Sphere Packing

The sphere packing problem in dimension 8

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In this paper we prove that no packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing.

THE SPHERE PACKING PROBLEM IN DIMENSION 24

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ABSTRACT. Building on Viazovska's recent solution of the sphere packing problem in eight dimensions, we prove that the Leech lattice is the densest packing of congruent spheres in twenty-four dimensions, and that it is the unique optimal periodic packing. In particular, we find an optimal auxiliary function for the linear programming bounds, which is an analogue of Viazovska's function for the eight-dimensional case.

- The Sphere Packing Problem
- In 3D and 2D
- Generalising to higher dimensions
- Applications
- The Kissing Number Problem

The Sphere Packing Problem

According to Wikipedia:

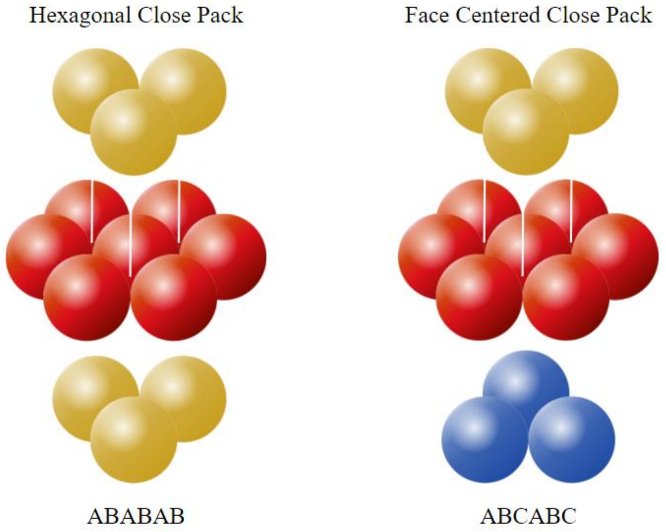
Sphere packing finds practical application in the stacking of oranges.

- Find densest arrangement for identical spheres within a containing space
- Spheres cannot overlap
- Density = fraction of space filled by the spheres

- *Asymptotic limit on density as containing space increases*

3-Dimensional Sphere Packing

Kepler's Conjecture (1611)
No arrangement of equally sized spheres filling space has a greater average density than that of the cubic close packing (face-centered) and hexagonal close packing arrangements. The density of these arrangements is $\frac{\pi}{3\sqrt{2}} \approx 74.05\%$.



Source: datagenetics.com

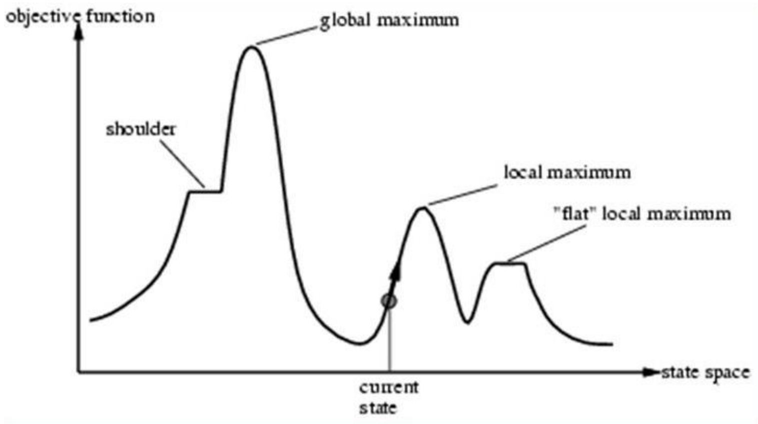
1901 One of Hilbert's 23 Problems

1998 Solution proposed by Hales "99% Certain"

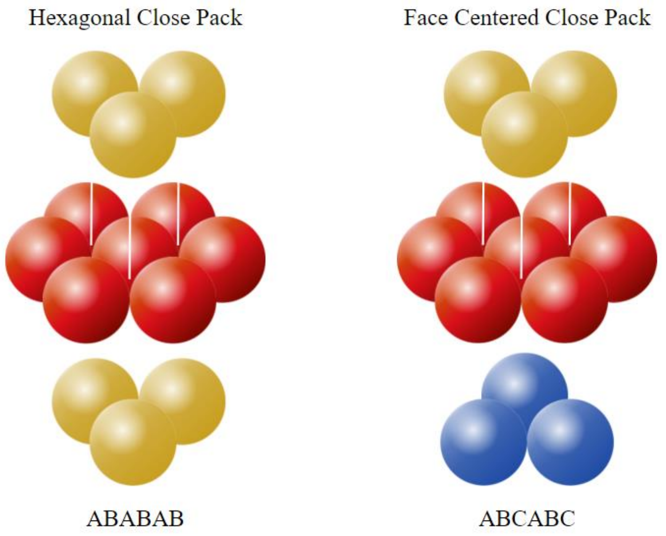
- Over 150 variables
- 250 pages of proof

Difficulties in the Sphere Packing Problem

- Not all sphere packings have a density that is readily computable
- Many local maxima
- Hard to rule out implausible sphere configurations
- Sometimes an infinite number of geometrically different packings with same density



Source: courses.cs.washington.edu



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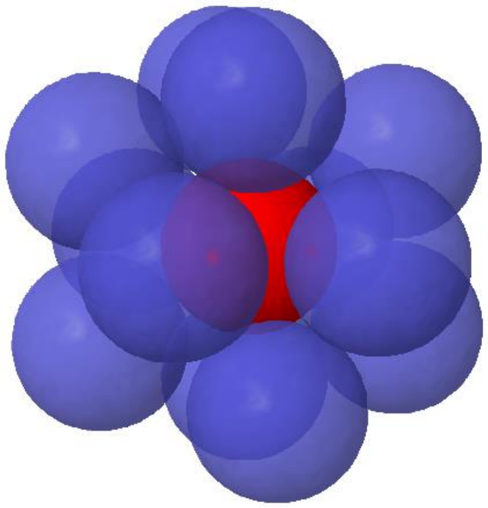
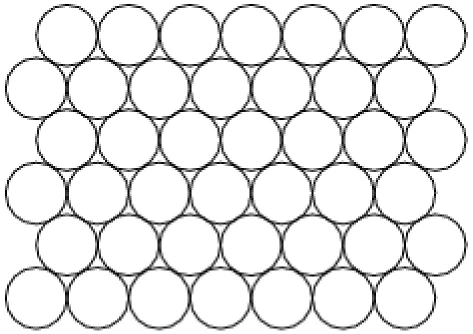

Source: Wikipedia

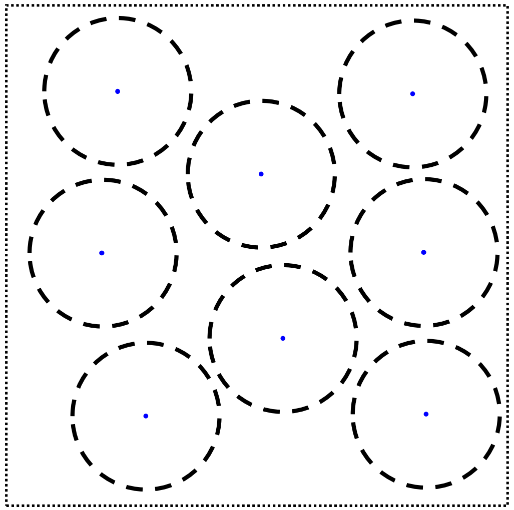
Circle Packing



hexagonal packing

Source: Wolfram MathWorld

Theorem: The hexagonal packing is the densest of all circle packings

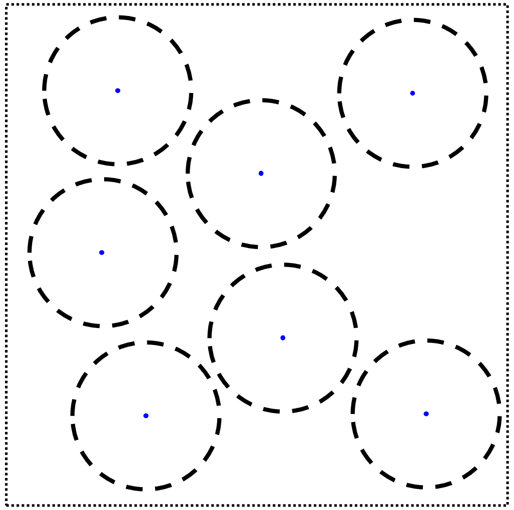


This circle configuration is saturated

Circle configuration: A set of points (that define the centres of unit-radius circles) such that the distance between any two points is >= 2.

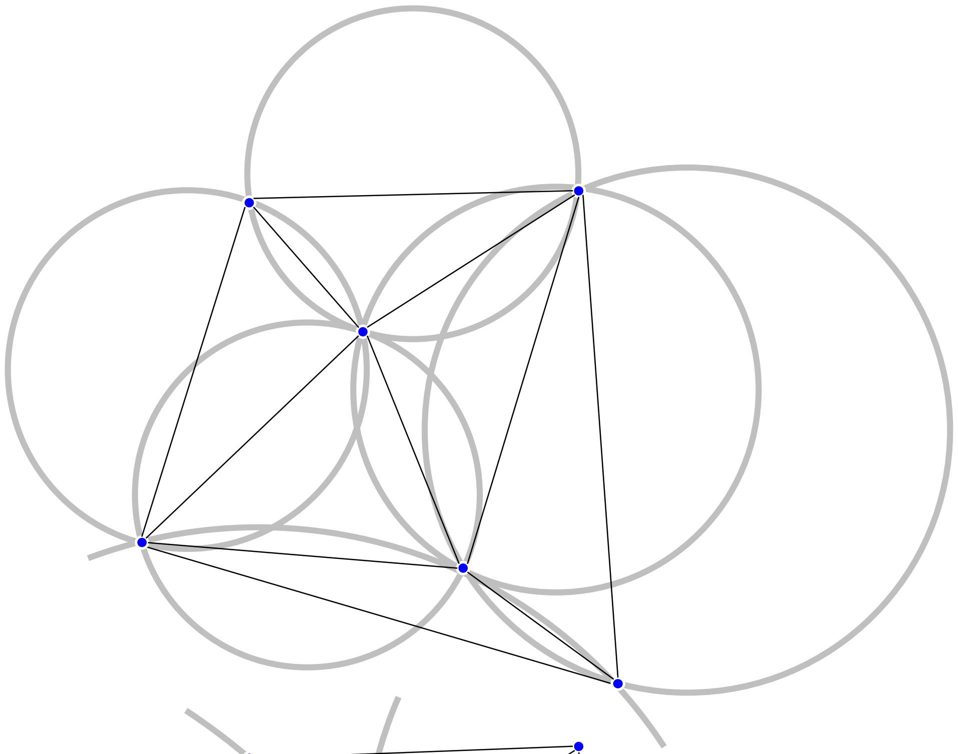
Saturated circle configuration: A circle configuration that is not a proper subset of another circle configuration

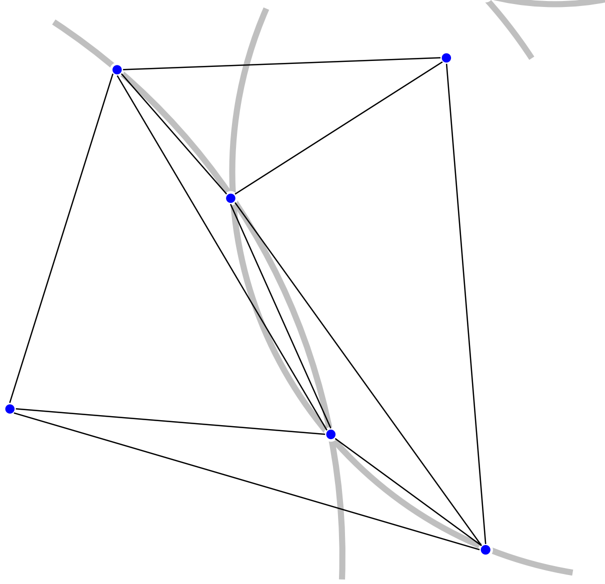
(So we only need to consider saturated configurations.)



This circle configuration is not saturated

Triangulations





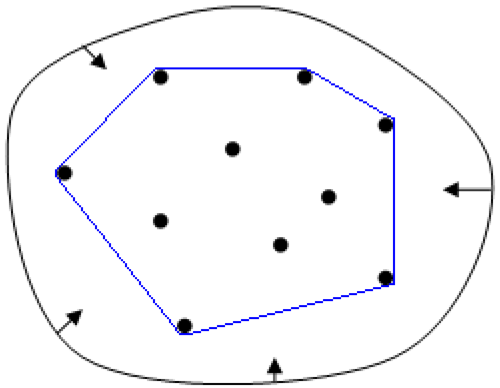
Triangulation (in $\mathbb{R}^2$): A set of edges (which do not cross) where between any two distinct points, p and q, the line-segment, pq, crosses some edge in the triangulation.

Delaunay Triangulation: A triangulation where no point is inside the circumcircle of any triangle of the triangulation.

(Optimal triangulation – smallest interior angle of any triangle is maximised)

- Does there exist a Delaunay triangulation for any set of points?
- Is a Delaunay triangulation unique for a set of points?

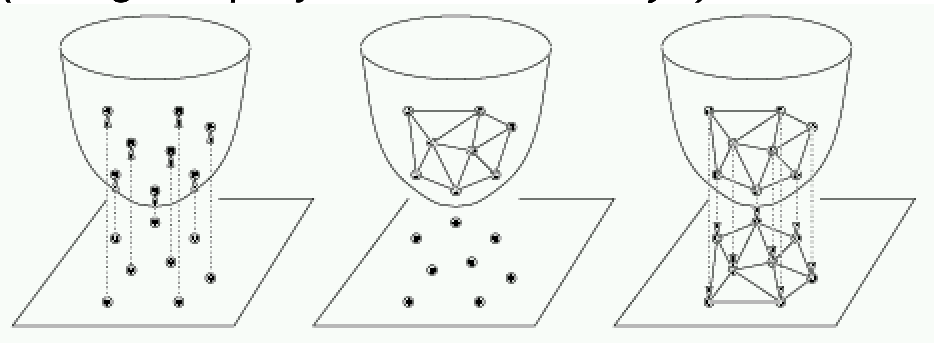
The Delaunay Triangulation of a Saturated Circle Configuration



Source: Wikipedia

Convex Hull (in $\mathbb{R}^3$): The three points, p, q and r form a face of the convex hull for a set of points, C, iff the plane passing through p, q and r has all points of C lying on one side.

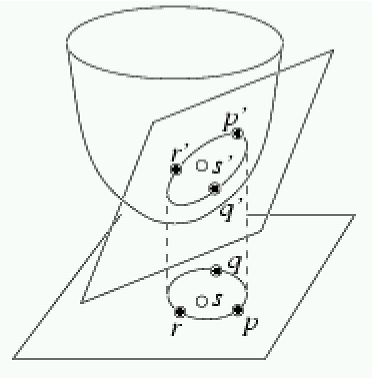
(Using the projection $z = x^2 + y^2$)



Project onto paraboloid. Compute convex hull. Project hull faces back to plane.

Source: http://www.cs.wustl.edu/

Lemma: Consider the four points, p, q, r and s. Let x_0 be the projection of the point x. The point s lies within the circumcircle of p, q and r iff s_0 lies below the plane passing through p_0, q_0 and r_0.

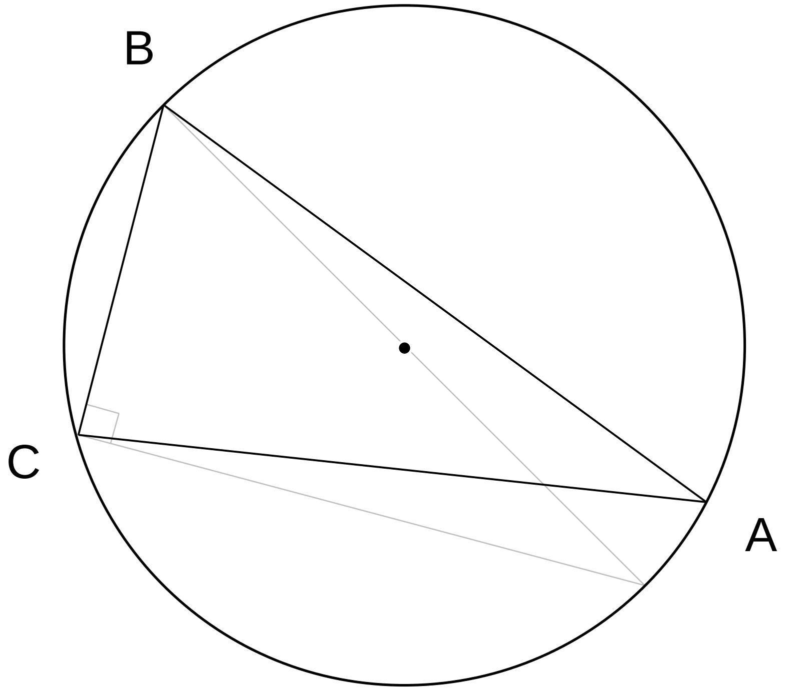


http://www.cs.wustl.edu/~pless/506/code/convex/convex.html

Optimal Circle Configurations

Lemma 1: Let θ be the largest internal angle of a ΔABC in a Delaunay triangulation for a saturated circle configuration, C. Then $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}$.

Lemma 2: The density of a triangle ΔABC in a Delaunay triangulation for a saturated circle configuration C is less than or equal to $\pi/\sqrt{12}$. The equality holds only for equilateral triangles with side-length 2.



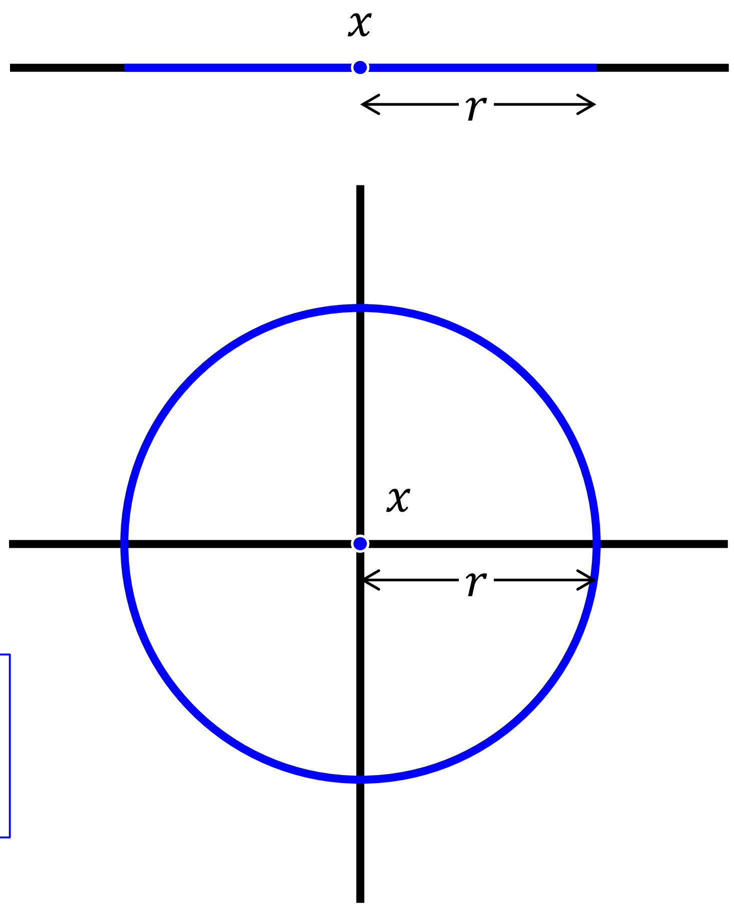
n-Dimensional Spheres

All points within a certain distance of centre

- $\mathbb{R}^n$ ordered set of real numbers $x = (x_1, x_2, \dots, x_n)$
- Distance between $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ is

  $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$

A solid sphere in $\mathbb{R}^n$ with radius $r > 0$ and centred at a point x consists of all points y in $\mathbb{R}^n$ whose distance to x is less than or equal to r

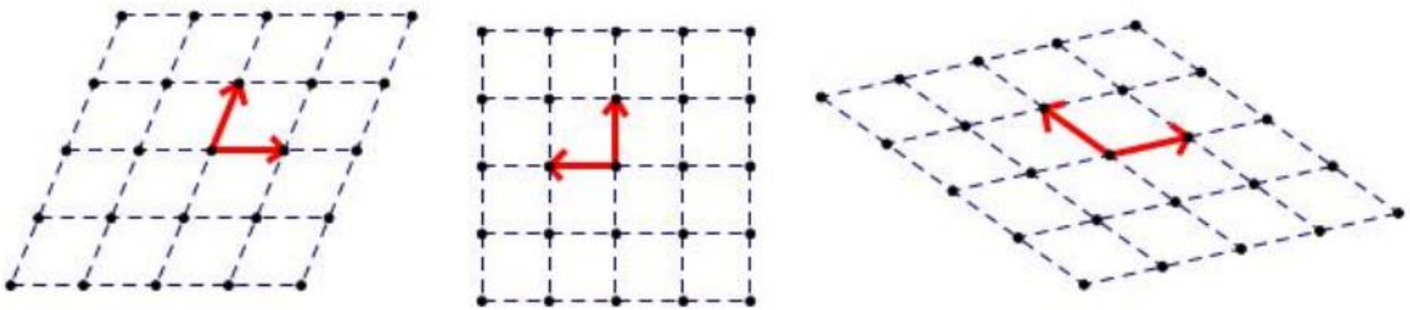


Higher Dimensional Sphere Packing: Lattices

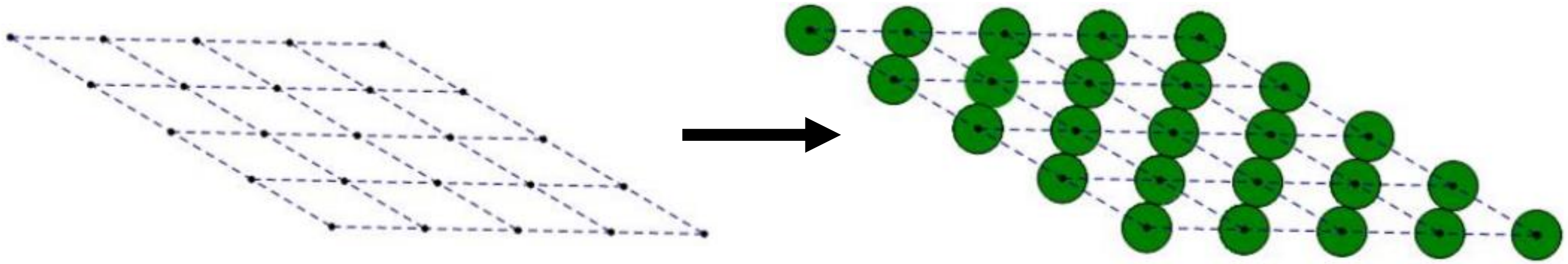
Closely related to number theoretic questions involving lattices in $\mathbb{R}^n$.

A lattice is the integral span of a vector space basis $(v_1, v_2, \ldots, v_n$ for $\mathbb{R}^n)$

I.e. $\Lambda = \{k_1 v_1 + k_2 v_2 + \cdots + k_n v_n : k_1, k_2, \ldots k_n \in \mathbb{Z}\}$



Source: https://www.math.washington.edu/



Minimum distance between lattice points = maximum radius of spheres

The Lattice Sphere Packing Problem

Need to know:

||∧|| = length of shortest non-zero vectors in ∧

(difficult – especially in high dimensions)

disc(∧) = volume of a lattice

(easy – determinant of generating matrix)

Density = Vol of ball with r=||∧|| / disc(∧)

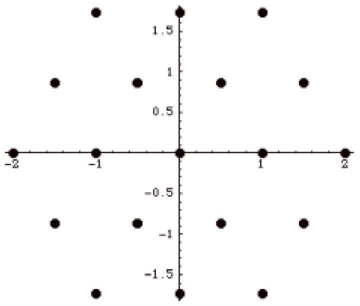


Figure: Hexagonal lattice generated by $\{(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\}$

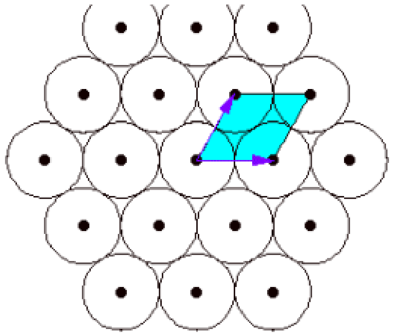


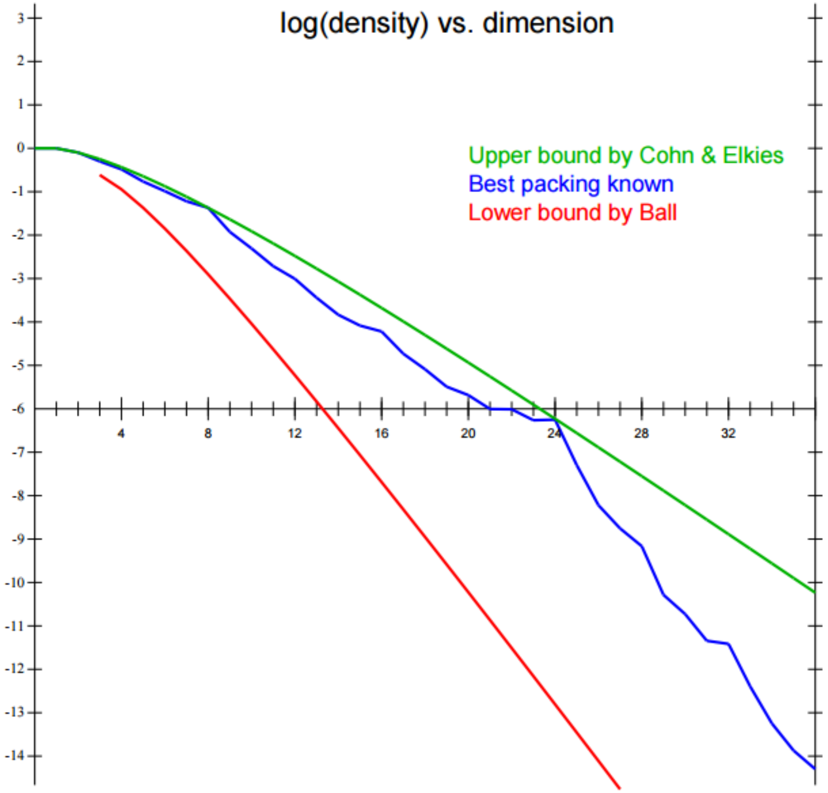
Figure: Hexagonal Lattice Sphere Packing and fundamental region

Source: https://www.math.washington.edu/

- Many (but not all!) of the densest known sphere packings are lattices
- Still very difficult!
- Densest known lattices have a lot of symmetries (large automorphism groups)
- E.g. root lattice, $\mathbb{E}_8$ and leech lattice, Λ_{24}

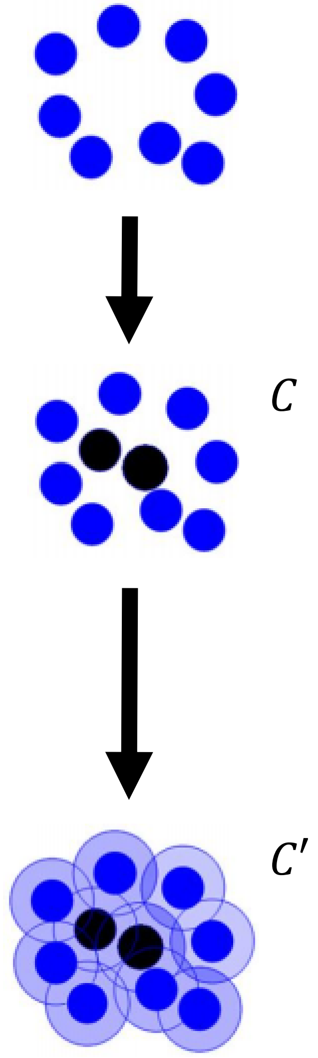
Lower Bounds

- Take a saturated sphere configuration, C
- Double radii keeping centres fixed to get C'
- We now have a cover of $\mathbb{R}^n$ (never >= 2r distance between spheres)



Source: http://www.ams.org/

C' has density >= 1
$\Rightarrow C$ has density >= $1/2^n$

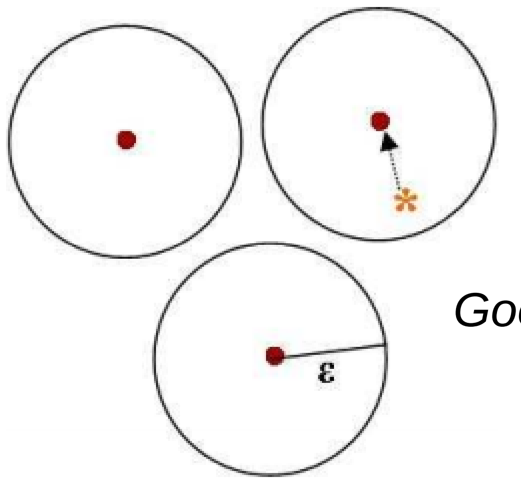


Applications of Sphere Packing – Hamming Code

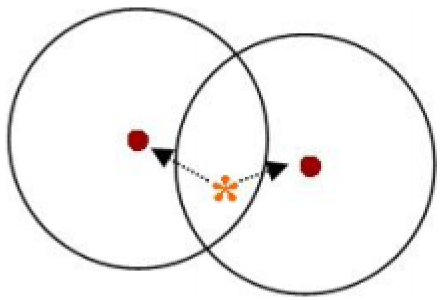
Sphere packings in $\mathbb{R}^n$ are the continuous analogue of error correcting codes of length n

- Let S denote a set of signals represented by points $x \in \mathbb{R}^n$ (e.g. $\mathbb{R}^7$ for ASCII)
- Message consists of a stream of signals (multiple characters)
- Sent over noisy channel so often $y \neq x$ is received at the end
- However it is likely that $|x - y| < \varepsilon$ where ε is a measure of the noise level of channel

We want to choose our points $x \in S$ such that the distance between any two signals is >= 2ε



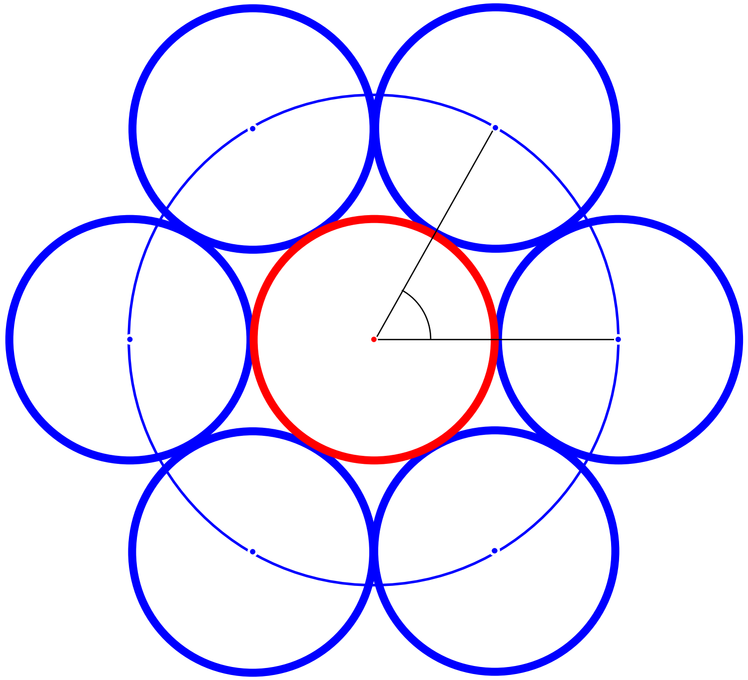
Good signal set



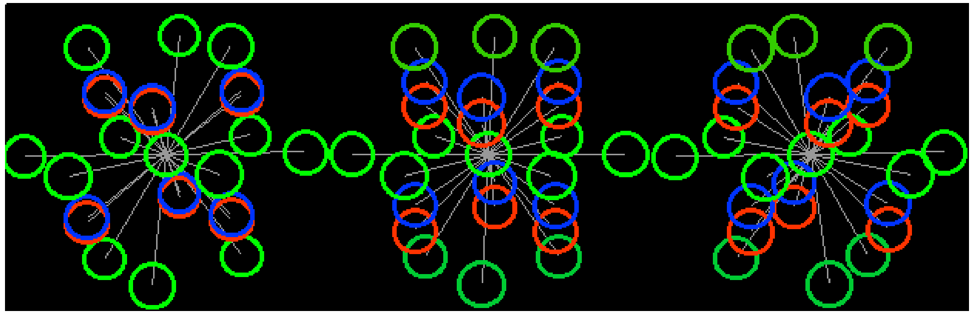
Signals too close

Source: https://www.math.washington.edu/

The Kissing Number Problem



Dimension	Lower bound	Upper bound
1	2	
2	6	
3	12	
4	24[6]	
5	40	44
6	72	78
7	126	134
8	240	



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Denis Auroux Sphere Packings, Crystals and Error-Correcting Codes