1. (A challenging problem) Write a computer program to solve the initial-value problem using the Taylor-Series method. Include terms in $h$, $h^2$, $h^3$ and continue to $t = 1$. Let $h = 0.01$.

$$\begin{cases} x_1' = \sin(x_1) + \cos(tx_2), & x_1(-1) = 2.37 \\ x_2' = t^{-1}\sin(tx_1), & x_1(-1) = -3.48 \end{cases}$$

   **Output:**

   **Code:**

2. (A practical problem) Solve and plot the resulting curve over the interval $[0, 5]$ for the ordinary differential equations. Make a video that show the dynamics of each mass-spring system. How do you interpret the results?

$$\begin{aligned} &(1)\;\; x'' + 192x = 0 \quad x(0) = \frac{1}{6}, \;\; x'(0) = 0 \\ &(2)\;\; x'' + x' + 192x = 0 \quad x(0) = \frac{1}{6}, \;\; x'(0) = 0 \\ &(3)\;\; x'' - x' + 192x = 0 \quad x(0) = \frac{1}{6}, \;\; x'(0) = 0 \end{aligned}$$

   **Solutions:**

   **(1)** $x'' + 192x = 0 \quad x(0) = \frac{1}{6}, \;\; x'(0) = 0$

   The characteristic equation is $m^2 + 192 = 0$ which can be solved for the characteristic roots: $m = \pm\sqrt{192}i$. The complete solution is

$$x(t) = x(t) = C_1\cos(\sqrt{192}t) + C_2\sin(\sqrt{192}t)$$

   Using initial conditions we can obtain $C_1$ and $C_2$:

$$x(t) = \frac{1}{6}\cos(\sqrt{192}t)$$

   This is an example of simple harmonic motion with amplitude $A = \frac{1}{6}$ and frequency $\omega = \sqrt{192}$.

   **(2)** $x'' + x' + 192x = 0 \quad x(0) = \frac{1}{6}, \;\; x'(0) = 0$

   The characteristic equation is $m^2 + m + 192 = 0$ which can be solved for the characteristic roots: $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{767}i$. The complete solution is

$$x(t) = e^{-\frac{t}{2}}\left(C_1\cos(\frac{\sqrt{767}}{2}t) + C_2\sin(\frac{\sqrt{767}}{2}t)\right)$$

   Using initial conditions we can obtain $C_1$ and $C_2$:

$$x(t) = e^{-\frac{t}{2}}\left(\frac{1}{6}\cos(\frac{\sqrt{767}}{2}t) + \frac{1}{6\sqrt{767}}\sin(\frac{\sqrt{767}}{2}t)\right)$$

The oscillations in this example are damped by a factor of $e^{-\frac{t}{2}}$ which causes the motion to decay as $t \to \infty$.

**(3)** $x'' - x' + 192x = 0 \quad x(0) = \frac{1}{6}, \quad x'(0) = 0$

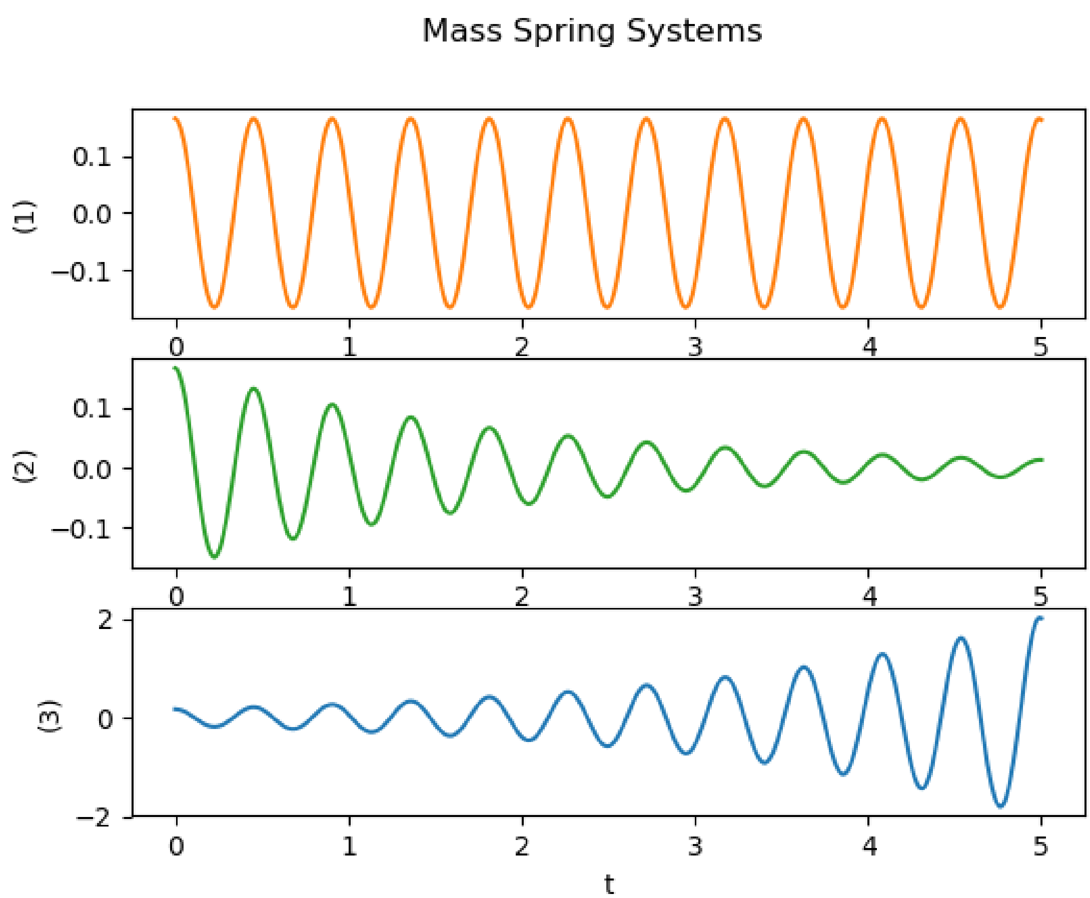
The characteristic equation is $m^2 + m + 192 = 0$ which can be solved for the characteristic roots: $m = \frac{1}{2} \pm \frac{1}{2}\sqrt{767}i$. The complete solution is

$$x(t) = e^{\frac{t}{2}} \left( C_1 \cos(\frac{\sqrt{767}}{2}t) + C_2 \sin(\frac{\sqrt{767}}{2}t) \right)$$

Using initial conditions we can obtain $C_1$ and $C_2$:

$$x(t) = e^{\frac{t}{2}} \left( \frac{1}{6} \cos(\frac{\sqrt{767}}{2}t) - \frac{1}{6\sqrt{767}} \sin(\frac{\sqrt{767}}{2}t) \right)$$

The damping constant in this example is less than zero which causes the oscillations of the system to increase without bound.




**Animation File:** See included file titled "HW3_P2_JordanSaethre_PaulMundt.MP4" for animation of dynamical behavior of solutions.

**Code for Animation:**

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation

fig = plt.figure()
ax1 = fig.add_subplot(1, 1, 1)

t = np.linspace(0, 5, 500)
x1 = (1/6)*np.cos(np.sqrt(192)*t)
x2 = np.exp((-1/2)*t)*((1/6)*np.cos((np.sqrt(767)/2)*t)
    + (1/(6*np.sqrt(767))*np.sin((np.sqrt(767)/2)*t)))
x3 = np.exp((1/2)*t)*((1/6)*np.cos((np.sqrt(767)/2)*t)
    - (1/(6*np.sqrt(767))*np.sin((np.sqrt(767)/2)*t)))

ax1.set_ylabel(u'cos(2\u03c0t)')
ax1.set_xlim(0, 5)
ax1.set_ylim(-1, 1)
plt.setp(ax1.get_xticklabels(),visible=True)

ax1.set_xlabel('t')


lines = []
for i in range(len(t)):
    head = i - 1
    head_slice = (t > t[i] - 1.0) & (t < t[i])
    line1,  = ax1.plot(t[:i], x1[:i], color='orange')
    line1a, = ax1.plot(t[head_slice], x1[head_slice],
        color='orange', linewidth=2)
    line1e, = ax1.plot(t[head], x1[head], color='orange',
        marker='o', markeredgecolor='orange')
    line2,  = ax1.plot(t[:i], x2[:i], color='green')
    line2a, = ax1.plot(t[head_slice], x2[head_slice],
        color='green', linewidth=2)
    line2e, = ax1.plot(t[head], x2[head], color='green',
     marker='o', markeredgecolor='g')
    line3,  = ax1.plot(t[:i], x3[:i], color='blue')
    line3a, = ax1.plot(t[head_slice], x3[head_slice],
        color='blue', linewidth=2)
    line3e, = ax1.plot(t[head], x3[head], color='blue',
        marker='o', markeredgecolor='b')
    lines.append([line1, line1a, line1e, line2, line2a,
        line2e, line3, line3a, line3e])

# Build the animation using ArtistAnimation function
```

```

ani = animation.ArtistAnimation(fig,lines,interval=50,
    blit=True)

fn = 'HW3_P2_JordanSaethre_PaulMundt'
ani.save('%s.mp4'%(fn),writer='ffmpeg')
```

3. (A theoretical problem) Find the exact solution of the two-point boundary-value problem $x'' = f(t), \quad x(0) = x(1) = 0$.

**Solutions:** First we find the kernal, i.e. the solution to the homogeneous equation $x'' = 0$. This has characteristic equation $m^2 = 0$ which yields the root of $m = 0$ with multiplicity 2. Hence our characteristic solutions are $u_1(t) = e^{0t} = 1$ and $u_2(t) = te^{0t} = t$. The difference between $u_1$ and $u_2$ is also a solution. Let $x_1(t) = t$ and $x_2(t) = 1 - t$. The Wronskian of $x_1$ and $x_2$ is

$$W(x_1, x_2)(t) = \det \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} = \det \begin{bmatrix} t & 1-t \\ 1 & -1 \end{bmatrix} = -1$$

Therefore the Green's function is

$$G(t,s) = \begin{cases} s(t-1) & if\ 0 \le s \le t \\ t(s-1) & if\ t \le s \le 1 \end{cases}$$

Hence

$$x(t) = \int_0^t sf(s)ds \cdot (t-1) + \int_t^1 (s-1)f(s)ds \cdot t$$

**Solution Using Laplace Transforms:**

$$\begin{aligned} x''(t) &= f(t) \\ \mathscr{L}\{x''(t)\} &= \mathscr{L}\{f(t)\} \\ s^2X(s) - sx(0) - x'(0) &= F(s) \\ s^2X(s) &= F(s) \\ X(s) &= \frac{1}{s^2}F(s) \end{aligned}$$

Let $G(s) = \frac{1}{s^2}$

$$\begin{aligned} X(s) &= G(s)F(s) \\ \mathscr{L}^{-1}\{X(s)\} &= \mathscr{L}^{-1}\{G(s)F(s)\} \\ x(t) = (g*f)(t) &= \int_0^t g(t-\tau)f(\tau)d\tau \end{aligned}$$

This is the convolution formula. In our case $g(t) = \mathscr{L}^{-1}\{G(s)\} = \mathscr{L}^{-1}\{\frac{1}{s^2}\} = t$.

$$\int_0^t (t-\tau)f(\tau)d\tau$$

If $f(t) = \delta(t-s)$ then we have

$$x(t) = \int_0^t (t-\tau)\delta(\tau - s)d\tau = \begin{cases} 0 \;\; if \;\; t < s \\ t - s \;\; if \;\; t \geq s \\ 0 \;\; otherwise \end{cases}$$