Homework 5

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1. Beck Exercise 6.9. Let $a, b \in \mathbb{R}^n$, $a \neq b$. For what values of μ is the set

$$S_\mu = \{x \in \mathbb{R}^n \colon \|x - a\|_2 \leq \mu \|x - b\|_2\}$$

convex?

Proof. Consider the point $z = \lambda x + (1 - \lambda) y$. For S_μ to be convex we need

$$\|z - a\|_2 \leq \mu \|z - b\|_2$$

Rewrite the left hand side

$$\begin{aligned} \|z - a\|_2 &= \|(\lambda x + (1 - \lambda) y) - b\|_2 \\ &\leq \lambda \|x - y\|_2 + \|y - a\|_2 \\ &\leq \lambda \|x - y\|_2 + \mu \|y - b\|_2 \end{aligned}$$

Similarly, rewrite the right hand side

$$\begin{aligned} \mu \|z - b\|_2 &= \mu \|(\lambda x + (1 - \lambda) y) - b\|_2 \\ &\leq \mu \left(\lambda \|x - y\|_2 + \|y - b\|_2 \right) \end{aligned}$$

Putting these together now we have

$$\lambda \|x - y\|_2 + \mu \|y - b\|_2 \leq \mu \left(\lambda \|x - y\|_2 + \|y - b\|_2 \right)$$

Solving for μ

$$\lambda \|x - y\|_2 \leq \mu \lambda \|x - y\|_2$$

$$\implies \frac{\lambda \|x - y\|_2}{\lambda \|x - y\|_2} \leq \mu$$

Therefore S_μ is convex if $\mu \geq 1$ $\square$

2. For $a \in \mathbb{R}^n$, what is the distance between the two parallel hyperplanes

$$\{x \in \mathbb{R}^n : a^t x = b_1\} \qquad \text{and} \qquad \{x \in \mathbb{R}^n : a^t x = b_2\}?$$

Let L be a line passing through a point x_1 that is on the hyperplane

$$H_1 = \{x \in \mathbb{R}^n : a^t x = b_1\}$$

and is perpendicular to H_1. The equation of this line is given by

$$L = x_1 + at \quad \forall t \in \mathbb{R}$$

The distance between the hyperplanes is the same as the distance between x_1 on H_1 and a point x_2 where the line L intersects $H_2 = \{x \in \mathbb{R}^n : a^t x = b_2\}$. To find this point x_2 we solve for t in the following

$$a^t(x_1 + at) = b_2 \implies t = \frac{b_2 - b_1}{a^t a}$$

Therefore

$$x_2 = x_1 + a\left(\frac{b_2 - b_1}{a^t a}\right)$$

The distance between x_1 and x_2 and therefore between the hyperplanes is

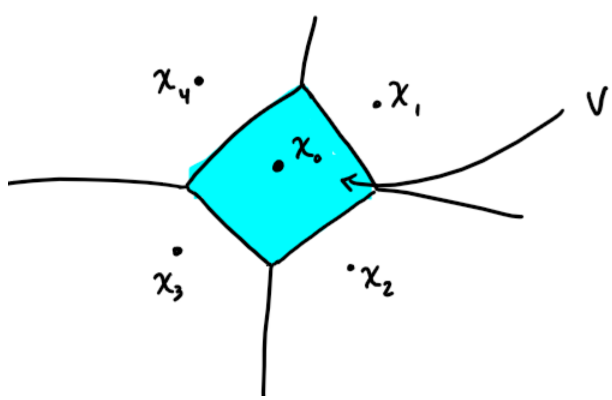
$$\begin{aligned} \|x_1 - x_2\|_2 &= \|x_1 - \left(x_1 + a\left(\frac{b_2 - b_1}{a^t a}\right)\right)\|_2 \\ &= \frac{|b_2 - b_1|}{\|a\|_2} \end{aligned}$$

3. (Voronoi sets and polyhedral decomposition) Let $x_0, \ldots, x_K \in \mathbb{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq \|x - x_i\|_2,\ i = 1, \ldots, K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, \ldots, x_K$.

(a) Illustrate a Voronoi region with a sketch.



(b) Show that V is a polyhedron. Express V in the form $V = \{x \colon Ax \leq b\}$.

$$\|x - x_0\|_2 \leq \|x - x_i\|_2$$

$$\|x - x_0\|_2^2 \leq \|x - x_i\|_2^2$$

$$(x - x_0)^t(x - x_0) \leq (x - x_i)^t(x - x_i)$$

$$x^t x - 2x_0^t x + x_0^t x_0 \leq x^t x - 2x_i^t x + x_i^t x_i$$

$$2(x_i - x_0)^t x \leq x_i^t x_i - x_0^t x_0$$

This last equality is of the form $Ax \preceq b$ where

$$A = 2\begin{bmatrix} x_1 - x_0 \\ \cdots \\ x_K - x_0 \end{bmatrix}, b = \begin{bmatrix} x_1^t x_1 - x_0^t x_0 \\ \cdots \\ x_K^t x_K - x_0^t x_0 \end{bmatrix}$$

(c) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \ldots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \ldots, x_K$.

Choose a point $x_0 \in P$ and reflect this point across each hyperplane

$$H_i = \{x | a_i^t x = b_i\}$$

Each of the points x_i are written as

$$x_i = x_0 + m a_i$$

where m is chosen so that the distance from x_0 to H_i is the same as the distance from x_i to H_i. Using this we can solve for m

$$\begin{aligned} b_i - a_i^t x_0 &= a_i^t x_i - b_i \\ &= a_i^t(x_0 + m a_i) - b_i \\ &= a_i^t x_0 + m a_i^t a_i - b_i \\ &\cdots \\ \implies m &= \frac{2}{\|a_i\|_2^2}(b_i - a_i^t x_0) \end{aligned}$$

Therefore the polyhedron P is the Voronoi region around x_0 with respect to the points

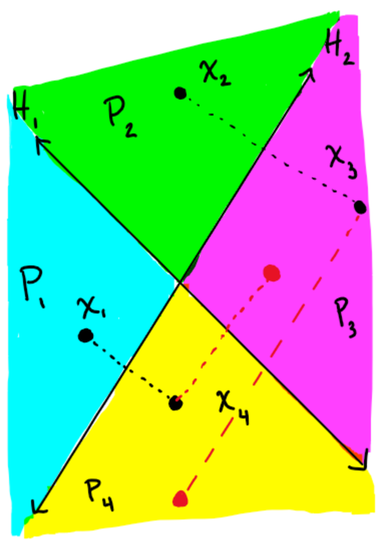
$$x_i = x_0 + \frac{2}{\|a_i\|_2^2}(b_i - a_i^t x_0) a_i \quad \text{for} \quad i = 1, \ldots, K$$

(d) We can also consider the sets

$$V_k = \{x \in \mathbb{R}^n \colon \|x - x_k\|_2 \leq \|x - x_i\|_2,\ i \neq k\}.$$

The set V_k consists of points in $\mathbb{R}^n$ for which the closest point in the set $\{x_0, \ldots, x_K\}$ is x_k. The sets $V_0, \ldots V_K$ give a polyhedral decomposition of $\mathbb{R}^n$. More precisely, the sets V_k are polyhedra, $\cup_{k=0}^{K} V_k = \mathbb{R}^n$, and $\mathrm{int}(V_i) \cap \mathrm{int}(V_j) = \emptyset$ for all $i \neq j$, *i.e.*, V_i and V_j intersect at most along a boundary. Suppose that $P_1, \ldots, P_m$ are polyhedra such that $\cup_{i=1}^{m} P_i = \mathbb{R}^n$, and $\mathrm{int}(P_i) \cap \mathrm{int}(P_j) = \emptyset$ for all $i \neq j$. Can this polyhedral decomposition of $\mathbb{R}^n$ be described as the Voronoi regions generated by an appropriate set of points?

Consider an example in $\mathbb{R}^2$. If we have two hyperplanes H_1 and H_2 we can arbitrarily pick a point x_1 in P_1 and x_2 in P_2. The position of the points x_3 and x_4 is then determined by reflecting x_1 and x_2 across H_2, but as can be seen in the diagram below x_1 and x_2 can be chosen so that the distance from x_4 to H_1 is not equal to the distance from x_3 to H_1. Therefore this polyhedral decomposition of $\mathbb{R}^n$ cannot be described as the Voronoi regions generated by a set of points.



4. Beck Exercise 6.10. Let $C \subset \mathbb{R}^n$ be a nonempty convex set. For each $x \in C$, define the *normal cone* of C at x by

$$N_C(x) = \{w \in \mathbb{R}^n \colon \langle w, y - x\rangle \leq 0 \text{ for all } y \in C\}$$

and define $N_C(x) = \varnothing$ when $x \notin C$. Show that $N_C(x)$ is a closed convex cone.

Proof. First we show that $N_C(x)$ is a convex cone. Consider the points $w_1, w_2 \in N_C(x)$.

$$\langle w_1 + w_2, y - x\rangle = \langle w_1, y - x\rangle + \langle w_2, y - x\rangle \leq 0$$

Next consider $\lambda \geq 0$

$$\langle \lambda w, y - x \rangle = \lambda \langle w, y - x \rangle \leq 0$$

Therefore $N_C(x)$ is a convex cone. $N_C(x)$ can be rewritten as

$$\cap_{y \in C} \{ w \in \mathbb{R}^n \colon \langle w, y - x \rangle \leq 0 \}$$

The intersection of a collection of closed sets is also closed. Hence $N_C(x)$ is a closed convex cone. □

5. Beck Exercise 7.4. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Show that for any $\varepsilon > 0$, the function

$$g_\varepsilon(x) = f(x) + \varepsilon \|x\|^2$$

is coercive.

Proof. Since f is a continuously differentiable convex function we know that

$$f(x) + \nabla f(x)^t (y - x) \leq f(y) \quad \forall x, y \in \mathbb{R}$$

$$\implies f(x) \leq f(y) - \nabla f(x)^t (y - x)$$
$$\implies f(x) + \varepsilon \|x\|^2 \leq f(y) - \nabla f(x)^t (y - x) + \varepsilon \|x\|^2$$
$$\implies g(x) \leq f(y) - \nabla f(x)^t (y - x) + \varepsilon \|x\|^2$$

Clearly as $\|x\| \to \infty$ we have $g(x) \to \infty$. Therefore $g(x)$ is coercive.

□