Homework 3

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1. Recall the example of the chess board Knight that travels along a random path on a 3×3 grid. Write the transition matrix for the path of the Knight, find $\bar{\pi}$ and $E\{T_{ii}^1\}$.

In [1]:
```
import numpy as np
```

In [2]:
```
P = np.array([[0.0,0.0,0.0,0.0,0.5,0.0,0.5,0.0],
              [0.0,0.0,0.0,0.0,0.0,0.5,0.0,0.5],
              [0.0,0.0,0.0,0.5,0.0,0.0,0.5,0.0],
              [0.0,0.0,0.5,0.0,0.0,0.0,0.0,0.5],
              [0.5,0.0,0.0,0.0,0.0,0.5,0.0,0.0],
              [0.0,0.5,0.0,0.0,0.5,0.0,0.0,0.0],
              [0.5,0.0,0.5,0.0,0.0,0.0,0.0,0.0],
              [0.0,0.5,0.0,0.5,0.0,0.0,0.0,0.0]])
```

In [3]:
```
d, v = np.linalg.eig(P)
```

In [4]:
```
np.round(d,3)
```

Out[4]:
```
array([-1.   , -0.707,  0.   ,  1.   ,  0.707, -0.707,  0.707,  0.   ])
```

In [5]:
```
np.round(v,3)
```

Out[5]:
```
array([[ 0.354, -0.5  ,  0.5  ,  0.354, -0.5  ,  0.021, -0.025, -0.133],
       [-0.354, -0.354, -0.   ,  0.354,  0.354, -0.339, -0.335, -0.482],
       [ 0.354,  0.   , -0.5  ,  0.354,  0.   , -0.5  ,  0.499,  0.133],
       [-0.354, -0.354, -0.   ,  0.354,  0.354,  0.368,  0.371,  0.482],
       [-0.354,  0.354,  0.   ,  0.354, -0.354, -0.368, -0.371,  0.482],
       [ 0.354,  0.   , -0.5  ,  0.354, -0.   ,  0.5  , -0.499,  0.133],
       [-0.354,  0.354,  0.   ,  0.354, -0.354,  0.339,  0.335, -0.482],
       [ 0.354,  0.5  ,  0.5  ,  0.354,  0.5  , -0.021,  0.025, -0.133]])
```

In [6]:
```
v_inv = np.linalg.inv(v)
np.round(v_inv,3)
```

Out[6]:
```
array([[ 0.354, -0.354,  0.354, -0.354, -0.354,  0.354, -0.354,  0.354],
       [-0.5  , -0.368, -0.021, -0.339,  0.339,  0.021,  0.368,  0.5  ],
       [ 0.5  , -0.138, -0.5  ,  0.138,  0.138, -0.5  , -0.138,  0.5  ],
       [ 0.354,  0.354,  0.354,  0.354,  0.354,  0.354,  0.354,  0.354],
       [-0.5  ,  0.371, -0.025,  0.336, -0.336,  0.025, -0.371,  0.5  ],
       [ 0.   , -0.354, -0.5  ,  0.354, -0.354,  0.5  ,  0.354,  0.   ],
       [ 0.   , -0.354,  0.501,  0.354, -0.354, -0.501,  0.354, -0.   ],
       [ 0.   , -0.519, -0.   ,  0.519,  0.519, -0.   , -0.519,  0.   ]])
```

In [7]:
```
# left eigen vector with eigen value 1 is 4th row of v_inv
st_dist = v_inv[3,:]
st_dist
```

Out[7]:
```
array([0.35355339, 0.35355339, 0.35355339, 0.35355339, 0.35355339,
       0.35355339, 0.35355339, 0.35355339])
```

In [8]:
```
# probability vector must sum to 1
st_dist = st_dist/st_dist.sum(axis = 0)
st_dist
```

Out[8]:
```
array([0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125])
```

$$E(T_{ii}^1) = \frac{1}{\pi(i)}$$

In [9]:
```
1/st_dist[0]
```

Out[9]:
```
8.0
```

2. Consider the random path for a chess board Rook on a 2 grid. Write the transition matrix for the path of the Rook, find $\bar{\pi}$ and $E\{T_{ii}^1\}$.

In [10]:
```
P = np.array([[0.0,0.5,0.0,0.5],
              [0.5,0.0,0.5,0.0],
              [0.0,0.5,0.0,0.5],
              [0.5,0.0,0.5,0.0]])
```

In [11]:
```
d, v = np.linalg.eig(P)
```

In [12]:
```
np.round(d,3)
```

Out[12]:
```
array([-1.,  0.,  1.,  0.])
```

In [13]:
```
np.round(v,3)
```

Out[13]:
```
array([[ 0.5  , -0.707,  0.5  ,  0.   ],
       [-0.5  , -0.   ,  0.5  , -0.707],
       [ 0.5  ,  0.707,  0.5  ,  0.   ],
       [-0.5  , -0.   ,  0.5  ,  0.707]])
```

In [14]:
```
v_inv = np.linalg.inv(v)
np.round(v_inv,3)
```

Out[14]:
```
array([[ 0.5  , -0.5  ,  0.5  , -0.5  ],
       [-0.707,  0.   ,  0.707,  0.   ],
       [ 0.5  ,  0.5  ,  0.5  ,  0.5  ],
       [-0.   , -0.707, -0.   ,  0.707]])
```

In [15]:
```
# left eigen vector with eigen value 1 is 3th row of v_inv
st_dist = v_inv[2,:]
st_dist
```
Out[15]: `array([0.5, 0.5, 0.5, 0.5])`

In [16]:
```
# probability vector must sum to 1
st_dist = st_dist/st_dist.sum(axis = 0)
st_dist
```
Out[16]: `array([0.25, 0.25, 0.25, 0.25])`

In [17]:
```
1/st_dist[0]
```
Out[17]: `4.000000000000001`

3. Show that the invariant probability $\pi(j) = (\bar{\pi}\mathcal{P})_j$ as given in the example for Urn Model 1 from Week 7.

   Suppose there is an urn with N balls. Each ball is either red or green. At each time step one ball is drawn at random and is either put back into the urn with probability 1/2 or exchanged for a ball of the opposite color. Let X_n be the number of red balls in the urn at time n. X_n forms an irreducible Markov Chain with state space $S = \{0, 1, \ldots, N\}$. If $X_{n-1} = j$ then 4 possible outcomes can happen:

   - Draw a red and replace $\implies X_n = j$
   - Draw a red and exchange $\implies X_n = j - 1$
   - Draw a green and replace $\implies X_n = j$
   - Draw a red and exchange $\implies X_n = j + 1$

   The transition probabilities are:

$$
\begin{aligned}
p(j,j) &= P(X_n = j | X_{n-1} = j) \\
&= P(\text{drew red and replace}) + P(\text{drew green and replace}) \\
&= P(R)P(replace) + P(G)P(replace) \\
&= \left(\frac{j}{N}\right)\left(\frac{1}{2}\right) + \left(\frac{N-j}{N}\right)\left(\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
p(j, j-1) &= P(X_n = j-1 | X_{n-1} = j) \\
&= P(\text{drew red and exchange}) \\
&= P(R)P(exchange) \\
&= \frac{j}{2N}
\end{aligned}
$$

$$
\begin{aligned}
p(j, j+1) &= P(X_n = j+1 | X_{n-1} = j) \\
&= P(\text{drew green and exchange}) \\
&= P(G)P(exchange) \\
&= \frac{N-j}{2N}
\end{aligned}
$$

All other transition probabilities are zero.

Therefore

$$
\mathcal{P} = \begin{bmatrix}
\frac{1}{2} & \frac{N}{2N} & 0 & 0 & \cdots & 0 \\
\frac{1}{2N} & \frac{1}{2} & \frac{N-1}{2N} & 0 & \cdots & 0 \\
0 & \frac{2}{2N} & \frac{1}{2} & \frac{N-2}{2N} & \cdots & 0 \\
0 & 0 & \frac{3}{2N} & \frac{1}{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & & \cdots & & \frac{1}{2} & \frac{1}{2N} \\
0 & & \cdots & & \frac{N}{2N} & \frac{1}{2}
\end{bmatrix}
$$

The invariant probability is given by the binomial distribution:

$$
\pi(j) = \lim_{n \to \infty} P(X_n = j)
$$

$$
= \binom{N}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{N-j}
$$

Now we show that $\pi(j) = (\bar{\pi}\mathcal{P})_j$:

Proof.

$$\begin{aligned}(\bar{\pi}\mathcal{P})_j &= \sum_{k=0}^{N} \pi(k)p(k,j) \\ &= \pi(j-1)p(j-1,j) + \pi(j)p(j,j) + \pi(j+1)p(j+1,j)\end{aligned}$$

$$\pi(j) = \binom{N}{j}\left(\frac{1}{2}\right)^{j}\left(\frac{1}{2}\right)^{N-j} = \binom{N}{j}\left(\frac{1}{2}\right)^{N}$$

$$\pi(j-1) = \binom{N}{j-1}\left(\frac{1}{2}\right)^{j-1}\left(\frac{1}{2}\right)^{N-(j-1)} = \binom{N}{j-1}\left(\frac{1}{2}\right)^{N} = \frac{j}{N-(j-1)}\binom{N}{j}\left(\frac{1}{2}\right)^{N}$$

$$\pi(j+1) = \binom{N}{j+1}\left(\frac{1}{2}\right)^{j+1}\left(\frac{1}{2}\right)^{N-(j+1)} = \binom{N}{j+1}\left(\frac{1}{2}\right)^{N} = \frac{N-j}{j+1}\binom{N}{j}\left(\frac{1}{2}\right)^{N}$$

$$\begin{aligned}\implies (\bar{\pi}\mathcal{P})_j &= \frac{j}{N-(j-1)}\binom{N}{j}\left(\frac{1}{2}\right)^{N}\left(\frac{N-(j-1)}{2N}\right) + \binom{N}{j}\left(\frac{1}{2}\right)^{N}\left(\frac{1}{2}\right) \\ &+ \frac{N-j}{j+1}\binom{N}{j}\left(\frac{1}{2}\right)^{N}\left(\frac{j+1}{2N}\right) \\ &= \binom{N}{j}\left(\frac{1}{2}\right)^{N}\left[\frac{j}{2N} + \frac{1}{2} + \frac{N-j}{2N}\right] \\ &= \binom{N}{j}\left(\frac{1}{2}\right)^{N}\end{aligned}$$

$\square$

4. Find the transition probabilities and construct $\mathcal{P}$ for the example in Urn Model 2 from week 7.

   Suppose there are 2 urns U_1 and U_2. There are N red balls and N green balls and each urn has N balls. At each time step a ball is chosen independently and at random from each urn and swapped into the other urn.

   Let X_n be the number of red balls in U_1 at time n. The distribution of red and green balls is entirely determined from knowing X_n.

If $X_n = j$ then

	U_1	U_2
Reds	j	$N-j$
Greens	$N-j$	j

If $X_{n-1} = j$ then 4 possibilities can happen:

- Swap red from U_1 and red from $U_2 \implies X_n = j$
- Swap red from U_1 and green from $U_2 \implies X_n = j-1$
- Swap green from U_1 and red from $U_2 \implies X_n = j+1$
- Swap green from U_1 and green from $U_2 \implies X_n = j$

These possibilities give the following transition probabilities

$$\begin{aligned}
p(j,j) &= P(X_n = j | X_{n-1} = j) \\
&= P(Swap(R,R)) + P(Swap(G,G)) \\
&= P(R\ from\ U_1)P(R\ from\ U_2) + P(G\ from\ U_1)P(G\ from\ U_2) \\
&= \left(\frac{j}{N}\right)\left(\frac{N-j}{N}\right) + \left(\frac{N-j}{N}\right)\left(\frac{j}{N}\right) \\
&= \frac{2j(N-j)}{N^2}
\end{aligned}$$

$$\begin{aligned}
p(j,j-1) &= P(X_n = j-1 | X_{n-1} = j) \\
&= P(Swap(R,G)) \\
&= P(R\ from\ U_1)P(G\ from\ U_2) \\
&= \left(\frac{j}{N}\right)^2
\end{aligned}$$

$$\begin{aligned}
p(j,j+1) &= P(X_n = j+1 | X_{n-1} = j) \\
&= P(Swap(G,R)) \\
&= P(G\ from\ U_1)P(R\ from\ U_2) \\
&= \left(\frac{N-j}{N}\right)^2
\end{aligned}$$

$$\mathcal{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & & \cdots & 0 \\ \left(\frac{1}{N}\right)^2 & \frac{2(N-1)}{N^2} & \left(\frac{N-1}{N}\right)^2 & 0 & & \cdots & 0 \\ 0 & \left(\frac{2}{N}\right)^2 & \frac{4(N-2)}{N^2} & \left(\frac{N-2}{N}\right)^2 & & \cdots & 0 \\ 0 & 0 & \left(\frac{3}{N}\right)^2 & \frac{6(N-3)}{N^2} & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & & \cdots & & \frac{4(N-2)}{N^2} & \left(\frac{2}{N}\right)^2 & 0 \\ 0 & & \cdots & & \left(\frac{N-1}{N}\right)^2 & \frac{2(N-1)}{N^2} & \left(\frac{1}{N}\right)^2 \\ 0 & & \cdots & & 0 & 1 & 0 \end{bmatrix}$$

5. Refer to the virus mutation problem introduced in class. Show that

$$P_n(0,0) = \frac{1}{N} + \left[1 - \frac{N\alpha}{N-1}\right]^n \left[1 - \frac{1}{N}\right]$$

Proof. In class it was shown that

$$P_n(0,0) = \frac{\alpha}{N-1} + \left[1 - \frac{N\alpha}{N-1}\right] P_{n-1}(0,0)$$

Let $a = \frac{\alpha}{N-1}$ and $b = 1 - \frac{N\alpha}{N-1}$ then we have $P_n(0,0) = a + bP_{n-1}(0,0)$. Now substituting recursively:

$$\begin{aligned} P_1(0,0) &= a + bP_0(0,0) = a + b \\ P_2(0,0) &= a + bP_1(0,0) = a + b(a+b) = a + ab + b^2 \\ P_3(0,0) &= a + bP_2(0,0) = a + ab + ab^2 + b^3 \\ &\vdots \\ P_n(0,0) &= a(1 + b + b^2 + \cdots + b^{n-1}) + b^n \\ &= \sum_{k=0}^{n-1} ab^k + b^n \\ &= a\left(\frac{1-b^n}{1-b}\right) + b^n \end{aligned}$$

Notice that $b = 1 - Na$

$$
\begin{aligned}
P_n(0,0) &= a\left(\frac{1-(1-Na)^n}{1-(1-Na)}\right) + b^n \\
&= a\left(\frac{1-(1-Na)^n}{Na}\right) + b^n \\
&= \frac{1}{N} - \frac{1}{N}b^n + b^n \\
&= \frac{1}{N} + \left[1 - \frac{N\alpha}{N-1}\right]^n \left[1 - \frac{1}{N}\right]
\end{aligned}
$$

□

6. Consider the characteristic polynomial of a linear difference equation: $bu^2 - u + a = 0$. We know that the roots of this equation are given by:

$$u_{1,2} = \frac{1 \pm \sqrt{1-4ab}}{2b}$$

Suppose $1 - 4ab \neq 0$ and $u_1 = x + iy$ and $u_2 = \bar{u_1}$. Show that the solution of form: $f(n) = \alpha_1 u_1^n + \alpha_2 u_2^n$ can be written in polar form as $f_n = r^n[c_1 \cos n\theta + c_1 i \sin n\theta]$.

Proof. Rewrite u_1 and u_2 in polar form:

$$
\begin{aligned}
u_1 = x + iy = r(\cos\theta_1 + i\sin\theta_1) \\
u_2 = x - iy = r(\cos\theta_2 + i\sin\theta_2)
\end{aligned}
$$

Where $r = |\sqrt{x^2+y^2}| = |\sqrt{x^2+(-y)^2}$. Also notice that since u_2 is the complex conjugate of u_1 we have $\theta_2 = -\theta_1$. Now we plug in the polar representations into the solution:

$$
\begin{aligned}
f(n) &= \alpha_1 u_1^n + \alpha_2 u_2^n \\
&= \alpha_1(r(\cos\theta_1 + i\sin\theta_1))^n + \alpha_2(r(\cos\theta_2 + i\sin\theta_2))^n \\
&= \alpha_1(r^n(\cos n\theta_1 + i\sin n\theta_1)) + \alpha_2(r^n(\cos n\theta_2 + i\sin n\theta_2))
\end{aligned}
$$

This last equality is due to DeMoivre's Theorem (the power rule for complex numbers). Now letting $\theta_2 = -\theta_1$ we have:

$$
\begin{aligned}
f(n) &= \alpha_1(r^n(\cos n\theta_1 + i\sin n\theta_1)) + \alpha_2(r^n(\cos(-n\theta_1) + i\sin(-n\theta_1))) \\
&= \alpha_1(r^n(\cos n\theta_1 + i\sin n\theta_1)) + \alpha_2(r^n(\cos(n\theta_1) - i\sin(n\theta_1)))
\end{aligned}
$$

Reorganizing terms we have:

$$f(n) = r^n[(\alpha_1 + \alpha_2)\cos n\theta_1 + (\alpha_1 - \alpha_2)i\sin n\theta_1]$$

Let $\theta_1 = \theta$, $(\alpha_1 + \alpha_2) = c_1$, and $(\alpha_1 - \alpha_2) = c_2$ and we have that

$$f(n) = r^n[c_1\cos n\theta + c_1 i\sin n\theta] = f_n$$

□

7. Suppose we want to generate the following discrete random variables from $U \sim Unif(0,1)$. Draw a graph and indicate $G(u)$ values for 4 different choices of u values in each case.

   (a) $X \sim Bern(p = 0.5)$

   (b) $X \sim Bin(n = 2, p = 0.25)$

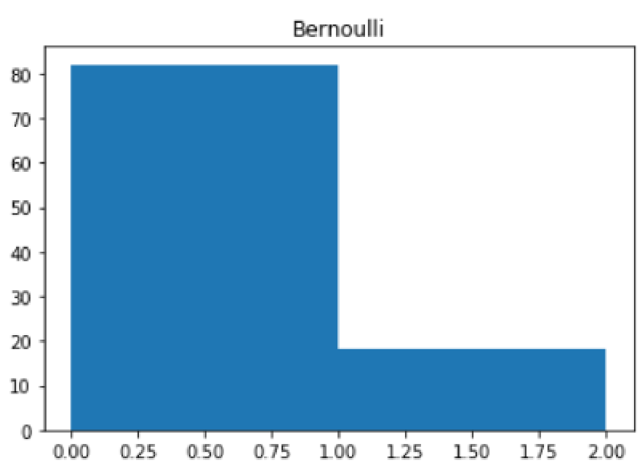
   (c) $X \sim Discrete\ \ Uniform\{1, 2, \ldots, 6\}$

8. Write a program in each of the 3 cases above to generate 100 samples from each distribution. Plot a histogram and compute the sample mean and variance in each case.

In [18]:

```python
p = 1/4
bernoulli = np.array([])
for i in range(100):
    u = random.uniform(0,1)
    if u <= p:
        x = 1
        bernoulli = np.append(x, bernoulli)
    else:
        x = 0
        bernoulli = np.append(x, bernoulli)

print('sample: ',bernoulli)
print('sample mean: ',np.round(bernoulli.mean(),3))
print('sample variance: ',np.round(bernoulli.var(),3))

plt.hist(bernoulli, bins = [0,1,2])
plt.title("Bernoulli")
plt.show()
```

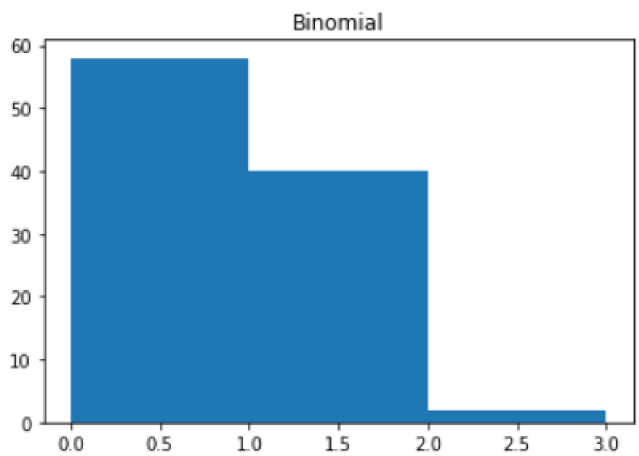
```
sample:  [0. 0. 1. 0. 0. 0. 0. 1. 1. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0.
0.
 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 1. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 1. 0. 0. 0. 1.
 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1.
 0. 0. 0. 0.]
sample mean:  0.18
sample variance:  0.148
```



In [19]:

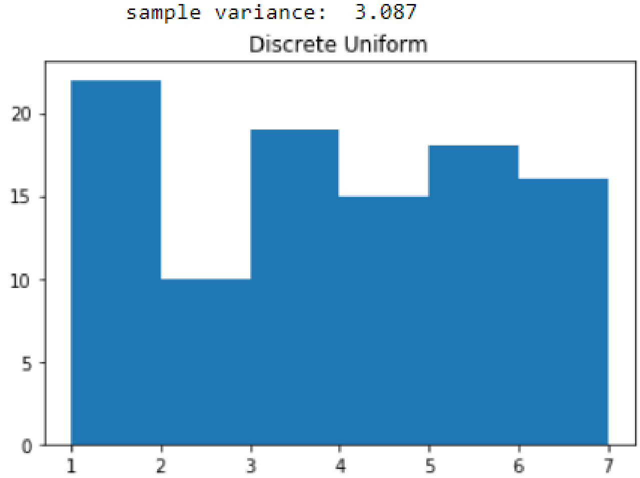
```
p1 = 9/16
p2 = 15/16

binomial = np.array([])

for i in range(100):
    u = random.uniform(0,1)
    if u < p1:
        x = 0
    elif u >= p1 and u < p2:
        x = 1
    else:
        x = 2
    binomial = np.append(x, binomial)

print('sample: ',binomial)
print('sample mean: ',np.round(binomial.mean(),3))
print('sample variance: ',np.round(binomial.var(),3))

plt.hist(binomial, bins = [0,1,2,3])
plt.title("Binomial")
plt.show()
```

```
sample:  [1. 0. 1. 1. 1. 1. 0. 0. 0. 1. 0. 0. 1. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0.
0.
 0. 1. 0. 0. 0. 0. 1. 1. 0. 0. 0. 2. 0. 0. 0. 0. 1. 1. 0. 1. 0. 0. 0. 0.
 1. 0. 0. 1. 1. 1. 1. 1. 2. 0. 1. 0. 1. 0. 1. 0. 1. 0. 1. 0. 0. 0. 0. 0.
 0. 1. 1. 0. 0. 0. 1. 1. 1. 1. 0. 0. 1. 1. 0. 1. 1. 0. 1. 0. 0. 0. 1. 1.
 0. 0. 1. 0.]
sample mean:  0.44
sample variance:  0.286
```



In [20]:
```python
p1 = 1/6
p2 = 2/6
p3 = 3/6
p4 = 4/6
p5 = 5/6

du = np.array([])

for i in range(100):
    u = random.uniform(0,1)
    if u < p1:
        x = 1
    elif u >= p1 and u < p2:
        x = 2
    elif u >= p2 and u < p3:
        x = 3
    elif u >= p3 and u < p4:
        x = 4
    elif u >= p4 and u < p5:
        x = 5
    else:
        x = 6
    du = np.append(x, du)

print('sample: ',du)
print('sample mean: ',np.round(du.mean(),3))
print('sample variance: ',np.round(du.var(),3))

plt.hist(du, bins = [1,2,3,4,5,6,7])
plt.title("Discrete Uniform")
plt.show()
```

```
sample:  [3. 5. 3. 1. 6. 4. 3. 1. 2. 4. 2. 4. 5. 3. 1. 4. 2. 5. 6. 5. 4. 6. 4.
5.
 4. 4. 3. 3. 5. 6. 6. 1. 4. 2. 1. 6. 3. 1. 5. 6. 1. 3. 6. 1. 6. 5. 3. 4.
 6. 2. 4. 3. 1. 1. 2. 1. 3. 3. 1. 6. 5. 6. 5. 1. 4. 5. 3. 4. 1. 5. 1. 2.
 1. 3. 5. 1. 5. 6. 3. 3. 3. 4. 1. 2. 5. 3. 1. 6. 3. 4. 2. 2. 6. 5. 1. 5.
 1. 1. 6. 5.]
sample mean:  3.45
sample variance:  3.087
```



9. Derive the expression to generate samples from Exponential ($\lambda = 2$) using a random generator that generates $U \sim Unif(0, 1)$. Write a program to generate samples from Exponential ($\lambda = 2$).

   To generate samples from an Exponential ($\lambda = 2$) we must find the inverse of the CDF:

$$F(x) = 1 - e^{-\lambda x}$$

$$F^{-1}(x) = \frac{-1}{\lambda}\ln(1 - x) = G(u)$$

In [21]:

```
lmda = 2
F_inverse = lambda x: -1/lmda*np.log(1-x)

exponential = np.array([])
for i in range(100):
    u = u = random.uniform(0,1)
    exponential = np.append(F_inverse(u), exponential)

print('sample: ',exponential)
print('sample mean: ',np.round(exponential.mean(),3))
print('sample variance: ',np.round(exponential.var(),3))
```

```
sample:  [0.31740502 0.75388906 0.40751261 0.09071278 0.52141162 0.42528648
 0.06749723 0.13999532 0.76798577 0.03956747 0.12084822 0.59811729
 0.13312581 0.74664999 0.02980341 0.22670248 0.42676389 0.51441168
 0.67807418 0.26468322 0.56185871 0.08395378 0.48920657 0.25820722
 2.00450899 0.16512961 1.22069364 0.2568203  0.1673751  1.61085331
 0.93161104 0.67087402 0.35720382 0.55882674 0.25771585 0.22813783
 0.18322908 0.71037208 0.0277622  0.00489348 0.06656656 0.17569063
 0.23119883 0.20946803 0.91240526 1.16423077 0.10257894 0.10379204
 0.13164676 0.49098252 0.76841227 0.20061104 1.13891501 0.20496555
 0.09618982 0.08459706 0.34428755 0.46554039 0.63635103 0.87418541
 0.26163092 0.43802517 0.29883206 0.88001898 1.05803415 0.09846601
 0.98452529 0.4925237  0.36810655 0.25336819 0.02517179 1.22397707
 0.11280182 2.01614472 0.50350532 0.38620033 0.20755442 1.40726979
 0.5983985  0.08586437 0.24514651 0.17927602 0.07898615 0.77386041
 0.21962174 0.19662905 0.10544915 0.91889274 0.20624855 0.05687981
 0.25005272 0.34527093 0.11389396 1.51428712 0.09798934 0.0242369
 0.73112286 0.17364737 1.34168751 0.26190027]
sample mean:  0.46
sample variance:  0.188
```

10. Consider a simple random walk in $\mathbb{Z}^2$. Derive an expression for $\sum_{n=0}^{\infty} P_{2n}(0, 0)$.

    Let the number of steps along each dimension be given by i and j. The random walker must take an equal number of steps in each of the 2 dimensions in order to return to the initial location, hence $i + j = n$. The number of steps taken can be represented by a Multinomial distribution:

$$\begin{bmatrix} I \\ J \\ I \\ J \end{bmatrix} \sim MN(2n, p = 1/4)$$

Therefore $P_{2n}(0,0)$ is the sum over all probabilities associated with each (i,j) combination such that $i+j=n$.

$$\begin{aligned}
P_{2n}(0,0) &= \sum_{i,j\geq 0,\quad i+j=n} \binom{2n}{i,j,i,j}\left(\frac{1}{4}\right)^{2n} \\
&= \sum_{i,j\geq 0,\quad i+j=n} \frac{(2n)!}{(i!j!)^2}\left(\frac{1}{4}\right)^{2n} \\
&= \sum_{i,j\geq 0,\quad i+j=n} \frac{(2n)!}{(n!n!)}\frac{n!n!}{(i!j!)^2}\left(\frac{1}{2}\right)^{2n}\left(\frac{1}{2}\right)^{2n} \\
&= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{i,j\geq 0,\quad i+j=n} \binom{n}{i,j}^2\left(\frac{1}{2}\right)^{2n}
\end{aligned}$$

Now consider what happens when summed over all n. There are two cases: $n=2m$ or $n=2m-1$, but it suffices to just show the case when $n=2m$ since this implies $i=j=m$ and is where the probability mass function is maximized. Let $n=2m$ then

$$\binom{n}{i,j} \leq \binom{2m}{m,m}$$

Then we have

$$\begin{aligned}
P_{2n}(0,0) &= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{i,j\geq 0,\quad i+j=n} \binom{n}{i,j}^2\left(\frac{1}{2}\right)^{2n} \\
&\leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{i,j\geq 0,\quad i+j=n} \binom{n}{i,j}\binom{2m}{m,m}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{n} \\
&= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\binom{2m}{m,m}\left(\frac{1}{2}\right)^{n} \sum_{i,j\geq 0,\quad i+j=n} \binom{n}{i,j}\left(\frac{1}{2}\right)^{n}
\end{aligned}$$

Notice that $\sum_{i,j\geq 0,\quad i+j=n}\binom{n}{i,j}\left(\frac{1}{2}\right)^n$ is the probability mass function of a Multinomial distribution with $p=1/2$ and since it is being summed over all n it is equal to just 1. Therefore

$$\sum_{n=0}^{\infty} P_{2n}(0,0) = \sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\binom{2m}{m,m}\left(\frac{1}{2}\right)^{n}$$