1. In this and the following problem, use $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ as the distance function on $\mathbb{R}^2$. Use ϵ-δ definition of continuity to prove that the multiplication map $\mathbb{R}^2 \to \mathbb{R}$ is continuous.

   Proof. The multiplication map is $f(x_1, x_2) = x_1x_2$. For $f(x_1, x_2)$ to be continuous we must show that $|x_1x_2 - y_1y_2| < \epsilon$ whenever $\max\{|x_1 - y_1|, |x_2 - y_2|\} < \delta$.

$$
\begin{aligned}
|x_1x_2 - y_1y_2| &= |x_1x_2 - x_1y_2 - y_1y_2 + x_1y_2| \\
&= |(x_1x_2 - x_1y_2) - (y_1y_2 - x_1y_2)| \\
&= |x_1(x_2 - y_2) - y_2(y_1 - x_1)| \\
&\leq |x_1(x_2 - y_2)| - |y_2(y_1 - x_1)| \\
&= |x_1||x_2 - y_2| - |y_2||y_1 - x_1| \\
&= |x_1||x_2 - y_2| - |y_2||x_1 - y_1| \\
&< \delta(|x_1| + |y_2|)
\end{aligned}
$$

   If $\delta < \frac{\epsilon}{|x_1|+|y_2|}$, then $|x_1x_2 - y_1y_2| < \epsilon$ as desired. □

2. Let $p_i : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the i-th coordinate. Prove that p_i is continuous.

   Proof. Assume the same distance function on $\mathbb{R}^2$ as in question (1).
   If the i^{th} projection is $p_i(x_1, x_2) = x_i$, then

$$
|p_i(x_1, x_2) - p_i(y_1, y_2)| = |x_i - y_i| < \delta
$$

   If $\delta < \epsilon$, then $|p_i(x_1, x_2) - p_i(y_1, y_2)| < \epsilon$ as desired. □

3. Let (X, d) be a metric space. Let $f : X \to \mathbb{R}^2$ be a map, and write $f(x) = (f_1(x), f_2(x))$ for every $x \in X$. In particular we have two functions $f_i : X \to \mathbb{R}$, $i = 1, 2$. Prove that f is continuous if and only if f_1 and f_2 are.

   Proof. Assume the same distance function on $\mathbb{R}^2$ as in question (1). If $f(x)$ is continuous then $|f(x) - f(y)| < \epsilon$ whenever $\max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\} < \delta$ for $\epsilon,\ \delta > 0$. Choose ϵ positive but less than δ then $|f_i(x) - f_i(y)| < \epsilon < \delta$.

   On the other hand, if $f_1(x)$ and $f_2(x)$ are continuous then $|f_i(x) - f_i(y)| < \epsilon$ whenever $|x - y| < \delta$. If both $|f_1(x) - f_1(y)| < \epsilon$ and $|f_2(x) - f_2(y)| < \epsilon$ then the $\max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\} < \delta$ if ϵ is chosen to be positive, but less than δ. This implies that $|f(x) - f(y)| < \epsilon$ whenever $\max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\} < \delta$.

   Therefore f is continuous if and only if f_1 and f_2 are. □

4. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$. Use the inductive definition $x^n = x \cdot x^{n-1}$ and previous exercises to prove that f is continuous.

   Proof. If $n = 1$ then $f(x) = x$. By the result from problem (2) above this is continuous. Assume that x^{n-1} is continuous and show that x^n is continuous. Since $x^n = x \cdot x^{n-1}$ by the result from question (1) above the multiplication of these functions is also continuous. $\square$

5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$. Prove that $\int_a^b f = 0$ implies $f(x) = 0$ for all $x \in [a, b]$.

   Proof. Assume $f(x) \not\equiv 0$ on $[a, b]$, then there is a point x_0 such that $f(x_0) > 0$. Since $f(x)$ is continuous, given $\epsilon, \delta > 0$, there exists an interval $[x_0 - \delta, x_0 + \delta]$ such that $d(f(x), f(x_0)) < \epsilon$. The integral of $f(x)$ is defined to be a number such that the greatest lower bound of the lower sums and least upper bound of the upper sums are equal. If the integral of $f(x)$ on $[x_0 - \delta, x_0 + \delta]$ is equal to zero then the least upper bound of the upper sums must also be zero. The upper sum is defined by $\sum_{i=0}^{n} M_i(x_i - x_{i-1})$ where M_i is the maximum value attained by $f(x)$ on the interval. Since the least upper bound of the upper sums is equal to zero this implies that $M_i = 0$, but we assumed that $f(x_0) > 0$, contradiction. Therefore $\int_a^b f = 0 \implies f(x) = 0\ \forall x \in [a, b]$. $\square$

6. Let (X, d) be a metric space. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in X. Prove that $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

   Proof.

$$
\begin{aligned}
d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_n) \\
d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\
d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_m, y_n) \\
d(d(x_n, y_n), d(x_m, y_m)) &\leq d(x_n, x_m) + d(y_m, y_n)
\end{aligned}
$$

   Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X this means for all $\epsilon > 0$ there exists a N such that $d(x_n, x_m) < \epsilon/2$ if $n, m > N$ and an M such that $d(y_n, y_m) < \epsilon/2$ if $n, m > M$.

$$d(d(x_n, y_n), d(x_m, y_m)) \leq d(x_n, x_m) + d(y_m, y_n) \leq \epsilon/2 + \epsilon/2 = \epsilon$$

   if $n, m > N + M$ which is the definition of Cauchy. $\square$

7. Let $K \subset \mathbb{R}$ be a set consisting of 0 and all $1/n$, $n = 1, 2, 3, \cdots$. Prove that K is compact directly using the definition, i.e. every open cover has a finite subcover.

Proof. First we prove that the sequence $\{a_n\} = 1/n$ converges to 0. Given any $\epsilon > 0$ then there must be a N such $d(1/n, 0) < \epsilon$ if $n > N$. Choose $N = 1/\epsilon$, then since $n > N$ this implies $d(1/n, 0) = 1/n < 1/N = \epsilon$. Therefore $\{a_n\} = 1/n$ converges to 0. Now to prove that K is compact consider $\bigcup_{i=0}^{N} B(i, \epsilon)$. Since N is finite and balls are open sets this a finite collection of open sets that fully contain K. In particular there is a ball around each point up to $1/N$ and then all the rest are contained in $B(0, \epsilon)$ which follows from the fact that the sequence $\{a_n\} = 1/n$ converges to 0. □

8. Let $F_1 \supseteq F_2 \supseteq \ldots$ be a descending sequence of non-empty compact subsets. Prove that $\cap_{n=1}^{\infty} F_n$ is non-empty.

Proof. Assume $\cap_{n=1}^{\infty} F_n = \emptyset$. For each n let $U_n = F_1 \setminus F_n$. Now $\cup U_n = F_1 \setminus (\cap F_n) = F_n$. All U_n are closed which means their complements are open. Since F_1 is compact and U_n is an open cover of F_1 there exists a finite subcover U_F. Let U_{maxF} be the largest most set containing all the other sets in the finite subcover. This means that $F_1 = U_{maxF}$, but this means that $U_{maxF} = F_1 \setminus U_F = \emptyset$, contradiction. □

9. Let (X, d) be a metric space and $\{f_n\}$ a sequence of continuous functions $f_n : X \to \mathbb{R}$ uniformly converging to f. Let $\{x_n\}$ be a sequence of points in X such that $\lim_n x_n = x \in X$. Prove that $\lim_n f_n(x_n) = f(x)$.

Proof. $\{f_n\}$ is a sequence of continuous functions which means given $\epsilon > 0$, there is a $\delta > 0$ such that $d(f_n(x), f_n(y)) < \epsilon$ whenever $x \in X$ and $d(x, y) < \delta$.

$\{f_n\}$ also converges uniformly to f which means given $\epsilon > 0$ there exists an N such that $d(f_n(x), f(x)) < \epsilon$ whenever $x \in X$ and $n > N$.

It is also true that given $\epsilon > 0$ and $x \in X$ there exists an N such that $d(x_n, x) < \epsilon$ whenever $n > N$.

Need to show that given $\epsilon > 0$ and $x \in X$ there exists an N such that $d(f_n(x_n), f(x)) < \epsilon$ whenever $n > N$.

First we show that if $\{f_n\}$ is a sequence of continuous functions converging to f that f is also continuous.

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

The first term and last term on the right hand side is less than ϵ since $f_n \to f$, the second term since each f_n is continuous. Therefore f is also less than ϵ and therefore continuous. Now to show that $f_n(x_n) \to f(x)$:

$$
\begin{aligned}
d(f_n(x_n), f(x)) &\leq d(f_n(x_n), f_n(y)) + d(f_n(y), f(x)) \\
&\leq d(f_n(x_n), f_n(y)) + d(f_n(y), f(y)) + d(f(y), f(x)) \\
&\leq d(f_n(x_n), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) + d(f(y), f(x)) \\
&\leq \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon
\end{aligned}
$$

The first term on the right hand side is less than $\epsilon/4$ due to f_n being continuous provided that $d(x_n, x) < \delta > 0$ which is true since we assumed that $x_n \to x$. The second is term is less than $\epsilon/4$ due to each f_n being continuous. The third term is less than $\epsilon/4$ due uniform convergence of $\{f_n\}$ and the last term is less than $\epsilon/4$ due to continuity of f. □

10. A subset $\mathbb{R}^n$ is convex if for any two points $x, y \in C$, the segment $[x, y]$ is contained in C. Prove that C is connected.

Proof. Assume that C is not connected, that is C can be expressed as the union of two disjoint non-empty open sets U and V. Pick a point $u \in U$ and $v \in V$ and consider the line segment that goes from u to v, $\{u + t(v - u) : t \in [0, 1]\}$. Since U and V are not connected this line must cross over the boundaries of U and V which implies it is not wholly contained in C, and therefore not convex. □