The DELAUNAY TRIANGULATION of a SATURATED CIRCLE CONFIGURATION

<u>Lemma</u>: Consider the four points, p, q, r and s. Let x_0 denote the projection of the point x. The point s lies within the circumcircle of p, q and r iff s_0 lies below the plane passing through p_0, q_0 and r_0.

Consider an arbitrary (nonvertical) plane in 3D space, tangent to the paraboloid at some point (a, b).

$$z = x^2 + y^2$$

so $\frac{dz}{dx} = 2x$ and $\frac{dz}{dy} = 2y$

So at the point (a, b, a^2+b^2), this gives $2a$ and $2b$

So the plane passing through this point has the form:

$$z = 2ax + 2by + k$$

$$\Rightarrow a^2 + b^2 = 2a^2 + 2b^2 + k \Rightarrow k = -(a^2+b^2)$$

i.e. $z = 2ax + 2by - (a^2+b^2)$

Shift this up by some positive r^2:

$$z = 2ax + 2by - (a^2+b^2) + r^2$$

$$x^2 + y^2 = 2ax + 2by - (a^2+b^2) + r^2$$

$$x^2 + a^2 - 2ax + y^2 + b^2 - 2by = r^2$$

$$(x-a)^2 + (y-b)^2 = r^2 \quad [\text{a circle!}]$$

Thus the intersection of an arbitrary lower half space with the parabaloid produces the interior of a circle

When we project p, q and r onto the parabaloid, and then back onto the x-y plane, p, q and r will lie on the circumference of the projected circle (the circumcircle of p, q and r)

Hence s lies within this circle if and only if its projection s_0 onto the parabaloid is within the lower half space. $\square$

OPTIMAL CIRCLE CONFIGURATIONS

<u>Lemma 1</u>: Let θ be the largest internal $\angle$ of a $\triangle ABC$ in a Delaunay triangulation for a saturated circle configuration, C. Then $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}$.

Largest $\angle$ of any $\triangle$ is always $\geq \frac{\pi}{3}$ $(60°)$

Suppose $\theta \geq \frac{2\pi}{3}$ and let A be the smallest $\angle$

$$\Rightarrow A \leq \frac{\pi}{3}.$$

$$\Rightarrow \sin A \leq \frac{1}{2}$$

Also $|BC| \geq 2.$

Let R be the circumradius of $\triangle ABC$.

$$2R = \frac{|BC|}{\sin A} \geq \frac{2}{\sin A} \geq 4.$$

Then the circumcenter of $\triangle ABC$ can be added to C.

But C is saturated. $\times$

$$\Rightarrow \quad \frac{\pi}{3} \leq \theta < \frac{2\pi}{3} \quad \text{as required}$$

<u>Lemma 2:</u> The density of a ΔABC in a Delaunay triangulation is $\leqslant \frac{\pi}{12}$.

[Since $A = \frac{1}{2} r^2 \theta$ for a sector]:

$$\text{density} = \frac{\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}(A+B+C)}{\text{Area of } \Delta ABC} = \frac{\pi/2}{\Delta ABC}.$$

Suppose that C is the largest internal $\angle$ of ΔABC.

By Lemma 1:

$$\Delta ABC = \frac{1}{2} \times AC \times BC \times \sin C \leqslant \frac{1}{2} \times 2 \times 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\Rightarrow \text{density} \leqslant \frac{\pi}{\sqrt{12}}$$

The equality only holds for equilateral Δs of side length 2.