Math 5610 Homework 1

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Root Finding and Fixed Point Methods:

- **Due:** 11 Sep 2019
- Implement the Bisection Method, Netwon's Method, and Secant Method to find roots of $f(x) = x^2 - e^{2-x^2}$.
- Implement the Fixed Point Method to find where $f(x) = (x+10)^{\frac{1}{4}}$.
- Check accuracy and order of convergence of each method for the specified examples.

Contents

1 Introduction

The following sections will describe the implementation of four numerical methods: Bisection, Netwon's, Secant, and Fixed Point. All methods were implemented with Python 3.7 and run in an Anaconda command line prompt. Each root finding method approximates the root of $f(x) = x^2 - e^{2-x^2}$ and the Fixed Point Method approximates the intersection of $f(x) = x$ and $g(x) = (x+10)^{\frac{1}{4}} = x$.

Order of convergence is approximated for each example by

$$p \approx \frac{\log \mid \frac{x_{n+1}-x_n}{x_n-x_{n-1}} \mid}{\log \mid \frac{x_n-x_{n-1}}{x_{n-1}-x_{n-2}} \mid}$$

2 Bisection Method

The following code defines a function called "find_root" that takes in a user specified interval, i.e. $[a, b]$, and an allowable error ϵ. A while loop is used to iterate until the calculated error is less than or equal to the allowable error.

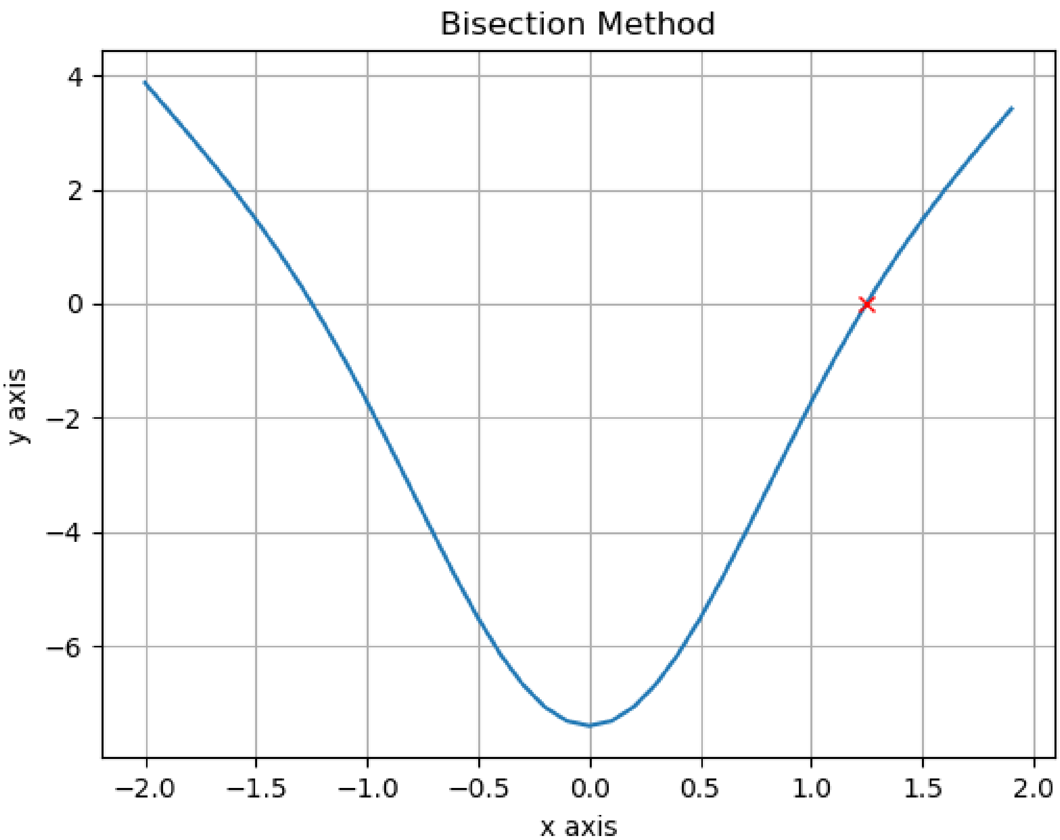
2.1 Code

```
import numpy as np
from matplotlib import pyplot as plt

my_function = lambda x: (x)**2-np.exp(2-x**2)
true_root = 1.24786

def find_root(a, b, epsilon):
    error = (b-a)/2
    c = (a+b)/2
    n = 0
    n_array = []
    true_error_array = []
    while (error > epsilon):
        if (my_function(b)*my_function(c) <= 0):
            a = c
            error = (b-a)/2
            c = (a+b)/2
            n = n + 1
            n_array.append(n)
            true_error = abs(true_root - c)
            true_error_array.append(true_error)
```

```
                print('iteration:', n, " - error:", error)
                continue
            else:
                b = c
                error = (b-a)/2
                c = (a+b)/2
                n = n + 1
                n_array.append(n)
                true_error = abs(true_root - c)
                true_error_array.append(true_error)
                print('iteration:', n, " - error:", error)
                continue
    print('error:', error)
    print('approximate root:', c)
    converge_rate = (np.log(abs((true_error_array[n-1] -
                        true_error_array[n-2])/
                    (true_error_array[n-2]-
                        true_error_array[n-3]))))/
                    (np.log(abs((true_error_array[n-2]-
                        true_error_array[n-3])/
                    (true_error_array[n-3]-
                        true_error_array[n-4]))))
    print("order of convergence:", converge_rate)
    return c, n_array, true_error_array


# Plot of function and root
root = find_root(-10,11,0.000001)
root_0 = 0
x = np.arange(-2,2,0.1)
y = (x)**2-np.exp(2-x**2)
plt.title("Bisection Method")
plt.xlabel("x axis")
plt.ylabel("y axis")
plt.plot(x, y)
plt.plot(root[0], root_0, 'rx')
plt.grid(True)
plt.show()


# Plot of error vs iterations
plt.title("Bisection Method Error")
plt.xlabel("n iterations")
plt.ylabel("true error")
plt.plot(root[1], root[2],'ro')
```

```
plt.grid(True)
plt.show()
```

2.2 Output



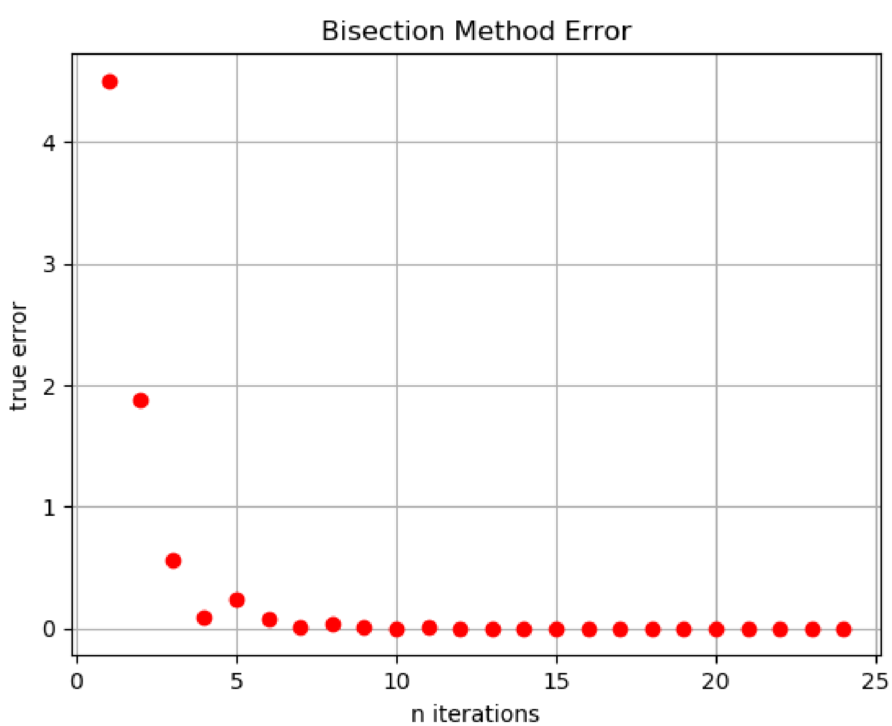
2.3 Accuracy and Order of Convergence

The last approximation for the root was with in 6.258487×10^{-7} of the true value. The approximated value for the root of the function is 1.247856. The order of convergence for this method is 1.0.

Anaconda Prompt

```
(base) C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610>python Bisection_Method.py
iteration: 1  - error: 5.25
iteration: 2  - error: 2.625
iteration: 3  - error: 1.3125
iteration: 4  - error: 0.65625
iteration: 5  - error: 0.328125
iteration: 6  - error: 0.1640625
iteration: 7  - error: 0.08203125
iteration: 8  - error: 0.041015625
iteration: 9  - error: 0.0205078125
iteration: 10  - error: 0.01025390625
iteration: 11  - error: 0.005126953125
iteration: 12  - error: 0.0025634765625
iteration: 13  - error: 0.00128173828125
iteration: 14  - error: 0.000640869140625
iteration: 15  - error: 0.0003204345703125
iteration: 16  - error: 0.00016021728515625
iteration: 17  - error: 8.0108642578125e-05
iteration: 18  - error: 4.00543212890625e-05
iteration: 19  - error: 2.002716064453125e-05
iteration: 20  - error: 1.0013580322265625e-05
iteration: 21  - error: 5.0067901611328125e-06
iteration: 22  - error: 2.5033950805664062e-06
iteration: 23  - error: 1.2516975402832031e-06
iteration: 24  - error: 6.258487701416016e-07
error: 6.258487701416016e-07
approximate root: 1.2478561103343964
order of convergence: 1.0

(base) C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610>
```

The distance of each approximation to the true value with each iteration n:



3 Newton's Method

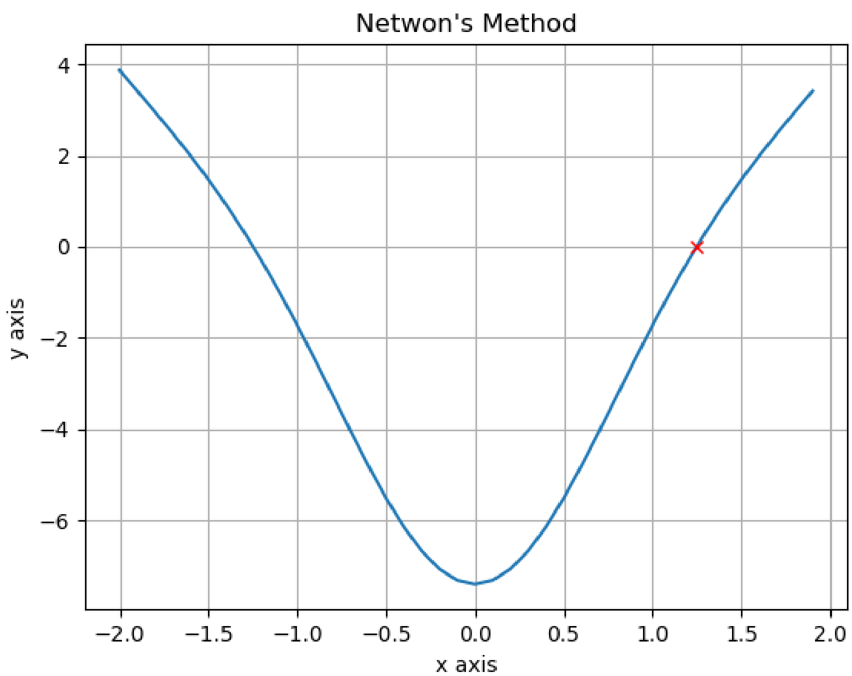
The following code was adapted from the Bisection Method. It was changed to only require one initial guess instead of an interval containing a root. A while loop iterates until $x_n = x_n + \frac{f(x_n)}{f'(x_n)}$ is within the user specified allowable error ϵ.

3.1 Code

```
import numpy as np
from matplotlib import pyplot as plt

my_function = lambda x: (x)**2-np.exp(2-x**2)
d_my_function = lambda x: 2*x*(1 + np.exp(2-x**2))
true_root = 1.24786

def find_root_newton(x_0, epsilon):
    error = 1
    n = 0
    n_array = []
    true_error_array = []
    while (error > epsilon):
        x_n = x_0 - (my_function(x_0)/d_my_function(x_0))
        error = abs(x_0 - x_n)
        x_0 = x_n
        n = n + 1
        n_array.append(n)
        true_error = abs(true_root - x_0)
        true_error_array.append(true_error)
        print('iteration:', n, " - error:", error)
        continue
    print('error:', error)
    print('approximate root:', x_n)
    converge_rate = (np.log(abs((true_error_array[n-1] -
                        true_error_array[n-2])/
                    (true_error_array[n-2]-
                        true_error_array[n-3]))))/
                    (np.log(abs((true_error_array[n-2]-
                        true_error_array[n-3])/
                    (true_error_array[n-3]-
                        true_error_array[n-4]))))
    print("order of convergence:", converge_rate)
    print("order of convergence:", converge_rate)
    return x_n, n_array, true_error_array
```

```


# Plot the function and root
root = find_root_newton(1.2,0.000001)
root_0 = 0
x = np.arange(-2,2,0.1)
y = (x)**2-np.exp(2-x**2)
plt.title("Netwon's Method")
plt.xlabel("x axis")
plt.ylabel("y axis")
plt.plot(x, y)
plt.plot(root[0], root_0, 'rx')
plt.grid(True)
plt.show()

# Plot of error vs iterations
plt.title("Bisection Method Error")
plt.xlabel("n iterations")
plt.ylabel("true error")
plt.plot(root[1], root[2],'ro')
plt.grid(True)
plt.show()
```

3.2 Output



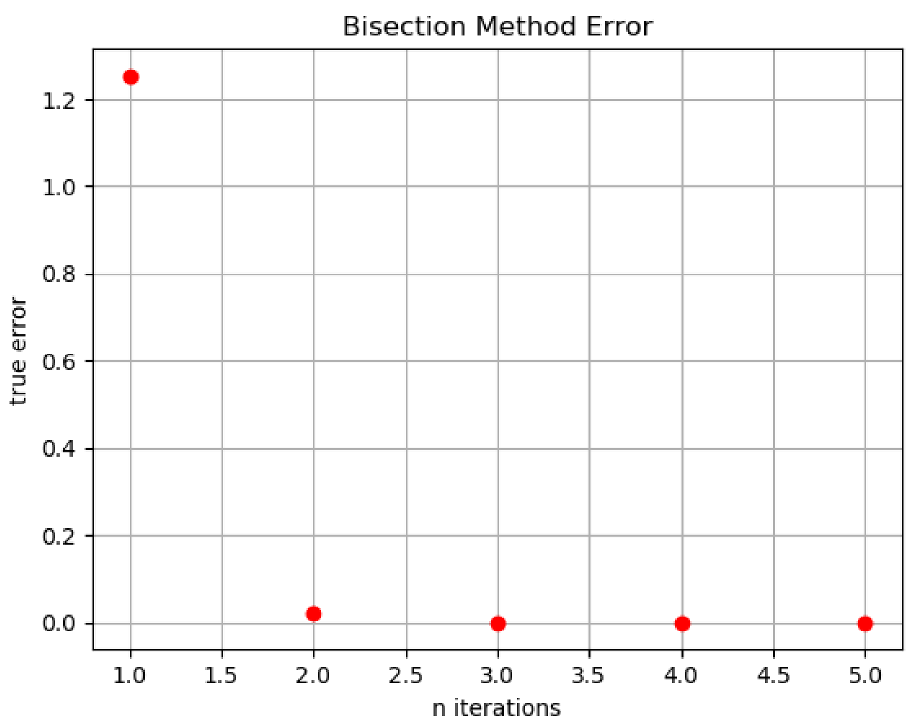
3.3 Accuracy and Order of Convergence

The last approximation for the root was with in 1.224921×10^{-8} of the true value. The approximated value for the root of the function is 1.247856. The order of convergence is 2.006120.

Anaconda Prompt

```
(base) C:\Users\saeth>cd C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610

(base) C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610>python Newtons_Method.py
iteration: 1  - error: 2.499999999733191
iteration: 2  - error: 1.229607740945944
iteration: 3  - error: 0.02272053813909114
iteration: 4  - error: 0.00018466816241025086
iteration: 5  - error: 1.2249209202508382e-08
error: 1.2249209202508382e-08
approximate root: 1.247856401593393
order of convergence: 2.0061201921518266
```

The distance of each approximation to the true value with each iteration n:



4 Secant Method

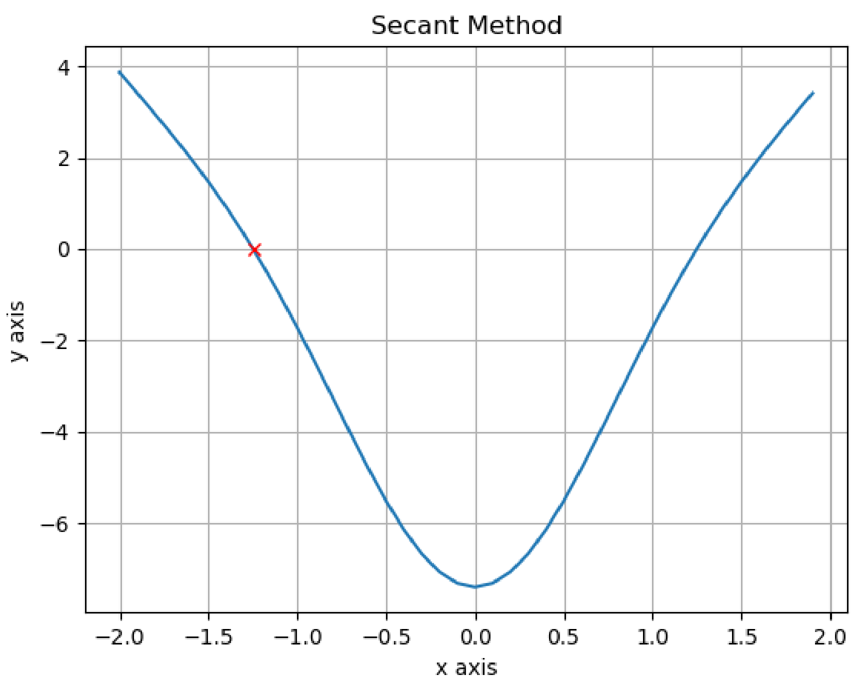
The Secant Method was adapted from Newton's Method by approximating $f'(x)$ with $\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$.

4.1 Code

```
import numpy as np
from matplotlib import pyplot as plt

my_function = lambda x: (x)**2-np.exp(2-x**2)
true_root = -1.24786

def find_root_secant(x_0, x_1, epsilon):
    error = 1
    n = 0
    n_array = []
    true_error_array = []
    while (error > epsilon):
        x_n = x_0 - (my_function(x_0)*(x_1 - x_0))/
        (my_function(x_1) - my_function(x_0))
        error = abs(x_0 - x_n)
        x_0 = x_1
        x_1 = x_n
        n = n + 1
        n_array.append(n)
        true_error = abs(true_root - x_0)
        true_error_array.append(true_error)
        print('iteration:', n, " - error:", error)
        continue
    print('error:', error)
    print('approximate root:', x_n)
    converge_rate = (np.log(abs((true_error_array[n-1] -
                        true_error_array[n-2])/
                    (true_error_array[n-2]-
                        true_error_array[n-3]))))/
                    (np.log(abs((true_error_array[n-2]-
                        true_error_array[n-3])/
                    (true_error_array[n-3]-
                        true_error_array[n-4]))))
    print("order of convergence:", converge_rate)
    return x_n, n_array, true_error_array


# Plot the function and the root
```

```
root = find_root_secant(-1.2,1,0.000001)
root_0 = 0
x = np.arange(-2,2,0.1)
y = (x)**2-np.exp(2-x**2)
plt.title("Secant Method")
plt.xlabel("x axis")
plt.ylabel("y axis")
plt.plot(x, y)
plt.plot(root[0], root_0, 'rx')
plt.grid(True)
plt.show()

# Plot of error vs iterations
plt.title("Secant Method Error")
plt.xlabel("n iterations")
plt.ylabel("true error")
plt.plot(root[1], root[2],'ro')
plt.grid(True)
plt.show()
```

4.2 Output



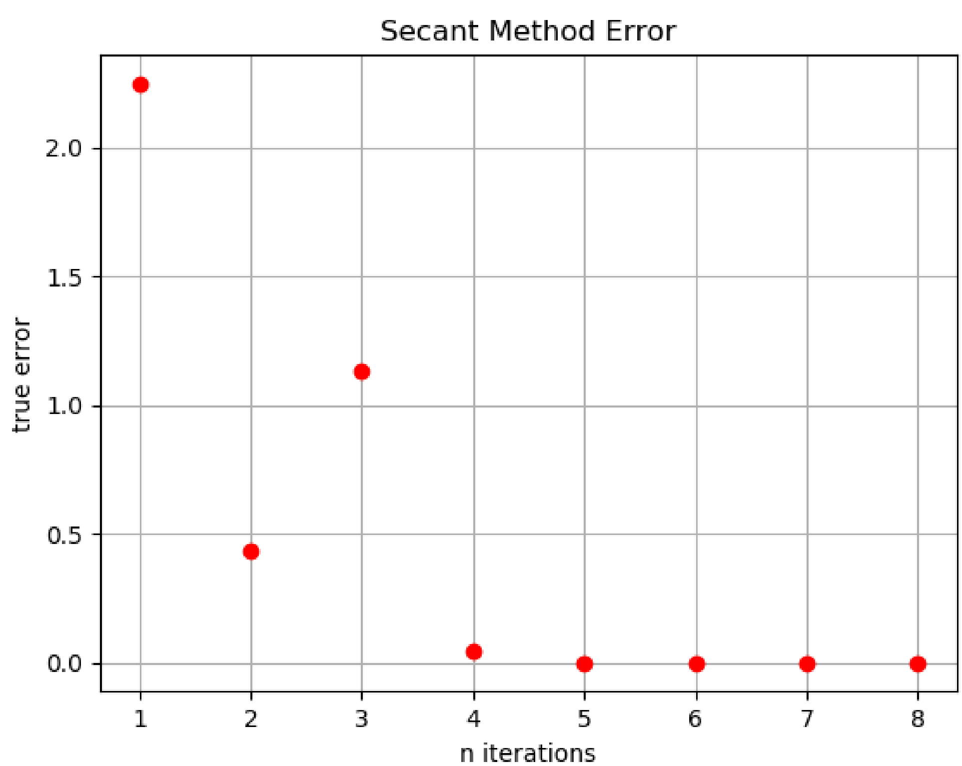
4.3 Accuracy and Order of Convergence

The last approximation for the root was with in 8.302587×10^{-9} of the true value. The approximated value for the root of the function is -1.247856. The order of convergence is 1.930753.

Anaconda Prompt

```
(base) C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610>python Secant_Method.py
iteration: 1  - error: 0.48556050814427665
iteration: 2  - error: 1.117820722186887
iteration: 3  - error: 0.39016790166402604
iteration: 4  - error: 1.1311915574356324
iteration: 5  - error: 0.04755619626864749
iteration: 6  - error: 0.0011558697265390272
iteration: 7  - error: 1.9991381849493450e-05
iteration: 8  - error: 8.302587284347851e-09
error: 8.302587284347851e-09
approximate root: -1.247856401593393
order of convergence: 1.9307527410602716
```

The distance of each approximation to the true value with each iteration n:



5 Fixed Point Method

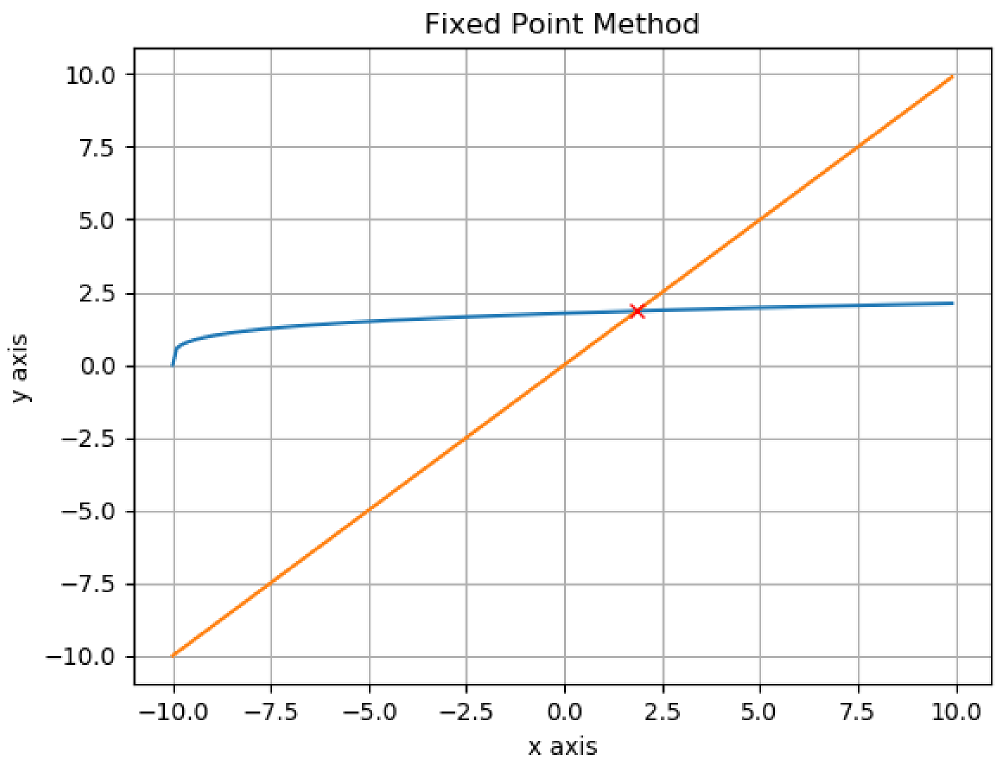
The Fixed Point Method approximates the value at with the function $g(x) = (x+10)^{\frac{1}{4}}$ intersects $f(x) = x$.

5.1 Code

```
import numpy as np
from matplotlib import pyplot as plt

my_function = lambda x: (x + 10)**(1/4)
true_val = 1.855584528640937863760250564887986956416868
             9613121880283886119879295797258305894223388
             7913894052622171808602448788374623181292369
             9694373052175808992515659707499952288816271
             1608283807536494470806341316485129583883331
             3556622920833781356730295328749934909901069
             0823009

def fixed_point(x_0, epsilon):
    error = 1
    n = 0
    n_array = []
    true_error_array = []
    while (error > epsilon):
        x_n = my_function(x_0)
        error = abs(x_0 - x_n)
        x_0 = x_n
        n = n + 1
        n_array.append(n)
        true_error = abs(true_val - x_0)
        true_error_array.append(true_error)
        print('iteration:', n, " - error:", error)
        continue
    print('error:', error)
    print('approximate fixed point:', x_n)
    converge_rate = (np.log(abs((true_error_array[n-1] -
                        true_error_array[n-2])/
                    (true_error_array[n-2]-
                        true_error_array[n-3]))))/
                    (np.log(abs((true_error_array[n-2]-
                        true_error_array[n-3])/
                    (true_error_array[n-3]-
                        true_error_array[n-4]))))
```

```
    print("order of convergence:", converge_rate)
    return x_n, n_array, true_error_array


# plot function and y = x
f_point = fixed_point(1,0.000001)
x = np.arange(-10,10,0.1)
y = (x + 10)**(1/4)
x_line = np.arange(-10,10,0.1)
y_line = x
plt.title("Fixed Point Method")
plt.xlabel("x axis")
plt.ylabel("y axis")
plt.plot(x, y)
plt.plot(x_line, y_line)
plt.plot(f_point[0], f_point[0], 'rx')
plt.grid(True)
plt.show()

# Plot of error vs iterations
plt.title("Fixed Point Method Error")
plt.xlabel("n iterations")
plt.ylabel("true error")
plt.plot(f_point[1], f_point[2],'ro')
plt.grid(True)
plt.show()
```

5.2 Output



5.3 Accuracy and Order of Convergence

The last approximation for the fixed point was with in 7.762710×10^{-8} of the true value. The approximated value for the fixed point is 1.855585. The order of convergence is 1.000000.

Anaconda Prompt

```
(base) C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610>python Fixed_Point_Method.py
iteration: 1  - error: 0.8211602868378718
iteration: 2  - error: 0.0330757894829774
iteration: 3  - error: 0.001295686596320511
iteration: 4  - error: 5.070105492266386e-05
iteration: 5  - error: 1.983880593137144e-06
iteration: 6  - error: 7.762709586245364e-08
error: 7.762709586245364e-08
approximate fixed point: 1.8555845254797814
order of convergence: 1.0000005140567554

(base) C:\Users\saeth\Documents\Courses\2019\Fall\MATH 5610>
```