Homework 2

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1. Let $a \in \mathbb{R}^n$ be a nonzero vector. Show that the maximum of $a^t x$ over $B[0,1] = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ is attained at $x = \frac{1}{||a||}a$ and that the maximal value is $||a||$.

   Proof. First note that $a^t x = \langle a, x \rangle$. By the Cauchy-Schwarz inequality we have $\langle a, x \rangle \leq ||a|| \cdot ||x||$. Since $||x|| \leq 1$ this implies that $\langle a, x \rangle \leq ||a||$. Therefore the maximal value of $a^t x$ over $B[0,1]$ is bounded above by $||a||$. It is easy to see that this upper bound is attained at $x = \frac{1}{||a||}a$ since $||a|| \cdot ||x|| = ||a|| \cdot ||\frac{1}{||a||}a|| = ||a||$.

   $\square$

2. Let $B \in \mathbb{R}^{n \times k}$ and let $A = BB^t$.

   (a) Prove that A is positive semidefinite.

   Proof. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite if $x^t A x \geq 0$ for all $x \in \mathbb{R}^n$. Now consider $A = BB^t$

   $$x^t B B^t x = (B^t x)^t (B^t x) = ||B^t x||^2 \geq 0$$

   $\square$

   (b) Prove that A is positive definite if and only if B has a full row rank.

   Proof. For A to be positive definite it must be that $||B^t x||^2 > 0$ for all real nonzero vectors x. Suppose B has full row rank and that $||B^t x||^2 = 0$ where $x \neq 0$. The matrix vector product inside the norm is a linear combination of the rows of B and since it is assumed that $x \neq 0$ this implies that at least one of the rows of B is linearly dependent on the rest which means that B cannot have full row rank. Contradiction! Therefore if B has full row rank it must be that $||B^t x||^2 > 0$, i.e. A is positive definite.

   Going the other way assume that $||B^t x||^2 > 0$ (A is positive definite) and that B doesn't have full row rank. This means that the rows of B are linearly dependent and that $||B^t x||^2 = 0$ for some nonzero x. This implies that A is not positive definite, a contradiction. Therefore if A is positive definite then B must have full row rank. $\square$

3. Let A^α be the $n \times n$ matrix $(n > 1)$ defined by

$$A^\alpha_{i,j} = \begin{cases} \alpha, & i = j \\ 1, & i \neq j \end{cases}$$

Show that A^α is positive semidefinite if and only if $\alpha \geq 1$.

Proof. Rewrite A^α as $ee^t + (\alpha - 1)I$ where e is a vector comprised of all 1's and I is the $n \times n$ identity matrix. Then

$$x^t A^\alpha x = x^t(ee^t + (\alpha - 1)I)x = (e^t x)^2 + (\alpha - 1)||x||^2$$

For A^α to be positive semidefinite this must hold for all x. Consider what happens when $x = [1, -1, 0, \cdots, 0]^t$. The first term in this final expression would vanish leaving us with $(\alpha - 1)||x||^2$ which is non-negative as long as $\alpha \geq 1$.

Going the other direction assume $\alpha \geq 1$ then $(e^t x)^2 + (\alpha - 1)||x||^2$ is a positive number plus a positive number, therefore A^α is positive semidefinite. $\square$

4. For each of the following functions, find all the stationary points and classify them according to whether they are saddle points, strict/nonstrict local/global minimum/-maximum points:

(a) $f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$

If $\nabla f(x^*) = 0$ then x^* is a stationary point.

$$\nabla f = \begin{bmatrix} f_{x_1} \\ f_{x_2} \end{bmatrix} = \begin{bmatrix} 6x_1x_2 \\ 6x_2^2 - 12x_2 + 3x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From $6x_1x_2 = 0$ either $x_1 = 0$, $x_2 = 0$, or both. This gives $(0, 0)$ as a stationary point. Substituting $x_2 = 0$ into the second equation we get

$$6x_2^2 - 12x_2 + 3x_1^2 = 6(0)^2 - 12(0) + 3x_1^2 = 0 \implies x_2 = 0$$

Substituting $x_1 = 0$ into the second equation we get

$$\begin{aligned} 6x_2^2 - 12x_2 + 3x_1^2 &= 6x_2^2 - 12x_2 + 3(0)^2 \\ &= 6x_2(x_2 - 2) = 0 \\ &\implies x_2 = 2 \end{aligned}$$

which gives a second stationary point $(0, 2)$.

To classify each of these points we consider the matrix

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$= 6 \begin{bmatrix} x_2 & x_1 \\ x_1 & 6x_2 - 2 \end{bmatrix}$$

$$\nabla^2 f(0,0) = 6 \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$y^t \nabla^2 f(0,0) y = -12y_2^2 < 0 \quad \forall y \in \mathbb{R}^2$$

y can be chosen such that $||y|| \neq 0$ but $-12y_2^2 = 0$ which proves that this matrix is *not negative definite*. Therefore this point cannot be a strict local maximum but could be a non-strict local maximum or a saddle point. To decide we look at points in the neighborhood of $(0,0)$. Consider the following

$$f(\alpha, \alpha) = 2\alpha^3 - 6\alpha^2 + 3\alpha^3 = \alpha^2(5\alpha - 6)$$

If α is small enough this value is negative.

$$f(-\alpha, \alpha) = 2\alpha^3 - 6\alpha^2 + 3\alpha^2(-\alpha) = -\alpha^2(\alpha - 6)$$

If α is small enough here the value is positive. Therefore $(0,0)$ is a saddle point.

$$\nabla^2 f(0,2) = 6 \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}$$

The trace of this matrix and it's determinant are both strictly positive therefore $\nabla^2 f(0,2) \succ 0$ which implies this point is a strict local minimum.

(b) $f(x_1, x_2) = x_1^4 + 2x_1^2 x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$
Solving the system of equations $\nabla f(x_1, x_2) = 0$ yields one stationary point $(1,3)$.

$$\nabla f(x_1, x_2) = \begin{bmatrix} 4x_1^3 + 4x_1 x_2 - 8x_1 - 8 \\ 2x_1^2 + 2x_2 - 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies (x_1, x_2) = (1,3)$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$= 2 \begin{bmatrix} 6x_1 + 2x_2 - 4 & 2x_1 \\ 2x_1 & 1 \end{bmatrix}$$

$$\nabla^2 f(1,3) = 2 \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix}$$

The trace and determinant of this matrix are both strictly positive therefore $\nabla^2 f(1,3) \succ 0$ which implies $(1,3)$ is a strict local minimum.