1 Equations

(from “Triangular” ideas. Idea 03, “The disk”. Version 010.3)

Red terms indicate a 2 dimensional system. Removing them would bring the system to 1D.

$$\ddot{x}_0 = -\frac{g}{m}\frac{-n_{1/2}}{x_0 - x_{1/2}}$$

$$\ddot{x}_i = -\frac{g}{m}\frac{n_{i-1/2} - n_{i+1/2}}{x_{i-1/2} - x_{i+1/2}}$$

$$0 < x_N < x_{N-1} < ... < x_1 < x_0 < \infty$$

with

$$\begin{aligned}
x_{i-1/2} &\equiv \frac{x_{i-1} + x_i}{2} \quad \text{for} \quad 0 < i \\
x_{i+1/2} &\equiv \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad i < N \\
x_{N+1/2} &\equiv \frac{x_N}{2} \\
n_{i-1/2} &\equiv n^{\delta t}\left(x^{\delta t}_{i-1/2}\right)\frac{x^{\delta t}_{i-1/2}}{x_{i-1/2}}\frac{x^{\delta t}_{i-1} - x^{\delta t}_i}{x_{i-1} - x_i} \\
n_{i+1/2} &\equiv n^{\delta t}\left(x^{\delta t}_{i+1/2}\right)\frac{x^{\delta t}_{i+1/2}}{x_{i+1/2}}\frac{x^{\delta t}_i - x^{\delta t}_{i+1}}{x_i - x_{i+1}} \quad \text{for} \quad i < N \\
n_{N+1/2} &\equiv n^{\delta t}\left(x^{\delta t}_{N+1/2}\right)\frac{x^{\delta t}_{N+1/2}}{x_{N+1/2}}\frac{x^{\delta t}_N}{x_N}
\end{aligned} \tag{1}$$

Forces can be rewritten as:

$$\begin{aligned}
\ddot{x}_i &= -\frac{g}{m}\frac{n_{i-1/2} - n_{i+1/2}}{x_{i-1/2} - x_{i+1/2}} \\
&= -\frac{g}{m}\frac{n^{\delta t}\left(x^{\delta t}_{i-1/2}\right)\frac{x^{\delta t}_{i-1/2}}{x_{i-1/2}}\frac{x^{\delta t}_{i-1}-x^{\delta t}_i}{x_{i-1}-x_i} - n^{\delta t}\left(x^{\delta t}_{i+1/2}\right)\frac{x^{\delta t}_{i+1/2}}{x_{i+1/2}}\frac{x^{\delta t}_i-x^{\delta t}_{i+1}}{x_i-x_{i+1}}}{x_{i-1/2} - x_{i+1/2}} \\
&= -2\frac{g}{m}\frac{n^{\delta t}\left(\frac{x^{\delta t}_{i-1}+x^{\delta t}_i}{2}\right)\frac{x^{\delta t}_{i-1}+x^{\delta t}_i}{x_{i-1}+x_i}\frac{x^{\delta t}_{i-1}-x^{\delta t}_i}{x_{i-1}-x_i} - n^{\delta t}\left(\frac{x^{\delta t}_{i+1}+x^{\delta t}_i}{2}\right)\frac{x^{\delta t}_i+x^{\delta t}_{i+1}}{x_i+x_{i+1}}\frac{x^{\delta t}_i-x^{\delta t}_{i+1}}{x_i-x_{i+1}}}{x_{i-1} - x_{i+1}} \\
&= \frac{\Phi^-_i}{(x_{i-1}+x_i)(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + \frac{\Phi^+_i}{(x_i+x_{i+1})(x_i-x_{i+1})(x_{i-1}-x_{i+1})}
\end{aligned} \tag{2}$$

where

$$
\begin{aligned}
\Phi_i^- &= -2\frac{g}{m} n^{\delta t} \left(\frac{x_{i-1}^{\delta t} + x_i^{\delta t}}{2} \right) \left(x_{i-1}^{\delta t} + x_i^{\delta t} \right) \left(x_{i-1}^{\delta t} - x_i^{\delta t} \right) \\
\Phi_i^+ &= +2\frac{g}{m} n^{\delta t} \left(\frac{x_{i+1}^{\delta t} + x_i^{\delta t}}{2} \right) \left(x_i^{\delta t} + x_{i+1}^{\delta t} \right) \left(x_i^{\delta t} - x_{i+1}^{\delta t} \right)
\end{aligned} \tag{3}
$$

are fixed parameters not depending on x and, thus, can be computed beforehand.

2 Boundary conditions

2.1 $i = 0$

$$
\begin{aligned}
\ddot{x}_0 &= -\frac{g}{m} \frac{-n_{1/2}}{x_0 - x_{1/2}} \\
&= -\frac{g}{m} \frac{-n^{\delta t} \left(x_{1/2}^{\delta t} \right) \frac{x_{1/2}^{\delta t}}{x_{1/2}} \frac{x_0^{\delta t} - x_1^{\delta t}}{x_0 - x_1}}{x_0 - x_{1/2}} \\
&= 2\frac{g}{m} \frac{n^{\delta t} \left(\frac{x_0^{\delta t} + x_1^{\delta t}}{2} \right) \frac{x_0^{\delta t} + x_1^{\delta t}}{x_0 + x_1} \frac{x_0^{\delta t} - x_1^{\delta t}}{x_0 - x_1}}{x_0 - x_1} \\
&= \frac{\Phi_0^+}{(x_0 + x_1)(x_0 - x_1)(x_0 - x_1)}
\end{aligned} \tag{4}
$$

2.2 $i = N$

$$
\begin{aligned}
\ddot{x}_N &= -\frac{g}{m} \frac{n_{N-1/2} - n_{N+1/2}}{x_{N-1/2} - x_{N+1/2}} \\
&= -\frac{g}{m} \frac{n^{\delta t} \left(x_{N-1/2}^{\delta t} \right) \frac{x_{N-1/2}^{\delta t}}{x_{N-1/2}} \frac{x_{N-1}^{\delta t} - x_N^{\delta t}}{x_{N-1} - x_N} - n^{\delta t} \left(x_{N+1/2}^{\delta t} \right) \frac{x_{N+1/2}^{\delta t}}{x_{N+1/2}} \frac{x_N^{\delta t}}{x_N}}{x_{N-1/2} - x_{N+1/2}} \\
&= -2\frac{g}{m} \frac{n^{\delta t} \left(\frac{x_{N-1}^{\delta t} + x_N^{\delta t}}{2} \right) \frac{x_{N-1}^{\delta t} + x_N^{\delta t}}{x_{N-1} + x_N} \frac{x_{N-1}^{\delta t} - x_N^{\delta t}}{x_{N-1} - x_N} - n^{\delta t} \left(x_N^{\delta t}/2 \right) \frac{x_N^{\delta t}}{x_N} \frac{x_N^{\delta t}}{x_N}}{x_{N-1}} \\
&= \frac{\Phi_N^-}{(x_{N-1} + x_N)(x_{N-1} - x_N) x_{N-1}} + \frac{2\frac{g}{m} n^{\delta t} \left(x_N^{\delta t}/2 \right) x_N^{\delta t} x_N^{\delta t}}{x_N x_N x_{N-1}}
\end{aligned} \tag{5}
$$

To simplify the code, we will set the numerator of the second fraction as a special value in Φ_N^+

3 Initial conditions

The superindex $f^{\delta t}$ indicates $f(t = \delta t)$

$$x_i^{\delta t} = \begin{cases} \Delta x \frac{i}{M} & \text{for} \quad 0 \leq i \leq M \\ \Delta x \, (i - M + 1) & \text{for} \quad M \leq i \leq N \end{cases} \tag{6}$$

$$n^{\delta t}(x) = \begin{cases} \frac{1}{9} \frac{m}{g} \left[\frac{x - (R_0 + 2V_\mu \delta t)}{\delta t} \right]^2 & \text{for} \quad R_0 - V_\mu \delta t \leq x \leq R_0 + 2V_\mu \delta t \\ n(0) & \text{for} \quad 0 < x \leq R_0 - V_\mu \delta t \end{cases} \tag{7}$$

$$\dot{x}^{\delta t}(x) = \begin{cases} 2V_\mu + \frac{2}{3} \frac{x - (R_0 + 2V_\mu \delta t)}{\delta t} & \text{for} \quad R_0 - V_\mu \delta t \leq x \leq R_0 + 2V_\mu \delta t \\ 0 & \text{for} \quad 0 < x \leq R_0 - V_\mu \delta t \end{cases} \tag{8}$$