CSc545 - fall 2018 - Homework #1.
Due: Sep 22 2018 11:59pm
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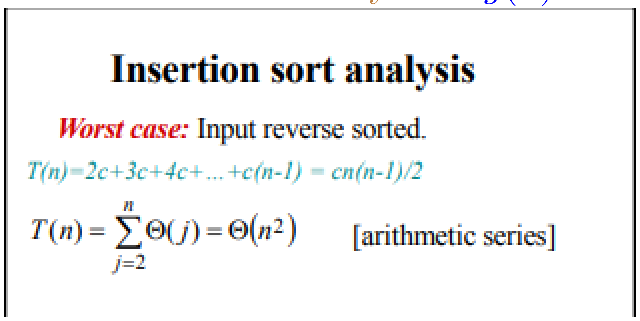
Instructions.

1. Solution may **not** be submitted by students in pairs.
2. You may submit a pdf of the homework, either printed or hand-written and scanned, as long as it is **easily** readable.
3. If your solution is illegible not clearly written, it might not be graded.
4. Unless otherwise stated, you should prove the correctness of your answer. A correct answer without justification may be worth less.
5. If you have discussed any problems with other students, mention their names clearly on the homework. These discussions are not forbidden and are actually **encouraged**. However, you must write your whole solution yourself.
6. Unless otherwise specified, all questions have the same weight.
7. You may refer to data structures or their properties studied in class without having to repeat details, and may reference theorems we have studied without proof. If your answer requires only modifications to one of the algorithms, it is enough to mention the required modifications, and the effect (if any) on the running time and on other operations that the algorithm performs.
8. In general, a complete solution should contain the following parts:
   (a) A high level description of the data structures (if needed). *E.g. We use a binary balanced search tree. Each node contains, a key and pointers to its children. We augment the tree so each node also contains a field...*
   (b) A clear description of the main ideas of the algorithm. You may include pseudocode in your solution, but this may not be necessary if your description is clear.
   (c) Proof of correctness (e.g. show that your algorithm always terminates with the desired output).
   (d) A claim about the running time, and a proof showing this claim.

1. Consider an infinite sequence of functions f_i. It is known that for every i, it is true that $f_i(n) = O(n)$. We define a new function $g(n) = \sum_{i=1}^{n} f_i(n)$. Is it true than $g(n) = O(n^2)$?

   Answer: *We can rewrite this as a summation series:*
   $g(n) = f_1(n) + f_2(n) + f_3(n) + ... + f_n(n)$
   And we know that $f_i(n) = O(n)$*,*
   so we can replace every $f_i(n)$ *with* $O(n)$*,*
   so we get $g(n) = \sum_{i=1}^{n} O(n)$
   and from the arithmetic series defined for insertion sort in the slides,
   it would be valid to say that $g(n) = O(n^2)$

2. Assume that we insert the keys $1, 2 \ldots n$ into a binary search tree T. The algorithm for inserting a key k_i is simple. Initially the tree is empty. Then for $i = 1, 2 \ldots n$, to insert key k_i we perform a search for k_i starting at the root of T, and once reaching a NULL pointer, we insert k_i as a new leaf. For simplicity, assume $k_i = i$.

   We do **not** not balance the tree after each insertion.

   As usual $h(T)$, the height of T denote the number of edges from the root of T to the leaf that is the most remote from the root.

   (a) warm-up

      i. show that the height of T is $\Theta(n)$.

      ii. Show that the time to perform all n insertions is $\Theta(n^2)$

      iii. next we insert the same keys into T but we could pick different permutations. Show that there is an order for which the time to perform all insertions is $O(n \log n)$

      iv. During the process of constructing the tree, we define random variable Y_{ij} to be 1 iff the keys i and j were compared to each other. (For every $1 \leq i, j \leq n$.) Otherwise $Y_{ij} = 0$. Show that the work to construct the tree is proportional to

      $$\sum_{1 \leq i,j \leq n} Y_{ij}.$$

      For example if j is the first key that was inserted, then it is in the root of T and it is compared to every other key inserted afterward.

   (b) Next assume that the keys are inserted in a random order, where each permutation is equally likely.

i. What is the expected time

$$E\left(\sum_{1\le i,j\le n} Y_{ij}\right)?$$

ii. Prove that the expected distance of a key j from the root is $O(\log n)$

iii. Prove that for large enough **constant** c, and large enough value of n, the probability that $h(T) \le c\log_2 n$ is very close to 1. You could pick what is c and how to formalize 'close to 1'

3. Show the preference list of a set of n men and n women, for which more than one single stable pairing exists.

   Answer: *We can consider the following preference lists for 3 men and 3 women*
   M_1 : *W1, W2 , W3* — *W1* : M_1; M_2; M_3
   M_2 : *W1, W3 , W2* — *W2* : M_2, M_3, M_1
   M_3 : *W2, W3 , W1* — *W3* : M_3, M_1, M_2
   Since M_1 and W1 are at the top of each others' preference lists, they will be matched with each other in any stable pairing to prevent them from going rogue.
   That leaves any combination of the rest of the two couples where Both M_2 and M_3 can be matched to either W2 or W3 without rogue couples.
   This gives us the following 2 stable pairings:
   P1 = (M_1, *W1*), (M_2, *W3*), (M_3, *W2*)
   P2 = (M_1, *W1*), (M_2, *W2*), (M_3, *W3*)

4. Suggest an $O(n^2)$-time algorithm that determines if, given the preference lists of all the men and women, there exists more than one stable matching.

5. Suggest a modification of the SkipList structure, such that in addition to the operations `insert`(x), `find`(x) and `delete`(x), you could also answer the operation $avg(x_1, x_2)$ which reports the average of all keys stores in the skip list whose value is $\ge x_1$ but $\le x_2$. The *expected* time for each operation should be $O(\log n)$.

   Hint: Start by storing at each node v of the SL another field, called *size*$[v]$, containing the number of keys in the SL between v and the next node at the same level as v. Note that maintaining the values of these fields might imply extra work while performing other operations.

6. (a) Let d be a fixed positive integer. Next consider a perfect SkipList constructed as follows: In order to create the ith level L_i of the SkipList, we scan the keys of level L_{i-1}, and promote to L_i every **d'th** key. So for example, the perfect SkipList discussed in class uses the value $d = 2$. The case $d = 3$ implies that every third key is promoted, and so on.

   i. What exactly is the worst case running time of `find`(x), as a function of both d and n ?

   ii. Which value(s) of d will do you think would lead to good performances, and which are poor choices? Why?

(b) In this question, consider a SkipList $\mathcal{L}$ created by inserting a set S of n keys into an (initially empty) SkipList. As seen in class, if a key x appears in level i, the probability that it also appears in level $i+1$ is $\frac{1}{2}$.
Assume that we re-create a SkipList by inserting the same keys, in the same order, but this time this probability p is 0.01. (if a key x appears in level i, the probability that it also appears in level $i+1$ is 0.01.) Will the expected time to perform *find*(x) operation increase or decrease, compared to the expected time for the same operation in the original SkipList $\mathcal{L}$?

7. What is the expected value of cnt after the following function is executed ?

```
cnt = 0; M = −∞ ;
for i = 1 to n {
    x_i = rand() ; Comment: returns a random float number of between 0 and 1.
    if (x_i > M) then {
        M = x_i;
        cnt + +;
    }
}
```

Hint: Define the random variable Y_i which is 1 if

$$x_i > \max\{x_1, x_2 \ldots x_{i-1}\}$$

and $Y_i = 0$ otherwise. What is $E(Y_1 + Y_2 + \cdots + Y_n)$.

Answer: *If we define the variable* Y_i *which is 1 if* $x_i > \max\{x_1, x_2 \ldots x_{i-1}\}$, *then on the first iteration of the loop, this variable is guaranteed to be 1 since anything is bigger than negative infinity, so count is at least one from the beginning. This means that* Y_i *is 1.*
For the next iterations of the loop, we're going to choose some number between 0 and 1, and whatever number we choose, we divide all numbers between 0 and 1 into 2 partitions using the random number between 0 and 1 that we got as reference point. One side of the partition represents $Y_i = 0$ *or* $Y_i = 1$.
If we assume that we get a nice even spread of random numbers, then each side that the random number lands on should be roughly $log(n)$, *because the odds of getting something higher than what we got last time is proportional to how many divisions(which side) in we are. In an arithmetic sequence, assuming the nice even spread of numbers, this would look something like:*
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} ... + \frac{1}{n} = log(n)$
Therefore, the expected value of cnt *is* $log(n)$.

8. Create a SkipList for a set of n keys $S = \{k_1 \ldots k_n\}$. The keys are known in advance, and are sorted. The height of the Skiplist is ≤ 3.

   The SkipList should be designed such that

$$\max_{k_i \in S} T(k_i)$$

   is as small as possible, where $T(k_i)$ is defined as the time for searching k_i, where the search is starting, as usual in the upper leftmost element.

9. Let L be a perfect SkipList constructed on a set of n keys, and let $X = \{k_1 \ldots k_m\}$ be another set of keys. Assume $m < n$. Suggest an $O(m \log(\frac{n}{m}))$ time algorithm for computing which key(s) of X appear in L. You could assume that X is sorted.

   You could **not** assume that the keys of X are equally spaced in L. Yet understanding this special case could shed some intuition.