Math 2250 Lab 10

21 NOV 2019

Name: ______________________________

uNID: ______________________________

<u>Instructions and due date:</u>

- **Due:** 5 Dec 2019 at the start of class.
- If extra paper is necessary, please staple it at the end of the packet.
- For full credit, show all of your work.
- Work together! However, your work should be your own (not copied from a group member).
- My email: yourname@math.utah.edu.

1. **Overspending at an Engineering Firm**
You are the lead mechanical engineer on a project that is building the assembly line machinery for a factory. Your scope of the project includes a mass-spring system. This is a government contract and you have been asked to provide justification for why you chose to use a part that costs $\$70,000$ more than what had been originally proposed.

   (a) The system in your project is described as such: You have a 1 kg mass that is connected to a spring with spring constant $k = \frac{101}{25}$. The damping coefficient is $c = \frac{2}{5}$. At time $t = 0$ the mass is at rest and is not stretching the spring. The original part that was in the proposal results in an external force $f(t)$. The external force is applied to the mass at time $t = 0$ and continues to be applied through out the whole process.

   Through-out this question use Laplace Tables and/or Wolfram Alpha/Computer Algebra software as necessary.

   (i) If $f(t) = e^{-t/5}\cos(2t)$ derive the second order differential equation that describes this system.

   Solution:

   $$x''(t) + \frac{2}{5}x'(t) + \frac{101}{25}x(t) = e^{-t/5}\cos(2t) \quad x'(0) = x(0) = 0$$

   (ii) Solve the differential equation using Laplace Transforms. **Hint:** Use Theorem 1 from section 10.3 of your textbook to evaluate the Inverse Laplace Transform ($\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$).

Solution: Take the Laplace Transform of all parts of the equation:

$$X(s) = \left(\frac{1}{s^2 - \frac{2}{5}s + \frac{101}{25}}\right) F(s)$$

$F(s)$ can be obtained from a Laplace Table.

$$X(s) = \left(\frac{1}{s^2 - \frac{2}{5}s + \frac{101}{25}}\right)\left(\frac{s + \frac{1}{5}}{(s + \frac{1}{5})^2 + 4}\right)$$

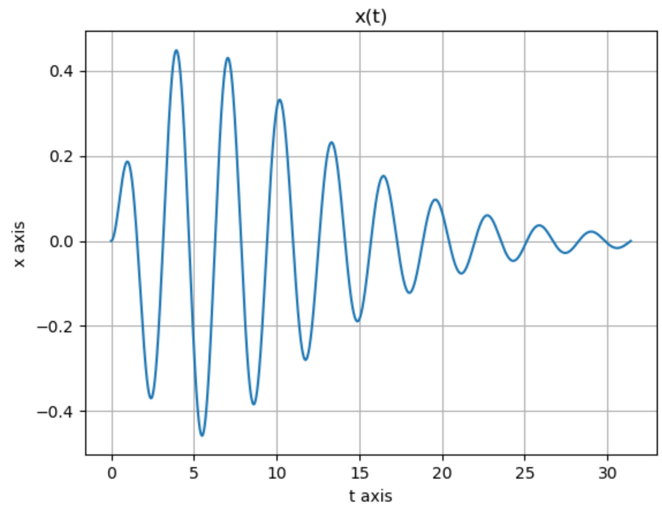
By completing the square for the denominator of the first term you can combine them into one term.

$$X(s) = \frac{s + \frac{1}{5}}{((s + \frac{1}{5})^2 + 4)^2}$$

Now using Theorem 1 in section 10.3 of the textbook $\mathscr{L}^{-1}\{F(s-a)\} = e^{at} f(t)$:

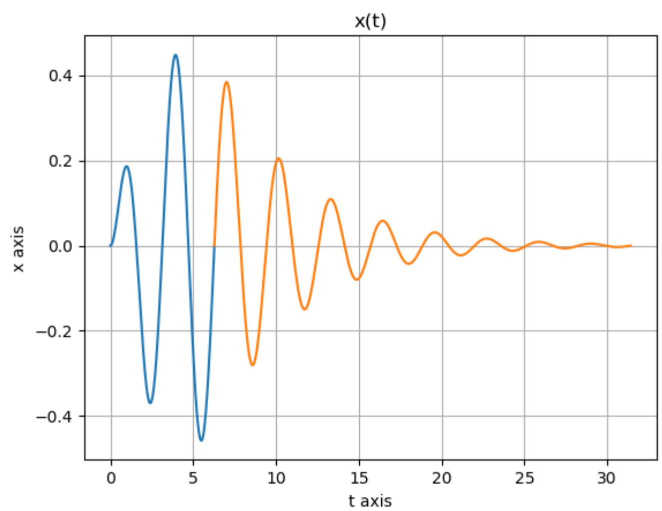
$$x(t) = \frac{1}{4} e^{-t/5} t \sin(2t)$$

(iii) Below is a plot of the solution from part (ii). Note that the solution's oscillations increase in amplitude and then die out. Some of the specifications in the contract include that it must reach a maximum amplitude of 0.46 by $t = 5$ and decrease to 0.33 by $t = 7$. Write down an equation for the amplitude of this solution. What is the maximum amplitude attained? At what time does this occur? What is the amplitude at $t = 7$? Does the originally proposed part meet the specifications in the contract?



Solution: The amplitude is defined by $A(t) = \frac{1}{4}e^{-t/5}t$. The maximum amplitude is $\frac{5}{4e} \approx 0.46$ which occurs at $t = 5$. The amplitude at $t = 7$ is $\frac{7}{4}e^{\frac{-7}{5}} \approx 0.43$. The original part proposed in the contract meets the first specification, but not the second which justifies why you couldn't use it.

(b) The part you ultimately ended up using has a driving force that is only applied for time $0 \le t < 2\pi$. At $t = 2\pi$ the force is stopped abruptly leaving the mass in motion unimpeded. The plot of the solution to this situation is below. Note that the oscillations for this solution die out slightly faster.



Using the step function defined by $u(t - a)$ we can get the new external force $g(t)$.

($f(t)$ is defined the same as above.)

$$u(t-a)=\begin{cases}0 & if \quad t<a\\ 1 & if \quad t\geq a\end{cases} \quad , \qquad g(t)=\begin{cases}f(t) & if \quad 0\leq t<2\pi\\ 0 & if \quad t\geq 2\pi\end{cases}$$

(i) Find a solution for this new situation that uses the more expensive part. Make use of calculations done in part (a).

Hint: Use the following fact.

$$\mathscr{L}\{u(t-a)f(t-a)\}=e^{-as}F(s)$$
$$\mathscr{L}^{-1}\{e^{-as}F(s)\}=u(t-a)f(t-a)$$

Solution:

$$f(t)=(1-u(t-2\pi))e^{-t/5}\cos(2t)$$

$$f(t)=e^{-t/5}\cos(2t)-u(t-2\pi)e^{-(t-2\pi)/5}\cos(2(t-2\pi))$$

Because of the periodicity of the cosine function this is equal to

$$f(t)=e^{-t/5}\cos(2t)-u(t-2\pi)e^{-(t-2\pi)/5}\cos(2(t)$$

$$\mathscr{L}\{f(t)\}=\mathscr{L}\{e^{-t/5}\cos(2t)-u(t-2\pi)e^{-(t-2\pi)/5}\cos(2t)\}$$

Use Laplace Tables get

$$F(s)=\frac{s+\frac{1}{5}}{(s+\frac{1}{5})^2+4}-e^{-2\pi s}\frac{s+\frac{1}{5}}{(s+\frac{1}{5})^2+4}$$

We can use the calculations we did in part (a)(ii). Replace $F(s)$ with the above equation.

$$X(s)=\frac{s+\frac{1}{5}}{((s+\frac{1}{5})^2+4)^2}-e^{-2\pi s}\frac{s+\frac{1}{5}}{((s+\frac{1}{5})^2+4)^2}$$

Now apply the hint:

$$x(t)=\frac{1}{4}e^{\frac{1}{5}t}t\sin(2t)-\frac{1}{4}u(t-2\pi)(t-2\pi)e^{\frac{1}{5}(t-2\pi)}$$

Which can be re-written as

$$x(t)=\begin{cases}\frac{1}{4}e^{-\frac{1}{5}t}t\sin(2t) & if \ t<2\pi\\ \frac{\pi}{2}e^{-\frac{1}{5}t}\sin(2t) & if \ t\geq 2\pi\end{cases}$$

(ii) Write down an equation for the amplitude of the new system. Does this new part meet the specifications of the contract, i.e. do you have proper justification for spending \$70,000 more for this different part?

Solution: Amplitude is given by

$$A(t) = \begin{cases} \frac{1}{4}e^{-\frac{1}{5}t}t & if\ t < 2\pi \\ \frac{\pi}{2}e^{-\frac{1}{5}t} & if\ t \geq 2\pi \end{cases}$$

The new part meets the first specification since

$$A(5) = \frac{5}{4e} \approx 0.46$$

The second specification requires that the amplitude decrease to 0.33 by $t = 7$

$$A(7) = \frac{\pi}{2}e^{-\frac{7}{5}} \approx 0.39$$

This more expensive part does not meet the specifications so you do not have proper justification for spending \$70,000 more. Time to call up your cost analyst to discuss damage control.