Stochastic Processes and Simulation

Homework 1
Jordan Saethre

September 16, 2020

1. Let A_0, A_1, A_2, A_3 be events of nonzero probability. Show that:

$$P(A_0 \cap A_1 \cap A_2 \cap A_3) = P(A_0)P(A_1|A_0)P(A_2|A_0 \cap A_1)P(A_3|A_0 \cap A_1 \cap A_2)$$

Proof. Recall conditional probability is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \implies P(A \cap B) = P(A|B)P(B)$$

Therefore

$$P(A_3 \cap (A_2 \cap A_1 \cap A_0)) = P(A_3|A_2 \cap A_1 \cap A_0)P(A_2 \cap A_1 \cap A_0)$$

$$P(A_2 \cap (A_1 \cap A_0)) = P(A_2|A_1 \cap A_0)P(A_1 \cap A_0)$$

$$P(A_1 \cap A_0) = P(A_1|A_0)(A_0)$$

Putting it all together you have

$$P(A_0 \cap A_1 \cap A_2 \cap A_3) = P(A_0)P(A_1|A_0)P(A_2|A_0 \cap A_1)P(A_3|A_0 \cap A_1 \cap A_2)$$

□

2. There are n indistinguishable keys in basket and only one opens the door. What is the probability that the door will open with the n-th trial (key) for $1 \leq n \leq N$?

Define A_i to be the event that the i-th key does not open the door for $1 \leq i \leq N$. Assume that keys are tried in sequence and discarded from the basket with each trial.

For i = 1:

$$P(\text{ first key opens the door}) = P(A_1^c) = \frac{1}{N}$$

$$P(\text{ first key does not open the door}) = P(A_1) = \frac{N-1}{N}$$

For i = 2:

$$P(\text{ second key opens the door}) = P(A_2^c) = \frac{1}{N-1}$$

$$P(\text{ second key does not open the door}) = P(A_2) = \frac{N-2}{N-1}$$

Continuing in this fashion for i = n:

$$P(\text{ n-th key opens the door}) = P(A_2^c) = \frac{1}{N-(n-1)}$$

$$P(\text{ n-th key does not open the door}) = P(A_2) = \frac{N-n}{N-(n-1)}$$

Each of the events A_i are independent of each other which means that the probability that all keys up to $n-1$ do not open the door and the n-th one does will be the product of probabilities.

$$P(A_n^c \cap A_{n-1} \cap \cdots \cap A_1) = P(A_n^c)P(A_{n-1}) \cdots P(A_1)$$

$$= \left(\frac{1}{N-(n-1)}\right)\left(\frac{N-(n-1)}{N-((n-1)-1)}\right)\left(\frac{N-(n-2)}{N-((n-2)-1)}\right)\cdots$$

$$\cdots\left(\frac{N-3}{N-2}\right)\left(\frac{N-2}{N-1}\right)\left(\frac{N-1}{N}\right)$$

Almost every term in this last expression cancels except for $\frac{1}{N}$. Therefore the probability that the n-th key opens the door and all keys tried before do not open the door is $\frac{1}{N}$.

3. Three state weather problem: Salt Lake City has only 3 possible states of the weather: Rain (R), Hot (H), and Snow (S). Suppose there are never two consecutive days of rain. A rainy day is just as likely to be followed by a snowy day as a hot day. There is however a half chance that a hot or a snowy day will be followed by another day of the same kind. On the other hand, a hot day that is not followed by another hot day is just as likely to be either rainy or snowy. And snowy that is not followed by another snowy day is 4 times more likely to be rainy than hot.

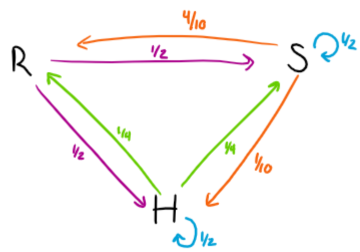
   (a) Suppose X_n is a random variable that represents the state of the weather on day n. Can it be represented as a Markov Chain? Why?

   Yes, this stochastic process can be represented as a Markov Chain since it has the Markov Property. This property essentially says that in order to make predictions about the future we only need to know about the present state, i.e.

$$P(X_n = i_n | X_{n-1} = i_{n-1}, \cdots, X_1 = i_1) = P(X_n = i_n | X_{n-1} = i_{n-1})$$

(b) Write the transition matrix of probabilities for this process.

Directed graph:



Transition Matrix:

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 4/10 & 1/10 & 1/2 \end{bmatrix}$$

4. Suppose the n-step transition probability of a Markov Chain is defined to be

$$P_{ij}^n = P(X_{n+m} = j | X_m = i), \quad \forall\, n \geq 0 \; and \; \forall\, i, j$$

Show that the following Chapman-Kolmogorov Equations hold

$$P_{ij}^{n+m} = \sum_{k \in S} P_{kj}^m P_{ik}^n, \quad \forall\, n, m \geq 0 \; and \; \forall\, i, j$$

Proof. The n-step transition probability is defined above. An m-step transition probability is

$$P_{ij}^m = P(X_m = j | X_0 = i), \quad \forall\, m \geq 0 \; and \; \forall\, i, j$$

Therefore a $(n+m)$-probability is

$$\begin{aligned} P_{ij}^{n+m} &= P(X_{n+m} = j | X_0 = i) \\ &= \sum_{k \in S} P(X_{n+m} = j | X_m = k) P(X_m = k | X_0 = i) \quad \text{by LTP} \\ &= P_{kj}^m P_{ik}^n \quad \text{by definition of n and m step probabilities} \end{aligned}$$

□

5. Recall Example 2 on page 10 of the textbook pertaining to a simple queuing model. We found the $(2,2)^{th}$ element of the transition matrix for the 3 state queuing system to be $P(1,1) = 1 - q(1-p) - p(1-q)$. Show that this is equivalent to the sum of the probabilities of all possible ways in which we could have $X_{n+1} = 1 | X_n = 1$.

In this 3 state queuing problem it is assumed that only one call can come in at a time and only one call can end at a time. This leaves us with only two possibilities for going from one caller at time n to one call at time $n+1$:

(a) There is one call that doesn't end and no new calls come in: $(1-q)(1-p)$

(b) One call is completed and one new call comes in: qp

Therefore the probability that the system goes from one call to one call is the sum of the probabilities of these two situations:

$$\begin{aligned}(1-q)(1-p)+qp &= 1-p-q+pq+pq \\ &= 1-(p-pq)-(q-pq) \\ &= 1-p(1-q)-q(1-p)\end{aligned}$$

6. Consider the following stochastic matrix:

$$\mathcal{P} = \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix}$$

(a) Find the determinant of $\mathcal{P}$.

$$\det(\mathcal{P}) = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{5}{12}$$

(b) Find the eigenvalues of $\mathcal{P}$ by solving $\det(\mathcal{P}-\lambda\mathbb{I}) = 0$.

$$\det(\mathcal{P}-\lambda\mathbb{I}) = \begin{bmatrix} 3/4-\lambda & 1/4 \\ 1/3 & 2/3-\lambda \end{bmatrix} = 12\lambda^2 - 17\lambda + 5$$

$$\implies \lambda = 1, \frac{5}{12}$$

(c) Find right and left eigenvectors associated with the eigenvalues found in part (b).

To find the right eigenvectors plug the eigenvalues found in part (b) into

$$(\mathcal{P}-\lambda\mathbb{I})x = 0$$

and solve for x.

When $\lambda = 1$ $x = [1,1]^t$ and when $\lambda = 5/12$ $x = [-3/4, 1]^t$. These two column vectors form the matrix Q such that $Q\mathcal{P}Q^{-1} = D$. The matrix D is a diagonal matrix where it's diagonal entries are the eigen values of $\mathcal{P}$. The rows of Q^{-1} are the left eigenvectors of $\mathcal{P}$.

$$Q^{-1} = \frac{1}{(1)(1)-(-1)(3/4)} \begin{bmatrix} 1 & 3/4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4/7 & 3/7 \\ -4/7 & 4/7 \end{bmatrix}$$

7. Recall conditions (1.9) and (1.10) that were discussed in class.

   (1.9) The left eigenvector can be chosen to have all nonnegative entries.

   (1.10) The eigenvalue 1 is simple and all other $|\lambda_i| < 1$.

   Diagonalize the following matrices and identify the conditions that fail, thus leading to lack of convergence. Compute with any software of your choice.

   (a) Stochastic matrix with periodicity:

   $$\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   Using the numpy.linalg.eig() and numpy.linalg.inv() in python we can find:

   $$Q = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix}$$

   Condition (1.10) is violated in this case since $\lambda_2 = -1$.

   (b) $\mathcal{P}$ from example 1 (p.18) pertaining to a simple random walk with reflecting boundaries.

   $Q =$
```
array([[ 0.447,  0.577,  0.577,  0.447, -0.577],
       [-0.447, -0.408, -0.   ,  0.447, -0.408],
       [ 0.447,  0.   , -0.577,  0.447, -0.   ],
       [-0.447,  0.408, -0.   ,  0.447,  0.408],
       [ 0.447, -0.577,  0.577,  0.447,  0.577]])
```

   $D =$ *(Note: The following is a vector of the diagonal of D.)*
```
array([-1.   , -0.707, -0.   ,  1.   ,  0.707])
```

   $Q^{-1} =$
```
array([[ 0.28 , -0.559,  0.559, -0.559,  0.28 ],
       [ 0.433, -0.612,  0.   ,  0.612, -0.433],
       [ 0.433, -0.   , -0.866, -0.   ,  0.433],
       [ 0.28 ,  0.559,  0.559,  0.559,  0.28 ],
       [-0.433, -0.612, -0.   ,  0.612,  0.433]])
```

   Condition (1.10) is violated in this case since $\lambda_1 = -1$.

   (c) Simple 3 state random walk with absorbing boundaries:

   $$\mathcal{P} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$Q =$
```
array([[0.   , 0.894, 0.   ],
       [1.   , 0.447, 0.447],
       [0.   , 0.   , 0.894]])
```

$D =$
```
array([0., 1., 1.])
```

$Q^{-1} =$
```
array([[-0.5  ,  1.   , -0.5  ],
       [ 1.118,  0.   ,  0.   ],
       [ 0.   ,  0.   ,  1.118]])
```

Condition (1.10) is violated since the eigenvalue 1 has multiplicity 2.

(d) $\mathcal{P}$ from example 2 (p.18) pertaining to a simple random walk with 5 states and absorbing boundaries.

$Q =$
```
array([[ 0.   ,  0.   ,  0.   ,  0.73 ,  0.   ],
       [ 0.5  ,  0.707, -0.5  ,  0.548,  0.183],
       [-0.707, -0.   , -0.707,  0.365,  0.365],
       [ 0.5  , -0.707, -0.5  ,  0.183,  0.548],
       [ 0.   ,  0.   ,  0.   ,  0.   ,  0.73 ]])
```

$D =$
```
array([-0.707,  0.   ,  0.707,  1.   ,  1.   ])
```

$Q^{-1} =$
```
array([[-0.146,  0.5  , -0.707,  0.5  , -0.146],
       [-0.354,  0.707, -0.   , -0.707,  0.354],
       [ 0.854, -0.5  , -0.707, -0.5  ,  0.854],
       [ 1.369,  0.   ,  0.   ,  0.   ,  0.   ],
       [ 0.   ,  0.   ,  0.   ,  0.   ,  1.369]])
```

Condition (1.10) is violated since the eigenvalue 1 has multiplicity 2.

(e) $\mathcal{P}$ from example 3 (p. 19) pertaining to a non-interacting sub-chains.

$Q =$
```
array([[-0.949, -0.707,  0.   ,  0.   ,  0.   ],
       [ 0.316, -0.707,  0.   ,  0.   ,  0.   ],
       [ 0.   ,  0.   ,  0.623,  0.881,  0.577],
       [ 0.   ,  0.   , -0.713,  0.128,  0.577],
       [ 0.   ,  0.   ,  0.322, -0.455,  0.577]])
```

$D =$

```
array([0.333, 1.   , 0.464, 0.786, 1.   ])
```

$Q^{-1} =$

```
array([[-0.791,  0.791, -0.   , -0.   , -0.   ],
       [-0.354, -1.061, -0.   , -0.   , -0.   ],
       [-0.   , -0.   ,  0.374, -0.857,  0.483],
       [ 0.   ,  0.   ,  0.664,  0.193, -0.857],
       [ 0.   ,  0.   ,  0.315,  0.63 ,  0.787]])
```

Condition (1.10) is violated since the eigenvalue 1 has multiplicity 2.