DUALITIES IN TRIANGULATED CATEGORIES

Overview. The goal of this lecture course is to give a modern point of view on some important duality theorems in algebra, from the point of view of triangulated categories. This perspective also enables one to view these dualities not just in an algebraic setting, but to transport them into other realms, such as geometry and topology. The main focus will be on Grothendieck's local duality theorem, which relates the Matlis dual of local cohomology to the ordinary functional dual. The course will give an introduction to triangulated categories, before turning to introducing local cohomology, firstly in the classical algebraic setting, and then in the triangulated realm and explaining how the latter recovers and generalises the former. We will then turn to exploring local duality in the triangulated setting, which naturally leads us to consider other duality theorems such as Greenlees-May duality, and Warwick duality. We will show how one can recover the classical statement of Grothendieck local duality from this more general triangulated version.

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Relevant literature.

(1) H. Krause. Localization theory for triangulated categories. In Triangulated categories, volume 375 of London Math. Soc. Lecture Note Ser., pages 161–235. Cambridge Univ. Press, Cambridge, 2010
(2) H. Krause. Homological theory of representations, volume 195 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2022
(3) M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. Mem. Amer. Math. Soc., 128(610):x+114, 1997
(4) T. Barthel, D. Heard, and G. Valenzuela. Local duality in algebra and topology. Adv. Math., 335:563–663, 2018
(5) W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. Amer. J. Math., 124(1):199–220, 2002
(6) C. Huneke. Lectures on local cohomology. Contemp. Math., 436, Interactions between homotopy theory and algebra, 51–99, Amer. Math. Soc., Providence, RI, 2007.
(7) C. Weibel. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics. 38. Cambridge: Cambridge University Press. xiv, 450 p. (1994).
(8) A. Neeman. Triangulated categories. Annals of Mathematics Studies. 148. Princeton, NJ: Princeton University Press. vii, 449 p. (2001).

Assessment. The final exam will be an oral exam. For zápočet, students will have to get at least 50% of marks on each of the 3 homework assignments.

1. What is duality?

There is no precise definition of what constitutes a duality, indeed, Atiyah said “Duality in mathematics is not a theorem, but a “principle””. Perhaps the closest to a precise formulation of duality is that it is a contravariant endofunctor $D\colon \mathcal{C} \to \mathcal{C}$ such that D^2 is the identity, either on $\mathcal{C}$ or on a convenient subcategory of it. However there are many forms of things which we call duality theorems which do not fit this mould; for example, there is even something called the covariant Grothendieck duality theorem! Another possible formulation of a duality is that it is a statement which relates a *co*variant functor to a *contra*variant functor. Let us investigate three examples of dualities to give a flavour of the meaning.

Example 1.1 (Complements of subsets)**.** Let A be a set, and $B \subseteq A$. Then taking the complement twice, we have $(B^c)^c = B$. In the above formulation, this amounts to taking $\mathcal{C}$ to be the category whose objects are subsets of A, with a morphism $B \to B'$ if and only if $B \subseteq B'$, and $D = (-)^c$.

Example 1.2 (Functional duality of vector spaces)**.** Let V be a vector space over a field k. The dual vector space V^* is defined to be the set of k-linear maps $V \to k$ with the obvious vector space structure, i.e., $V^* = \mathrm{Hom}_k(V, k)$. There is a natural map $f\colon V \to V^{**}$ defined via

$$f(v)\colon \mathrm{Hom}_k(V,k) \to k \qquad f(v)(g) = g(v).$$

One checks that this map is k-linear, and that moreover, if V is finite dimensional, then f is an isomorphism. Rephrasing this categorically, we take $\mathcal{C}$ to be the category of k-vector spaces, $D = (-)^*$, and D^2 is isomorphic to the identity on the subcategory of finite dimensional vector spaces. There is an important note to be made here: as finite dimensional vector spaces are determined by their dimension, one may check that for V finite dimensional, we have that V is isomorphic to V^*. However, this isomorphism is *not* natural since it relies on a choice of basis. On the other hand, V is naturally/canonically equivalent to its double dual V^{**}. As such, in the prototype definition of duality given above, we actually want to require that D^2 is *naturally* isomorphic to the identity.

Example 1.3 (Grothendieck local duality)**.** We now turn to stating the main duality theorem of this course. We will not define all of the terms in the statement; we will make them precise throughout the course. We will focus on the statement in commutative algebra, but one selling point of the language which we will study in this course, is that it allows for a statement of Grothendieck local duality to be made in a broad range of settings.

Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring; recall that a ring is Gorenstein if it has finite injective dimension as a module over itself. Then Grothendieck local duality asserts that

$$\mathrm{Ext}^i_R(M, R^\wedge_\mathfrak{m}) = H^{\dim(R)-i}_\mathfrak{m}(M)^\vee$$

for all R-modules M.

In this statement, $H^*_\mathfrak{m}(-)$ denotes the *local cohomology*. This is a much used tool in commutative algebra and beyond. One example of where local cohomology can be used is in answering questions about how many generators one needs to generate an ideal up to radical. Recall that for an ideal I, the radical is

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n\}.$$

For example, in the polynomial ring $k[x,y]$ the ideal $I=(x^2,xy,y^2)$ can actually be generated up to radical by only two elements; that is, $\sqrt{I}=\sqrt{(x^2,y^2)}$. This example is very small, but for larger rings and ideals, local cohomology provides a structured way to attack such questions.

Grothendieck local duality does not fit the mould for duality theorems as we 'defined' above. However, note that it deserves the title of a duality since it relates the contravariant functor $\mathrm{Ext}^i_R(-,R^\wedge_\mathfrak{m})$ to the covariant functor $H^{\dim(R)-i}_\mathfrak{m}(-)$.

Grothendieck local duality is a powerful tool since it enables one to replace questions about local cohomology with questions about Ext-groups. Another reason Grothendieck local duality is useful arises when trying to pass to local problems: $H^*_\mathfrak{m}(-)_\mathfrak{p}$ is always zero if $\mathfrak{p}\neq\mathfrak{m}$, so one cannot simply localize local cohomology directly. Instead one may pass to the world of Ext-groups, localize there, and then translate back to local cohomology.

2. Triangulated categories

2.1. **The axioms.** Loosely speaking, a triangulated category consists of an additive category T, together with two extra pieces of data:

(1) an equivalence of categories $\Sigma\colon \mathsf{T}\xrightarrow{\sim}\mathsf{T}$ called the *shift*;
(2) a collection of *triangles* $X\to Y\to Z\to\Sigma X$ satisfying various axioms which ensure good behaviour.

Definition 2.1. Let T be an additive category and $\Sigma\colon \mathsf{T}\xrightarrow{\sim}\mathsf{T}$ be an additive equivalence of categories. A *candidate triangle* is a diagram

$$X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}\Sigma X$$

such that the composites $g\circ f$, $h\circ g$, and $\Sigma f\circ h$ are all zero. A morphism of candidate triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'. \end{tikzcd}$$

Considering a smaller class of candidate triangles which satisfy certain properties leads to the notion of a triangulated category. The first four axioms are easy to justify, but the final axiom is harder to motivate. It is convenient to develop the theory assuming only these first four axioms, and then add in the final one once it becomes relevant. Nonetheless, we'll give both definitions now, so that we can consider an example before embarking on the abstract theory.

Definition 2.2. A *pretriangulated category* T is an additive category together with an additive equivalence of categories $\Sigma\colon \mathsf{T}\xrightarrow{\sim}\mathsf{T}$, and a subclass of candidate triangles called *distinguished triangles* which satisfy the following axioms:

(TR0) Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle, and for all $X\in\mathsf{T}$ the candidate triangle

$$X\xrightarrow{1}X\to 0\to\Sigma X$$

is distinguished.

(TR1) For all $f\colon X\to Y$ in T, there exists a distinguished triangle $X\xrightarrow{f}Y\to Z\to\Sigma X$.

(TR2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a candidate triangle. This is distinguished if and only if the candidate triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

(TR3) For any commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] & \Sigma X \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'.
\end{tikzcd}$$

in which the rows are triangles, there exists a map $w\colon Z \to Z'$ (which need *not* be unique) making

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'.
\end{tikzcd}$$

commute.

Remark 2.3. It is standard to drop the adjective 'distinguished', and just refer to them as *triangles*. For candidate triangles, we will never drop the adjective.

Remark 2.4. By combining (TR2) with (TR3), one sees that there always exists fillers in the first and second column too.

Definition 2.5. A *triangulated category* T is a pretriangulated category satisfying the following additional axiom:

(TR4) Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z' \xrightarrow{h} \Sigma X$, $Y \xrightarrow{u} Z \xrightarrow{v} X' \xrightarrow{w} \Sigma Y$ and $X \xrightarrow{uf} Z \xrightarrow{a} Y' \xrightarrow{b} \Sigma X$ are distinguished triangles. Then there exists a distinguished triangle

$$Z' \xrightarrow{\alpha} Y' \xrightarrow{\beta} X' \xrightarrow{\gamma} \Sigma Z'$$

such that $v = \beta a$, $h = b\alpha$, $\gamma = \Sigma g \circ w$, $w\beta = \Sigma f \circ b$ and $\alpha g = au$.

Pictorially this axiom can be represented by the following commuting diagram:

$$\begin{tikzcd}
 & & Y' \arrow[ddll, dashed, "b"] \arrow[drr, "\beta"] & & \\
Z' \arrow[urr, "\alpha"] \arrow[d, dashed, "h"'] & & & & X' \arrow[llll, dashed, "\Sigma g \circ w"] \arrow[ddll, dashed, "w"] \\
X \arrow[rrrr, "uf"] \arrow[drr, "f"'] & & & & Z \arrow[u, "v"'] \arrow[uull, "a"'] \\
 & & Y \arrow[urr, "u"'] \arrow[uull, "g"] & &
\end{tikzcd}$$

The dotted maps are of degree 1 (i.e., $f\colon X \dashrightarrow Y$ represents a map $f\colon X \to \Sigma Y$), and composites of the form $\to\to\dashrightarrow$ are triangles. The red maps are the extra data, together with the condition that they form a triangle. In order to remember this, note that the primed letters are the cones of maps, and that every triangle contains an X, Y, and a Z (primed or otherwise). In light of the shape of the above diagram, (TR4) is often referred to as the *octahedral axiom*.

Alternatively, one can give the following pictorial representation.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[r, "g"] \arrow[d, "u"] & Z' \arrow[r, "h"] \arrow[d, dashed, "\alpha"] & \Sigma X \arrow[d, "1"] \\
X \arrow[r, "uf"] \arrow[d, "f"] & Z \arrow[r, "a"] \arrow[d, "1"] & Y' \arrow[r, "b"] \arrow[d, dashed, "\beta"] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, "u"] \arrow[d, "g"] & Z \arrow[r, "v"] \arrow[d, "a"] & X' \arrow[r, "w"] \arrow[d, "1"] & \Sigma Y \arrow[d, "\Sigma g"] \\
Z' \arrow[r, dashed, "\alpha"] & Y' \arrow[r, dashed, "\beta"] & X' \arrow[r, dashed, "\gamma"] & \Sigma Z'
\end{tikzcd}$$

The first three rows are the given triangles, and (TR4) then asserts the existence of the dotted arrows making the diagram commute, so that the bottom row is also a triangle.

Let's briefly discuss the axioms and provide some motivation for them. If one thinks as triangles as a generalisation of short exact sequences, then in a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ you should think of Z as the (homotopy coherent) cokernel of f, and X as the (homotopy coherent) kernel of g. The axioms then mean the following.

(TR0) The kernel and cokernel of the identity is zero.
(TR1) Every map has a kernel and cokernel.
(TR2) Up to sign, every map is the kernel of its cokernel and vice versa.
(TR3) Kernels and cokernels are almost functorial.
(TR4) One can interpret the given triangles as saying $Z' \simeq Y/X$, $X' \simeq Z/Y$ and $Y' \simeq Z/X$, and then the axiom asserts that $X' \simeq Y'/Z'$, i.e., $(Z/X)/(Y/X) \simeq Z/Y$.

Remark 2.6. Given a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$, it is common to call Z the *cofibre* (or *cone*) of f, and X the *fibre* (or *cocone*) of g. Sometimes it is customary to write triangles as $X \to Y \to Z$ and drop the map to the shift. We will sometimes subscribe to this later on the course for brevity, but we warn the reader that it is important not to forget this map. For example, a morphism of triangles requires the square including the map $Z \to \Sigma X$ to commute.

We will give two detailed examples of triangulated categories: the homotopy category of a ring and the derived category of a ring. These are intrinsically algebraic examples; there are many more examples of triangulated categories, including plenty coming from topology and geometry, but these require more background to describe, so we will focus on these algebraic cases. Here's a list of some examples we won't describe, included for the interested reader:

- the stable homotopy category, and equivariant and chromatic versions of this,
- the derived category of quasi-coherent sheaves over a nice enough scheme,
- various 'mixed' versions of these, such as the motivic stable homotopy category,
- the stable module category of a finite group.

2.2. **Aside: preliminaries from homological algebra.** Homological algebra is built upon taking resolutions of modules. Therefore, one seeks a category which contains precisely the homological information of modules, so that objects are resolutions, and a module is isomorphic to any resolution of it. This utopian category will the derived category of the ring in question.

Let R be a ring. A chain complex of R-modules M is a collection of R-modules $\{M_i\}_{i\in\mathbb{Z}}$ together with maps called differentials, $d_i\colon M_i \to M_{i-1}$ for all $i \in \mathbb{Z}$, satisfying $d_i \circ d_{i+1} = 0$. We write $\mathrm{Ch}(R)$ for the category of chain complexes.

Recall that the condition on the differential ensures that $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$, so that we may consider the homology groups $H_i(M) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1})$ which measure how far away from being exact a sequence is. A map $f\colon M \to N$ of chain complexes is a collection of levelwise maps $f_i\colon M_i \to N_i$ for each $i \in \mathbb{Z}$ which commute with the differentials. Such a map is often called a chain map. A chain map $f\colon M \to N$ is said to be a *quasi-isomorphism* if

$$H_i(f)\colon H_iM \to H_iN$$

is an isomorphism for all $i \in \mathbb{Z}$. Recall that given any short exact sequence of complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

there is a connecting map $\delta\colon H_nC \to H_{n-1}A$ so that the induced sequence

$$\cdots \to H_nB \to H_nC \to H_{n-1}A \to H_{n-1}B \to \cdots$$

is long exact.

In this language, one may rephrase the definition of projective resolution of a module M as being a complex P consisting of projective modules, together with a chain map $P \to M[0]$ which is a quasi-isomorphism. Note here that we view the *module* M, as a *complex* $M[0]$ by putting M in degree 0 and zeroes everywhere else. (Henceforth we will just write M for $M[0]$ and leave it implicit that modules are viewed as complexes in degree 0.)

Given chain maps $f, g\colon M \to N$, a *chain homotopy* from f to g is a collection of maps $h_n\colon M_n \to N_{n+1}$ such that $f_n - g_n = d^N_{n+1}h_n + h_{n-1}d^M_n$ as demonstrated by the diagram

$$\begin{tikzcd}
M_{n+1} \arrow[r, "d^M_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & M_n \arrow[r, "d^M_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & M_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \\
N_{n+1} \arrow[r, "d^N_{n+1}"'] & N_n \arrow[r, "d^N_n"'] & N_{n-1}
\end{tikzcd}$$

We say that two complexes M, N are *chain homotopy equivalent* if there exists maps $f\colon M \to N$ and $g\colon N \to M$ so that gf is chain homotopic to the identity on M and fg is chain homotopic to the identity on N.

Let $f\colon M \to N$ be a map of chain complexes. The mapping cone of f denoted $C(f)$ is the complex with $C(f)_n = M_{n-1} \oplus N_n$ and differential $d(m, n) = (-dm, dn - fm)$. Sometimes it can be convenient to write this differential as the matrix

$$\begin{pmatrix} -d_M & 0 \\ -f & d_N \end{pmatrix}.$$

The *suspension* (or *shift*) of a complex M, denoted ΣM, is defined by $(\Sigma M)_n = M_{n-1}$ with differential $-d$. A way to remember which way the shift moves, is to note that the shift is opposite to the differential. By including the M factor and sending the N factor to 0 one obtains a chain map $c_f\colon C(f) \to \Sigma M$, $c_f(m,n) = m$, as demonstrated in the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & M_n \oplus N_{n+1} \arrow[r, "{(-d_M,\ d_N - f)}"] \arrow[d, "{(\mathrm{id},0)}"'] & M_{n-1} \oplus N_n \arrow[r] \arrow[d, "{(\mathrm{id},0)}"] & \cdots \\
\cdots \arrow[r] & M_n \arrow[r, "-d_M"'] & M_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Similarly, there is a chain map $i_f\colon N \to C(f)$ given by $i_f(n) = (0,n)$.

The tensor product of chain complexes is defined by

$$(M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j$$

with differential $d(m \otimes n) = (dm \otimes n) + (-1)^{|m|}(m \otimes dn)$. The internal hom of chain complexes is defined by

$$\mathrm{Hom}_R(M,N)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$$

with differential $d(f) = d^N \circ f - (-1)^{|f|} f \circ d^M$. If R is a commutative ring, then for any complex of R-modules M, the tensor product functor $-\otimes_R M\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is left adjoint to the internal hom functor $\mathrm{Hom}_R(M,-)$.

2.3. **Example: the homotopy category of a ring.** Fix a ring R. Before we can define our desired category in which quasi-isomorphisms are inverted, it is convenient to introduce a stepping stone towards this category, called the homotopy category of complexes. In this category we kill the null homotopic maps. Recall that a map $f\colon M \to N$ is null homotopic if it is chain homotopic to the zero map. We write $\mathrm{Null}(M,N)$ for the subgroup of $\mathrm{Hom}_{\mathrm{Ch}(R)}(M,N)$ consisting of the null homotopic maps.

The *homotopy category* $\mathsf{K}(R)$ is defined by having objects the chain complexes of R-modules, and morphisms given by the homotopy classes of chain maps, i.e.,

$$\mathrm{Hom}_{\mathsf{K}(R)}(M,N) = \mathrm{Hom}_{\mathrm{Ch}(R)}(M,N)/\mathrm{Null}(M,N).$$

Equivalently, the morphisms are the chain maps up to the equivalence relation of chain homotopy equivalence. The distinguished triangles in the homotopy category are the triangles which are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopic) to those of the form

$$M \xrightarrow{f} N \to C(f) \to \Sigma M$$

for some map of chain complexes $f\colon M \to N$.

Let us now prove that $\mathsf{K}(R)$ is a triangulated category. Due to the number of axioms, this is a bit of a slog, but it is worthwhile seeing the details spelled out.

Theorem 2.7. *Let R be a ring. The category $\mathsf{K}(R)$ with the triangles those which are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopic) to those of the form*

$$M \xrightarrow{f} N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M$$

for some map of chain complexes $f\colon M \to N$ is a triangulated category.

Proof of (TR0). For (TR0) it suffices to prove that $C(1_M)$ is null homotopic, in other words, the identity map on M is null homotopic for all M. Recall that $C(1_M)_n = M_n \oplus M_{n+1}$ with differential $(-d, d-1)$. Define $h_n \colon M_n \oplus M_{n+1} \to M_{n+1} \oplus M_{n+2}$ by $h_n(x,y) = (-y, 0)$. One easily verifies that this defines a chain homotopy from the identity on the cone of the identity to the zero map. Hence $C(1_M)$ is null homotopic. □

Proof of (TR1). This is immediate from the definition of the triangles in $\mathsf{K}(R)$. □

Proof of (TR2). Consider the triangle

$$M \xrightarrow{f} N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M.$$

We must show that the candidate triangle

$$N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M \xrightarrow{-\Sigma f} \Sigma N$$

is also a triangle. (The argument for rotating the other direction is analogous so we omit it.) So we show that taking the cone of i yields a triangle which is isomorphic to this candidate triangle.

Firstly we verify that $C(i)$ and ΣM are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopy equivalent). We will then show that this is compatible with the triangles. Recall that $i \colon N \to C(f)$ is defined by $n \mapsto (0, n)$. Therefore, $C(i)_n = N_{n-1} \oplus C(f)_n = N_{n-1} \oplus M_{n-1} \oplus N_n$ with differential

$$\begin{pmatrix} -d_N & 0 \\ -i & d_{C(f)} \end{pmatrix} = \begin{pmatrix} -d_N & 0 & 0 \\ 0 & -d_M & 0 \\ -\mathrm{id}_N & -f & d_N \end{pmatrix}.$$

Firstly we define a map $\alpha \colon \Sigma M \to C(i)$ by $\alpha(m) = (-f(m), m, 0)$. To verify this is a chain map we must show that

$$\begin{tikzcd}
M_{n-1} \arrow[r, "-d"] \arrow[d, "\begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix}"'] & M_{n-2} \arrow[d, "\begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix}"] \\
C(i)_n \arrow[r, "\begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ -1 & -f & d \end{pmatrix}"'] & C(i)_{n-1}
\end{tikzcd}$$

commutes. One calculates that

$$\begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ -1 & -f & d \end{pmatrix} \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} df \\ -d \\ 0 \end{pmatrix}$$

and therefore the square commutes so that α is indeed a chain map. We also define a map $\beta \colon C(i) \to \Sigma M$ by $(n, m, n') \mapsto m$ (i.e., the matrix $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$). One easily verifies that this is a chain map.

The composite $\beta\alpha$ is the identity since $\beta(\alpha(m)) = \beta(-f(m), m, 0) = m$. On the other hand, $\alpha\beta(n, m, n') = \alpha(m) = (-f(m), m, 0)$ so $\alpha\beta$ is not the identity. However, let us show that it is indeed chain homotopic to the identity. Define $h_j \colon C(i)_j \to C(i)_{j+1}$ by $h_j(n, m, n') = (n', 0, 0)$.

One then calculates that $dh + hd = \alpha\beta - 1$ so that $\alpha\beta$ is chain homotopic to the identity. Therefore $\alpha\colon \Sigma M \to C(i)$ is an isomorphism in $\mathsf{K}(R)$. Therefore it only remains to verify that the diagram

$$\begin{array}{ccccccc}
N & \xrightarrow{i} & C(f) & \xrightarrow{c} & \Sigma M & \xrightarrow{-\Sigma f} & \Sigma N \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} \\
N & \xrightarrow[i]{} & C(f) & \xrightarrow[\text{inc}]{} & C(i) & \xrightarrow[c']{} & \Sigma N
\end{array}$$

commutes in $\mathsf{K}(R)$ since then the top row is isomorphic (as a candidate triangle) to the bottom row which is a triangle. Hence it is by definition also a triangle as required.

The left hand square clearly commutes on the nose. The right most square also commutes on the nose, since $c'\alpha(m) = c'(-f(m), m, 0) = -f(m)$. For the middle square, $\alpha c(m, n') = \alpha(m) = (-f(m), m, 0)$ whereas $\text{inc}(m, n') = (0, m, n')$. Therefore the middle square does not commute in the category of chain complexes, but we will show that it does commute in $\mathsf{K}(R)$. Since $\alpha\beta$ is the identity in $\mathsf{K}(R)$ (as proved above), it suffices to verify that $\beta \circ \text{inc} = c$ instead, which is easy from the definitions. This completes the proof of (TR2). $\square$

Proof of (TR3). By definition of the triangles, we may assume that we are given the following solid diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{f} & N & \xrightarrow{i_f} & C(f) & \xrightarrow{c_f} & \Sigma X \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\
M' & \xrightarrow[g]{} & N' & \xrightarrow[i_g]{} & C(g) & \xrightarrow[c_g]{} & \Sigma X'.
\end{array}$$

in which the first square commutes, and we must define the dotted map w. The first square commuting, means that vf and gu are chain homotopic. Therefore, for each n there exists maps $h_n\colon N_n \to M'_{n+1}$ such that $gu - vf = dh + hd$. Define a map $w\colon C(f) \to C(g)$ by the matrix

$$\begin{pmatrix} u & 0 \\ h & v \end{pmatrix}.$$

We need to check that this is indeed a chain map, and that it makes both squares commute in the diagram. These are easy calculations which are left to the reader. $\square$

Proof of (TR4). We may assume that the triangles are 'standard' ones, so we assume the existence of a commuting diagram as follows, whose rows are triangles.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i_f} & C(f) & \xrightarrow{c_f} & \Sigma X \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{uf} & Z & \xrightarrow{i_{uf}} & C(uf) & \xrightarrow{c_{uf}} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma f} \\
Y & \xrightarrow{u} & Z & \xrightarrow{i_g} & C(u) & \xrightarrow{c_u} & \Sigma Y \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Sigma g} \\
C(f) & \xrightarrow{\alpha} & C(uf) & \xrightarrow{\beta} & C(u) & \xrightarrow{\gamma} & \Sigma C(f)
\end{array}$$

We must construct dotted maps so that the bottow row is also a triangle. By functoriality of the mapping cone as proved in (TR3), we have maps $\alpha\colon C(f) \to C(uf)$ and $\beta\colon C(uf) \to C(u)$ making the whole diagram commute. We define $\gamma\colon C(u) \to \Sigma C(f)$ (note that this equates to maps $Y_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1}$) by $(y, z) \mapsto (0, y)$. This clearly makes the bottom right square commute also.

Therefore it remains to verify that $C(f) \xrightarrow{\alpha} C(uf) \xrightarrow{\beta} C(u) \xrightarrow{\gamma} \Sigma C(f)$ is a triangle. Define $w\colon C(\alpha) \to C(u)$ by $w(x, y, x', z) = (y + f(x'), z)$, and define $\widetilde{w}\colon C(u) \to C(\alpha)$ by $\widetilde{w}(y, z) = (0, y, 0, z)$. Consider the diagrams

$$\begin{tikzcd} C(uf) \arrow[r, "\beta"] \arrow[d, "1"'] & C(u) \\ C(uf) \arrow[r, "i_\alpha"] & C(\alpha) \arrow[u, "w"'] \end{tikzcd} \qquad \begin{tikzcd} C(u) \arrow[r, "\gamma"] \arrow[d, "\widetilde{w}"'] & \Sigma C(f) \arrow[d, "1"] \\ C(\alpha) \arrow[r, "c_\alpha"] & \Sigma C(f) \end{tikzcd}$$

where $\beta(x, z) = (fx, z)$ (as in the proof of (TR3)). One easily checks from the definitions that both of these diagrams commute. Therefore it only remains to prove that w is a chain homotopy equivalence. The composite $w\widetilde{w}$ is equal to the identity. For the other composite, define

$$h_n\colon C(\alpha)_n = X_{n-1} \oplus Y_n \oplus X_n \oplus X_{n+1} \to C(\alpha)_{n+1} = X_n \oplus Y_{n+1} \oplus X_{n+1} \oplus X_{n+2}$$

by $h_n(x, y, x', z) = (x', 0, 0, 0)$. We leave it to the reader to check that this defines a chain homotopy showing that $\widetilde{w}w$ is homotopic to the identity. □

We end our discussion of the homotopy category of a ring, by proving it has a universal property. There is a functor $h\colon \mathrm{Ch}(R) \to \mathsf{K}(R)$ defined to be the identity on objects, and to send a map f to its equivalence class $[f]$ under the relation of chain homotopy.

Proposition 2.8. *Let $F\colon \mathrm{Ch}(R) \to \mathcal{C}$ be a functor which is homotopy invariant, i.e., if $f \simeq g$ then $F(f) = F(g)$. Then there exists a unique functor $\bar{F}\colon \mathsf{K}(R) \to \mathcal{C}$ making the diagram*

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, "F"] \arrow[d, "h"'] & \mathcal{C} \\ \mathsf{K}(R) \arrow[ur, dashed, "\bar{F}"'] & \end{tikzcd}$$

commute.

Proof. Uniqueness is immediate from the fact that h is the identity on objects, and is full. For existence, define $\bar{F}(M) = F(M)$ on objects, and $\bar{F}([f]) = F(f)$ on maps. We note that this is well-defined since F is homotopy invariant by assumption. □

We'd next like to define a category in which the quasi-isomorphisms are inverted. Before we can do this, we establish some more basic properties of general triangulated categories.

2.4. **Basic properties of triangulated categories.** A nice tool in triangulated categories which we will use throughout this course, is a version of the 5 lemma which we will prove in this section.

Definition 2.9. Let T be a pretriangulated category and A be an abelian category. An additive functor $H\colon \mathsf{T} \to \mathsf{A}$ is *homological* if for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, the induced sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

is exact. Such a functor which is contravariant is said to be *cohomological*.

The following lemma shows that one can also rephrase the definition of homological to be those additive functors which turn triangles into long exact sequences.

Lemma 2.10. *If $H\colon \mathsf{T} \to \mathsf{A}$ is a homological functor, then for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, applying H yields a long exact sequence*

$$\cdots \to H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \to \cdots$$

Proof. By (TR2), $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is also a triangle, so

$$HY \xrightarrow{H(g)} HZ \xrightarrow{H(h)} H(\Sigma X)$$

is exact. Repeating this procedure gives the claim. □

Lemma 2.11. *Let T be a pretriangulated category, and $A \in \mathsf{T}$. The functor* $\mathrm{Hom}_{\mathsf{T}}(A,-)\colon \mathsf{T} \to \mathsf{Ab}$ *is homological, and the functor* $\mathrm{Hom}_{\mathsf{T}}(-,A)\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ *is cohomological.*

Proof. The second claim follows from the first by duality so we prove only the first. So suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle. We need to show that

$$\mathrm{Hom}_{\mathsf{T}}(A,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathsf{T}}(A,Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathsf{T}}(A,Z)$$

is exact where $f_*(\theta) = f \circ \theta$ and similarly for g_*. Since $gf = 0$, it is clear that the image of f_* is contained in the kernel of g_*. Conversely, suppose that $g \circ \theta = 0$ where $\theta\colon A \to Y$. We have a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\theta"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-1"] & \Sigma A \arrow[d, "\Sigma\theta"] \\
Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & \Sigma X \arrow[r, "-\Sigma f"'] & \Sigma Y
\end{tikzcd}$$

in which the left square commutes as $g \circ \theta = 0$. The bottom row is a triangle by (TR2), and the top row is a triangle by (TR0) together with (TR2). Therefore, by (TR3) there exists a map $\Psi\colon \Sigma A \to \Sigma X$ making the diagram commute. Applying Σ^{-1}, one obtains that $\theta = f \circ \Sigma^{-1}\Psi$. Therefore θ is in the image of f_*, which completes the proof. □

In order to prove that certain candidate triangles are in fact triangles, it is helpful to introduce a certain subclass of homological functors, and a subclass of candidate triangles which interacts well with these homological functors.

Definition 2.12. A homological functor $H\colon \mathsf{T} \to \mathsf{A}$ is *decent* if A satisfies (AB4*) (that is, A has products and these products are exact), and H preserves products.

The relevant subclass of candidate triangles is then the following.

Definition 2.13. A candidate triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a *pretriangle* if for every decent homological functor $H\colon \mathsf{T} \to \mathsf{A}$ the induced sequence

$$\cdots \to H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}h)} HX \xrightarrow{H(f)} HY \xrightarrow{H(g)} HZ \xrightarrow{H(h)} H(\Sigma X) \to \cdots$$

is exact.

Note that every triangle is a pretriangle (Lemma 2.10) but the converse is not true.

Lemma 2.14. *Let* T *be a pretriangulated category, and*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'. \end{tikzcd}$$

be a morphism of pretriangles. If u *and* v *are isomorphisms, then so is* w.

Proof. Let $H\colon \mathsf{T} \to \mathsf{A}$ be a decent homological functor. Applying H gives a commutative diagram

$$\begin{tikzcd} HX \arrow[r, "H(f)"] \arrow[d, "H(u)"] & HY \arrow[r, "H(g)"] \arrow[d, "H(v)"] & HZ \arrow[r, "H(h)"] \arrow[d, "H(w)"] & H(\Sigma X) \arrow[r, "H(\Sigma f)"] \arrow[d, "H(\Sigma u)"] & H(\Sigma Y) \arrow[d, "H(\Sigma v)"] \\ HX' \arrow[r, "H(f')"'] & HY' \arrow[r, "H(g')"'] & HZ' \arrow[r, "H(h')"'] & H(\Sigma X') \arrow[r, "H(\Sigma f')"'] & H(\Sigma Y') \end{tikzcd}$$

whose rows are exact since H is decent. All of the columns except for the middle are isomorphisms, and therefore by the 5 lemma, $H(w)$ is also an isomorphism. For all $A \in \mathsf{T}$, the functor $\mathrm{Hom}_{\mathsf{T}}(A, -)\colon \mathsf{T} \to \mathsf{Ab}$ is a decent homological functor by Lemma 2.11, and therefore $\mathrm{Hom}_{\mathsf{T}}(A, w)$ is an isomorphism for all $A \in \mathsf{T}$ by the previous paragraph. As such, by the Yoneda lemma w is an isomorphism as required. □

Proposition 2.15 (The 5 lemma)**.** *Let* T *be a pretriangulated category, and*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'. \end{tikzcd}$$

be a morphism of triangles. If any two of u*,* v*, and* w *are isomorphisms, then so is the third.*

Proof. By applying (TR2) to rotate the triangles, it suffices to prove that w is an isomorphism when u and v are both isomorphisms. This is then an immediate corollary of Lemma 2.14. □

Beyond proving the 5 lemma for triangles, the 5 lemma for pretriangles provides a neat way to construct new triangles from old. Recall that by (TR2) we can always produce new triangles by rotation (up to sign), and we will now see that (co)products of triangles (when the (co)products exist termwise) also yield new triangles.

Lemma 2.16. *Let* T *be a pretriangulated category. Suppose that* $X_i \to Y_i \to Z_i \to \Sigma X_i$ *is a triangle for all* i*. If the products exist, then the induced diagram*

$$\prod X_i \to \prod Y_i \to \prod Z_i \to \Sigma \prod X_i$$

is a triangle.

Proof. Firstly note that the above diagram makes sense since Σ commutes with all limits as it is an equivalence of categories. We next show that this diagram is a pretriangle, so fix a decent homological functor $H\colon \mathsf{T} \to \mathsf{A}$.

For each i, there is a long exact sequence

$$\cdots \to H(\Sigma^{-1}Z_i) \to H(X_i) \to H(Y_i) \to H(Z_i) \to H(\Sigma X_i) \to \cdots$$

in A. Since products are exact in A as it is (AB4*),

$$\cdots \to \prod H(\Sigma^{-1}Z_i) \to \prod H(X_i) \to \prod H(Y_i) \to \prod H(Z_i) \to \prod H(\Sigma X_i) \to \cdots$$

is also exact. Since H commutes with products, we conclude that $\prod X_i \to \prod Y_i \to \prod Z_i \to \Sigma \prod X_i$ is a pretriangle. Let us now show that it is infact a triangle.

By (TR1), we may extend the map $\prod X_i \to \prod Y_i$ to a triangle

$$\prod X_i \to \prod Y_i \to C \to \Sigma \prod X_i.$$

So by usual projection onto factors, we have for each i, a commutative diagram

$$\begin{tikzcd}
\prod X_i \arrow[r] \arrow[d] & \prod Y_i \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & \Sigma \prod X_i \arrow[d] \\
X_i \arrow[r] & Y_i \arrow[r] & Z_i \arrow[r] & \Sigma X_i
\end{tikzcd}$$

so the dashed filler exists by (TR3). By universal property of the product, the maps $C \to Z_i$ assemble to give map $C \to \prod Z_i$, thus giving a commutative diagram

$$\begin{tikzcd}
\prod X_i \arrow[r] \arrow[d] & \prod Y_i \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & \Sigma \prod X_i \arrow[d] \\
\prod X_i \arrow[r] & \prod Y_i \arrow[r] & \prod Z_i \arrow[r] & \Sigma \prod X_i
\end{tikzcd}$$

Both rows are pretriangles, and therefore by Lemma 2.14 we see that $\bigoplus Z_i \to C$ is an isomorphism. Therefore $\prod X_i \to \prod Y_i \to \prod Z_i \to \Sigma \prod X_i$ is isomorphic to a triangle, and hence is itself a triangle by (TR0). □

Remark 2.17. A dual argument shows that coproducts of triangles are again triangles.

Finally we have the following result, which is a standard trick for showing that a map in a triangulated category is an isomorphism.

Proposition 2.18. *Let $\theta \colon X \to Y$ be a map in T. Then θ is an isomorphism if and only if there is a triangle $X \xrightarrow{\theta} Y \to 0 \to \Sigma X$.*

Proof. This is part of Exercise A.3. □

2.5. **Functors and subcategories.** We now turn to what the 'correct' notion of functors between, and subcategories of, triangulated categories are.

Definition 2.19. Let T and U be triangulated categories. A *triangulated functor* is an additive functor $F \colon \mathsf{T} \to \mathsf{U}$ together with a natural isomorphism $\phi \colon F\Sigma \xrightarrow{\sim} \Sigma F$ such that for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in T, the candidate triangle

$$FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \xrightarrow{\phi_X \circ F(h)} \Sigma FX$$

is a triangle in U.

Example 2.20. The shift functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$ is a triangulated functor. The identity gives a natural isomorphism $\Sigma^2 \xrightarrow{\sim} \Sigma^2$, and

$$\Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2 X$$

is a triangle by three applications of (TR2). It is straightforward to see that this triangle is isomorphic to the candidate triangle

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \xrightarrow{\Sigma h} \Sigma^2 X$$

and hence the latter is also a triangle.

Definition 2.21. A full additive subcategory $\mathcal{S}$ of T is a *triangulated subcategory* if it is closed under isomorphisms, shifts, and triangles.

2.6. **Example: the derived category of a ring.** Returning to our motivation then, we want a category which contains all the resolutions of modules, and in which quasi-isomorphisms are isomorphisms. Since injective resolutions and projective resolutions point in opposite directions, we consider all chain complexes (i.e., rather than just those bounded above or below 0). The derived category of a ring R is the universal category in which quasi-isomorphisms of complexes are inverted. We denote this category by $\mathsf{D}(R) := \mathsf{K}(R)[\text{quasi isos}^{-1}]$. We give a precise construction of this below.

Remark 2.22. This construction of the derived category leads to some set-theoretic discussions, namely, why are the hom sets actually sets? For the purposes of this course we ignore this, and just remark that one can give alternative constructions bypassing this issue.

We can now make the definition of the derived category. We will verify that everything is well-defined afterwards.

Definition 2.23. The objects of $\mathsf{D}(R)$ are the same as the objects of $\mathsf{K}(R)$, that is, they are the chain complexes of R-modules. The morphisms in $\mathsf{D}(R)$ are equivalence classes of rooves, defined as follows. Let $M, N \in \mathsf{D}(R)$. A *roof* from M to N is a pair of chain maps

$$\begin{tikzcd} & Z \arrow[dl, "s"', "\simeq"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where $Z \in \mathsf{D}(R)$ and s is a quasi-isomorphism. Two rooves $(M \leftarrow Z \to N)$ and $(M \leftarrow Z' \to N)$ are equivalent if there exists another roof $(M \leftarrow W \to N)$ and maps $W \to Z$ and $W \to Z'$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ M & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & N \\ & Z' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. The hom sets of $\mathsf{D}(R)$ are rooves up to this equivalence relation. Composition of rooves is defined below after Lemma 2.25.

Throughout this section we use the following trivial observation: if $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a triangle in $\mathsf{K}(R)$, then f is a quasi-isomorphism if and only if Z is acyclic (i.e., $H_*(Z) = 0$). To see this, recall that f is a quasi-isomorphism if and only if the mapping cone $C(f)$ is acyclic. Since any triangle in $\mathsf{K}(R)$ is isomorphic to one in which the third term is the mapping cone, the observation follows.

Lemma 2.24 (Cancellation)**.** *For maps $f, g\colon X \to Y$ in $\mathsf{K}(R)$, the following are equivalent:*

(1) $sf = sg$ for some quasi-isomorphism s with source Y;
(2) $ft = gt$ for some quasi-isomorphism t with codomain X.

Proof. Given a quasi-isomorphism $s\colon Y \to Y'$ with $sf = sg$, we have a triangle $Z \xrightarrow{k} Y \xrightarrow{s} Y' \to \Sigma Z$ by (TR1) (and (TR2)). The functor $\operatorname{Hom}_{\mathsf{K}(R)}(X, -)$ is a homological functor by Lemma 2.11, so

$$\operatorname{Hom}_{\mathsf{K}(R)}(X, Z) \xrightarrow{k_*} \operatorname{Hom}_{\mathsf{K}(R)}(X, Y) \xrightarrow{s_*} \operatorname{Hom}_{\mathsf{K}(R)}(X, Y)$$

is exact. Since $s(f - g) = 0$, that is, $f - g \in \ker(s_*)$, there is a map $h\colon X \to Z$ such that $f - g = kh$. Consider the triangle $X' \xrightarrow{t} X \xrightarrow{h} Z \to \Sigma X'$ which exists by (TR1). As $ht = 0$, we have $(f - g)t = kht = 0$, and hence $ft = gt$ as required. So it remains to see that t is a quasi-isomorphism. Since s is a quasi-isomorphism, Z is acyclic by its defining triangle, and hence t is also a quasi-isomorphism. The other direction is analogous. □

Lemma 2.25 (The Ore Condition)**.** *Given a quasi-isomorphism $s\colon Y' \to Y$ and a map $f\colon X \to Y$ in $\mathsf{K}(R)$, there exists a commutative diagram in $\mathsf{K}(R)$*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "s'"', "\simeq"] & Y' \arrow[d, "\simeq"', "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in which s' is also a quasi-isomorphism.

Proof. By (TR1) there is a triangle $Y' \xrightarrow{s} Y \xrightarrow{u} Z \to \Sigma Y'$. By (TR1) together with (TR2) there is also a triangle $X' \xrightarrow{s'} X \xrightarrow{uf} Z \to \Sigma X'$. By (TR3), there is a map $f'\colon X' \to Y'$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "s'"] \arrow[d, dashed, "f'"] & X \arrow[r, "uf"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "\mathrm{id}"] & \Sigma X' \arrow[d, dashed, "\Sigma f'"] \\ Y' \arrow[r, "s"'] & Y \arrow[r, "u"] & Z \arrow[r] & \Sigma X' \end{tikzcd}$$

commutes. Since s is a quasi-isomorphism, Z is acyclic, and hence s' is also a quasi-isomorphism. □

Lemma 2.26. *The relation on rooves defined above is an equivalence relation.*

Proof. Reflexivity and symmetry are clear, so it suffices to prove transitivity. Suppose that we have rooves $R_1 \sim R_2$ and $R_2 \sim R_3$. Spelling this out, we have two commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ L & W \arrow[u, "p"] \arrow[d, "q"] & M \\ & Z' \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ L & U \arrow[u, "r"] \arrow[d, "v"] & M \\ & V \arrow[ul, "u"] \arrow[ur, "h"'] & \end{tikzcd}$$

with sp, tq, tr and uv all quasi-isomorphisms.

By the Ore Condition (Lemma 2.25), we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "z"'] & U \arrow[d, "tr"] \\ W \arrow[r, "sp"'] & L \end{tikzcd}$$

in which z is a quasi-isomorphism.

Note that $tqz = spz = tra$, and hence by Lemma 2.24 there is a quasi-isomorphism $\theta\colon H \to R$ such that $qz\theta = ra\theta$. Now consider the diagram

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ L & H \arrow[u, "pz\theta"] \arrow[d, "va\theta"] & M \\ & V \arrow[ul, "u"] \arrow[ur, "h"'] & \end{tikzcd}$$

The left hand square commutes since

$$spz\theta = tqz\theta = tra\theta = vua\theta$$

and the right hand square commutes as

$$fpz\theta = gqz\theta = gra\theta = hva\theta.$$

Finally, note that $spz\theta$ is a quasi-isomorphism since sp, z, and θ are. Hence $R_1 \sim R_3$ as required. □

Using the previous lemma we may now define composition in $\mathsf{D}(R)$. Given two rooves $(L \leftarrow Z \xrightarrow{f} M)$ and $(M \leftarrow Z' \xrightarrow{g} N)$, their composite gf is the roof $(L \leftarrow P \to N)$ defined via the diagram

$$\begin{tikzcd} & P \arrow[r, "f'"] \arrow[d, "\simeq"] & Z' \arrow[r, "g"] \arrow[d, "\simeq"'] & N \\ L & Z \arrow[l, "\simeq"] \arrow[r, "f"'] & M & \end{tikzcd}$$

where the existence of the commutative diagram is provided by the Ore condition (Lemma 2.25).

By tedious diagram chasing (left to the interested reader), one may check that the composition operation defined above is well-defined, i.e., unique up to equivalence of rooves, associative, and that the roof $X \leftarrow X \rightarrow X$ with all maps the identity, is the identity map on X. This shows that $\mathsf{D}(R)$ is a category, but we want to now show that it is infact a *triangulated* category. First we need to deal with the additive structure. In order to do this, we note a convenient way to compare rooves.

Lemma 2.27. *Let $R_i = (X \leftarrow Z_i \rightarrow Y_i)$ be a finite collection of rooves. Then there exists a quasi-isomorphism $Z \rightarrow X$ such that each R_i is equivalent to the roof $(X \leftarrow Z \rightarrow Y_i)$.*

Proof. In the case when we have two rooves, the Ore condition (Lemma 2.25) says that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "t"'] & Z_2 \arrow[d, "s_2"] \\ Z_1 \arrow[r, "s_1"'] & X \end{tikzcd}$$

in which t is a quasi-isomorphism. It is straightforward to see that the roof $X \leftarrow Z \rightarrow Y_i$ where $Z \rightarrow X$ is the composite $s_1 t$ does the trick. The case with more rooves follows by induction. $\square$

Proposition 2.28. *The derived category $\mathsf{D}(R)$ is additive.*

Proof. Let us just give a sketch of how to define a group operation on the set of rooves from X to Y up to equivalence. Given rooves $(X \leftarrow Z_1 \rightarrow Y)$ and $(X \leftarrow Z_2 \rightarrow Y)$ we may replace them up to equivalence by rooves $(X \leftarrow Z \xrightarrow{f_1} Y)$ and $(X \leftarrow Z \xrightarrow{f_2} Y)$ as in Lemma 2.27. The addition of these rooves is now defined by $(X \leftarrow Z \xrightarrow{f_1+f_2} Y)$. We leave to the reader that this indeed defines an additive structure. $\square$

We are now almost ready to prove that $\mathsf{D}(R)$ is a triangulated category. The missing ingredient is the following functor and a couple of its properties. We define $Q\colon \mathsf{K}(R) \rightarrow \mathsf{D}(R)$ as the functor which is the identity on objects, and which takes a morphism $f\colon X \rightarrow Y$ to the roof $X = X \xrightarrow{f} Y$.

Lemma 2.29. *The functor $Q\colon \mathsf{K}(R) \rightarrow \mathsf{D}(R)$ sends quasi-isomorphisms to isomorphisms.*

Proof. Let $f\colon X \rightarrow Y$ be a quasi-isomorphism. We claim that $R = (f, \mathrm{id}_X)\colon Y \leftarrow X = X$ is an inverse to $Q(f)$. It is easy to check that $R \circ Q(f) = \mathrm{id}_X$. Now one checks that $Q(f) \circ R = (f, f)\colon Y \leftarrow X \rightarrow Y$. The diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f"] & \\ Y & & Y \\ & Y \arrow[ul, "\mathrm{id}_Y"] \arrow[ur, "\mathrm{id}_Y"'] & \end{tikzcd}$$

shows that the roof $Q(f) \circ R$ is equivalent to id_Y, and hence $Q(f)$ is an isomorphism as claimed. $\square$

Lemma 2.30. *Let $f, g\colon M \rightarrow N$ in $\mathsf{K}(R)$. Then $Q(f) = Q(g)$ if and only if there exists a quasi-isomorphism t with codomain M such that $ft = gt$, if and only if there exists a quasi-isomorphism s with domain N such that $sf = sg$.*

Proof. If $Q(f) = Q(g)$, then there is a roof $M \leftarrow Z \rightarrow N$ and maps $p, q\colon Z \rightarrow M$ such that the diagram

$$\begin{tikzcd}
& M \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\
M & Z \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "p"'] \arrow[d, "q"] & N \\
& M \arrow[ul, "\mathrm{id}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Commutativity of the left square shows that $p = q$ and that p is a quasi-isomorphism. Therefore we have a quasi-isomorphism $p\colon Z \rightarrow M$ such that $fp = gp$. Therefore by the cancellation property (Lemma 2.24) there is a quasi-isomorphism s with domain M such that $sf = sg$. The converse is analogous. $\square$

Theorem 2.31. *Let R be a ring. Then the derived category $\mathsf{D}(R)$ in which the distinguished triangles are the triangles which are isomorphic in $\mathsf{D}(R)$ to those of the form*

$$M \xrightarrow{f} N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M$$

for some map of chain complexes $f\colon M \rightarrow N$, is a triangulated category. In other words, the triangles are those which are isomorphic to the images of triangles under Q.

Define a shift functor on objects as in $\mathsf{K}(R)$ and on rooves by shifting each leg of the roof. Let us now go through each of the axioms in turn.

Proof of (TR0). This is immediate since (TR0) holds in $\mathsf{K}(R)$. $\square$

Proof of (TR1). Let $f\colon X \rightarrow Y$ in $\mathsf{D}(R)$. Choose a presentation $(s, a)\colon X \leftarrow Z \rightarrow Y$ of f as a roof. By the Ore condition (Exercise A.5), we obtain a commutative square

$$\begin{tikzcd}
Z \arrow[r, "a"] \arrow[d, "s"'] & Y \arrow[d, "t"] \\
X \arrow[r, "b"'] & U
\end{tikzcd}$$

in which t is also a quasi-isomorphism. Using (TR1) and (TR3) in $\mathsf{K}(R)$, we have a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "a"] \arrow[d, "s"] & Y \arrow[r, "i_a"] \arrow[d, "t"] & C(a) \arrow[r, "c_a"] \arrow[d, "u"] & \Sigma Z \arrow[d, "\Sigma s"] \\
X \arrow[r, "b"'] & U \arrow[r, "i_b"'] & C(b) \arrow[r, "c_b"'] & \Sigma X
\end{tikzcd}$$

in which the rows are distinguished triangles in $\mathsf{K}(R)$. Taking the long exact sequence in homology proves that u is a quasi-isomorphism since both s and t are. We now apply Q to this diagram, and may consider the diagram

$$\begin{tikzcd}
Z \arrow[r, "Q(a)"] \arrow[d, "Q(s)"] & Y \arrow[r, "Q(i_a)"] \arrow[d, "\mathrm{id}"] & C(a) \arrow[r, "Q(c_a)"] \arrow[d, "Q(u)"] & \Sigma Z \arrow[d, "\Sigma Q(s)"] \\
X \arrow[r, "f"'] & Y \arrow[r, "Q(i_b \circ t)"'] & C(b) \arrow[r, "Q(c_b)"'] & \Sigma X
\end{tikzcd}$$

in $\mathsf{D}(R)$. The left hand square commutes since (s, a) is a presentation of f, and the other squares commute as they are the images of commuting squares under Q. Each of the vertical maps is an isomorphism since s and u are quasi-isomorphisms, and hence we have shown that f fits into a triangle which is isomorphic to a standard one, and hence is distinguished. ☐

Proof of (TR2). Rotation of triangles is immediate from rotation in $\mathsf{K}(R)$. ☐

Proof of (TR3). The existence of fillers is not impacted by isomorphic triangles, so we may assume that we are given a diagram

$$\begin{tikzcd}
X \arrow[r, "Q(f)"] \arrow[d, "u"] & Y \arrow[r, "Q(g)"] \arrow[d, "v"] & Z \arrow[r, "Q(h)"] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r, "Q(f')"'] & Y' \arrow[r, "Q(g')"'] & Z' \arrow[r, "Q(h')"'] & \Sigma X'.
\end{tikzcd}$$

in which the left square commutes, the rows are distinguished triangles, and we need to construct a dashed map making the diagram commute. We may write the vertical maps as rooves, to give the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\
A \arrow[u, "s"] \arrow[d, "a"] & B \arrow[u, "t"] \arrow[d, "b"] & & \Sigma A \arrow[u, "\Sigma u"'] \arrow[d, "\Sigma a"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

in $\mathsf{K}(R)$. By the Ore condition (Lemma 2.25), we have a commutative square

$$\begin{tikzcd}
A \arrow[r, "fs"] & Y \\
A' \arrow[u, "t'"] \arrow[r, "c"'] & B \arrow[u, "t"']
\end{tikzcd}$$

in which t' is a quasi-isomorphism.

It is easy to see that the rooves $(s, a)\colon X \leftarrow A \rightarrow X'$ and $(st', at')\colon X \leftarrow A' \rightarrow X'$ are equivalent so we may replace the left hand column in the diagram and consider the new diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
A' \arrow[u, "st'"] \arrow[r, "c"] \arrow[d, "at'"'] & B \arrow[u, "t"'] \arrow[d, "b"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

The top square commutes by definition of c and t', but the bottom square need not commute in $\mathsf{K}(R)$. However, let us show that it commutes after applying Q. Since Q sends quasi-isomorphisms to isomorphisms by Lemma 2.29 we have

$$Q(f') \circ Q(a) \circ Q(t') = Q(f') \circ Q(a) \circ Q(s)^{-1} \circ Q(s) \circ Q(t').$$

Since $vQ(f) = Q(f')u$ this is in turn equal to

$$Q(b) \circ Q(t)^{-1} \circ Q(f) \circ Q(s) \circ Q(t') = Q(b) \circ Q(t)^{-1} \circ Q(t) \circ Q(c) = Q(b) \circ Q(c).$$

Therefore by Lemma 2.30, there exists a quasi-isomorphism $w\colon A'' \to A'$ such that $fat'w = bcw$. We may now again replace the left hand column in the diagram and apply (TR1) in $\mathsf{K}(R)$ to obtain the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\
A'' \arrow[u, "stw'"] \arrow[d, "atw'"'] \arrow[r, "cw"] & B \arrow[u, "t"'] \arrow[d, "b"] \arrow[r, "i"] & C \arrow[r, "j"] & \Sigma A \arrow[u, "\Sigma u"'] \arrow[d, "\Sigma a"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

in which the left square is commutative in $\mathsf{K}(R)$. Therefore, by (TR3) in $\mathsf{K}(R)$, there are fillers as indicated in the diagram below:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\
A'' \arrow[u, "stw'"] \arrow[d, "atw'"'] \arrow[r, "cw"] & B \arrow[u, "t"'] \arrow[d, "b"] \arrow[r, "i"] & C \arrow[u, dashed, "\alpha"'] \arrow[d, dashed, "\beta"] \arrow[r, "j"] & \Sigma A \arrow[u, "\Sigma u"'] \arrow[d, "\Sigma a"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

The only thing that remains is to check α is a quasi-isomorphism, so that $(\alpha, \beta)\colon Z \leftarrow C \to Z'$ does represent a map in $\mathsf{D}(R)$. Just as in the proof of (TR2), taking the long exact sequence in homology shows that α is a quasi-isomorphism since stw and t are both quasi-isomorphisms. □

Proof of (TR4). We omit this for brevity. □

Lemma 2.32. *Consider a roof $(s, f)\colon M \leftarrow Z \to N$. Prove that (s, f) is an isomorphism in $\mathsf{D}(R)$ if and only if f is a quasi-isomorphism, in which case $(s, f)^{-1} = (f, s)$.*

Proof. This is Exercise A.6. □

We may now say the precise way in which this construction inverts isomorphisms.

Theorem 2.33. *The functor $Q\colon \mathsf{K}(R) \to \mathsf{D}(R)$ is triangulated and has the property that $Q(s)$ is an isomorphism if s is a quasi-isomorphism. Moreover, given any functor $F\colon \mathsf{K}(R) \to \mathsf{T}$ such that $F(s)$ is an isomorphism if s is a quasi-isomorphism, there exists a unique functor $F'\colon \mathsf{D}(R) \to \mathsf{T}$ such that $F' \circ Q = F$. Moreover, if F is triangulated, then F' is also triangulated.*

Proof. The functor Q is triangulated by definition, and sends quasi-isomorphisms to isomorphisms by Lemma 2.29. For the universal property, we first prove uniqueness, so suppose that there exists such an F'. Since Q is the identity on objects, the value of F' on objects is determined by F. For maps, consider a roof (s, f) in $\mathsf{D}(R)$. Then

$$(s, f) = (\mathrm{id}, f) \circ (s, \mathrm{id}) = (\mathrm{id}, f) \circ (\mathrm{id}, s)^{-1} = Q(f) \circ Q(s)^{-1}$$

where we used Lemma 2.32 for the second equality. Therefore $F'(s, f) = F'(Q(f)) \circ F'(Q(s)^{-1}) = F(f) \circ F(s)^{-1}$, so the value of F' on maps is also determined by F.

We now show existence. Define $F'(M) = F(M)$, and $F'(s,f) = F(s)^{-1} \circ F(f)$ where we used the assumption that F sends quasi-isomorphisms to isomorphisms to deduce the existence of $F(s)^{-1}$. This is functorial since F is. (Technically we need to check that $F'(s,f)$ is invariant under equivalence of rooves; we omit this.)

We finally show that F' is triangulated if F is. Firstly, note that $(F\Sigma)' = F'\Sigma$: by uniqueness it suffices to verify that $F'\Sigma\colon \mathsf{D}(R) \to \mathsf{T}$ sends quasi-isomorphisms to isomorphisms, and satisfies $F'\Sigma Q = F\Sigma$, both of which are clear. Similarly, $(\Sigma F)' = \Sigma F'$. Therefore, $\Sigma F' = F'\Sigma$, so that F' behaves well with the shift. So suppose that

$$L \xrightarrow{Qf} M \xrightarrow{Qg} N \xrightarrow{Qh} \Sigma L$$

is a triangle in $\mathsf{D}(R)$; recall that by definition, all triangles in $\mathsf{D}(R)$ take this form. We need to check that the image under F' is a triangle in T. However, the image under T may be identified with

$$FL \xrightarrow{Ff} FM \xrightarrow{Fg} FN \xrightarrow{Fh} F\Sigma L = \Sigma FL$$

since $F'Q = F$. This is a triangle in T since F is assumed to be triangulated, and hence F' is also triangulated. □

Remark 2.34. Everything above can be vastly generalized to the case of a triangulated category T and a set S of morphisms in T which is closed under composition, and satisfies cancellation and the Ore condition. In other words, after we had verified that $\mathsf{T} = \mathsf{K}(R)$ and $S = \{\text{quasi-isomorphisms}\}$ satisfied cancellation and the Ore condition, we never used that we were working with complexes again.

The previous universal property makes precise the statement that the derived category is the universal home for homological algebra: any operation on complexes which inverts quasi-isomorphisms factors uniquely through the derived category. One can also combine this with Proposition 2.8 to construct functors on derived categories. We will use this later on when we discuss tensor-triangulated categories.

Lemma 2.35. *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules, then there is a map $h\colon N \to \Sigma L$ such that $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L$ is a triangle in $\mathsf{D}(R)$, where L, M and N are viewed as complexes in degree 0.*

Proof. The complex $C(f) = (L \xrightarrow{-f} M)$ in degrees 1 and 0 by definition. The diagram

$$\begin{tikzcd} L \arrow[r, "-f"] \arrow[d, "0"'] & M \arrow[d, "g"] \\ 0 \arrow[r] & N \end{tikzcd}$$

commutes by exactness, and hence the vertical maps define a chain map $\phi\colon C(f) \to N$. This chain map is a quasi-isomorphism: $H_1(C(f)) = \ker(-f) = 0$ by exactness, and again by exactness, we have $H_0(C(f)) = M/\ker(g)$ which is isomorphic by g to N. Hence ϕ is a quasi-isomorphism.

Therefore we may define a map $h\colon N \to \Sigma L$ in $\mathsf{D}(R)$ by $c \circ \phi^{-1}$ where $c\colon C(f) \to \Sigma L$ is the canonical map. The diagram

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "i"] \arrow[d, equal] & C(f) \arrow[r, "c"] \arrow[d, "\phi"] & \Sigma L \arrow[d, equal] \\
L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r, "h"'] & \Sigma L
\end{tikzcd}$$

commutes by construction, and hence the bottom row is a triangle in $\mathsf{D}(R)$. $\square$

Remark 2.36. The same argument may be generalised to prove that short exact sequences of complexes (rather than just of modules) give triangles in the derived category.

2.7. **Calculating maps in the derived category.** We end with some fundamental results about the maps in the derived category. Before we do so we introduce truncation functors; these allow inductive arguments by gluing together complexes one piece at a time. There are two forms of truncations: the so-called smart and brutal truncations.

Definition 2.37 (Brutal truncation)**.** For a complex $M \in \mathsf{D}(R)$ and integer n, the brutal truncation above n is $(t_{\geq n}M)_i = M_i$ if $i \geq n$ and 0 otherwise. Similarly, we define $(t_{\leq n}M)_i = M_i$ if $i \leq n$ and 0 otherwise. There is a short exact sequence $0 \to t_{\leq n}M \to M \to t_{\geq n+1}M \to 0$ of complexes, and hence a triangle $t_{\leq n}M \to M \to t_{\geq n+1}M \to \Sigma t_{\leq n}M$ in $\mathsf{D}(R)$. Note that this does not behave well with respect to homology, for example, the canonical map $t_{\leq n}M \to M$ does not induce an isomorphism in homology in degree n.

Definition 2.38 (Smart truncation)**.** To rectify the poor behaviour of the brutal truncation with respect to homology, it is convenient to consider an alternative form of truncation called the smart truncation. Fix a complex M and an integer M. We define complexes $\tau_{\leq n}M$ and $\tau_{\geq n}M$ as follows:

$$(\tau_{\geq n}M)_i = \begin{cases} M_i & i \geq n+1 \\ \ker(d_n\colon M_n \to M_{n-1}) & i = n \\ 0 & \text{otherwise} \end{cases}$$

$$(\tau_{\leq n}M)_i = \begin{cases} M_i & i \leq n-1 \\ \operatorname{coker}(d_{n+1}\colon M_{n+1} \to M_n) & i = n \\ 0 & \text{otherwise} \end{cases}$$

There are canonical maps $\tau_{\geq n}M \to M$ and $M \to \tau_{\leq n}M$ which induce isomorphisms on homology in degrees $\geq n$ and $\leq n$ respectively. There is a triangle

$$\tau_{\geq n}M \to M \to \tau_{\leq n-1}M \to \Sigma\tau_{\geq n}M$$

in $\mathsf{D}(R)$. This last claim needs justification. Consider the complex Q defined by

$$Q_i = \begin{cases} M_n/\ker(d_n) & i = n \\ M_i & i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

The obvious map $\phi\colon Q \to \tau_{\leq n-1}M$ is a quasi-isomorphism. Then there is an evident short exact sequence of complexes $0 \to \tau_{\geq n}M \to M \to Q \to 0$ which gives rise to a triangle

$$\tau_{\geq n}M \to M \to Q \xrightarrow{\theta} \Sigma\tau_{\geq n}M$$

in $\mathsf{D}(R)$ by Lemma 2.35. We may then define a map $\tau_{\leq n-1}M \to \Sigma\tau_{\geq n}M$ by $\theta\phi^{-1}$. The diagram

$$\begin{tikzcd}
\tau_{\geq n}M \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & Q \arrow[r, "\theta"] \arrow[d, "\phi"] & \Sigma\tau_{\geq n}M \arrow[d, equal] \\
\tau_{\geq n}M \arrow[r] & M \arrow[r] & \tau_{\leq n-1}M \arrow[r, "\theta\phi^{-1}"'] & \Sigma\tau_{\geq n}M
\end{tikzcd}$$

commutes, and the top row is triangle; hence the bottom row is also a triangle as required. Note that $\tau_{\geq n}\tau_{\leq n}M \simeq H_nM[n]$, so that a special case of the above triangle is $H_nM[n] \to \tau_{\leq n}M \to \tau_{\leq n-1}M$.

With truncations introduced, we may proceed with calculating maps in the derived category.

Lemma 2.39. *Let I be a bounded above complex of injective R-modules, and X be a bounded above complex. If X is acyclic, then any map $f\colon X \to I$ is null homotopic.*

Proof. Left as an exercise (Exercise A.9). □

Lemma 2.40. *Let I be a bounded above complex of injective R-modules, and X be a bounded above complex. If $s\colon I \to X$ is a quasi-isomorphism, then there is a chain map $t\colon X \to I$ such that $ts = \mathrm{id}$ in $\mathsf{K}(R)$.*

Proof. Consider the triangle $I \xrightarrow{s} X \xrightarrow{i} C(s) \xrightarrow{c} \Sigma I$. Since s is a quasi-isomorphism, the mapping cone $C(s)$ is acyclic, and hence by Lemma 2.39 the map $c\colon C(s) \to \Sigma I$ is null homotopic. Since c is null homotopic we have maps $h_n\colon C(s)_n \to (\Sigma I)_{n+1}$, which we may write as matrices $\begin{pmatrix} a_n & b_n \end{pmatrix}$ where $a_n\colon I_{n-1} \to I_n$ and $b_n\colon X_n \to I_n$. Since c is null, we have $c = d_{\Sigma I}h + hd_{C(s)}$. Writing this out in matrices, we have

$$\begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} -da & -db \end{pmatrix} + \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} -d & 0 \\ s & d \end{pmatrix} = \begin{pmatrix} bs - da - ad & bd - db \end{pmatrix}.$$

The second component says that $bd = db$ so that $b\colon X \to I$ is a chain map. The first component says that $bs - \mathrm{id} = da + ad$ and hence bs is chain homotopic to the identity as required. □

Proposition 2.41. *The localisation functor $Q\colon \mathsf{K}(R) \to \mathsf{D}(R)$ induces an isomorphism on hom groups*

$$\mathrm{Hom}_{\mathsf{K}(R)}(M, I) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}(R)}(M, I)$$

for all $M \in \mathsf{K}(R)$ whenever I is a bounded above complex of injective R-modules.

Proof. For injectivity, if $Q(f) = Q(g)$ then there is a quasi-isomorphism $s\colon I \to X$ such that $sf = sg$ by Lemma 2.30. However, by Lemma 2.40, there exists a map $t\colon X \to I$ such that $ts = \mathrm{id}_X$ in $\mathsf{K}(R)$. (Here note that we may assume that X is bounded above; it has to be bounded above in homology as I is, so write $\sup(X)$ for the highest integer for which the homology is non-zero. Then the canonical map $X \to \tau_{\leq \sup(X)}X$ is a quasi-isomorphism, so we may replace X with $\tau_{\leq \sup(X)}$.) Therefore $f = tsf = tsg = g$ so the map is injective.

For surjectivity, take any roof (here we use Exercise A.7)

$$\begin{tikzcd}
 & X & \\
M \arrow[ur, "f"] & & I \arrow[ul, "s"', "\sim"]
\end{tikzcd}$$

By Lemma 2.40 there exists a quasi-isomorphism $t\colon X \to I$ such that $ts = \mathrm{id}$ (as before we may assume that X is bounded above). Therefore the diagram

$$\begin{tikzcd} & X \arrow[dd, "t"] & \\ M \arrow[ur, "f"] \arrow[dr, "tf"'] & & I \arrow[ul, "s"', "\sim"] \arrow[dl, "\mathrm{id}"] \\ & M & \end{tikzcd}$$

shows that our starting roof is equivalent to $Q(tf)$. Hence the map is surjective as required. $\square$

Lemma 2.42. *Let M be an R-module and I be a bounded above complex of injectives. Then*

$$\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i M, I) = \frac{\ker\left(\mathrm{Hom}_R(M, I_i) \xrightarrow{(d_i)_*} \mathrm{Hom}_R(M, I_{i-1})\right)}{\mathrm{im}\left(\mathrm{Hom}_R(M, I_{i+1}) \xrightarrow{(d_{i+1})_*} \mathrm{Hom}_R(M, I_i)\right)}.$$

Proof. By Proposition 2.41 we have $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i M, I) = \mathrm{Hom}_{\mathsf{K}(R)}(\Sigma^i M, I)$. The latter of these consists of homotopy classes of chain maps

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & I_{i+1} \arrow[r, "d_{i+1}"'] & I_i \arrow[r, "d_i"'] & I_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

A morphism of R-modules $f\colon M \to I_i$ gives such a chain map if and only if the composite $M \xrightarrow{f} I_i \xrightarrow{d_i} I_{i-1}$ is zero. Therefore

$$\mathrm{Hom}_{\mathrm{Ch}(R)}(\Sigma^i M, I) = \ker\left(\mathrm{Hom}_R(M, I_i) \xrightarrow{(d_i)_*} \mathrm{Hom}_R(M, I_{i-1})\right).$$

Now this chain map is null homotopic if and only if there exists a map $h\colon M \to I_{i+1}$ such that $f = d_{i+1}h$. Therefore the subgroup of $\mathrm{Hom}_{\mathrm{Ch}(R)}(\Sigma^i M, I)$ consisting of null homotopic maps is the image of $(d_{i+1})_*\colon \mathrm{Hom}_R(M, I_{i+1}) \to \mathrm{Hom}_R(M, I_i)$. This completes the proof of the claim. $\square$

Proposition 2.43. *Let R be a ring and M be a complex of R-modules. Then*

$$H_i(M) = \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, M).$$

Proof. Let I be an injective resolution of M, i.e., we have an isomorphism $M \xrightarrow{\sim} I$ in $\mathsf{D}(R)$. We then have $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, M) = \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, I)$. By Lemma 2.42,

$$\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, I) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}) = H_i(I) = H_i(M)$$

as required. $\square$

Proposition 2.44. *Let M and N be R-modules viewed as complexes in degree 0. Then*

$$\mathrm{Ext}^i_R(M, N) = \mathrm{Hom}_{\mathsf{D}(R)}(M, \Sigma^i N).$$

Proof. Let I be an injective resolution of N. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}(R)}(M,\Sigma^i N) &= \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^{-i}M, I) && \text{as } N \simeq I \\
&= \frac{\ker\left(\mathrm{Hom}_R(M,I_{-i}) \xrightarrow{(d_{-i})_*} \mathrm{Hom}_R(M,I_{-i-1})\right)}{\mathrm{im}\left(\mathrm{Hom}_R(M,I_{-i+1}) \xrightarrow{(d_{-i+1})_*} \mathrm{Hom}_R(M,I_{-i})\right)} && \text{by Lemma 2.42} \\
&= H_{-i}\mathrm{Hom}_R(M,I) && \text{by definition of } H_{-i} \\
&= \mathrm{Ext}^i_R(M,N) && \text{by definition of Ext}
\end{aligned}$$

as required. ∎

3. Finiteness in triangulated categories

In this section we introduce and study various notions of what it means for an object of a triangulated category to be small or finite. These are fundamental notions, and much of the theory of triangulated categories is dependent on these ideas.

3.1. Compact objects.

In this section we study certain types of 'small' objects in triangulated categories. It is often the case that the whole triangulated category is generated by small objects, and much of the theory of triangulated categories relies upon such assumptions.

Definition 3.1. Let T be a triangulated category which has coproducts. An object $X \in \mathsf{T}$ is said to be *compact* if the natural map

$$\bigoplus \mathrm{Hom}_{\mathsf{T}}(X, Y_i) \to \mathrm{Hom}_{\mathsf{T}}(X, \bigoplus Y_i)$$

is an equivalence for every set of objects $\{Y_i\}$. We write T^c (or T^ω) for the full subcategory of T consisting of the compact objects.

Before we can give some examples of compact objects, and criteria for detecting them, we must introduce some terminology.

Definition 3.2. Let T be a triangulated category which has coproducts.

1. A full subcategory $\mathcal{S}$ of T is *thick* if it is closed under retracts and is triangulated.
2. A full subcategory $\mathcal{S}$ of T is *localizing* if it is thick, and closed under coproducts.

Given a set of objects $\mathcal{X}$ of T, we write $\mathrm{Thick}(\mathcal{X})$ (resp., $\mathrm{Loc}(\mathcal{X})$) for the smallest thick (resp., localizing) subcategory of T containing $\mathcal{X}$. Note that these are well defined since the intersection of thick/localizing subcategories is again thick/localizing.

Example 3.3. Let $X \in \mathsf{T}$. The full subcategory $\{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i X, Y) \simeq 0 \text{ for all } i \in \mathbb{Z}\}$ is thick. It is localizing if X is compact.

Definition 3.4. Let $\mathcal{X}$ be a set of objects of T. We say that $\mathcal{X}$ *generates* T if $\mathrm{Loc}(\mathcal{X}) = \mathsf{T}$. If each element of $\mathcal{X}$ is compact, then we say that $\mathcal{X}$ *compactly generates* T.

If T is compactly generated, then we can characterise the compact objects in terms of building operations.

Proposition 3.5. *Let T be a triangulated category which is compactly generated by a set G. Then $\mathsf{T}^c = \mathrm{Thick}(\mathsf{G})$.*

Proof. The implication that $X \in \mathsf{T}^{\mathrm{c}}$ implies that $X \in \mathrm{Thick}(\mathsf{G})$ requires some work so we omit it. For the converse, consider the set

$$\{Y \in \mathsf{T} \mid \bigoplus \mathrm{Hom}_{\mathsf{T}}(Y, Z_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(Y, \bigoplus Z_i) \text{ for all sets } \{Z_i\}\}.$$

This is a thick subcategory of T, and contains G. Hence it contains $\mathrm{Thick}(\mathsf{G})$ since this is the *smallest* thick subcategory of T containing G. □

Example 3.6. For a ring R, the compact objects in $\mathsf{D}(R)$ are the *perfect complexes*; that is, those complexes which are quasi-isomorphic to a bounded complex of finitely generated projectives. We will give a proof of this below in Example 3.14.

3.2. **Brown representability and consequences.** Brown Representability is a remarkably powerful result in triangulated categories. The initial statement was in stable homotopy theory where it concerns cohomology theories being represented by objects called spectra, but since then various versions have been proved in general triangulated categories. It has many striking consequences, such as providing criteria to determine existence of adjoints, a way to check compact generation, and a way to construct 'designer' objects. We will not give the proof of Brown Representability here, but we will prove a special case as Theorem 3.21.

Recall that a functor $H \colon \mathsf{T} \to \mathsf{Ab}$ is homological if it sends triangles to long exact sequences.

Example 3.7. Let X be an object of T. Then $\mathrm{Hom}_{\mathsf{T}}(X, -)$ is homological and $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is cohomological by Lemma 2.11.

Theorem 3.8 (Brown Representability)**.** *Let T be a compactly generated triangulated category. If $H \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ is a cohomological functor which takes coproducts in T to products in Ab, then H is representable, i.e., it is isomorphic to $\mathrm{Hom}_{\mathsf{T}}(-, X)$ for some $X \in \mathsf{T}$.*

Recall that compactly generated triangulated categories are assumed to have coproducts by definition. The first consequence of Brown representability which we give is that in fact they also have products.

Proposition 3.9. *If T is a compactly generated triangulated category, then T has products.*

Proof. Let $\{X_i\}$ be a set of objects in T and consider the functor $H \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ defined by $H = \prod_i \mathrm{Hom}_{\mathsf{T}}(-, X_i)$. This is cohomological and sends coproducts in T to products. Hence there exists an object $Z \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{T}}(-, Z) = \prod_i \mathrm{Hom}_{\mathsf{T}}(-, X_i)$. It is an exercise to verify that Z has the universal property of the product of the X_i. □

In light of the previous proposition, we may now state a dual version of Brown representability.

Theorem 3.10 (Brown Representability for the dual)**.** *Let T be a compactly generated triangulated category. If $H \colon \mathsf{T} \to \mathsf{Ab}$ is a homological functor which takes products in T to products in Ab, then H is corepresentable, i.e., it is isomorphic to $\mathrm{Hom}_{\mathsf{T}}(Y, -)$ for some $Y \in \mathsf{T}$.*

Remark 3.11. It is important to note that despite the appearance and the terminology, the dual form of Brown representability is *not* a formal consequence of the former. Indeed, the opposite of a compactly generated triangulated category need not be compactly generated, and moreover, the existence of compact objects in the opposite category is extremely rare.

The next consequence is a powerful adjoint functor theorem.

Theorem 3.12. *Let $F\colon \mathsf{T} \to \mathsf{U}$ be a triangulated functor, and suppose that T is compactly generated.*

(1) If F is coproduct preserving then F has a right adjoint.
(2) If F is product preserving then F has a left adjoint.

Proof. We prove (1); the proof of (2) is similar. Since F is triangulated and coproduct preserving, the functor $\mathrm{Hom}_{\mathsf{U}}(F(-), Y)$ is cohomological and takes coproducts to products. Hence by Theorem 3.8, there exists an object $G(Y) \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{U}}(F(-), Y) = \mathrm{Hom}_{\mathsf{T}}(-, G(Y))$. Given a map $Y \to Y'$, one obtains a map $G(Y) \to G(Y')$ via the Yoneda lemma, namely, we have

$$\mathrm{Hom}_{\mathsf{T}}(-, G(Y)) = \mathrm{Hom}_{\mathsf{U}}(F(-), Y) \to \mathrm{Hom}_{\mathsf{U}}(F(-), Y') = \mathrm{Hom}_{\mathsf{T}}(-, G(Y'))$$

which by the Yoneda lemma must come from a map $G(Y) \to G(Y')$. One checks that this makes G into a functor which is right adjoint to F. $\square$

Finally we give a consequence which allows one to check when a given set of compact objects is in fact a set of generators.

Proposition 3.13. *Let T be a triangulated category with coproducts, and $\mathcal{S}$ be a set of compact objects of T. Then the following are equivalent:*

(1) $\mathcal{S}$ generates T;
(2) if $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, X) = 0$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$, then $X \simeq 0$.

Proof. For the implication (1) $\Rightarrow$ (2), suppose that $\mathcal{S}$ generates T, and that $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, X) = 0$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$. Consider the set

$$\mathcal{X} = \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i Y, X) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

This is a localizing subcategory of T, and contains $\mathcal{S}$ by assumption. Hence $\mathsf{T} = \mathrm{Loc}(\mathcal{S}) \subseteq \mathcal{X}$, i.e., $\mathrm{Hom}_{\mathsf{T}}(Z, X) \simeq 0$ for all $Z \in \mathsf{T}$, and so by Yoneda we have $X \simeq 0$.

For the converse, consider the localizing subcategory $\mathrm{Loc}(\mathcal{S})$ of T. Since $\mathcal{S}$ consists of compact objects, the inclusion $i\colon \mathrm{Loc}(\mathcal{S}) \hookrightarrow \mathsf{T}$ has a right adjoint Γ by Theorem 3.12. The counit of the adjunction gives a natural map $\Gamma X \to X$ which we may complete to a triangle

$$\Gamma X \to X \to Y$$

by axiom (TR1). We will show that $Y \simeq 0$, so that $\Gamma X \simeq X$ by Proposition 2.18, and hence $X \in \mathrm{Loc}(\mathcal{S})$. If we apply $\mathrm{Hom}_{\mathsf{T}}(S, -)$ to the triangle $\Gamma X \to X \to Y$ we get a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathsf{T}}(S, \Gamma X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(S, X) \to \mathrm{Hom}_{\mathsf{T}}(S, Y) \to \mathrm{Hom}_{\mathsf{T}}(\Sigma^{-1} S, \Gamma X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(\Sigma^{-1} S, X) \to \cdots$$

where the isomorphisms hold by the adjunction $i \dashv \Gamma$. Therefore $\mathrm{Hom}_{\mathsf{T}}(S, Y) \simeq 0$, and by the same argument shifted, we have $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, Y) \simeq 0$ for all i. Therefore $Y \simeq 0$ by assumption as required. $\square$

Example 3.14 ($\mathsf{D}(R)^{\mathrm{c}} = \mathrm{Perf}(R)$)**.** Let us show that the compact objects in $\mathsf{D}(R)$ are exactly the perfect complexes. To do this, recall from Proposition 3.5 that $\mathsf{D}(R)^{\mathrm{c}} = \mathrm{Thick}(R)$. Recall that $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, -) = H_i(-)$ by Proposition 2.43. From this one checks that R is compact, and moreover R can be seen to be a generator using this together with Proposition 3.13. Since $\mathrm{Perf}(R)$ is thick (Exercise A.15) and contains R, we must have $\mathrm{Thick}(R) \subseteq \mathrm{Perf}(R)$ as $\mathrm{Thick}(R)$ is the *smallest* thick subcategory containing R. So it suffices to show that any perfect complex

P is in Thick(R). Since Thick(R) is closed under isomorphisms by definition, we may assume that P be a bounded complex of finitely generated projectives

$$P = (\cdots \to 0 \to P_a \to P_{a-1} \to \cdots \to P_b \to 0 \to \cdots).$$

We argue by induction on $a-b$. When $a-b=0$, the complex P is infact a finitely generated projective module concentrated in a single degree. Since P is finitely generated we have a surjection $f\colon R^n \to P$ and therefore a short exact sequence $0 \to \ker(f) \to R^n \to P \to 0$. Since any short exact sequence ending in a projective splits, we see that P is a summand of R^n and hence is in Thick(R). Suppose that the claim is true for $a-b=k-1$, and now fix $a-b=k$. There is a triangle

$$P_b[b] \to t_{\geq b}P \to t_{\geq b+1}P$$

by taking the brutal truncation, see Definition 2.37. By inductive hypothesis $t_{\geq b+1}P$ is in Thick(R), and we have already seen that $P_b[b]$ is also in Thick(R). Hence P is in Thick(R), which completes the proof.

3.3. **Homotopy colimits.** In this section we introduce some technical results which give constructions of objects in terms of smaller building blocks. In an ordinary category, one thinks of colimits as a way of constructing new objects from smaller pieces. However, in general triangulated categories do not admit true colimits, so the universal properties one encounters in this setting are weaker than the usual universal properties of colimits.

Before we give the key definition, let us give some motivation. Suppose we have a system of abelian groups and group homomorphisms $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$. The map

$$\bigoplus_{i=0}^{j-1} A_i \xrightarrow{1-f} \bigoplus_{i=0}^{j} A_i$$

which on the nth component is given by

$$A_n \xrightarrow{a \mapsto (a, -f_n(a))} A_n \oplus A_{n+1} \xrightarrow{\text{inc}} \bigoplus_{i=0}^{j} A_i$$

gives rise to an exact sequence

$$0 \to \bigoplus_{i=0}^{j-1} A_i \xrightarrow{1-f} \bigoplus_{i=0}^{j} A_i \to A_j \to 0.$$

Taking the direct limit of this (and since direct limits are exact in abelian groups), we obtain an exact sequence

$$0 \to \bigoplus A_i \xrightarrow{1-f} \bigoplus A_i \to \varinjlim A_i \to 0. \tag{3.15}$$

We may mimic this exact sequence in a triangulated category to define the homotopy colimit.

Definition 3.16. Let T be a triangulated category with coproducts, and let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system of maps in T. The *homotopy colimit* of this system, denoted $\mathrm{hocolim} X_i$, is the cone of the map

$$\bigoplus X_i \xrightarrow{1-f} \bigoplus X_i$$

which on the nth component is given by

$$X_n \xrightarrow{x \mapsto (x, -f_n(x))} X_n \oplus X_{n+1} \xrightarrow{\text{inc}} \bigoplus X_i.$$

Lemma 3.17. *Let T be a triangulated category with coproducts, and let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system of maps in T. Let $Y \in \mathsf{T}$, and suppose that there exists maps $g_n \colon X_n \to Y$ such that $g_n = g_{n+1} f_n$ for all n. Then there exists a map $\overline{g} \colon \mathrm{hocolim} X_i \to Y$ such that $g = \overline{g} q$ where q is the induced map $q \colon \bigoplus X_i \to \mathrm{hocolim} X_i$.*

Proof. By universal property of the coproduct, we obtain a map $g \colon \bigoplus X_i \to Y$ by assembling the g_i. Note that $g(1-f) = 0$ since on the nth component we have $g(1-f)(x) = g(x_n, -f_n(x_n)) = g_n(x_n) - g_{n+1} f_n(x_n) = 0$. So consider the diagram

$$\begin{tikzcd}
\bigoplus X_i \arrow[r, "1-f"] \arrow[d] & \bigoplus X_i \arrow[r, "q"] \arrow[d, "g"] & \mathrm{hocolim} X_i \arrow[r] & \Sigma \bigoplus X_i \arrow[d] \\
0 \arrow[r] & Y \arrow[r, "\mathrm{id}"'] & Y \arrow[r] & 0.
\end{tikzcd}$$

The top row is a triangle by definition of the homotopy colimit, and the bottom row is a triangle by (TR0). The left hand square commutes since $g(1-f) = 0$, so by (TR3) there exists a map $\overline{g} \colon \mathrm{hocolim} X_i \to Y$ making the diagram commute. $\square$

Lemma 3.18. *Let T be a triangulated category with coproducts. Let C be a compact object and let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system in T. Then the canonical map*

$$\varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i)$$

is an isomorphism.

Proof. By definition of the homotopy colimit and since $\mathrm{Hom}(C, -)$ is a homological functor by Lemma 2.11 we obtain a long exact sequence,

$$\cdots \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus \Sigma X_i) \to \cdots.$$

By shifting the exact sequence (3.15), and comparing with the long exact sequence above, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) \arrow[r] \arrow[d] & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) \arrow[r] \arrow[d] & \varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \arrow[r] & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \arrow[r] & \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) \arrow[r] & \cdots
\end{tikzcd}$$

The first two columns are isomorphisms since C is compact, and so we see that the map $\mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i)$ is injective. Similarly, we may apply the same argument to ΣX_i to deduce that $\mathrm{Hom}_{\mathsf{T}}(C, \bigoplus \Sigma X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus \Sigma X_i)$ is also injective. Therefore the long exact sequence collapses into short exact sequences. We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) \arrow[r] \arrow[d] & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) \arrow[r] \arrow[d] & \varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \arrow[r] & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \arrow[r] & \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) \arrow[r] & 0
\end{tikzcd}$$

in which the first two columns are isomorphisms as C is compact. The result then follows. $\square$

Corollary 3.19. *Let T be a compactly generated triangulated category. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system in T, and let $Y \in \mathsf{T}$ such that there exists maps $g_n \colon X_n \to Y$ with*

$g_n = g_{n+1} f_n$ for all n. If the canonical map

$$\varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, Y)$$

is an isomorphism for all $C \in \mathsf{T}^c$, then $Y \simeq \mathrm{hocolim} X_i$.

Proof. By Lemma 3.17 we obtain a map $\overline{g}\colon \mathrm{hocolim} X_i \to Y$. Since T is compactly generated, to prove this is an isomorphism it suffices to check that the induced map

$$\mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, Y)$$

is an isomorphism for all $C \in \mathsf{T}^c$. This is immediate from Lemma 3.18. $\square$

Before we can prove some consequences of this definition, we need the following construction. The following construction is the key behind the proof of the Brown representability theorem. Although we won't use it to prove the full version of Brown representability stated in the previous section, Theorem 3.21 can be interpreted as a special case.

Construction 3.20. Let T be a triangulated category with coproducts, and suppose that $\mathcal{K} = \{K_i\}$ is a set of compact objects in T. Let $X \in \mathsf{T}$ be arbitrary. We will build a tower whose homotopy colimit defines a colocalization functor, and hence a localization functor (we haven't introduced this language yet, but we will recast this construction in that terminology later on in the course). Let $X_0 = X$ and

$$A_0 = \bigoplus_{Z \in \Sigma^? \mathcal{K}} \bigoplus_{Z \to X} Z$$

where the question mark indicates that this ranges across all possible shifts. There is a map $A_0 \to X_0$ by universal property of the coproduct, and we write X_1 for the cofibre of this map. Iterating the same procedure, we set

$$A_i = \bigoplus_{Z \in \Sigma^? \mathcal{K}} \bigoplus_{Z \to X_i} Z$$

and a triangle $A_i \to X_i \to X_{i+1}$. Write F_i for the fibre of the map $X \to X_i$.

Therefore we have triangles $F_i \to X \to X_i$ and $A_i \to X_i \to X_{i+1}$ for each i. Note that the map $X \to X_{i+1}$ factors as $X \to X_i \to X_{i+1}$. Therefore by applying the octahedral axiom to the triangles $X \to X_i \to \Sigma F_i$, $X_i \to X_{i+1} \to \Sigma A_i$, and $X \to X_{i+1} \to \Sigma F_{i+1}$ we obtain a triangle $\Sigma F_i \to \Sigma F_{i+1} \to \Sigma A_i$. Pictorially, this is

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "1"] & X_i \arrow[r] \arrow[d] & \Sigma F_i \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "1"] \\
X \arrow[r] \arrow[d] & X_{i+1} \arrow[r] \arrow[d, "1"] & \Sigma F_{i+1} \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d] \\
X_i \arrow[r] \arrow[d] & X_{i+1} \arrow[r] \arrow[d] & \Sigma A_i \arrow[r] \arrow[d, "1"] & \Sigma X_i \arrow[d] \\
\Sigma F_i \arrow[r, dashed] & \Sigma F_{i+1} \arrow[r, dashed] & \Sigma A_i \arrow[r, dashed] & \Sigma^2 F_i
\end{tikzcd}$$

Shifting we therefore have a system of maps $\cdots \to F_i \to F_{i+1} \to \cdots$. We define $\Gamma_{\mathcal{K}} X$ to be the homotopy colimit of the system of F_i's. Note that by applying Lemma 3.17, we have a map $\Gamma_{\mathcal{K}} X \to X$.

Theorem 3.21. *With notation as in Construction 3.20, we have $\Gamma_{\mathcal{K}}X \in \mathsf{Loc}(\mathcal{K})$, and the induced map $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma_{\mathcal{K}}X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is an isomorphism for all $Z \in \mathsf{Loc}(\mathcal{K})$. In particular, $\Gamma_{\mathcal{K}}X \to X$ is an isomorphism if $X \in \mathsf{Loc}(\mathcal{K})$.*

Proof. For the first claim it suffices to show that each $F_i \in \mathsf{Loc}(\mathcal{K})$ by definition of the homotopy colimit. We do this by induction. The base case is clear as $F_0 \simeq 0$, so suppose $F_i \in \mathsf{Loc}(\mathcal{K})$. By construction, there is a triangle $F_i \to F_{i+1} \to A_i$ and A_i is a coproduct of suspensions of elements of $\mathcal{K}$. Therefore $F_{i+1} \in \mathsf{Loc}(\mathcal{K})$ as required.

For the second claim, by a localizing subcategory argument it suffices to prove it when $Z \in \mathcal{K}$.

Let us first deal with surjectivity. For any $i > 0$, the map $\mathrm{Hom}_{\mathsf{T}}(Z, X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X_i)$ is zero by construction: any such Z embeds into A_{i-1} and hence by the triangle $A_{i-1} \to X_{i-1} \to X_i$ the map is zero. Therefore the map $\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is surjective by the exact sequence associated to the triangle $F_i \to X \to X_i$ by Lemma 2.11. From the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}_{\mathsf{T}}(Z, X) \\
\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) \arrow[ur] &
\end{tikzcd}$$

it follows that $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is surjective. (Recall that if gf is surjective, then g is surjective.)

To complete the proof of the second claim, we need to verify that the map is injective. So suppose that $f\colon Z \to \Gamma X$ is such that $Z \to \Gamma X \to X$ is zero. Now $f \in \mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) = \varinjlim \mathrm{Hom}_{\mathsf{T}}(Z, F_i)$ by Lemma 3.18, and so there exists an $f_i\colon Z \to F_i$ such that f factors as

$$Z \xrightarrow{f_i} F_i \to \Gamma X.$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(Z, F_i) \arrow[d, "\alpha"'] \arrow[dr] & \\
\mathrm{Hom}(Z, \Gamma X) \arrow[r, "\beta"'] & \mathrm{Hom}(Z, X).
\end{tikzcd}$$

By definition $\alpha(f_i) = f$, and by construction $\beta(f) = 0$. Therefore f_i is in the kernel of the map $\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$. By exactness of the sequence associated to the triangle $F_i \to X \to X_i$,

$$\ker(\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)) = \mathrm{im}(\mathrm{Hom}_{\mathsf{T}}(\Sigma^{-1}Z, X_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, F_i)).$$

Therefore there exists a map $h\colon Z \to \Sigma^{-1}X_i$ such that the composite $Z \to \Sigma^{-1}X_i \to F_i$ is f_i. By definition of A_i and the triangle

$$\Sigma^{-1}A_i \xrightarrow{\delta} F_i \xrightarrow{g} F_{i+1},$$

the map h factors through δ and hence $gf_i = 0$. Finally write $\bar{g}$ for the map $F_{i+1} \to \Gamma X$. We then have

$$f = (\bar{g} \circ g) \circ f_i = \bar{g} \circ (g \circ f_i) = 0$$

so that the map is injective as required. This completes the proof of the second claim.

For the final claim, if $X \in \mathsf{Loc}(\mathcal{K})$, then as $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma_{\mathcal{K}} X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is an isomorphism for all $Z \in \mathsf{Loc}(\mathcal{K})$ we see that $\Gamma_{\mathcal{K}} X \to X$ is an isomorphism by the Yoneda lemma. ☐

Remark 3.22. An extended version of the argument used in the proof of the previous theorem can be used to give a proof of Brown representability (Theorem 3.8).

We obtain the following interesting consequence of this construction which gives a characterisation of the localizing subcategory generated by a set of compact objects.

Corollary 3.23. *Let T be a triangulated category with coproducts. Suppose that $\mathcal{K}$ is a set of compact objects in T. Then the following are equivalent for an arbitrary object $X \in \mathsf{T}$:*

(1) X is the homotopy colimit of a sequence

$$0 = F_0 \to F_1 \to F_2 \to \cdots$$

such that the cofibre of $F_i \to F_{i+1}$ for each i is a coproduct of suspensions of elements of $\mathcal{K}$;

(2) $X \in \mathsf{Loc}(\mathcal{K})$.

Proof. The implication (1) ⇒ (2) holds similarly to Theorem 3.21. For (2) ⇒ (1), we use Construction 3.20. By Theorem 3.21, $\Gamma_{\mathcal{K}} X \to X$ is an isomorphism as $X \in \mathsf{Loc}(\mathcal{K})$. As $\Gamma_{\mathcal{K}} X$ is a homotopy colimit of the required form, this completes the proof. ☐

We will use this construction extensively later on, since it provides a recipe for constructing objects with convenient properties from small building blocks.

4. Tensor-triangulated categories and finiteness

Triangulated categories abound, but often there is extra structure floating around which it is useful to remember; we consider when the triangulated category is also equipped with a closed symmetric monoidal structure which is suitably compatible with the triangulation. Such considerations have exploded into the field of *tensor-triangular geometry* whose starting point was the observation that a scheme $\mathcal{X}$ cannot be recovered from the structure of $\mathsf{D}(\mathrm{QCoh}(\mathcal{X}))$ as a triangulated category alone, but it can be recovered once the tensor product is remembered.

Recall that a symmetric monoidal structure on a category $\mathcal{C}$ is the data of a bifunctor

$$- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

called the *tensor product* (or *monoidal product*) together with an unit object $\mathbb{1} \in \mathcal{C}$ satisfying:

- unitality: $\mathbb{1} \otimes X \simeq X$ for all $X \in \mathcal{C}$;
- associativity: $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- symmetry: $X \otimes Y \simeq Y \otimes X$ for all $X, Y \in \mathcal{C}$;

all satisfying various coherences that we won't make precise here. Such a monoidal structure is moreover called *closed*, if there is a bifunctor $\underline{\mathrm{Hom}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ called the *internal hom* such that $- \otimes X$ is left adjoint to $\underline{\mathrm{Hom}}(X, -)$. Again, this is all subject to various coherences that we will not make explicit. Note that $\underline{\mathrm{Hom}}(\mathbb{1}, X) \simeq X$ for all $X \in \mathcal{C}$.

Definition 4.1. A *tensor-triangulated category* T is a triangulated category which also has a closed symmetric monoidal structure, such that the tensor product and internal hom are triangulated functors in both variables.

4.1. **Example: the derived category.** Let R be a *commutative* ring. Recall the tensor product of chain complexes of R-modules is given by

$$(M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j$$

with differential $d(m \otimes n) = (dm \otimes n) + (-1)^{|m|}(m \otimes dn)$. However, this cannot be a tensor product on the derived category since it is not invariant under quasi-isomorphism as the following example shows.

Example 4.2. Let $R = \mathbb{Z}/4$ and define $P = (\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \cdots)$. There is a quasi-isomorphism $P \xrightarrow{\sim} 0$. However, this is not preserved by the functor $\mathbb{Z}/2 \otimes_{\mathbb{Z}/4} -$. Indeed $\mathbb{Z}/2 \otimes_{\mathbb{Z}/4} P$ is the complex

$$\cdots \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \cdots$$

and hence has homology $\mathbb{Z}/2$ in each degree.

So we now explain how to give a construction of a tensor product which is invariant under quasi-isomorphism, thus making the derived category into a tensor-triangulated category.

We begin by investigated induced maps on derived categories. For now, let R and S be rings (not necessarily commutative).

Proposition 4.3. *Let $F\colon \mathrm{Ch}(R) \to \mathrm{Ch}(S)$ be an additive functor.*

(1) There exists a unique functor $\bar{F}\colon \mathsf{K}(R) \to \mathsf{K}(S)$ such that

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, "F"] \arrow[d, "h_R"'] & \mathrm{Ch}(S) \arrow[d, "h_S"] \\ \mathsf{K}(R) \arrow[r, "\bar{F}"'] & \mathsf{K}(S) \end{tikzcd}$$

commutes.

(2) If F preserves quasi-isomorphisms, there exists a unique functor $\widetilde{F}\colon \mathsf{D}(R) \to \mathsf{D}(S)$ such that

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, "F"] \arrow[d, "Q_R h_R"'] & \mathrm{Ch}(S) \arrow[d, "Q_S h_S"] \\ \mathsf{D}(R) \arrow[r, "\widetilde{F}"'] & \mathsf{D}(S) \end{tikzcd}$$

commutes. Moreover, if F is exact, then $\widetilde{F}$ is a triangulated functor.

Proof. Since additive functors are homotopy invariant, the first part is a consequence of Proposition 2.8 applied to the functor $h_S F\colon \mathrm{Ch}(R) \to \mathsf{K}(S)$. The second part follows by applying Theorem 2.33 to the functor $Q_S\bar{F}\colon \mathsf{K}(R) \to \mathsf{D}(S)$. For the final claim, by Theorem 2.33 it suffices to show that $Q_S\bar{F}\colon \mathsf{K}(R) \to \mathsf{D}(R)$ is triangulated. As Q_S is triangulated, we can reduce further to checking that $\bar{F}\colon \mathsf{K}(R) \to \mathsf{K}(S)$ is triangulated. Since triangles in $\mathsf{K}(R)$ are of the form

$$M \xrightarrow{f} N \to C(f) \to \Sigma M$$

this follows from exactness, since exact functors preserve mapping cones. (Exercise A.26 asks you to make this precise.) ∎

It is important to note that if P is a complex of projective R-modules, even though $\operatorname{Hom}_R(P,-)$ is an exact functor, it does not preserve quasi-isomorphisms. For example, consider $R = \mathbb{Z}/4$ and the complex

$$P = \cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \cdots .$$

The quasi-isomorphism $P \xrightarrow{\sim} 0$ is not preserved by $\operatorname{Hom}_{\mathbb{Z}/4}(P,-)$. As such, we need to introduce refined notions of projective and injective objects in order to construct functors between derived categories. So as to not get bogged down in homological algebra which is unrelated to the triangulated structure, we will omit many of the proofs of the following claims.

Definition 4.4. A complex P of R-modules is *dg-projective* if any of the following equivalent conditions hold:

(i) $\operatorname{Hom}_R(P,-)$ preserves surjective quasi-isomorphisms;
(ii) $\operatorname{Hom}_R(P,-)$ is exact and preserves quasi-isomorphisms;
(iii) P is a complex of projective R-modules and $\operatorname{Hom}_R(P,-)$ preserves acyclic complexes.
(iv) for any chain map $f\colon P \to N$ and any surjective quasi-isomorphism $g\colon M \to N$, there exists a chain map $\alpha\colon P \to M$ so that $g\alpha = f$.

The following gives the appropriate notion of a projective resolution in this context.

Theorem 4.5. *For any complex of R-modules M, there exists a dg-projective complex P together with a surjective quasi-isomorphism $P \xrightarrow{\sim} M$.*

In a similar way, we have the following for injectives instead.

Definition 4.6. A complex I of R-modules is *dg-injective* if any of the following equivalent conditions hold:

(i) $\operatorname{Hom}_R(-,I)$ sends injective quasi-isomorphisms to surjective quasi-isomorphisms;
(ii) $\operatorname{Hom}_R(-,I)$ is exact and preserves quasi-isomorphisms;
(iii) I is a complex of injective R-modules and $\operatorname{Hom}_R(-,I)$ preserves acyclic complexes.
(iv) for any chain map $f\colon M \to I$ and any injective quasi-isomorphism $g\colon M \to N$, there exists a chain map $\alpha\colon N \to I$ so that $\alpha g = f$.

Theorem 4.7. *For any complex of R-modules M, there exists a dg-injective complex I together with an injective quasi-isomorphism $M \xrightarrow{\sim} I$.*

There is also the case for flat objects.

Definition 4.8. A complex F of R-modules is *dg-flat* if any of the following equivalent conditions hold:

(i) $F \otimes_R -$ sends injective quasi-isomorphisms to surjective quasi-isomorphisms;
(ii) $F \otimes_R -$ is exact and preserves quasi-isomorphisms;
(iii) F is a complex of flat R-modules and $F \otimes_R -$ preserves acyclic complexes.

Theorem 4.9. *For any complex of R-modules M, there exists a dg-flat complex F together with a quasi-isomorphism $F \xrightarrow{\sim} M$.*

With all these preliminaries set up, we can now discuss derived tensor products and derived homs. Henceforth we assume that R is a commutative ring. One can make some of the following statements for non-commutative rings if one worries about left vs right module structures, but we will focus only on the commutative case.

Construction 4.10. Let $M \in \mathsf{D}(R)$. Using Theorem 4.5, there exists a dg-projective complex P together with a quasi-isomorphism $P \to M$. The functor $P \otimes_R -\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is exact and preserves quasi-isomorphisms, and therefore by Proposition 4.3, there exists a functor $M \otimes^{\mathsf{L}}_R -\colon \mathsf{D}(R) \to \mathsf{D}(R)$.

The following result is fundamental and we will use it throughout.

Proposition 4.11. *The derived tensor product $M \otimes^{\mathsf{L}}_R -$ constructed in Construction 4.10 may be computed using a dg-flat replacement F of M, i.e., $M \otimes^{\mathsf{L}}_R - = F \otimes_R -$, and is independent of the choice of such a replacement.*

Proof. Suppose that F is a dg-flat replacement of M, and write $G\colon \mathsf{D}(R) \to \mathsf{D}(R)$ for the induced functor making the diagram

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, "F\otimes_R -"] \arrow[d, "Q_R h_R"'] & \mathrm{Ch}(S) \arrow[d, "Q_S h_S"] \\ \mathsf{D}(R) \arrow[r, "G"'] & \mathsf{D}(S) \end{tikzcd}$$

commute. Since $M \otimes^{\mathsf{L}}_R -$ satisfies a uniqueness property, it suffices to check that the diagram

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, "P\otimes_R -"] \arrow[d, "Q_R h_R"'] & \mathrm{Ch}(S) \arrow[d, "Q_S h_S"] \\ \mathsf{D}(R) \arrow[r, "G"'] & \mathsf{D}(S) \end{tikzcd}$$

commutes.

By Definition 4.4(iv), there exists a quasi-isomorphism $P \xrightarrow{\phi} F$. For any $X \in \mathrm{Ch}(R)$, we claim that $\phi \otimes_R X\colon P \otimes_R X \to F \otimes_R X$ is a quasi-isomorphism. Indeed, take a dg-flat replacement $\psi\colon F(X) \to X$ and consider the commutative diagram

$$\begin{tikzcd} P \otimes_R F(X) \arrow[r, "P\otimes_R \psi"] \arrow[d, "\phi\otimes_R F(X)"'] & P \otimes_R X \arrow[d, "\phi\otimes_R X"] \\ F \otimes_R F(X) \arrow[r, "F\otimes_R \psi"] & F \otimes_R X. \end{tikzcd}$$

The left vertical and both horizontals are quasi-isomorphisms since tensoring with a semi-flat complex preserves quasi-isomorphisms. Therefore the right most vertical is also a quasi-isomorphism as claimed. Since Q sends quasi-isomorphisms to isomorphisms, we see that $Q_S h_S(P \otimes_R -) = Q_S h_S(F \otimes_R -)$, and hence by uniqueness, $G = M \otimes^{\mathsf{L}}_R -$. $\square$

In a similar way we construct the derived hom functor.

Construction 4.12. Let $M \in \mathsf{D}(R)$. By Theorem 4.5, there exists a dg-projective replacement P of M. The functor $\mathrm{Hom}_R(P, -)\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is exact and preserves quasi-isomorphism, and therefore by Proposition 4.3, there exists a triangulated functor $\mathsf{R}\mathrm{Hom}_R(M, -)\colon \mathsf{D}(R) \to \mathsf{D}(R)$.

Construction 4.13. Let $N \in \mathsf{D}(R)$. By Theorem 4.7, there exists a dg-injective replacement I of N. The functor $\mathrm{Hom}_R(-, I)\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)^{\mathrm{op}}$ is exact and preserves quasi-isomorphisms, and hence there exists a triangulated functor $\mathsf{R}\mathrm{Hom}_R(-, N)\colon \mathsf{D}(R) \to \mathsf{D}(R)^{\mathrm{op}}$.

In a similar way to Proposition 4.11, $\mathsf{R}\mathrm{Hom}_R(M,-)$ is independent of the choice of dg-projective replacement, and $\mathsf{R}\mathrm{Hom}_R(-,N)$ is independent of the choice of dg-injective replacement. We warn the reader that $\mathsf{R}\mathrm{Hom}_R(M,-)$ may *not* be computed using semi-flat replacements.

Proposition 4.14. *Let $M, N \in \mathsf{D}(R)$. Let P be a dg-projective replacement of M, and I be a dg-injective replacement of N. Then $\mathrm{Hom}_R(P,N)$ and $\mathrm{Hom}_R(M,I)$ are quasi-isomorphic. As such, there is no ambiguity in the definition of $\mathsf{R}\mathrm{Hom}_R(-,-)$: it can be computed by resolving either factor.*

Proof. We have quasi-isomorphisms $\mathrm{Hom}_R(P,N) \to \mathrm{Hom}_R(P,I)$ and $\mathrm{Hom}_R(M,I) \to \mathrm{Hom}_R(P,I)$ by definition of dg-projective and dg-injective. ☐

With all these preliminaries we may prove that $\mathsf{D}(R)$ (for R commutative) is a tensor-triangulated category.

Proposition 4.15. *Let R be a commutative ring. Then $\mathsf{D}(R)$ is a tensor-triangulated category.*

Proof. We skip over most of the technical coherence details one should check in the definition of a symmetric monoidal category. We have a functor $- \otimes^{\mathsf{L}}_R -$ by Construction 4.10 which equips $\mathsf{D}(R)$ with a symmetric monoidal structure. We see that $M \otimes^{\mathsf{L}}_R -$ is left adjoint to $\mathsf{R}\mathrm{Hom}_R(M,-)$, since we may take a semi-projective replacement P of M, and compute both using this. It is standard that $P \otimes_R -$ is left adjoint to $\mathrm{Hom}_R(P,-)$ at the level of chain complexes, and the claim follows from this. Finally both the derived tensor and derived hom functors are triangulated in both variables by construction. ☐

4.2. **Rigid objects.** When T does not (necessarily) have a tensor product, we saw that compactness is a measure of smallness. When T is tensor-triangulated, there are various other notions of smallness in play. Under reasonable assumptions these notions are closely related as we will see below.

Definition 4.16. Suppose that T is a tensor-triangulated category with coproducts.

(1) An object $X \in \mathsf{T}$ is said to be *rigid* if the natural map
$$\nu_{X,Y} \colon F(X,\mathbb{1}) \otimes Y \to F(X,Y)$$
is an equivalence for all $Y \in \mathsf{T}$. To spell it out, the natural map is the composite
$$F(X,\mathbb{1}) \otimes Y \xrightarrow{\text{coev}} F(X, X \otimes F(X,\mathbb{1}) \otimes Y) \xrightarrow{F(X,\text{ev})} F(X,Y).$$

(2) An object $X \in \mathsf{T}$ is said to be *F-compact* if the natural map
$$\bigoplus F(X,Y_i) \to F(X, \bigoplus Y_i)$$
is an equivalence for every set $\{Y_i\}$.

In order to study the properties of rigid objects it is helpful to consider an alternative definition (which we will show is in fact equivalent).

Definition 4.17. Let T be a tensor-triangulated category, and $X \in \mathsf{T}$. The object X is *dualizable* if and only if there exists a map $\eta\colon \mathbb{1} \to X \otimes DX$ such that the diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[dr, "\mathrm{coev}"'] & X \otimes DX \arrow[d, "\nu_{X,X}"] \\ & F(X,X) \end{tikzcd}$$

commutes.

We write $D\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ for the functor $D = F(-,\mathbb{1})$. This is called the *functional dual* (or sometimes Spanier-Whitehead dual), as justified by the following results which shows that it gives an equivalence on the full subcategory of dualizable objects.

Firstly, note that there is a natural map $\rho\colon X \to D^2X$ given by the composite

$$X \xrightarrow{\mathrm{coev}} F(DX, X \otimes DX) \xrightarrow{F(DX,\mathrm{ev})} F(DX, \mathbb{1})$$

for all $X \in \mathsf{T}$. Many of the proofs of the following statements follow by tedious diagram chases, so we provide only an outline of the proofs.

Lemma 4.18. *Let T be a tensor-triangulated category and $X \in \mathsf{T}$. If X is dualizable then DX is dualizable.*

Proof. We define a map $\mathbb{1} \to DX \otimes D^2X$ via the composite

$$\mathbb{1} \xrightarrow{\eta} X \otimes DX \xrightarrow{\rho \otimes DX} D^2X \otimes DX.$$

It remains to check that the required diagram commutes. That is, we want to check that the outer square in the diagram

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\eta"] \arrow[dr, "\mathrm{coev}"] \arrow[dd, "\mathrm{coev}"'] & & X \otimes DX \arrow[dl, "\nu_{X,X}"'] \arrow[dd, "\rho \otimes DX"] \\ & F(X,X) \arrow[dl, "\alpha"] & \\ F(DX,DX) & & D^2X \otimes DX \arrow[ll, "\nu_{DX,DX}"] \end{tikzcd}$$

commutes, where α is defined to be the composite

$$F(X,X) \xrightarrow{F(X,\rho)} F(X, D^2X) \simeq F(X \otimes DX, \mathbb{1}) \simeq F(DX, DX).$$

The top triangle commutes since X is dualizable, so it suffices to prove that the other subdiagrams commute. This can be checked by unravelling the definitions of α and ρ; we leave this as a (tedious) exercise for the reader. $\square$

Lemma 4.19. *Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is dualizable. Then the natural map $\rho\colon X \to D^2X$ is an equivalence.*

Proof. We claim that the composite

$$D^2X \simeq \mathbb{1} \otimes D^2X \xrightarrow{\eta \otimes D^2X} X \otimes DX \otimes D^2X \xrightarrow{X \otimes \mathrm{ev}} X$$

is an inverse to ρ. This can be checked by diagram chasing. $\square$

Proposition 4.20. *Let T be a tensor-triangulated category, and $X \in \mathsf{T}$. Then X is rigid if and only if it is dualizable.*

Proof. The forward direction is clear since we may take $\eta = \nu_{X,X}^{-1} \circ \mathrm{coev}$. For the reverse direction, we claim that the composite

$$F(X,Y) \simeq F(X,Y) \otimes \mathbb{1} \xrightarrow{F(X,Y)\otimes\eta} F(X,Y) \otimes X \otimes DX \xrightarrow{\mathrm{ev}\otimes DX} Y \otimes DX$$

is an inverse to $\nu_{X,Y}$. Again, one may verify this by diagram chasing. ☐

Using Lemma 4.18 and Lemma 4.19, we therefore see that rigid objects are closed under D, and that D is a duality on rigid objects. One other important property of rigid objects is the following.

Lemma 4.21. *Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is rigid. Then X is a retract of $X \otimes DX \otimes X$.*

Proof. Recall that $X \otimes -$ is left adjoint to $F(X,-) \simeq DX \otimes -$ as X is rigid. By the triangle identity (on the unit $\mathbb{1}$), the diagram

$$\begin{tikzcd} X \arrow[r, "X\otimes\eta"] \arrow[dr, equal] & X \otimes DX \otimes X \arrow[d, "\varepsilon"] \\ & X \end{tikzcd}$$

commutes, which is exactly the claim that X is a retract of $X \otimes DX \otimes X$. ☐

Is it natural to ask how the different notions of 'smallness' in Definition 3.1 and Definition 4.16 are related. The following answers this.

Proposition 4.22. *Let T be a tensor-triangulated category, and suppose that T has a set of rigid generators G.*

(1) For $X \in \mathsf{T}$ we have

$$X \in \mathrm{Thick}(\mathsf{G}) \implies X \textit{ is rigid} \iff X \textit{ is } F\textit{-compact.}$$

(2) If the elements of G are compact, then for $X \in \mathsf{T}$ we have

$$X \textit{ is compact} \iff X \in \mathrm{Thick}(\mathsf{G}) \implies X \textit{ is rigid} \iff X \textit{ is } F\textit{-compact.}$$

(3) If the elements of G are compact, and the unit $\mathbb{1}$ of T is compact, then for $X \in \mathsf{T}$ we have

$$X \textit{ is compact} \iff X \in \mathrm{Thick}(\mathsf{G}) \iff X \textit{ is rigid} \iff X \textit{ is } F\textit{-compact.}$$

Proof. For (1), firstly note that the set of rigid objects in T is thick (Exercise A.19). By assumption it contains G, so we have $\mathrm{Thick}(\mathsf{G}) \subseteq \{\text{rigid objects}\}$, which proves the first implication. That rigid objects are F-compact is an immediate consequence of the definitions. For the remaining implication, suppose X is F-compact, and consider the set

$$\mathcal{L} = \{Y \in \mathsf{T} \mid DX \otimes Y \xrightarrow{\sim} F(X,Y)\}.$$

The set $\mathcal{L}$ is localizing as X is F-compact, and contains G by Exercise A.20. Therefore $\mathcal{L} = \mathsf{T}$, and hence X is rigid.

For (2), it suffices to prove that X is compact if and only if $X \in \mathrm{Thick}(\mathsf{G})$, which was the content of Proposition 3.5. We leave (3) as an exercise. $\square$

Definition 4.23. When T satisfies the assumptions of Proposition 4.22(3) (that is, T has a set of rigid and compact generators and the unit is compact), we say that T is a rigidly-compactly generated tensor-triangulated category.

Example 4.24. Let R be a commutative ring. Then the derived category $\mathsf{D}(R)$ is a rigidly-compactly generated tensor-triangulated category. To see this, note that $\mathsf{D}(R)$ is generated by R by Proposition 2.43 and Proposition 3.13, and that R is both compact and rigid. So this follows from Proposition 4.22(3). More generally, suppose that T is a tensor-triangulated category which is compactly generated by the tensor unit $\mathbb{1}$. Then T is a rigidly-compactly generated tensor-triangulated category.

Appendix A. Exercises

Exercise A.1. Prove that in the definition of a triangulated category it is not necessary to assume that distinguished triangles are candidate triangles. In other words, prove that any 'triangle' satisfying (TR0)-(TR3) is necessarily a candidate triangle.

Exercise A.2. This exercise is designed to convince you that signs can be important in triangulated categories (e.g. in (TR2)). Consider the multiplication by 3 map on $\mathbb{Z}$, and the cone of this map $C(\cdot 3)$. Consider the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "1"'] & C(\cdot 3) \arrow[r] \arrow[d, "1"] & \Sigma\mathbb{Z} \arrow[r, "\Sigma(\cdot 3)"] \arrow[d, dashed, "f"] & \Sigma\mathbb{Z} \arrow[d, "1"] \\
\mathbb{Z} \arrow[r] & C(\cdot 3) \arrow[r] & \Sigma\mathbb{Z} \arrow[r, "-\Sigma(\cdot 3)"'] & \Sigma\mathbb{Z}
\end{tikzcd}$$

Show that for no such map f making the diagram commute (in $\mathsf{K}(R)$) can exist.

Exercise A.3. Let T be a pretriangulated category.

(1) Suppose that $X \to Y \to Z \to \Sigma X$ and $X' \to Y' \to Z' \to \Sigma Z'$ are candidate triangles. Show that if their sum is a distinguished triangle, then so is each summand.
(2) Show that for any map $\theta\colon X \to Y$ in T, θ is an isomorphism if and only if there is a triangle $X \xrightarrow{\theta} Y \to 0 \to \Sigma X$.
(3) Show any triangle of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{0} \Sigma X$ is split, that is, isomorphic to a triangle of the form $X \to X \oplus Z \to Z \to \Sigma X$.

Exercise A.4. Let $F\colon \mathsf{T} \to \mathsf{U}$ be a triangulated functor. Prove that the kernel of F is a triangulated subcategory of T.

Exercise A.5. Prove that $\mathsf{K}(R)$ satisfies the following dual Ore condition: given a quasi-isomorphism $s\colon X \to X'$ and any map $f\colon X \to Y$, then there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "s'"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

in which s' is also a quasi-isomorphism.

Exercise A.6. Prove Lemma 2.32.

Exercise A.7. What we have called rooves so far are infact *left rooves*. In an analogous way, one may define right rooves to be diagrams

$$\begin{tikzcd} & Z & \\ X \arrow[ur] & & Y \arrow[ul, "\sim"'] \end{tikzcd}$$

The Ore condition associates to any right roof a left roof. Show that this gives a bijection between equivalence classes of right and left rooves.

Exercise A.8. Consider the full subcategory $\mathsf{D}(R)_{\geq n}$ consisting of the complexes M such that $H_i(M) = 0$ for all $i < n$. Prove that $\tau_{\geq n}$ is right adjoint to the inclusion functor $\mathsf{D}(R)_{\geq n} \hookrightarrow \mathsf{D}(R)$.

Exercise A.9. Prove Lemma 2.39.

Exercise A.10. Show that there exists a map $\alpha\colon \mathbb{Z}/p \to \Sigma\mathbb{Z}/p$ in $\mathsf{D}(\mathbb{Z})$ which is not null homotopic, but which has $H_*(\alpha) = 0$.

Exercise A.11. Give an example of a quasi-isomorphism which does not have an inverse in the category of chain complexes (i.e., find a quasi-isomorphism $f\colon M \to N$ for which there can be no chain map $g\colon N \to M$ such that $H_*(f)$ and $H_*(g)$ are inverses.)

Exercise A.12. Let I be an injective R-module. Prove that for any $M \in \mathsf{D}(R)$ and $n \in \mathbb{Z}$ there is an isomorphism $H_n(\mathrm{Hom}_R(M, I)) = \mathrm{Hom}_R(H_{-n}M, I)$.

Exercise A.13. Let k be a field. Prove that the category of vector spaces over k may be given a triangulated structure in which the shift functor is the identity, and the distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$ are the exact sequences $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y$.

Exercise A.14.

(i) Show that the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ of $\mathbb{Z}$-modules does not give rise to a triangle in $\mathsf{K}(\mathbb{Z})$.

(ii) Prove that if $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a split exact sequence of R-modules, then there is a map $h\colon N \to \Sigma L$ such that $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L$ is a triangle in $\mathsf{K}(R)$.

Exercise A.15. A complex $M \in \mathsf{D}(R)$ is said to be *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projectives. Write $\mathrm{Perf}(R)$ for the full subcategory of $\mathsf{D}(R)$ consisting of the perfect complexes. Show that $\mathrm{Perf}(R)$ is a thick subcategory.

Exercise A.16. Which of the following are compact objects?

(1) $\mathbb{Z}/p \in \mathsf{D}(\mathbb{Z})$
(2) $\mathbb{Q} \in \mathsf{D}(\mathbb{Z})$
(3) $\mathbb{Q} \in \mathsf{D}(\mathbb{Q}[x])$
(4) $\mathbb{Q} \in \mathsf{D}(\mathbb{Q}[x]/x^2)$

Exercise A.17. Let $F : \mathsf{T} \rightleftarrows \mathsf{U} : G$ be an adjunction between triangulated categories.

(1) Prove that if G preserves coproducts, then F preserves compact objects.
(2) Suppose that T is compactly generated and that F preserves compacts. Show that G preserves coproducts.

Exercise A.18. Let T be a compactly generated triangulated category. A map $f\colon X \to Y$ is said to be *phantom* if the induced map $\mathrm{Hom}_{\mathsf{T}}(C,X) \to \mathrm{Hom}_{\mathsf{T}}(C,Y)$ is zero for all compacts C. Prove that a coproduct and product preserving triangulated functor preserves phantom maps.

Exercise A.19. Prove that the full subcategory of rigid objects of a tensor-triangulated category T is thick. Prove that the full subcategory of F-compact objects of T is thick.

Exercise A.20. Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is rigid. Prove that the natural map

$$F(Y,\mathbb{1}) \otimes X \to F(Y,X)$$

is an equivalence for all $Y \in \mathsf{T}$.

Exercise A.21. Let X be a rigid object of T. Prove that the functor $X \otimes -\colon \mathsf{T} \to \mathsf{T}$ commutes with products.

Exercise A.22. Prove Proposition 4.22(3).

Exercise A.23. This exercise concerns the construction of Brown-Comenetz duals in tensor-triangulated categories. These are certain 'designer' objects which play an important role in stable homotopy theory. Let T be a rigidly-compactly generated tensor-triangulated category.

(1) Let $C \in \mathsf{T}^{\mathrm{c}}$. Show that there exists an object $\mathbb{I}_C \in \mathsf{T}$ such that
$$\mathrm{Hom}_{\mathsf{T}}(-,\mathbb{I}_C) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathsf{T}}(C,-),\mathbb{Q}/\mathbb{Z}).$$
(2) Define a functor $I_C\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ by $I_C(-) := F(-,\mathbb{I}_C)$. Prove that $I_C(X) \simeq F(F(C,X),\mathbb{I}_{\mathbb{1}})$.
(3) Let $X \in \mathsf{T}$. Prove that if $I_{\mathbb{1}}(X) \simeq 0$, then $X \simeq 0$. (*Hint:* Recall that $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for abelian groups, so that if $M \in \mathsf{Mod}(\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) \simeq 0$, then $M \simeq 0$.)
(4) Let $\mathcal{X}$ be a set of a compact objects and suppose that if $X \in \mathcal{X}$ and $C \in \mathsf{T}^{\mathrm{c}}$, then $C \otimes X \in \mathcal{X}$. Consider the set
$$\mathcal{X}^{\perp_{\mathbb{Z}}} := \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i X, Y) \simeq 0 \text{ for all } X \in \mathcal{X} \text{ and } i \in \mathbb{Z}\}.$$
Show that if $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$, then $X \otimes Y \simeq 0$. Deduce that if $Y \in \mathcal{X}^{\perp_{\mathbb{Z}}}$, then $I_C(Y) \in \mathcal{X}^{\perp_{\mathbb{Z}}}$ for all $C \in \mathsf{T}^{\mathrm{c}}$.
(5) Consider $\mathsf{T} = \mathsf{D}(R)$ for a commutative ring R. What is $\mathbb{I}_R$?

Exercise A.24. Let T be a tensor-triangulated category.

(1) Prove that if X is compact and Y is rigid, then $X \otimes Y$ is compact.
(2) Prove that if X and Y are rigid, then $X \otimes Y$ is rigid.

Exercise A.25. Let T be a tensor-triangulated category, and let $\mathcal{S}$ be a thick $\otimes$-ideal of T. Prove that $\mathcal{S}$ is radical, i.e., for all $X \in \mathsf{T}$, if $X^{\otimes n} \in \mathcal{S}$ for some n, then $X \in \mathcal{S}$.

Exercise A.26. Let $F\colon \mathrm{Ch}(R) \to \mathrm{Ch}(S)$ be an exact functor. Prove that there exists a natural isomorphism $\phi\colon F\Sigma \xrightarrow{\sim} \Sigma F$, and for every $f\colon M \to N$ in $\mathrm{Ch}(R)$, there is an isomorphism $\theta\colon F(C(f)) \xrightarrow{\sim} C(F(f))$ making the diagram

$$\begin{array}{ccccc}
F(N) & \xrightarrow{F(i_f)} & F(C(f)) & \xrightarrow{F(c_f)} & F(\Sigma M) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi_M} \\
F(N) & \xrightarrow[i_{F(f)}]{} & C(F(f)) & \xrightarrow[c_{F(f)}]{} & \Sigma F(M)
\end{array}$$

commute.