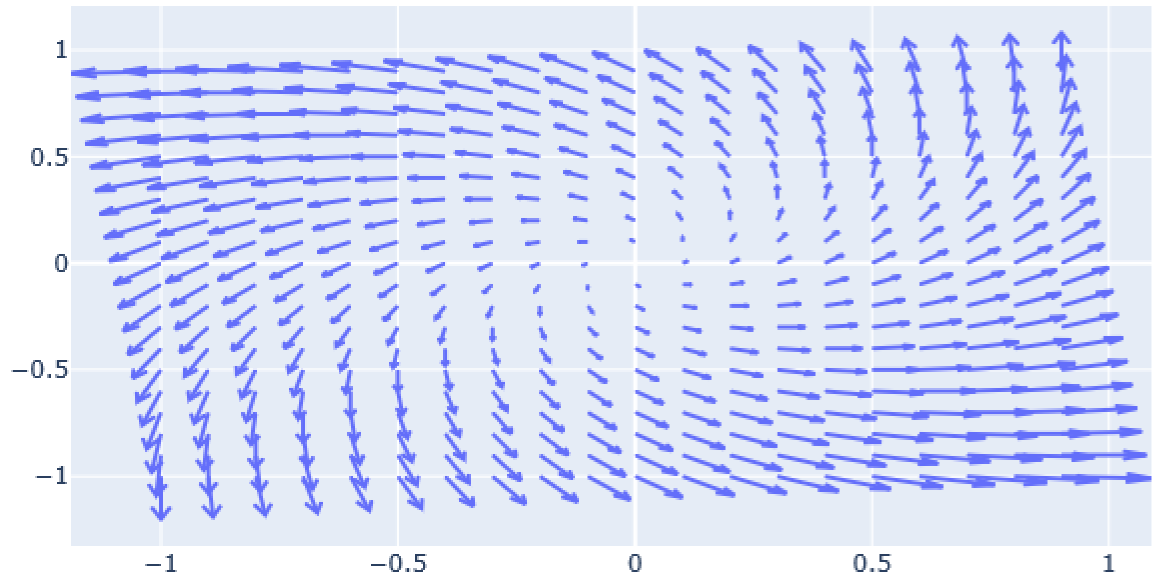
3. $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) + 1 = \lambda^2 = 0 \implies \lambda = 0$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x + y \; and \; y' = -x - y$$



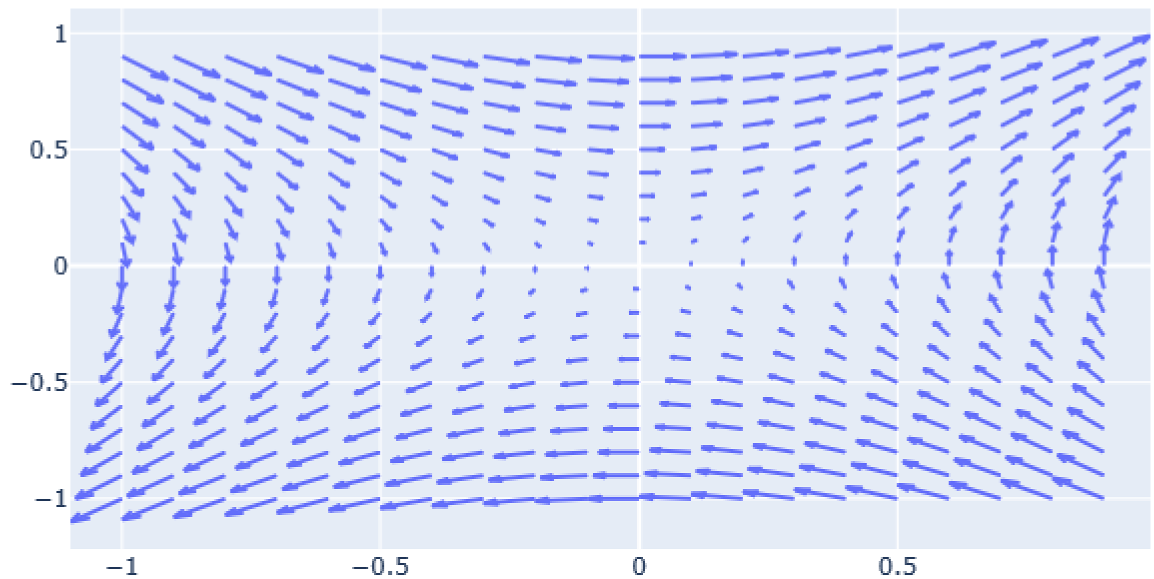
4. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)+1 = \lambda^2 - 2\lambda + 2 = 0 \implies \lambda = 1 \pm i$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x - y \; and \; y' = x + y$



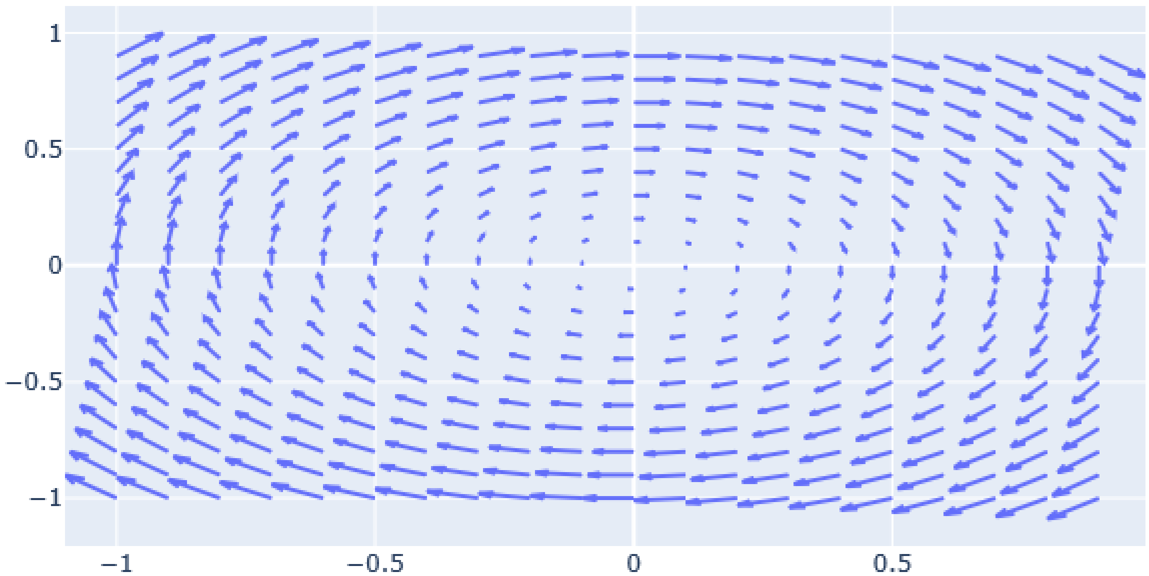
5. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: $det \begin{bmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} = (-\lambda)(-\lambda) - 1 = \lambda^2 - 1 = 0 \implies \lambda = \pm 1$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = y \; and \; y' = x$



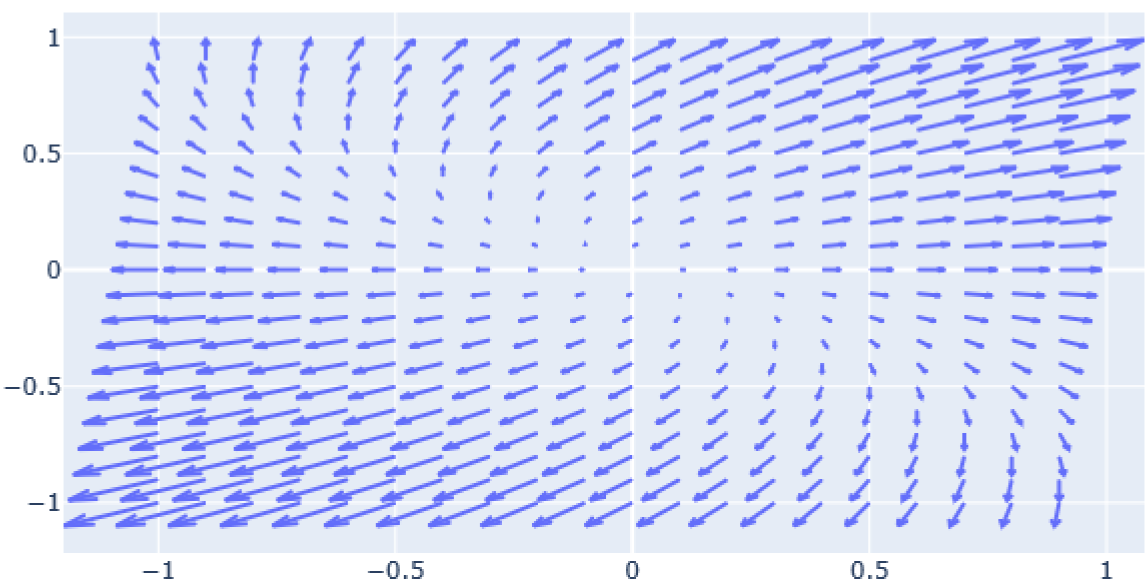
6. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: $det \begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix} = (-\lambda)(-\lambda) + 1 = \lambda^2 + 1 = 0 \implies \lambda = \pm i$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = y \text{ and } y' = -x$$



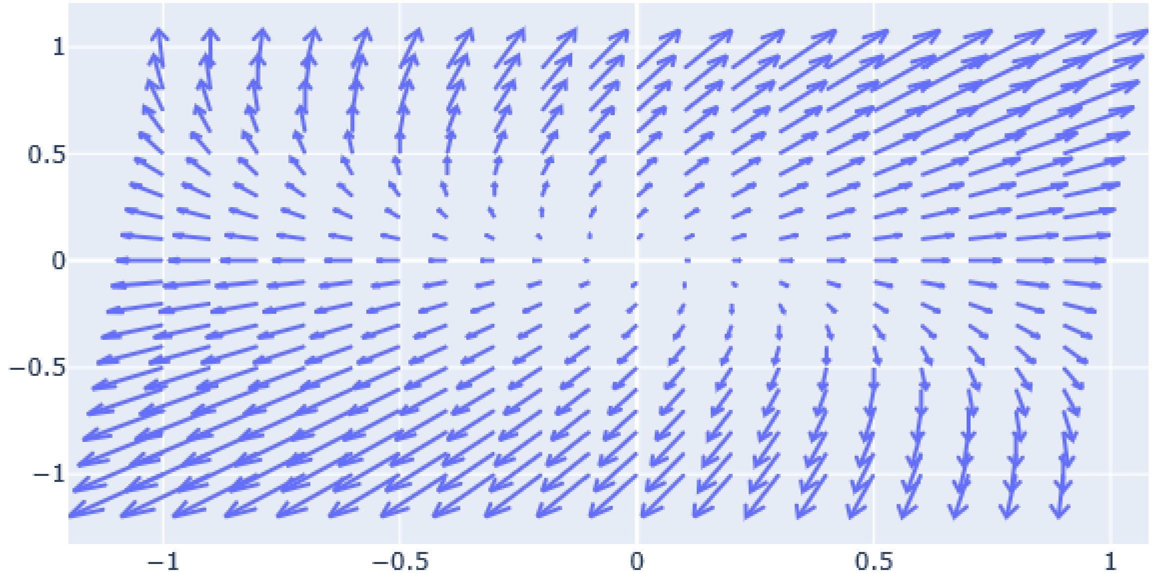
7. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 0 = (1-\lambda)^2 = 0 \implies \lambda = 1 \text{ multiplicity } 2$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x + y \text{ and } y' = y$$



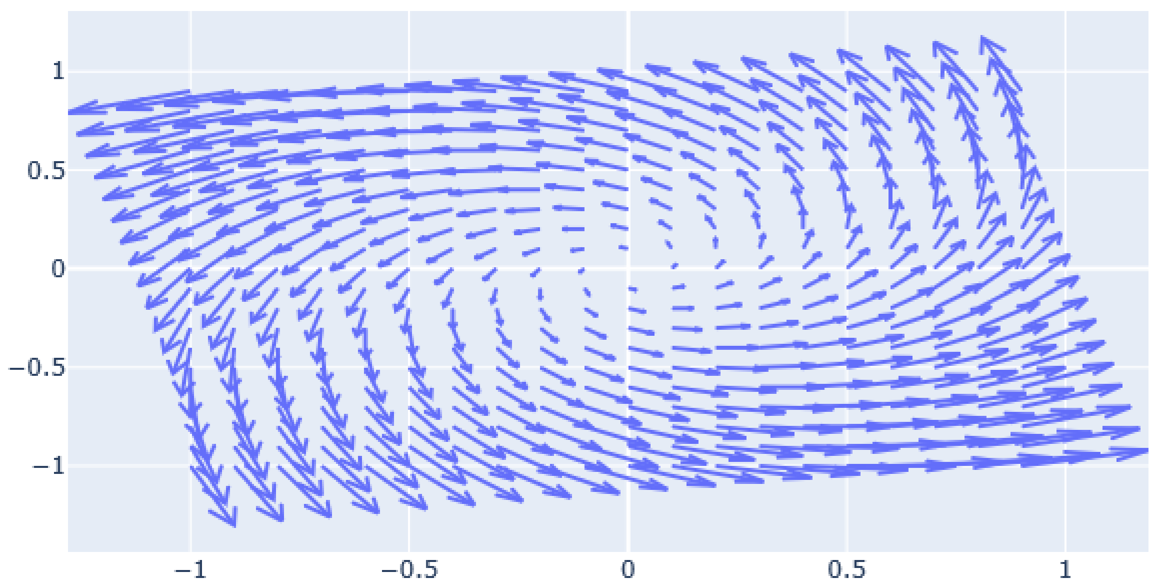
8. $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 0 = \lambda^2 - 3\lambda + 2 = 0 \implies \lambda = 1,\ 2$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x + y \textit{ and } y' = 2y$$



9. $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$: $det \begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) + 4 = \lambda^2 - 2\lambda + 5 = 0 \implies \lambda = 1 \pm 2i$

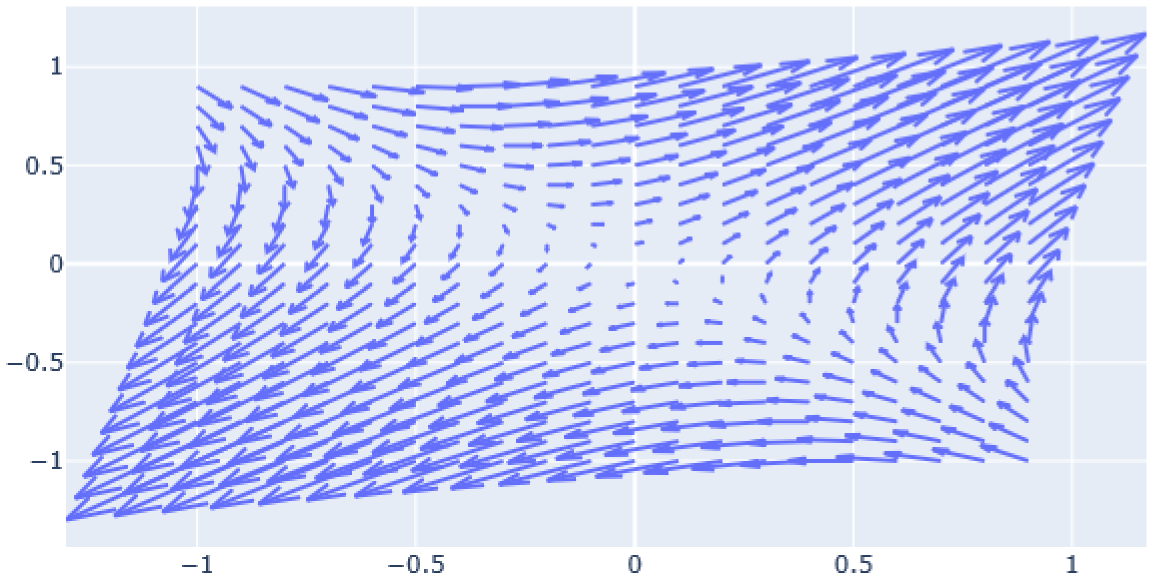
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x - 2y \textit{ and } y' = 2x + y$$



10. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

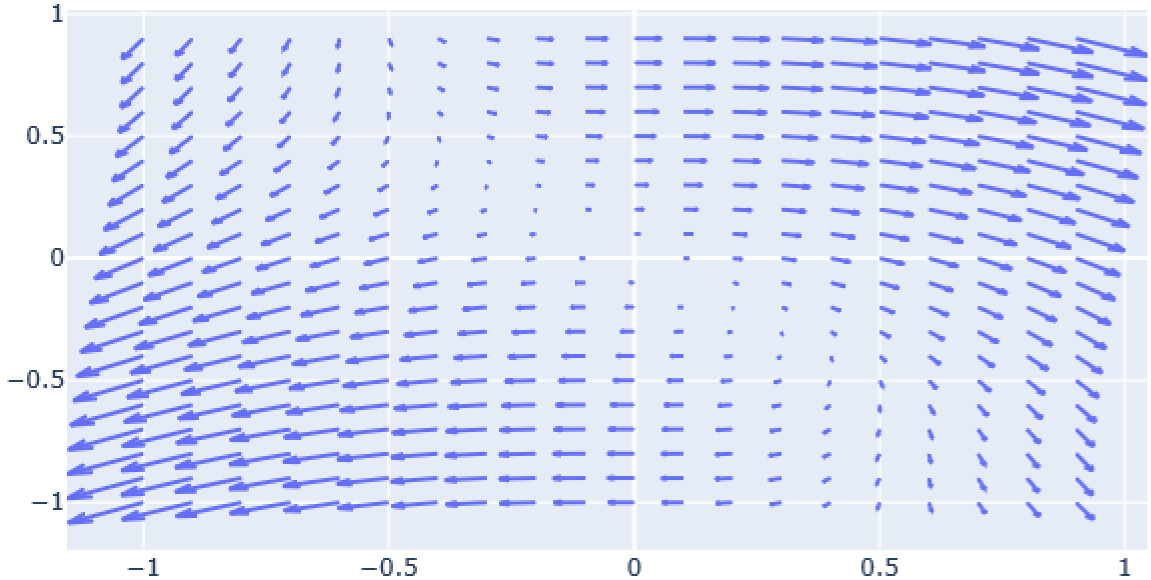
    Eigenvalues: $det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)-4 = \lambda^2-2\lambda-3 = 0 \implies \lambda = -1,\ 3$

    $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies x' = x + 2y \ and \ y' = 2x + y$



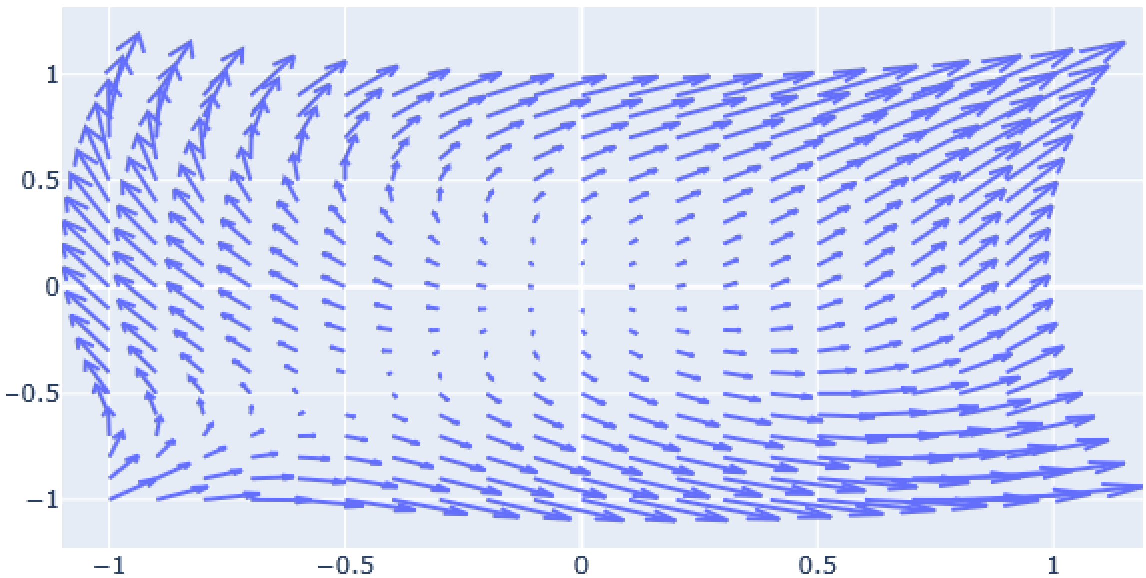
Exercise 25 Part II: For each of the following show that θ is an equilibrium point, locally linearize there and discuss the behavior of the "linear part".

1. $f(x(t), y(t)) = \begin{bmatrix} x + sin(y) \\ cos(x) - 1 \end{bmatrix} \implies \nabla f = \begin{bmatrix} 1 & cos(y) \\ -sin(x) & 0 \end{bmatrix}$



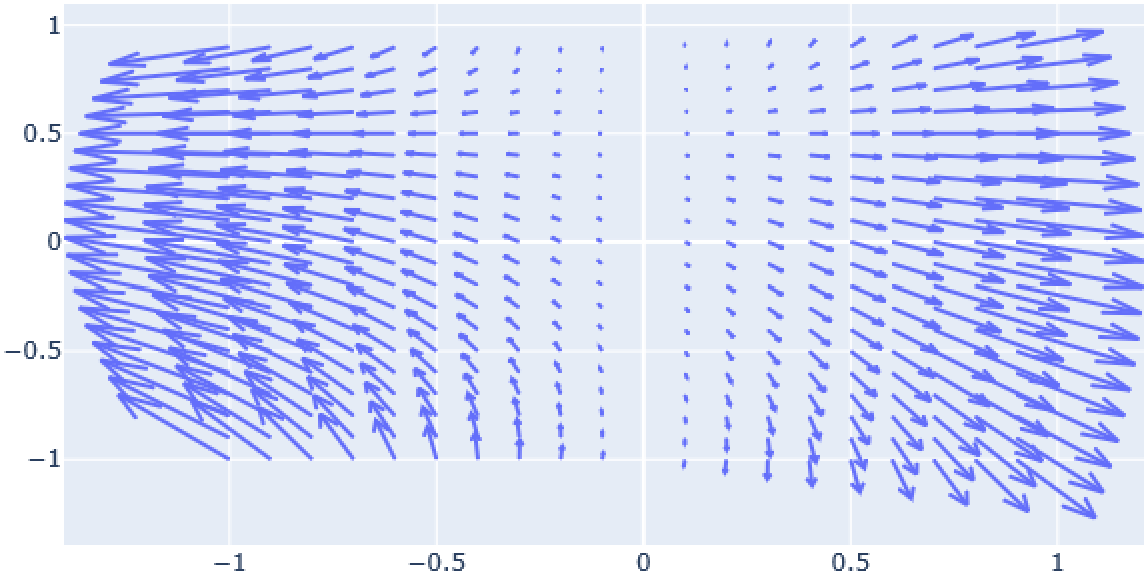
The local linearization is unstable so we are unable to conclude anything about the non-linear system.

2. $f(x(t), y(t)) = \begin{bmatrix} x + y^2 \\ y + x^2 \end{bmatrix} \implies \nabla f = \begin{bmatrix} 1 & 2y \\ 2x & 1 \end{bmatrix}$



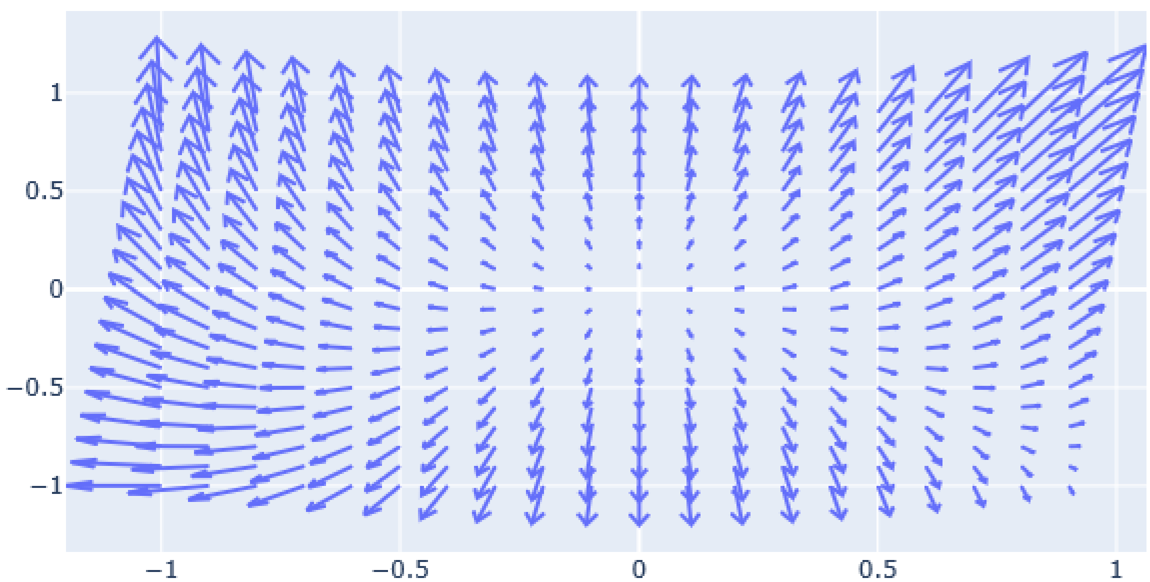
The local linearization is unstable so we are unable to conclude anything about the non-linear system.

3. $f(x(t), y(t)) = \begin{bmatrix} x cos(y) + x^3 \\ y^2 - x \end{bmatrix} \implies \nabla f = \begin{bmatrix} cos(y) + 3x^2 & -x sin(y) \\ -1 & 2y \end{bmatrix}$



The local linearization is unstable so we are unable to conclude anything about the non-linear system.

4. $f(x(t), y(t)) = \begin{bmatrix} x + x^2 y \\ x^2 + 2y \end{bmatrix} \implies \nabla f = \begin{bmatrix} 1 + 2xy & x^2 \\ 2x & 2 \end{bmatrix}$



The local linearization is unstable so we are unable to conclude anything about the non-linear system.