Math 5610/6860: Assignment 5

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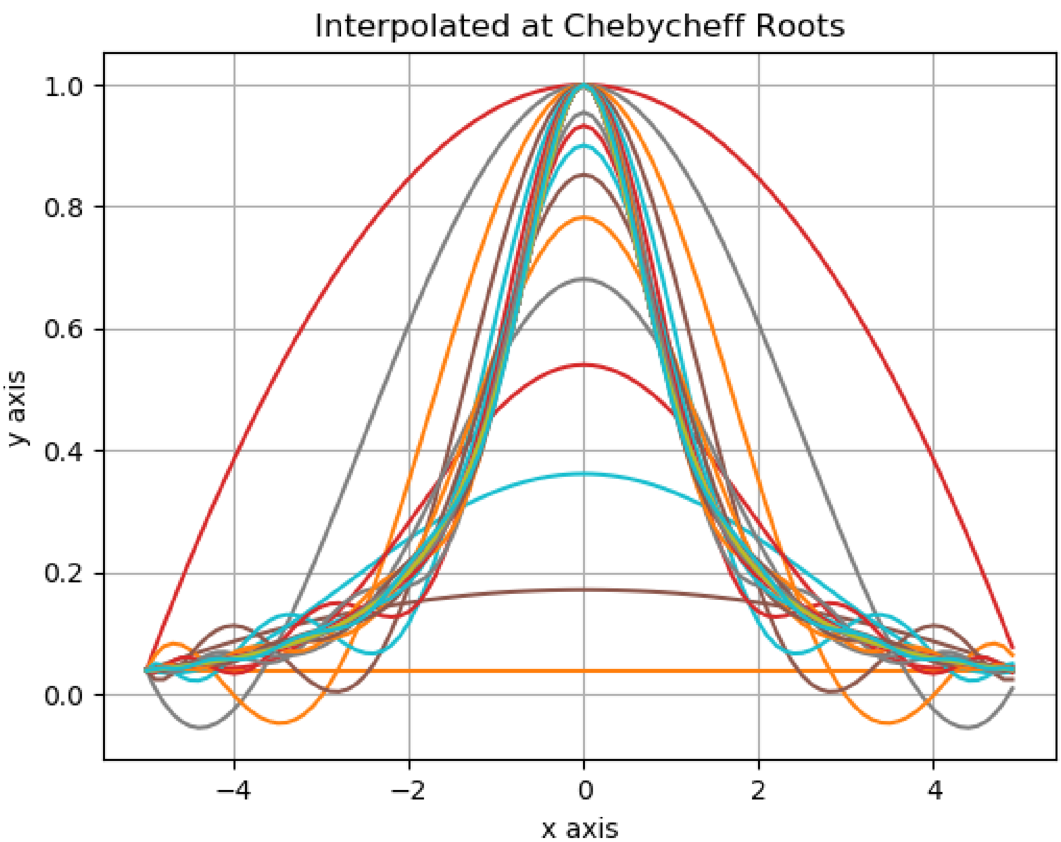
1. Repeat Question 4 in Assignment 4 except that you interpolate at the roots of the Chebycheff polynomials, i.e.,

$$x_i = 5\cos\frac{i\pi}{n}, \quad i = 0, 1, \ldots, n.$$



Code for Chebycheff Polynomials

```
import numpy as np
from matplotlib import pyplot as plt
import scipy
from scipy import interpolate

# Define the function that we will interpolate:
my_function = lambda x: 1/(1 + x**2)

# Get the x_j's
def find_xjs(n):
    xj = np.array([])
    for j in range(0, n + 1):
        xj = np.append(xj, 5*np.cos(j*np.pi/n))
    return xj

# Get the f_xj's
def find_f_xjs(n):
    f_xj = np.array([])
    for j in find_xjs(n):
        f_xj = np.append(f_xj, my_function(j))
    return f_xj

def find_true_f_xjs():
    f_xj = np.array([])
    for j in np.arange(-5,5,0.1):
        f_xj = np.append(f_xj, my_function(j))
    return f_xj


def plot_poly(n):
    poly_x = np.arange(-5,5,0.1)
    xj = find_xjs(n)
    f_xj = find_f_xjs(n)
    poly = scipy.interpolate.lagrange(xj,f_xj)
    poly_points = np.array([])
    for x in poly_x:
        poly_points = np.append(poly_points, poly(x))
    plt.title("Interpolated at Chebycheff Roots")
    plt.xlabel("x axis")
    plt.ylabel("y axis")
    f = find_true_f_xjs()
    plt.plot(poly_x, f)
    plt.plot(poly_x,poly_points)
    plt.grid(True)

def plot_on_one(n):
```

```
    for i in range(1,n+1):
        plot_poly(i)
    plt.show()

plot_on_one(20)
```

2. Suppose you are given symmetric data

$$(x_i, y_i), \quad i = -n, -n+1, \cdots, n-1, n,$$

such that

$$x_{-i} = -x_i, \quad \text{and} \quad y_{-i} = -y_i \quad i = 0, 1, \cdots, n.$$

What is the required degree of the interpolating polynomial p? Show that the interpolating polynomial is odd, i.e.,

$$p(x) = -p(-x)$$

for all real numbers x.

Begin by writing out this out as a system of equations

$$\begin{bmatrix} 1 & x_{-n} & x_{-n}^2 & \cdots & x_{-n}^{n-1} & x_{-n}^n \\ 1 & x_{-(n-1)} & x_{-(n-1)}^2 & \cdots & x_{-(n-1)}^{n-1} & x_{-(n-1)}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} & x_{n-1}^n \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} y_{-n} \\ y_{-(n-1)} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

Since the data is symmetric this is the same as:

$$\begin{bmatrix} 1 & -x_n & -x_n^2 & \cdots & -x_n^{n-1} & -x_n^n \\ 1 & -x_{n-1} & -x_{n-1}^2 & \cdots & -x_{n-1}^{n-1} & -x_{n-1}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} & x_{n-1}^n \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} -y_n \\ -y_{n-1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

The polynomial $a_0 + x + x^2 + \cdots + x^n = y$ must be odd for the above system to have a unique solution.

3. Assume you are given the data

$$\begin{array}{ccccc} x_i: & 1 & 2 & 4 & 8 \\ y_i: & 1 & 2 & 3 & 4 \end{array}$$

Construct the interpolating polynomial using **a.** the power form, **b.** the Lagrange form, **c.** the Newton form,and show that they all yield the same polynomial.

The Power Form:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \implies \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 2 \\ 1 & 4 & 16 & 64 & 3 \\ 1 & 8 & 64 & 512 & 4 \end{array}\right] \implies \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 1 \\ 0 & 3 & 15 & 63 & 2 \\ 0 & 7 & 63 & 511 & 3 \end{array}\right]$$

$$\implies \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 1 \\ 0 & 0 & 1 & 7 & -1/6 \\ 0 & 0 & 42 & 462 & -4 \end{array}\right] \implies \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 1 \\ 0 & 0 & 1 & 7 & -1/6 \\ 0 & 0 & 0 & 1 & 1/56 \end{array}\right] \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -10/21 \\ 0 & 1 & 0 & 0 & 7/4 \\ 0 & 0 & 1 & 0 & -7/24 \\ 0 & 0 & 0 & 1 & 1/56 \end{array}\right]$$

$$\implies \quad p(x) = \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}$$

The Lagrange Form

$$p(x) = \frac{(x-2)(x-4)(x-8)}{(1-2)(1-4)(1-8)}(1) + \frac{(x-1)(x-4)(x-8)}{(2-1)(2-4)(2-8)}(2)$$

$$+ \frac{(x-1)(x-2)(x-8)}{(4-1)(4-2)(4-8)}(3) + \frac{(x-1)(x-2)(x-4)}{(8-1)(8-2)(8-4)}(4)$$

$$\implies \quad p(x) = \left(-\frac{1}{21}\right)(x-2)(x-4)(x-8) + \left(\frac{1}{6}\right)(x-1)(x-4)(x-8)$$

$$+ \left(-\frac{1}{8}\right)(x-1)(x-2)(x-8) + \left(\frac{1}{42}\right)(x-1)(x-2)(x-4)$$

$$\implies p(x) = \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}$$

The Newton Form

$$p(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) + c_4(x - x_1)(x - x_2)(x - x_3)$$

$$p(x) = c_1 + c_2(x-1) + c_3(x-1)(x-2) + c_4(x-1)(x-2)(x-4)$$

$$p(1) = c_1 + c_2(1-1) + c_3(1-1)(1-2) + c_4(1-1)(1-2)(1-4) \implies c_1 = 1$$

$$p(2) = 1 + c_2(2-1) + c_3(2-1)(2-2) + c_4(2-1)(2-2)(2-4) \implies c_2 = 1$$

$$p(4) = 1 + (4-1) + c_3(4-1)(4-2) + c_4(4-1)(4-2)(4-4) \implies c_3 = -\frac{1}{6}$$

$$p(8) = 1 + (8-1) - \frac{1}{6}(8-1)(8-2) + c_4(8-1)(8-2)(8-4) \implies c_4 = \frac{3}{168}$$

$$p(x) = 1 + (x-1) - \frac{1}{6}(x-1)(x-2) + \frac{3}{168}(x-1)(x-2)(x-4)$$

$$\implies p(x) = \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}$$

4. Suppose that for some reason you wish to use a differentiation formula of the form

$$f'(a) = \frac{1}{h}\left[\alpha_0 f(a) + \alpha_1 f(a + \frac{h}{3}) + \alpha_2 f(a + \frac{h}{2}) + \alpha_3 f(a+h)\right]$$

where the α's are chosen so as to make the formula exact for polynomials of degree as high as possible. What are the α's?

Choose test functions $f(x) = 1,\ x,\ x^2,\ x^3$ and some nice values for a and h to create a system of equations and then solve for the α's.

With $a = h = 1$ we get the following equations:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_0 + \left(\frac{4}{3}\right)\alpha_1 + \left(\frac{3}{2}\right)\alpha_2 + 2\alpha_3 = 2$$

$$\alpha_0 + \left(\frac{4}{3}\right)^2\alpha_1 + \left(\frac{3}{2}\right)^2\alpha_2 + 2^2\alpha_3 = 2$$

$$\alpha_0 + \left(\frac{4}{3}\right)^3\alpha_1 + \left(\frac{3}{2}\right)^3\alpha_2 + 2^3\alpha_3 = 2$$

Solving this system of equations we get

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 4/3 & 3/2 & 2 & 1 \\ 1 & 16/9 & 9/4 & 4 & 2 \\ 1 & 64/27 & 27/8 & 8 & 3 \end{array}\right] \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & 27/2 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right]$$

5. Let the inner product (f, g) be defined by

$$(f, g) = \int_a^b w(x)f(x)g(x)\ dx$$

(where w is a positive weight function). Prove that the sequence of polynomials defined by

$$\begin{aligned} Q_n &= (x - a_n)Q_{n-1} - b_n Q_{n-2} \\ \text{with}\quad Q_0 &= 1, \quad Q_1 = x - a_1, \\ a_n &= (xQ_{n-1}, Q_{n-1})/(Q_{n-1}, Q_{n-1}) \\ b_n &= (xQ_{n-1}, Q_{n-2})/(Q_{n-2}, Q_{n-2}) \end{aligned}$$

is orthogonal to each other with respect to the inner product. Note that the proof of this fact uses the property

$$(xf, g) = (f, xg).$$

Proof. First it is shown that $(Q_0, Q_1) = 0$, where $Q_0 = 1$ and $Q_1 = x - \frac{(x,1)}{(1,1)}$:

$$\begin{aligned} (Q_0, Q_1) &= \left(1, x - \frac{(x,1)}{(1,1)}\right) \\ &\implies (1, x) - \left(1, \frac{(x,1)}{(1,1)}\right) \\ &\implies (1, x) - \frac{(x,1)}{(1,1)}(1,1) \\ &\implies (1, x) - (x, 1) \\ &\implies (1, x) - (1, x) = 0 \end{aligned}$$

By induction, assume it is true up to $n - 1$ and show that

$$(Q_n, Q_{n-1}) = (Q_n, Q_{n-2}) = 0$$

By the induction hypothesis a_{n-i} and b_{n-i} for $i = 0, 1, \cdots, n$ are all zero.

$$\begin{aligned} (Q_n, Q_{n-1}) &= ((x - a_n)Q_{n-1} - b_n Q_{n-2}, (x - a_{n-1})Q_{n-2} - b_{n-1}Q_{n-3}) \\ &= (xQ_{n-1}, xQ_{n-2}) \\ &= 0 \end{aligned}$$

Similarly for (Q_n, Q_{n-2}),

$$\begin{aligned} (Q_n, Q_{n-2}) &= ((x - a_n)Q_{n-1} - b_n Q_{n-2}, (x - a_{n-2})Q_{n-3} - b_{n-2}Q_{n-4}) \\ &= (xQ_{n-1}, xQ_{n-3}) \\ &= 0 \end{aligned}$$

$\square$

6. Consider the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\ dx.$$

Use the recurrence relation described in question 5 to compute Q_i for $i = 0, 1, 2, 3, 4, 5$.

$$Q_0 = 0$$

$$Q_1 = x - a_1 = x - \frac{(xQ_0, Q_0)}{(Q_0, Q_0)} = x - \frac{(x,1)}{(1,1)} = x - \frac{\int_{-1}^{1} xdx}{\int_{-1}^{1} 1dx} = x$$

$$Q_2 = (x - a_2)Q_1 - b_2 Q_0 = \left(x - \frac{(xQ_1, Q_1)}{(Q_1, Q_1)}\right) Q_1 - \frac{(xQ_1, Q_0)}{(Q_0, Q_0)} Q_0$$

$$= \left(x - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx}\right) Q_1 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1dx} Q_0$$

$$= x^2 - \frac{1}{3}$$

$$Q_3 = (x - a_3)Q_2 - b_3 Q_1 = \left(x - \frac{(xQ_2, Q_2)}{(Q_2, Q_2}\right) Q_2 - \frac{(xQ_2, Q_1)}{(Q_1, Q_1)} Q_1$$

$$= \left(x - \frac{\int_{-1}^{1} x(x^2 - \frac{1}{3})^2 dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx}\right) (x^2 - \frac{1}{3}) - \frac{\int_{-1}^{1} x(x^2 - \frac{1}{3}) dx}{\int_{-1}^{1} x^2 dx} x$$

$$= x^3 - \frac{3}{5}$$

$$Q_4 = (x - a_4)Q_3 - b_4 Q_2 = \left(x - \frac{(xQ_3, Q_2)}{(Q_2, Q_2}\right) Q_3 - \frac{(xQ_2, Q_1)}{(Q_1, Q_1)} Q_1$$

$$= \left(x - \frac{\int_{-1}^{1} x(x^3 - \frac{3}{5})^2 dx}{\int_{-1}^{1} (x^3 - \frac{3}{5})^2 dx}\right) (x^3 - \frac{3}{5}) - \frac{\int_{-1}^{1} x(x^3 - \frac{3}{5})(x^2 - \frac{1}{3}) dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx} (x^2 - \frac{1}{3})$$

$$= x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$$Q_5 = (x - a_5)Q_4 - b_5 Q_3 = \left(x - \frac{(xQ_4, Q_3)}{(Q_3, Q_3}\right) Q_4 - \frac{(xQ_3, Q_2)}{(Q_2, Q_2)} Q_2$$

$$= \left(x - \frac{\int_{-1}^{1} x(x^4 - \frac{6}{7}x^2 + \frac{3}{35})^2 dx}{\int_{-1}^{1} (x^4 - \frac{6}{7}x^2 + \frac{3}{35})^2 dx}\right) (x^4 - \frac{6}{7}x^2 + \frac{3}{35}) - \frac{\int_{-1}^{1} x(x^3 - \frac{3}{5})(x^4 - \frac{6}{7}x^2 + \frac{3}{35}) dx}{\int_{-1}^{1} (x^3 - \frac{3}{5})^2 dx} (x^3 - \frac{3}{5})$$

$$= x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$

7. Use the Gram-Schmidt Process to find a basis of

$$\text{span}\{1, x, e^x\}$$

that is orthonormal with respect to the inner product

$$(f, g) = \int_0^1 f(x)g(x)\ dx.$$

Start by finding an orthogonal basis and then normalize each of the elements.

$$\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= x - \frac{(x,1)}{(1,1)} = x - \frac{\int_0^1 x dx}{\int_0^1 dx} = x - \frac{1}{4} \\
\phi_2 &= e^x - \frac{(e^x,1)}{(1,1)} - \frac{(e^x, x-\frac{1}{4})}{(x-\frac{1}{4}, x-\frac{1}{4})} = e^x - \frac{\int_0^1 e^x dx}{\int_0^1 dx} - \frac{\int_0^1 e^x(x-\frac{1}{4})dx}{\int_0^1 (x-\frac{1}{4})^2 dx} \\
&= e^x + \frac{5}{7}e - \frac{53}{7}
\end{aligned}$$

Now find $e_0 = \frac{\phi_0}{||\phi_0||}$, $e_1 = \frac{\phi_1}{||\phi_1||}$, and $e_2 = \frac{\phi_2}{||\phi_2||}$.

$$\begin{aligned}
e_0 &= 1 \\
e_1 &= \left(\frac{x - \frac{1}{4}}{\sqrt{(\int_0^1 (x-\frac{1}{4})^2 dx}}\right) = \frac{48}{7}\left(x - \frac{1}{4}\right) \\
e_2 &= \left(\frac{e^x + \frac{5}{7}e - \frac{53}{7}}{\sqrt{(\int_0^1 (e^x + \frac{5}{7}e - \frac{53}{7})^2 dx}}\right) = \frac{1}{\sqrt{\frac{1}{98}(7053 - 2684e + 239e^2)}}\left(e^x + \frac{5}{7}e - \frac{53}{7}\right)
\end{aligned}$$

8. Show that the function basis defined below

$$\varphi_i(t) = 1, \cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos nt, \sin nt, \cdots$$

is an orthogonal basis.

Proof. With out loss of generality, assume $m < n$ and $a = -b$. It needs to be shown that

$$(\cos(nt), \sin(mt)) = (\cos(nt), \cos(mt)) = (\sin(nt), \sin(mt)) = 0$$

For $(\cos(nt), \sin(mt)) = \int_{-b}^{b} \cos(nt)\sin(mt)dx$, it should be noticed that $\cos(nt)\sin(mt)$ is an even function multiplied by an odd function which results in an odd function. An odd function evaluated over a symmetric domain is equal to zero.

For $(\cos(nt), \cos(mt)) = \int_{-b}^{b} \cos(nt)\cos(mt)dx$, use a trigonometric identity to rewrite this as

$$\begin{aligned}
\int_{-b}^{b} \cos(nt)\cos(mt)dx &= \frac{1}{2}\int_{-b}^{b} \cos((m+n)t) + \cos((m-n)t)dt \\
&= \frac{1}{2}\left[\frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n}\right]_{-b}^{b} \\
&= 0
\end{aligned}$$

For $(\sin(nt), \sin(mt)) = \int_{-b}^{b} \sin(nt)\sin(mt)dx$, use a trigonometric identity to rewrite this as

$$\begin{aligned}
\int_{-b}^{b} \sin(nt)\sin(mt)dx &= \frac{1}{2}\int_{-b}^{b} \cos((m-n)t) - \cos((m+n)t)dt \\
&= \frac{1}{2}\left[\frac{\sin((m-n)t)}{m-n} - \frac{\sin((m+n)t)}{m+n}\right]_{-b}^{b} \\
&= 0
\end{aligned}$$

$\square$