# Spectral quasi-monotonicity under renormalization

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In [2], the authors obtain that when $\mu$ is a log-concave probability distribution in $\mathbb{R}^d$, then the Poincaré constant $C_P(\mu * \gamma_s)$ is nondecreasing in $s$, where $\gamma_s$ denotes the Gaussian distribution with zero mean and covariance $sI_d$. Here, using the renormalization semigroup method of [1], we extend this result in case where $\mu$ is not log-concave, but verifies a generalized curvature assumption.

Let us briefly present the framework set up in [1, Section 2.1]. Let $(C_t)_{t\geq 0}$ be a family of semidefinite matrices on $\mathbb{R}^d$. We assume that the $C_t$ matrices increase continuously as quadratic forms from $C_0 = 0$ to a matrix $C_\infty$. We also assume the family to be twice differentiable with respect to the parameter $t$. We denote by $\gamma_{C_t}$ the (possibly degenerate) Gaussian measure with covariance $C_t$ and mean zero.

Let

$$\nu_0 = \frac{1}{Z} e^{-<x, C_\infty x> - V_0(x)} dx$$

be a probability distribution on $\mathbb{R}^d$, where $Z$ is a normalization constant and $V_0 : \mathbb{R}^d \to \mathbb{R}$ is a smooth function. The part $e^{-<x, C_\infty x>}$ represents the log-concave part of the measure, while $e^{-V_0}$ represents the non log-concave part of $\nu_0$.

We define

$$\begin{aligned} V_t &:= -\log\left(\gamma_{C_t} * e^{-V_0}\right) \\ P_{s,t} f &:= e^{V_t}\, \gamma_{C_t - C_s} * f e^{-V_s} \\ \nu_t &:= e^{V_\infty}(0)\,(\gamma_{C_\infty - C_s} * e^{-V_t})(x)\, dx = e^{-<x,\,(C_\infty - C_t)x> - V_t(x) + V_\infty(0)} dx \end{aligned}$$

where $*$ denotes the convolution.

As in [1, Section 2.1], we assume that $\mathbb{E}_{\nu_t} g\left(P_{0,t} f\right)$ is continuous in $t$, so that

$$\mathbb{E}_{\nu_t} g\left(P_{0,t} f\right) \underset{t\to\infty}{\to} \mathbb{E}_{\nu_0} g(f).$$

We have that $V_t$ satisfies the Polchinski equation (see [1, Proposition 2.1])

$$\partial_t V_t = \frac{1}{2}\left(\frac{1}{2}\Delta_{C_t'} + L_t\right) V_t,$$

where

$$L_t := \frac{1}{2}\Delta_{C_t'} - < \nabla V_t, \nabla >_{C_t'},$$

and the index $C_t'$ in the Laplacian or the dot product denotes that these operations are computed with respect to the metric tensor $C_t'$ on $\mathbb{R}^d$, i.e.

$$< U, V >_{C_t'} := \sum_{i,j} (C_t')_{i,j} U_i V_j \quad \text{and} \quad \Delta_{C_t'} f := \sum_{i,j} (C_t')_{i,j} \frac{\partial^2 f}{\partial x_{i,j}}.$$

We make the following generalized curvature assumption (see [1, Proposition 2.1]).

**Assumption 1.** *For all $t \geq 0$, there exists $\lambda'_t \in \mathbb{R}$ (possibly negative) such that*

$$\forall x \in \mathbb{R}^d, \quad C'_t \nabla^2 V_t C'_t \geq C''_t + \lambda'_t C'_t,$$

*in the sense of quadratic forms. The function $t \mapsto \lambda'_t$ is assumed to be locally integrable.*

When $V_0 = 0$, then $V_t = 0$, and one takes $C_t = C_\infty^{-1} - C_\infty^{-1} e^{-tC_\infty}$. Then $C''_t = -C_\infty C'_t$ is negative semidefinite, and then Assumption 1 is satisfied with $\lambda'_t = 0$.

When $\nabla^2 V_0 \geq 0$, then $\nabla^2 V_t \geq 0$ for all $t \geq 0$ (see [1, Example 1.3]). Hence by taking $C_t = C_\infty^{-1} - C_\infty^{-1} e^{-tC_\infty}$ as in the case $V_0 = 0$, Assumption 1 is satisfied again with $\lambda'_t = 0$.

In all the sequel, we will denote $\lambda_t := \int_0^t \lambda'_s ds$.

Using the same approach as in [2, Section 2], we obtain the following result.

**Theorem 1.** *Under all the above conditions, we have that for all $0 \leq s \leq t$,*

$$C_P(\nu_s) \leq e^{\lambda_s - \lambda_t} C_P(\nu_t).$$

Let us underline the following.

1. When $\nu_0$ is log-concave, then we have seen that $\lambda_t$ is zero, and hence we recover [2, Theorem 1.1].

2. Contrary to [2, Theorem 1.1], the measure $\nu_t$ satisfies a weighted Poincaré inequality. Indeed, the metric is adapted to $\nu_t$, and the Poincaré constant is defined as the smaller constant $K_t$ such that for all $\nu_t$-centered and compactly supported $\phi$,

$$\int_{R^d} \phi^2 d\nu_t \leq K_t \int_{\mathbb{R}^d} |\nabla \phi|^2_{C'_t} d\nu_t.$$

The sequel is devoted to sketch the proof of Theorem 1.

**Lemma 2.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth and compactly supported. We denote by $\phi_t := P_{0,t}\phi$. Then the Raylegh quotient*

$$R_\phi(t) := \frac{\mathbb{E}_{\nu_t} |\nabla \phi_t|^2_{C'_t}}{\mathbb{E}_{\nu_t} \phi_t^2}$$

*satisfies for all $t > s$,*

$$R_\phi(t) \leq R_\phi(s) e^{\lambda_s - \lambda_t}.$$

**Proof** First, let us underline that $\mathbb{E}_{\nu_t} \phi_t = 0$, justifying that $\mathbb{E}_{\nu_t} \phi_t^2$ is the $\nu_t$-variance of $\phi_t$. We can compute that (see [1, Proposition 2.1])

$$\frac{\partial}{\partial t} \mathbb{E}_{\nu_t} |\nabla \phi_t|^2_{C'_t} = \mathbb{E}_{\nu_t} \left[ 2 < \nabla L_t \phi_t, \nabla \phi_t >_{C'_t} - L_t |\nabla \phi_t|^2_{C'_t} + |\nabla \phi_t|^2_{C''_t} \right]$$

and

$$\frac{\partial}{\partial t} \mathbb{E}_{\nu_t} \phi_t^2 = \mathbb{E}_{\nu_t} |\nabla \phi_t|^2_{C'_t}.$$

Hence

$$\left(\mathbb{E}_{\nu_t} \phi_t^2\right)^2 R'(t) = \mathbb{E}_{\nu_t} \phi_t^2 \mathbb{E}_{\nu_t} \left[ 2 < \nabla L_t \phi_t, \nabla \phi_t >_{C'_t} - L_t |\nabla \phi_t|^2_{C'_t} + |\nabla \phi_t|^2_{C''_t} \right] - \left( \mathbb{E}_{\nu_t} |\nabla \phi_t|^2_{C'_t} \right)^2.$$

But the Bochner-type formula for the flow $(L_t)_t$ gives (see [2, Proposition 2.3, Formulas (32) and (33)])

$$\mathbb{E}_{\nu_t} \phi_t^2 \mathbb{E}_{\nu_t} \left[ 2 < \nabla L_t \phi_t, \nabla \phi_t >_{C'_t} - L_t |\nabla \phi_t|^2_{C'_t} \right] = \mathbb{E}_{\nu_t} \left[ ||\nabla^2 \phi_t||^2_{C'_t} + 2 < \nabla^2 V_t C'_t \nabla \phi_t, \nabla \phi_t >_{C'_t} \right].$$

So using assumption 1,

$$C_t' \nabla^2 V_t C_t' \geq C_t'' + \lambda_t' C_t'$$

one gets

$$R_\phi'(t) \leq -\lambda_t' R_\phi(t).$$

Finaly the Gronwall lemma gives

$$R(t) \leq R(s) e^{\lambda_s - \lambda_t}.$$

□

**Proof of Theorem** 1 We derive the quasi monotonicity of the spectral gap of $(\nu_t)_t$ by using the same reasonning as in the proof of Theorem 1.1 in [2].

By definition of the spectral gap, one has

$$\frac{1}{C_P(\nu_0)} = \inf\{R_\phi(0) \,|\, \phi \in C_c^\infty(\mathbb{R}^d)\},$$

where $C_c^\infty(\mathbb{R}^d)$ denotes the set of all smooth compactly supported functions on $\mathbb{R}^d$. Let then $t \geq 0$. For all $\varepsilon > 0$, there exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

$$R_\phi(0) < \frac{1}{C_P(\nu_0)} + \varepsilon.$$

By Lemma 2, it follows that

$$e^{\lambda_t} R_\phi(t) < \frac{1}{C_P(\nu_0)} + \varepsilon.$$

But by the definition of the Poincaré constant, $\frac{1}{C_P(\nu_t)} \leq R_\phi(t)$, so we get

$$\forall \varepsilon > 0, \quad \frac{e^{\lambda_t}}{C_P(\nu_t)} < \frac{1}{C_P(\nu_0)} + \varepsilon,$$

from which we get

$$C_P(\nu_t) \geq e^{\lambda_t} C_P(\nu_0)$$

by let $\varepsilon$ go to zero. Theorem 1 is therefore proven thanks to the semigroup property. □

# References


[1] Roland Bauerschmidt and Thierry Bodineau. Log-Sobolev inequality for the continuum sine-Gordon model. *Comm. Pure Appl. Math.*, 74(10):2064–2113, 2021.

[2] Bo'az Klartag and Eli Putterman. Spectral monotonicity under gaussian convolution, 2021.