Econ 171
Decisions Under Uncertainty
Lecture Notes and Problem Sets

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1 Chapter 1: Basics of Decision-Making Under Uncertainty

Imagine you hold an object and then drop it. What will happen? We know for sure it will fall down. This is an example of an action with a certain outcome.

Economic decisions, and many decisions in life, for that matter, are not like dropping an object. The outcomes are not certain. Instead, several possible outcomes are usually possible, and we take actions without knowing a priori what the result will be. You go to a new restaurant without knowing how much you will like the food. An entrepreneur opens a business without certainty about how it will perform. A doctor prescribes a drug to a patient without being sure whether it will be effective or whether the patient will experience some or all of the known side effects.

Because choices lacking certainty are ubiquitous, a branch of economics and decision theory is devoted to the study of decisions under uncertainty (the lack of certainty). In this course, you will be introduced to the main concepts and ideas of choice under uncertainty from the perspective of economics. Other fields also study decisions under uncertainty, including psychology, neuroscience, and computer science. More generally, decisions under uncertainty belong to the broader field of decision theory.

1.1 What Kinds of Uncertainty Are There?

Not all uncertain problems that we face are the same. Sometimes the probabilities of different outcomes are well-defined; other times they are just a subjective perception, and sometimes it feels like a combination of both. In 1921, Frank Knight identified two types of uncertainty: "risk" or what can be fully measured, and "uncertainty" or what can't be measured in a useful way. In modern times, we distinguish between three types of uncertain environments:

1. **Objective Risk** is when the chances of each outcome are known and agreed upon. For example, if you roll a fair six-sided die, you know each face has exactly a 1-in-6 chance of appearing. Another example: the weather forecast predicts that it will rain tomorrow with a 20% probability. A medicine for headaches is known (through studies) to be effective with a probability of 98%. In these cases, the randomness is objective and quantifiable.

2. **Subjective Risk** happens when the probabilities are not known for sure, but the individual forms their own beliefs or guesses consistently, that is, they treat these beliefs as if they were the real probabilities. For example, when betting on horse races, each bettor probably believes their horse is the most likely to win.

3. **Ambiguity** (or true uncertainty in Knight's words) arises when several probability distributions of the outcomes are possible, and the individual does not have more information. For example, a firm is deciding whether to open a branch in country X. If the president of country X is friendly to foreign investors, the firm will succeed with probability 80% and fail with probability 20%. If the president is hostile to foreign investors, the success chances are only 40% and the failure chances are 60%. Because the president was just elected, it is unknown whether she is friendly or hostile. Hence, the firm does not know whether it operates under the (80%, 20%) chances or the (40%, 60%) chances. We refer to ambiguity as a situation of multiple possible distributions or multiple priors.

Exercise: Think about these scenarios and decide which type of uncertainty each represents:

- Going on a date at the park. Google Weather reports a 20% chance of rain.
- Going on a date at the park. Your grandma looks at the sky and says, "The shape of the clouds makes me think it's more likely to rain than not."
- Accepting a job in a new industry, where the probability of success depends on whether your skills are a good match (which is unknown).
- A student cheats on an exam because they believe the chance of getting caught is very low.
- Drawing another card in blackjack when your current total is 17 points.

Which ones involve objective risk? Subjective risk? Ambiguity?

1.2 Elements of a Decision Problem

In a decision problem under uncertainty, the decision maker takes one of a set of possible actions. Each action leads to one of many possible outcomes, depending on the state of the world (or just "state"). Each state occurs with some probability.

Four elements are identified in the previous description:

Actions. What the decision maker does. It is the only part under their control.
Outcomes. The consequences or payoffs of the decision-maker's actions.
States. The variables that determine which outcome is realized.
Probabilities. The chances of each state occurring.

Formally, we will denote actions with the letter a, that is, a_1, a_2, a_3, etc., are possible actions. We will denote states with the Greek letter θ, that is $\theta_1, \theta_2, \theta_3$, etc., are possible states of the world, with probabilities $p(\theta_1), p(\theta_2)$, etc. (all the probabilities lie between 0 and 1). Finally, we will denote the outcomes with the letter z, that is, z_1, z_2, etc., are possible outcomes.

For each action and each possible state, there is a corresponding outcome. Hence, if we have K possible actions and M possible states, a decision problem will define $K \times M$ outcomes (although some outcomes may be repeated).

Consider this example: You have two job offers on the table. One comes from a fast-growing tech startup, promising exciting opportunities but with some risks. The other comes from a stable government agency that offers steady work and a reliable salary.

Now, you need to decide which job to accept, but there is uncertainty about how the economy will perform in the next year, something that will affect your salary at the startup.

Your prospects are as follows:

- **If you accept the tech startup job** and the economy grows strongly, the company strives, and your total compensation will be high ($120,000, including salary and stock options).

- **But if the economy falls into a recession**, the startup may struggle. Your salary will be reduced to \$60,000.
- **On the other hand, if you accept the government job**, you get a steady \$80,000 salary regardless of how the economy performs.

Your estimation is a 60% chance of economic growth and a 40% chance of recession.

Now, let's use this story to identify the parts of this decision problem:

Actions: Accepting the startup job or accepting the government job.

States of the World: How the economy performs next year. Either it grows or falls into recession.

Outcomes: Your actual compensation, which depends on both the job you choose and the economic conditions.

Probabilities: Your belief that there is a 60% chance of strong growth and a 40% chance of recession.

Note: The actions of a decision problem are **mutually exclusive** and **collectively exhaustive**. Mutually exclusive means that if a participant has a set of possible actions $\{a_1, a_2, a_3\}$, only one action can be taken at a time (never more than one action). Collectively exhaustive means that the set of actions available to the decision maker is the set of all the actions that the decision maker can take (no hidden actions).

Similarly, the states of the world are **mutually exclusive** (only one state of the world can occur), and **collectively exhaustive** (the set of states of the world includes all the states that could influence an outcome -no hidden states).

Hence, in a decision problem, the decision maker takes one (and only one) of the possible actions. After this, one (and only one) state of the world occurs and determines the outcome.

1.3 Graphical Representation

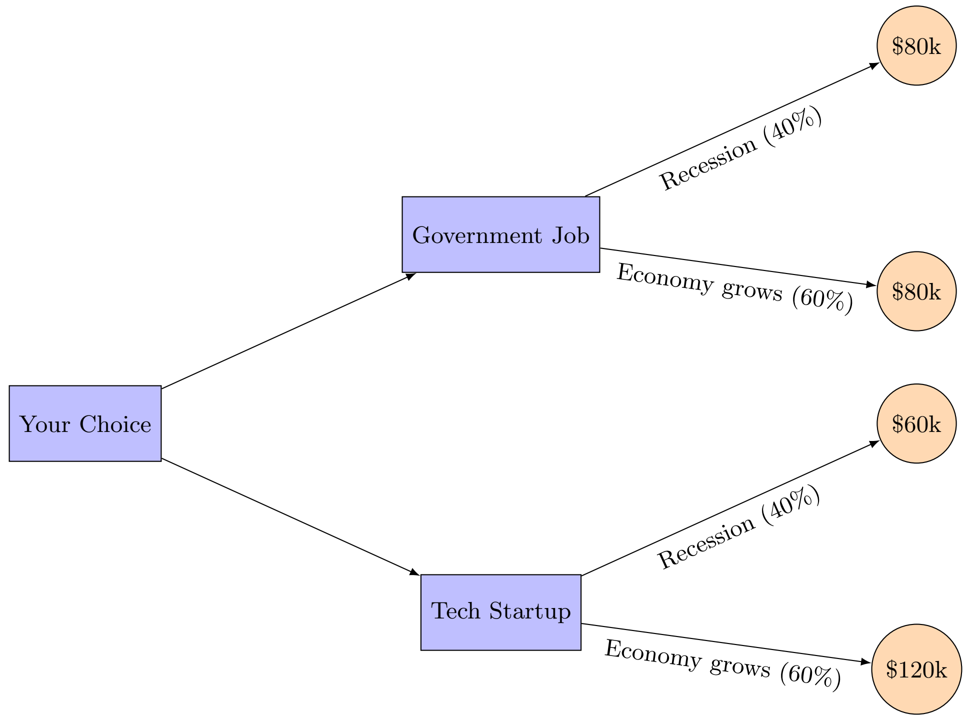
A helpful way to represent decisions under uncertainty is the graphical approach. We will cover two types: decision trees and payoff matrices.

These representation techniques are also used in Game Theory. However, here the entire representation refers to the choice problem of a single decision-maker.

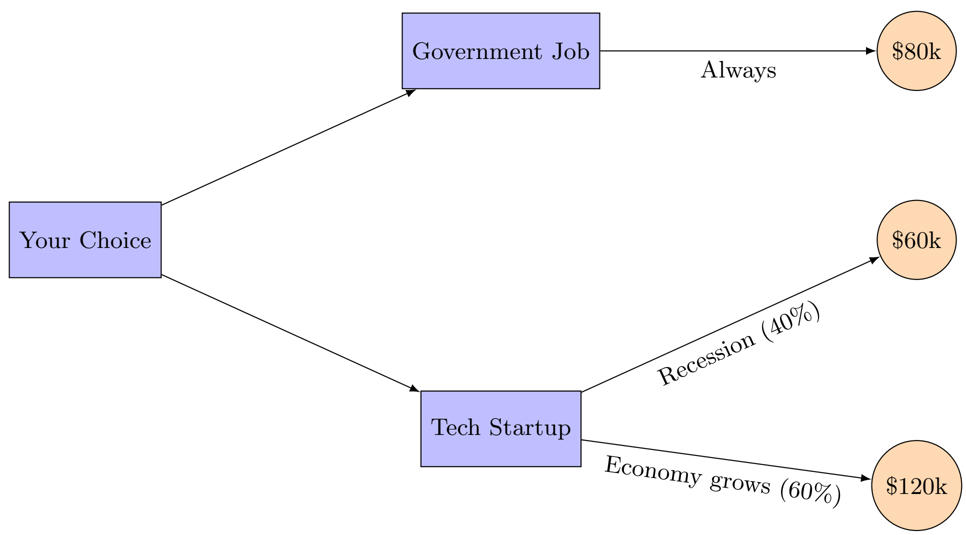
Decision Trees

Think of a decision tree as a road map that starts with your actions (choices). Each action branches into arrows representing states with some probability, and each state (arrow) leads to an outcome.

Below is a decision tree representing our job example. The purple squares represent the actions (the jobs you choose), the arrows that come out of those squares are the possible states of the world with their probabilities, and the circles at the end show the outcomes you will receive for each possible state:



Notice that, because you have two actions and two states, the corresponding number of outcomes is 4 (one per combination of action-state). However, the government job pays the same regardless of the state (economic performance), so we could have included a single outcome in that case, but not for the tech startup, whose payoff depends on the state.



Payoff Matrix

Another useful way to represent the decision problem is through a payoff matrix. This matrix organizes the individual's possible actions and the states in a table, where each cell shows the outcome (payoff) of choosing a certain action under a particular state.

Here's how to read the matrix below:

- Each **row** represents an action you can take (job choice). - Each **column** represents a state of the world (how the economy performs). - The **cell** at the intersection shows the outcome you get if you pick that action and that state occurs.

Job Choice	**Economy Grows (60%)**	**Economy Recession (40%)**
Tech startup	$120,000	$60,000
Government agency	$80,000	$80,000

For example, if you choose the tech startup job (top row) and the economy grows (left column), you will receive $120,000. If the economy instead falls into recession (right column), your outcome will be $60,000.

Exercise. Represent the following problem in a decision tree and a payoff matrix.

A student can choose to study all weekend for next Tuesday's midterm exam, or go to a music festival instead. The midterm could be easy or hard.

- If the student goes to the festival and the exam is hard, the student will get a C-, but will have enjoyed the festival.
- If the student goes to the festival and the exam is easy, the student will get an A-, and will have enjoyed the festival.
- If the student does not go to the festival and the exam is hard, the student will get an A-, but with a boring weekend.
- If the student does not go to the festival and the exam is easy, the student will get an A, but with a boring weekend.

2 Chapter 2: Random Variables and Decision Criteria

In the previous chapter, we introduced problems with uncertainty, cases in which the outcome of an action is not known a priori but depends on some state of the world. In this chapter, we will explore how to describe that uncertainty mathematically, using the concept of **random variables** from statistics. Economists often call random variables *lotteries* as they represent the chance of different outcomes occurring. I will use the word lottery to refer to random variables or problems with uncertainty.

A random variable (usually represented by capital letters, like X) is simply a rule that assigns a number to every possible outcome of a random situation.

For example, if you flip a coin and earn \$10 for heads and \$0 for tails, we can describe that situation using a random variable X, where:

- $X = 10$ with probability 0.5 (heads)
- $X = 0$ with probability 0.5 (tails)

Random variables can be discrete or continuous.

2.1 Discrete vs. Continuous Random Variables

Discrete Random Variables

Discrete random variables have a countable number of possible values. Imagine a lottery that gives you:

Outcome (x)	Probability ($P(x)$)
\$0	0.4
\$50	0.3
\$100	0.3

This is a discrete random variable, it can only result in three possible outcomes. You can list them all.

Continuous Random Variables

These random variables can take values from a whole range, often infinitely many possibilities. A common example is the amount of rainfall in a day or the return on a stock. In a single day it may rain 15.3mm, 7.008mm, 17.922mm, etc.

For continuous random variables, the probability of a single outcome is zero (infinitesimally small), we compute instead the probability that the outcome is lower than or equal to a certain amount.

2.2 Key Properties of Random Variables

From now on, we will focus our study on discrete random variables since many of our examples come from discrete environments.

Our working example will be the one we introduced earlier:

Variable X with the following payoff structure:

Outcome (x)	Probability ($P(x)$)
\$0	0.4
\$50	0.3
\$100	0.3

Understanding a random variable means understanding its distribution: the possible outcomes and their likelihood of occurring. Here are some important properties of random variables.

Probability Distribution Function (PDF)

The PDF is a complete description of the random variable, it lists all the possible outcomes and their respective probability of occurring. It's ordered from the lowest to the highest possible outcome.

We typically denote the PDF as f_X, where X is the random variable.

For our example lottery, the PDF is given by:

- \$0 has probability 0.4
- \$50 has probability 0.3
- \$100 has probability 0.3

We can also write $f_X(0) = 0.4$, $f_X(50) = 0.3$, and $f_X(100) = 0.3$.

An important characteristic of the PDF is that all the probabilities must be non-negative and must add up to one. This guarantees three things:

- No outcome happens with negative probability (negative probabilities do not make sense).
- The outcomes are collectively exhaustive (the PDF includes all the possible outcomes).
- The outcomes are mutually exclusive (the probability that two outcomes occur at the same time is zero).

Cumulative Distribution Function (CDF)

The CDF tells us the probability that the outcome is **less than or equal** to a certain number. It is not about the probability of a particular value, but rather the probability of a range "this value or less".

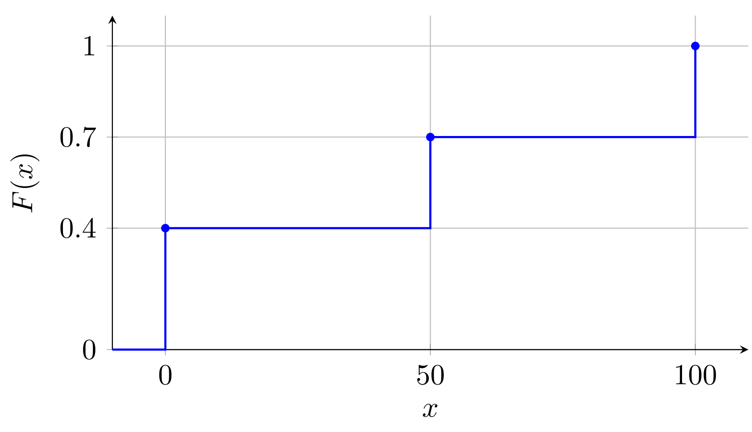
The CDF is typically denoted as F_X.

Let's compute the CDF of our example lottery:

- $F_X(0) = P(X \leq 0) = 0.4$. The only outcome smaller or equal to 0 in the lottery is $X = 0$, which happens with probability 40%.
- $F_X(50) = P(X \leq 50) = 0.4 + 0.3 = 0.7$. Two outcomes are smaller or equal to 50, $X = 0$ and $X = 50$. The probability that X is either 0 or 50 is equal to $P(X = 0) + P(X = 50) = 0.4 + 0.7$[1]
- $F_X(100) = P(X \leq 100) = 0.4 + 0.3 + 0.3 = 1$. All the outcomes in our example lottery are equal to or smaller than 100, so the probability that the outcome is smaller than or equal to 100 must be 1.
- What about something in between, like $F_X(70)$? That's the probability that $X \leq 70$. Both 0 and 50 (but not 100) are smaller than or equal to 70, so the probability of $X \leq 70$ is equal to $P(X = 0) + P(X = 50) = 0.4 + 0.3 = 0.7$. Notice that $f_x(70) = 0$ because no outcome is equal to 70.

Later in this class, we will study other concepts involving CDFs. For this, it is helpful to know how to graph them. CDFs graphs have the potential outcomes in the x-axis and the probabilities in the y-axis. The graph is always weakly increasing in x and the values they can take go from 0 to 1.

The figure below shows the CDF for our example lottery. It matches every possible outcome (0, 50, and 100) with it's respective cumulative probability that we just computed.



[1]Because all the outcomes are mutually exclusive, the probability that $X = 0$ or $X = 50$ is just the sum of the probabilities $P(X = 0) + P(X = 50)$. If they were not mutually exclusive, we would need to subtract the probability of the intersection, that is $P(X = 0) + P(X = 50) - P(X = 0 \text{and} X = 50)$.

Mean (or Expected Value)

It is the sum of every possible value multiplied by its respective probability.

The formula is:

$$\mathbb{E}[X] = \sum_{i=1,2,3,\ldots} x_i \times P(X = x_i)$$

For our lottery example:

$$\mathbb{E}[X] = 0 \cdot 0.4 + 50 \cdot 0.3 + 100 \cdot 0.3 = 0 + 15 + 30 = 45$$

So the average (or expected) outcome is \$45.

Variance and Standard Deviation

The variance and standard deviation are measures of how spread out the outcomes are. The variance is the average of the squared distance between each possible value and the mean of the variable.

The formal definition is $VAR(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$

The formula is:

$$\text{Var}(X) = \sum_{i=1,2,\ldots} P(x_i) \cdot (x_i - \mathbb{E}[X])^2$$

$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$

Another way to denote the variance is σ_X^2 and the standard deviation, σ_X
For our example lottery:

Outcome	Probability	Deviation from Mean	Squared
\$0	0.4	-45	2025
\$50	0.3	5	25
\$100	0.3	55	3025

Now compute:

$$\text{Var}(X) = 0.4 \cdot 2025 + 0.3 \cdot 25 + 0.3 \cdot 3025 = 810 + 7.5 + 907.5 = 1725$$

$$\text{SD}(X) = \sqrt{1725} \approx 41.5$$

This tells us that on average, the outcomes deviate from the mean by about \$41.50 units.

Min, Max, and Range

The min, the max, and the range provide information about the extreme values of a random variable.

- **Min:** It is the lowest possible outcome for X.
- **Max:** It is the highest possible outcome for X.
- **Range:** It is the distance between the Max and the Min.

For our example, the values are:

- Min = lowest possible outcome → \$0
- Max = highest possible outcome → \$100
- Range = Max – Min = \$100 – \$0 = **\$100**

Skewness and Kurtosis

Skewness and Kurtosis are other properties of a distribution typically studied in statistics classes. I will only briefly cover skewness.

Skewness refers to the "asymmetries" of the distribution function of a random variable.

For example, the random variable given by $X = \{1, 2, 3\}$ with $P(1) = 1/3, P(2) = 1/3, P(3) = 1/3$ is fully symmetrical, because each possible value happens with the same chance. However, the random variable $Y = \{1, 2, 3\}$ with $P(1) = 0.5, P(2) = 0.25, P(3) = 0.25$ is "skewed" to the right, because it is more likely to get a lower value (1) than a higher value (3). On the other hand, the random variable $Y = \{1, 2, 3\}$ with $P(1) = 0.1, P(2) = 0.2, P(3) = 0.7$, is "skewed" to the left because the greater value (3) is much more likely than the lower value (1).

Another example for practice

Consider the following lottery:

Outcome (x)	Probability ($P(x)$)
\$20	0.5
\$80	0.5

The CDF is given by:

- $P(X \leq 0) = 0$
- $P(X \leq 20) = 0.5$
- $P(X \leq 80) = 1$

The expected value is given by:

$$\mathbb{E}[X] = 20 \cdot 0.5 + 80 \cdot 0.5 = 10 + 40 = 50$$

The variance is given by:

Outcome	Probability	Deviation from Mean	Squared
$20	0.5	20 - 50 = -30	900
$80	0.5	80 - 50 = 30	900

Now compute:

$$\text{Var}(X) = 0.5 \cdot 900 + 0.5 \cdot 900 = 450 + 450 = 900$$

And the standard deviation is:

$$\text{SD}(X) = \sqrt{900} = 30$$

This means the expected deviation from the mean is $30.

The mix, max, are range are:

Min = $20, Max = $80, Range = $60

Measures of Variability: Why Variance is a Better Measure than Range

If many applications of decisions under uncertainty, like finance, the variability or dispersion of an object (financial asset, project strategy, etc.) is often used as a measure of risk.

Both **variance** (or standard deviation) and **range** measure how spread out a lottery's possible outcomes are (variability), but they do it differently:

- **Range** is just the difference between the highest and lowest outcomes.
- **Variance** takes into account how far each outcome is from the mean, weighted by how likely each outcome is.

Consider these two lotteries:

- Lottery A: outcomes -$100, $0, and $100 with probabilities 0.01, 0.98 and 0.01, respectively.
- Lottery B: outcomes -$30, 0, and $30 with probabilities 0.3, 0.4 and 0.3, respectively.

Both lotteries has mean $0, so they are equally good in terms of expected value.

Lottery A has a range of (100-(-100) =200), which is larger than the range of Lottery B of (30-(-30)=60). The difference between the extreme values is less extreme in lottery B, which makes it seem less spread.

However, the variance for lottery A is

$$\sigma_A^2 = 0.01 \cdot (100 - 0)^2 + 0.98 \cdot (0 - 0)^2 + 0.01 \cdot (100 - 0)^2 = 100 + 0 + 100 = 200$$

And the variance of lottery B is

$$\sigma_B^2 = 0.3 \cdot 900 + 0.4 \cdot 0 + 0.3 \cdot 900 = 270 + 0 + 270 = 540$$

which is more than lottery A.

Which lottery exhibits more variability?

While Lottery A has a greater range, the extreme outcomes of -100 and 100 occur with very low probability. Most of the time, the outcome is 0, making the experience of playing Lottery A feel relatively stable despite its wide range.

In contrast, Lottery B presents outcomes of -30, 0, and 30 with relatively balanced probabilities. Getting 0 is only slightly more likely than getting either of the other two outcomes. All three outcomes feel "on the table," contributing to a greater sense of variability.

Another way to put it is this: If someone plays Lottery A repeatedly, the outcome will most often be 0, and the rare large deviations will have limited influence on the overall experience. On the other hand, someone repeatedly playing Lottery B will routinely observe all three outcomes, with fluctuations being more noticeable from round to round.

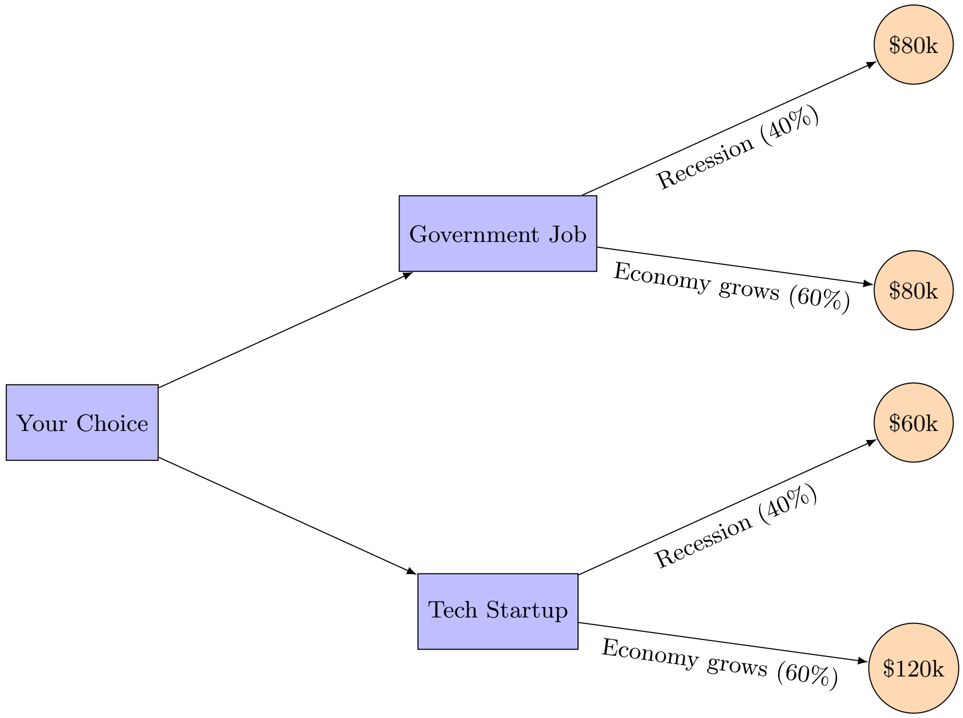
Thus, although Lottery A has the greater range, Lottery B exhibits more practical or experienced variability, which is captured more accurately by the variance measure. The extreme values in Lottery A lose importance precisely because they are so unlikely.

In summary: while the range only measures the distance between extremes, the variance better captures the *typical* variability of outcomes compared to the mean accounting for how likely each outcome is. This makes variance a preferred measure of risk.

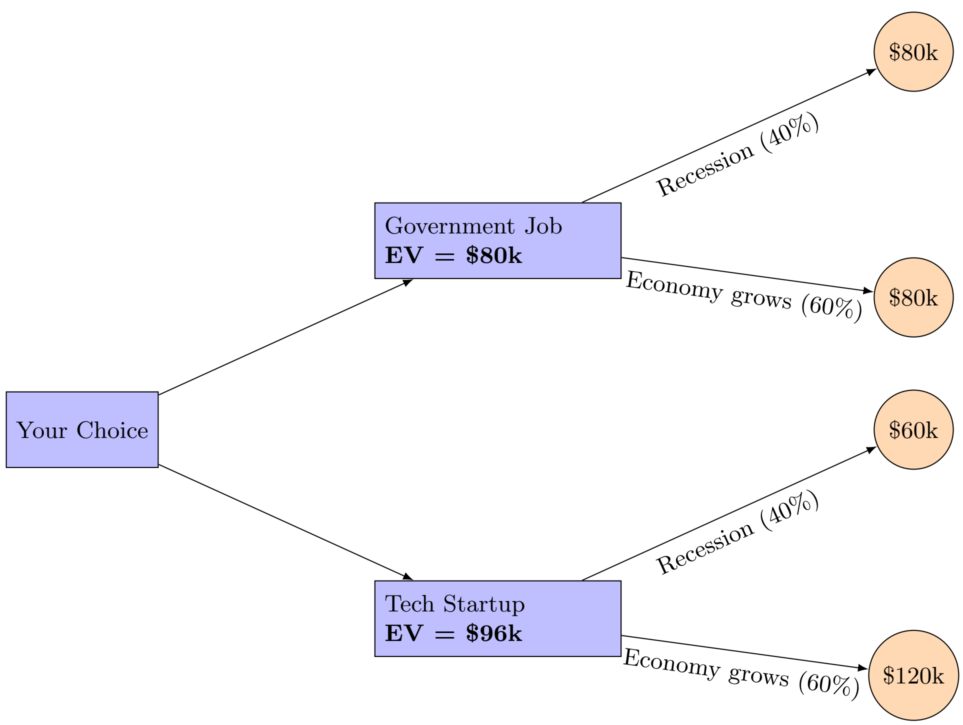
Expected Value in Decision Trees

Let's go back to the choice between two jobs example that we developed in the last chapter to illustrate the decision tree, but now we will complete it with the expected value of each option.

We will compute the expected income of each job and add that value on top of each job node. Below is our original representation of the problem:



We can now calculate the **expected value (EV)** of each action by multiplying the outcomes by their respective probabilities and adding them. Adding these EVs above each action helps visualize which action might be better on average.



Notice the EV for the Tech Startup is $\$120k \times 0.6 + \$60k \times 0.4 = \$96k$, and for the Government Job it is $\$80k \times 1 = \$80k$, since the government pays $\$80k$ regardless of the state. This representation helps us compare the average expected earnings from each choice.

2.3 Decision Criteria: How to Choose under Uncertainty

So far, we focused on representing uncertainty and reviewing the basic statistical concepts related to the random variables equivalent to the lotteries in uncertainty problems.

How do people make choices in environments with uncertainty? Is there any "correct prescription" about how to behave? Can we predict how people make decisions when facing uncertainty?

These questions delve into an important distinction in economics between positive and normative theories:

- Positive Theories: Try to describe the world as it is. How do people behave? How do they make choices? What are their preferences? These theories only care about how the agent's behavior **is**, not about how it should be.
- Normative Theories: Try to provide a framework for how behavior should be. It serves as a reference point. Normative theories usually use terminology like "optimal behavior" or "utility or profit maximization" because they have that type of behavior as a normative standard.

We are going to study different criteria that people can use to choose when dealing with uncertainty. We will not claim that all decision makers actually behave like that, or at least not all the time, instead we will provide a benchmark of the type: "if people behaved following this criterion, their choices should be like this".

We can broadly divide the criteria into two groups:

- **Non-probabilistic criteria:** These rules make no use of probabilities. Instead, they focus on the best or worst possible outcomes, or on minimizing regret.

  An important **advantage** of these criteria is that, because they do not rely on probabilities, they remain applicable even in situations of high ambiguity[2].

  They include:

  - Maximax
  - Maximin
  - Minimax Regret

- **Probabilistic criteria:** These rules use probabilities to define rules that will guide decisions.

[2] Note that these criteria are overly simplistic, even for cases of ambiguity. If a decision maker considers multiple possible probability distributions, they can incorporate them into more sophisticated decision rules. We will touch on some of these later in the course, time permitting.

An important **advantage** of these criteria is that they incorporate more information about the outcomes-generating process into the decision rules.

They include:

- Expected Value
- Mean-Variance
- First-Order Stochastic Dominance
- (Expected Utility – covered in the next chapter)

Each criterion reflects a different attitude toward risk and uncertainty. Some are more optimistic, others are more cautious, and some strike a balance depending on your risk tolerance.

We will use again the example of a decision maker considering which job offer to take: a startup offer or a government offer. There are two states of the world: good economy and recession, and the outcomes for each outcome and state are summarized in the payoff matrix below:

	Good Economy	**Recession**
Startup	\$120k	\$60k
Government	\$80k	\$80k

2.3.1 Non-probabilistic criteria

Maximax Criterion

This criterion assumes that the best state will happen and chooses the option with the *highest possible payoff* :

- Startup: max payoff = \$120k (since when the economy grows, it yields \$120k, which is greater than \$60k in recession)
- Government: max payoff = \$80k (constant regardless of the economy).

Since \$120k > \$80k, **Maximax chooses Startup**. This reflects optimism, aiming for the best-case scenario.

Maximin Criterion

This criterion assumes that the worst state will happen and chooses the option with the *best worst-case payoff*:

- Startup: minimum payoff = \$60k (the payoff if the economy is in recession)

- Government: minimum payoff = \$80k (stable payoff in all states).

Since \$80k > \$60k, **Maximin chooses Government**. This reflects caution, aiming to protect against the worst possible outcome.

Minimax Regret Criterion

First, construct the *regret table*, which shows how much you lose compared to the best option in each state:

	Good Economy	**Recession**
Startup	\$0k	\$20k
Government	\$40k	\$0k

Explanation:

- In a good economy, Startup yields the highest payoff (\$120k), so if you choose Startup, there is nothing to regret (the regret for choosing Startup is \$0). Government yields \$80k, so if you choose Government, your regret is \$40k (the extra you could have won by choosing Startup).
- In a recession, the Government yields the highest payoff (\$80k), so its regret is \$0; Startup yields \$60k, so the regret of choosing the Startup instead of the Government in this case is \$20k (the extra you could have won by choosing Government).

Now, find the maximum regret for each option:

- Startup: max regret = \$20k
- Government: max regret = \$40k

Minimax regret chooses Startup since it minimizes the maximum potential regret.

Summary of Non-Probabilistic Criteria

- **Maximax:** Focus on the *best possible outcome.*
- **Maximin:** Focus on the *worst possible outcome.*
- **Minimax Regret:** Focus on minimizing the *maximum regret.*

2.3.2 Probabilistic Criteria

Expected Value Criterion (EV)

This criterion relies on one important assumption: **monotonicity in expectation**. Although we have not formally discussed preferences, we need to assume that the decision maker prefers an option with a higher expected outcome over one with a lower expected outcome.

Choice rule: Choose the action with the greatest expected value.

For our example,

Remember that the probabilities are:

- Good Economy: 60%
- Recession: 40%

Calculate expected values (EV):

$$\text{EV}_{Startup} = 0.6 \times 120 + 0.4 \times 60 = 72 + 24 = 96k$$

$$\text{EV}_{Government} = 0.6 \times 80 + 0.4 \times 80 = 48 + 32 = 80k$$

It chooses Startup, as it has the higher expected value (average payoff).

Mean-Variance Criterion

The mean-variance criterion takes into account not only the expected value (mean) but also the variance or standard deviation.

This criterion relies on two assumptions:

- All else being equal, people prefer greater expected outcomes (monotonicity in expectation).
- All else being equal, people prefer options with lower outcome variance.

This criterion is important in finance. It is common to consider the variance of an asset (bond, stock, etc.) as a measure of risk. In other words, the variability of an asset's returns is interpreted as its associated risk.

For our example:

- Startup: EV = $96k, but has a higher standard deviation (since payoffs vary between $120k and $60k).

- Government: EV = \$80k, standard deviation = 0 (payoff is constant).

There is a trade-off: Startup has a greater expected value but also brings more variability. Government pays less, but brings more certainty (less variability). If someone cares about both expected value and variability, it is not obvious which one to choose. We need to incorporate other elements into the decision, which we will cover in the next chapters.

Additional Example: Three financial assets.

Imagine you have three assets:

- Asset 1: Returns 10% of each dollar you invest with probability 30%, and 0% with probability 70%.
- Asset 2: Returns 1% of each dollar you invest with probability 30%, 3% with probability 30%, and 5% with probability 40%.
- Asset 3: Returns 40% of each dollar you invest with probability 5%, 25% with probability 5%, and 0% with probability 90%.

The average return of Asset 1 is $EV(A_1) = 0.1 \times 0.3 = 0.03$.

The average return of Asset 2 is $EV(A_2) = 0.01 \times 0.3 + 0.03 \times 0.3 + 0.05 \times 0.4 = 0.032$.

The average return of Asset 3 is $EV(A_3) = 0.4 \times 0.05 + 0.25 \times 0.05 = 0.0325$.

Asset 3 has the highest expected return, followed by Asset 2, and Asset 1 is last.

In terms of variance:

$\sigma^2_{A_1} = 0.0021$

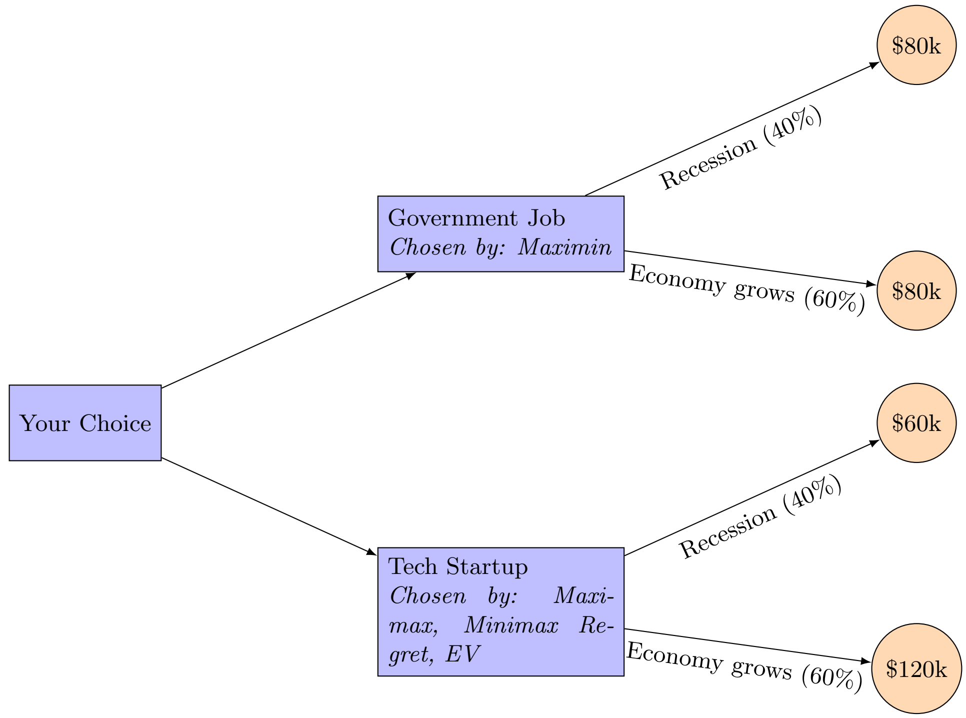
$\sigma^2_{A_2} = 0.000276$

$\sigma^2_{A_3} = 0.01$

So Asset 2 has the lowest variance (it's less volatile/risky), followed by Asset 1, and the most risky is Asset 3.

Asset 2 **clearly dominates** Asset 1 in the sense that it offers a greater expected return and less volatility. However, between Asset 1 and Asset 3, there is a mean/variance trade-off. The same happens between Asset 2 and Asset 3. Again, in those cases we need to look at other aspects that we will cover later.

Decision Tree Illustrating Choice by Criterion



First-Order Stochastic Dominance (FOSD)

First-Order Stochastic Dominance is a powerful way to compare lotteries (random variables). It relies on the assumption of simple monotonicity (*more of a good is better*).

We say that **Lottery A *first-order stochastically dominates* Lottery B** if, at every possible value, the probability that lottery A yields an outcome lower than or equal to that value is *never greater* than the same probability for lottery B — and strictly less for some levels.

In simpler terms:

- For every threshold value, the chance that A gives an outcome *below* that value is less than or equal to the chance that B gives an outcome *below* that value.
- This means that **A always offers outcomes that are as good or better than B**.
- Because this holds at every point, **all decision makers who prefer more to less will prefer Lottery A over Lottery B**, regardless of how they feel about risk.

Important: The following equivalence is useful: **Lottery A *first-order stochastically dominates* Lottery B** if the CDF of lottery A is below the CDF of lottery B for every possible outcome.

Example:
Consider these two lotteries with their probability distributions:

Outcome (x)	0	50	100
Lottery A:	0.5	0.3	0.2
Lottery B:	0.7	0.2	0.1

Let's compute their cumulative distribution functions (CDFs):

$$F_A(x) = P(X \leq x) \qquad F_B(x) = P(X \leq x)$$

- At $x = 0$:
$$F_A(0) = 0.5, \quad F_B(0) = 0.7$$
Since $0.5 < 0.7$, A has lower chance of lower outcomes here.

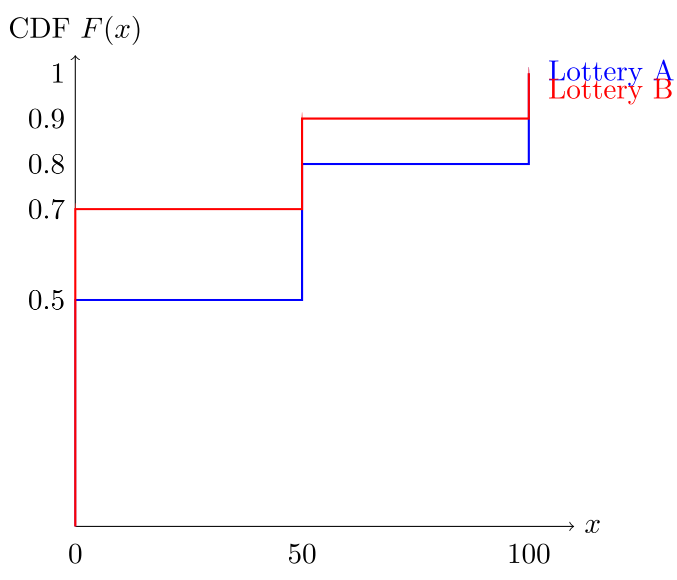
- At $x = 50$:
$$F_A(50) = 0.5 + 0.3 = 0.8, \quad F_B(50) = 0.7 + 0.2 = 0.9$$
Again, $0.8 < 0.9$.

- At $x = 100$:
$$F_A(100) = 1.0, \quad F_B(100) = 1.0$$

So they have equal chance of lower outcomes for the value of 100.

Because $F_A(x) \leq F_B(x)$ for all x, and strictly less for some x, Lottery A FOSD Lottery B.

Graphical Illustration:



Interpretation: The blue curve (Lottery A) is always *below or equal* to the red curve (Lottery B), which means the probability of getting a low outcome is always less or equal under A. Thus, you would prefer Lottery A regardless of your attitude towards risk, as long as you prefer more money to less.

2.4 Problem Set 1

Problem 1: Case

A company is considering how to accommodate commuting for its new employees. One option is simply to expand the parking, which would cost an estimated \$60K.

Alternatively, the company could offer a commuting service by contracting a bus company. They can hire either one bus for \$20K or two buses for \$35K.

The bus contracts must be signed early in the year, before the new employees start work mid-year. This means the company will not know in advance how many employees will use the service. According to company policy, if the service is offered, it must be available to *all* new employees.

If all new employees take the commuting service and the company has contracted two buses, everyone will be accommodated. However, if they have only one bus, there will be insufficient capacity. In that case, the company must still fulfill its commitment and will need to hire taxis at a cost of \$55K.

If only the employees who live nearby choose the commuting service, they will all fit in a single bus, and no parking expansion will be necessary.

Finally, it is possible that no new employees will use the service, in which case the company will have to expand the parking, regardless of the bus options.

Based on job interviews, HR estimates the estimation of the demand for buses as follows:

- Nobody opts for the bus service: 10% chance.
- Only nearby employees opt for the bus service: 60% chance.
- Everyone opts for the bus service: 30% chance.

Based on the previous case:

(a) Identify the actions, states of the world, and probabilities.

(b) Identify the outcomes (hint: What is the firm trying to minimize?).

(c) Represent the problem in a decision tree.

(d) Represent the problem in a payoff matrix.

(e) What would someone following the maximax criterion choose? What about someone following the maximin criterion?

(f) What would someone following the minimax-regret criterion choose? (Draw a regret table).

(g) What would someone following the EV criterion choose?

(h) Is any option dominant over the other one with respect to the mean-variance criterion?

(i) Is any option dominant over the other one with respect to the FOSD criterion?

Problem 2: Case

An author signs a contract with a publisher to produce books over a three-year period. The author can commit to producing 2, 3, 4, or 5 books during this time.

At the end of the three years, the publisher will review the actual number of books published and award a bonus based on the author's commitment.

If the author produced at least as many books as they committed to, they receive a bonus equal to $2,000 multiplied by the number of books they originally committed to produce. For example, if the author committed to producing 4 books, they will receive a bonus of $2000*4=$8000 if they produce 4 books or more.

If the author produced fewer books than their commitment, the bonus will be $0.

The author's estimation about the total number of books they can write is:

- 2 books with probability 25%.
- 3 books with probability 40%.
- 4 books with probability 20%.
- 5 books with probability 15%

Based on the previous case:

(a) Identify the actions, outcomes, states of the world, and probabilities.

(b) Represent the problem in a payoff matrix.

(c) What would the author choose if they followed the maximax criterion? What if they followed the maximin criterion?

(d) What would the author choose if they followed the minimax-regret criterion?

(e) What would the author do if they followed the EV criterion?

(f) Is any option dominant over the other one with respect to the FOSD criterion?

Problem 3 (*challenging*):

Consider the following payoff matrix:

	θ_1	θ_2
a_1	5+β	0
a_2	10-β	8
a_3	3	-2

Assuming β can be greater than or equal to 0:

(a) For which values of β is a_2 the choice under the maximax criterion?

(b) Is there any set of values of β for which a_1 is the choice under all three criteria: maximax, maximin, and minimax regret?

Problem 4 (*very challenging*):

You are deciding how much to invest in a financial project. You can invest any amount between \$0 and \$2000.

The returns depend on whether the project turns out to be promising (with a 50% chance) or mediocre (also 50%).

If you invest \$$M$ and the project is promising, your return (in dollars) will be:

$$W(M) = 10M - 0.002M^2 - 2000$$

If you invest\$$M$ and the project is mediocre, your return (in dollars) will be:

$$W(M) = 3M - 0.004M^2 - 100$$

.

Based on the previous case:

(a) Identify the actions, outcomes, states of the world, and probabilities.

(b) Why are the payoff matrix and decision tree less useful to represent this problem? (Hint: how many possible actions do you have?)

(c) Following the EV criterion, how much should you invest in this project?

(d) What is the expected return if you follow the EV criterion?

3 Chapter 3: More Complex Environments: Problems with Layers

In the previous chapters, we explored basic scenarios with uncertainty and how we can use random variables to represent and evaluate lotteries. However, uncertainty often comes in layers: decisions unfold over time, and outcomes depend on sequences of events or decisions. In this chapter, we explore how to deal with this layered complexity.

3.1 Some Relevant Concepts on Probability

Conditional Probability and Dependence

To understand complex lotteries and multi-step decision problems, we first need to understand how to compute probabilities when multiple random events are involved.

Definition: Conditional Probability

The **conditional probability** of an event A given that another event B has occurred is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0.$$

This measures the probability that A happens once we know that B has occurred.

Dependent and Independent Events

Using conditional probability, we can distinguish between dependent and independent events:

- **Dependent Events:** Events A and B are **dependent** if the occurrence of one affects the probability of the other. That is, either

$$P(A \mid B) \neq P(A) \quad \text{or} \quad P(B \mid A) \neq P(B) \quad \text{or both.}$$

- **Independent Events:** Events A and B are **independent** if the occurrence of one does not affect the probability of the other. Equivalently:

$$P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B).$$

Probability of the Intersection

From the definition of conditional probability, we can always express the probability that both A and B occur as:

$$P(A \cap B) = P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$$

- If A and B are independent, then $P(B \mid A) = P(B)$ and $P(A \mid B) = P(A)$, so this simplifies to:

$$P(A \cap B) = P(A) \cdot P(B)$$

Example 1: Flipping Two Coins

- Event A: First coin shows heads $\Rightarrow P(A) = 0.5$
- Event B: Second coin shows heads $\Rightarrow P(B) = 0.5$

Since the two events are independent:

$$P(A \cap B) = P(A) \cdot P(B) = 0.5 \cdot 0.5 = 0.25$$

It is also true that:

$$P(B = \text{heads} \mid A = \text{heads}) = P(B = \text{heads}) = 0.5$$

and

$$P(A = \text{heads} \mid B = \text{heads}) = P(A = \text{heads}) = 0.5,$$

so the outcome of one coin does not influence the outcome of the other.

In contrast, for dependent events, the conditional probability changes: for example, drawing cards from a deck without replacement, the probability of drawing an Ace on the second draw depends on whether the first card was an Ace or not.

3.2 Compound States

A **compound lottery** is a situation where randomness unfolds in stages. That is, the final outcomes depend on several events happening in sequence, each with its own probabilities. These types of situations are very common in the real world.

Real-life examples:

- In a competition, you may or may not get selected to participate. If selected for participation, you have a chance to win the prize.
- Applying for a job: first, you may pass the first round selection, then you may be selected in the final round.
- Investing in a startup: the startup has to survive the first year, and if so, then succeed to be worthwhile of pursuing.

Example: Lottery With Entry and Prize

- Stage 1: You are selected to participate in a raffle with probability $P(E) = 0.7$
- Stage 2 (if selected):
  - You win \$100 with probability 0.4. That is $P(W_{100}|E) = 0.4$.
  - You win \$10 with probability 0.6. That is$P(W_{10}|E) = 0.6$.
- If not selected ($P(\neg E) = 0.3$): you win \$0.

Final Outcome Probabilities:

- Win \$100: $P(E \cap W_{100}|E) = 0.7 \cdot 0.4 = 0.28$ (to win \$100, you need both to be selected and to win that prize).
- Win \$10: $P(E \cap W_{10}|E) = 0.7 \cdot 0.6 = 0.42$ (to win \$10, you need both to be selected and to win that prize).
- Win \$0: $P(\neg E) = 0.3$ (If you are not selected, you get \$0 for sure).

Tabular Representation:

Selected?	Prize	Combined Probability
Yes	\$100	$0.7 \cdot 0.4 = 0.28$
Yes	\$10	$0.7 \cdot 0.6 = 0.42$
No	\$0	0.3

Zero Probability States

Notice that the events of being selected to participate and winning the prize are **not independent**. In particular:

$$P(\text{win} \mid \text{not selected}) = 0, \quad P(\text{lose} \mid \text{not selected}) = 0.$$

Hence, when considering the states of the world, it is important to ignore states that happen with zero probability (impossible states). Including them could lead to calculation errors.

Also, notice that the problem already provides the probabilities of winning or losing *conditional on being selected*. Therefore, when assessing the probability of the compound states like (selected to participate $\cap$ winning), we simply multiply the given probabilities: 0.7×0.4. Similarly for being selected and losing.

This may not always be the case, so pay attention to whether the problem already gives you the conditional probabilities or whether you need to compute them.

Expected Value

For our example,

$$E[X] = 100 \cdot 0.28 + 10 \cdot 0.42 + 0 \cdot 0.3 = 28 + 4.2 = \boxed{32.2}$$

Example 2: Probabilities Not Explicitly Given.

A jar contains 10 balls, 5 of which are winning balls. You draw one ball:

- If that ball is a winning ball, that ball is removed, and you draw a second ball. Otherwise, you get \$0.

- If only the first ball is winning, you get \$10.
- If both balls are winning, you get \$70.

The states of the world here are:

1. First ball not winning.
2. First ball winning and second ball not winning.
3. First and second balls winning.

Again, notice that we do not include outcomes for the second ball if the first ball is not winning, because these states are impossible.

The probabilities are:

$$P(\text{1st ball not winning}) = \frac{5}{10}$$

$$P(\text{1st ball winning, 2nd not winning}) = \frac{5}{10} \cdot \frac{5}{9}$$

$$P(\text{1st and 2nd balls winning}) = \frac{5}{10} \cdot \frac{4}{9}.$$

Notice that here the conditional probabilities are not given directly and require an interpretation of the setting (the winning ball is removed, so the total number of balls changes). The computation of the expected prize from this task is left as an exercise.

Problem 3: Compounded Lotteries with Fully Independent Events.

Imagine two class projects:

- **Option 1:** Success probability 0.8; grade 85% if successful, 50% if unsuccessful.
- **Option 2:** Success probability 0.6; grade 95% if successful, 50% if unsuccessful.

The projects are independent, so the success of one does not influence the success of the other.

Which are the states of the world here?

- Both succeed: probability 0.8×0.6
- Both fail: probability 0.2×0.4
- Only Option 1 succeeds: probability 0.8×0.4

- Only Option 2 succeeds: probability 0.2×0.6

Because the events are independent, no combination happens with zero probability.

For expected value purposes, we only need the success probability of the project being assessed (we can safely ignore intersections):

$$EV(\text{Project 1}) = 85 \cdot 0.8 + 50 \cdot 0.2 = 78$$

$$EV(\text{Project 2}) = 95 \cdot 0.6 + 50 \cdot 0.4 = 77$$

Then, why should we care about the joint states?

The EV does not need to consider each of the four states separately, but if we want to apply a **minimax regret** comparison, we need to know the exact possible combinations of outcomes that can occur and their outcomes.

Which project will the minimax regret criterion choose?

3.3 Multi-Stage Decisions and Backward Induction

Not only do final outcomes sometimes depend on sequences of events, but real-life decisions often unfold in multiple stages. In some situations, the choices we make today affect the prospects (further actions, outcomes, and probabilities) we may face in the future.

Think of situations like:

- Choosing a job offer today that will influence your future promotion chances.
- Choosing a major today that will influence your chances of becoming an entrepreneur in the future.
- Choosing a group of friends today that will influence the type of romantic prospects you will have access to later.

These are typical problems for which **decision trees** are a great tool—another reason why we introduced them in the previous chapters. The technique used to solve these problems is known as **backward induction**. We'll illustrate this through an investment-related example.

Example: Choosing a Business Project

A young entrepreneur decides whether to develop an app that connects restaurants with food suppliers or start a physical business supplying food to restaurants.

The app has a small initial cost, but the success rate is only 25%. The physical business has a large initial cost, but the success rate is 80%.

If either project is successful, they can either merge with a restaurant chain or continue working independently. For the app, the probability of success if they merge is 40%. For the physical store, the probability of success if they merge is 50%.

The payoffs are as follows:

If the app fails, the profit is $-\$600$. If the physical store fails, the profit is $-\$3,000$. If the app succeeds and they don't merge, the profit is \$3,200. If they merge and succeed, the profit is \$5,000. If they merge and do not succeed, the profit is \$750. For the physical store, if they succeed and don't merge, the profit is \$500. If they merge and succeed, the profit is \$4,500. If they merge and fail, the profit is \$1,000.

Which project should the young entrepreneur choose under the EV criterion?

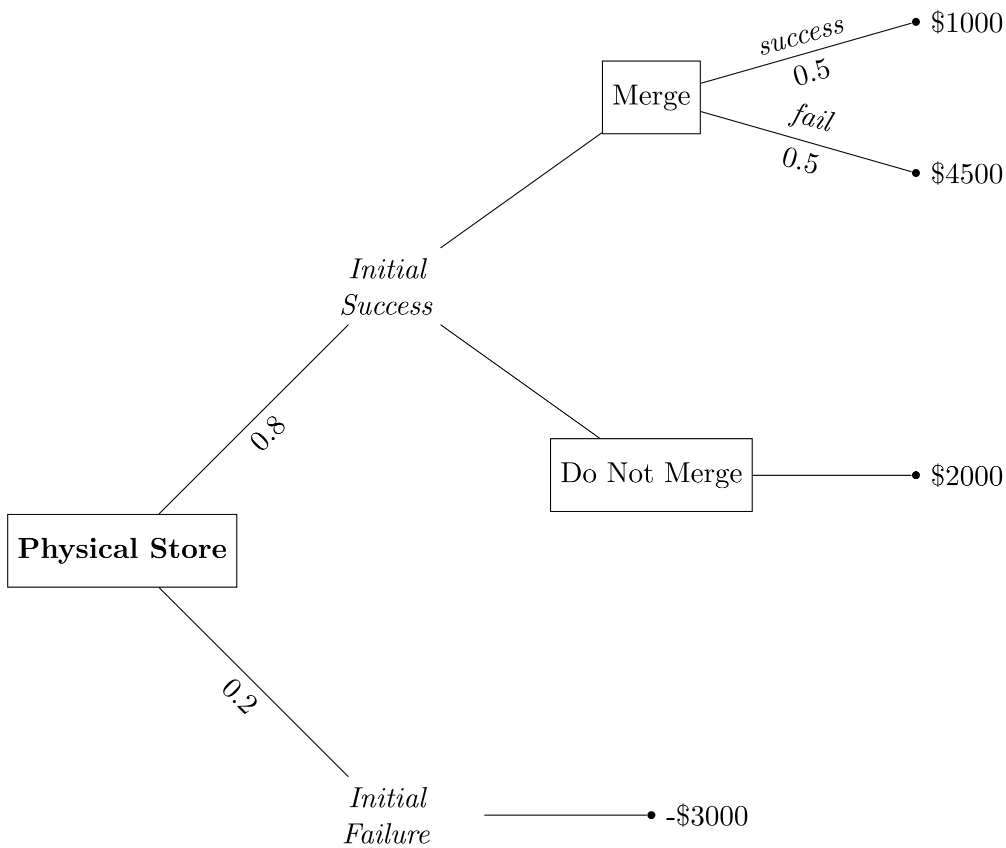
This problem looks complicated because there is a lot of information and many cases to consider, so we need to be careful about how we organize the information to find the solution.

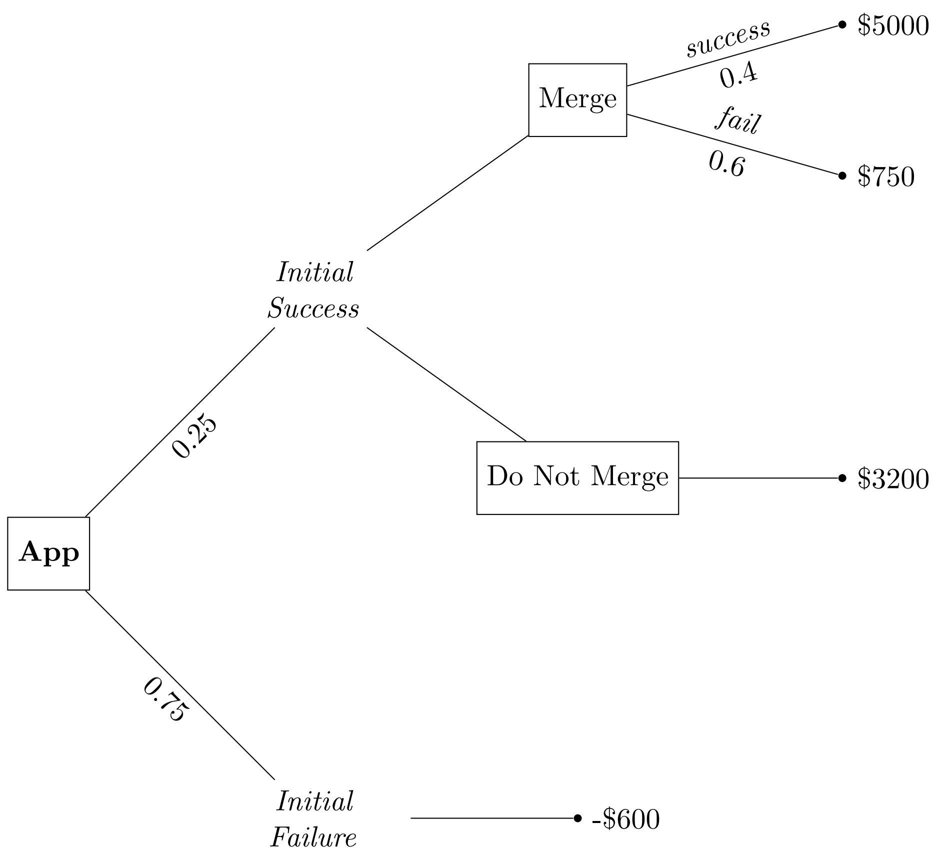
First, let's identify the actions, states, and outcomes in this problem.

- **Action 1:** Choose between *app* and *physical store.*
  - **State 1:** Initial success
    - App with probability 0.25
    - Physical store with probability 0.8
  - **State 2:** Initial failure
    - App with probability 0.75 (outcome: -600)
    - Physical store with probability 0.2 (outcome: -3000)

Notice that the success/failure of each project is independent of the other.

- **Action 2:** If the project succeeds, they may want to merge with a restaurant chain or stay independent.
  - If they merge:
    - **State 1:** Success
      - For the app: success with 60% probability (outcome: 10,000)
      - For the physical store: success with 50% probability (outcome: 4,500)
    - **State 2:** Failure
      - For the app: failure with 40% probability (outcome: 750)
      - For the physical store: failure with 50% probability (outcome: 1,000)
  - If they don't merge:
    - For the app: (outcome: 2,000)
    - For the physical store: (outcome: 500)


Merge
success
0.5
$1000
fail
0.5
$4500
Initial
Success
Do Not Merge
$2000
0.8
Physical Store
0.2
Initial
Failure
-$3000


Merge
success
0.4
$5000
fail
0.6
$750
Initial
Success
Do Not Merge
$3200
0.25
App
0.75
Initial
Failure
-$600

Initially, this problem looks complex—there are many branches in the tree and many numbers to consider. Let's use backward induction to solve it.

Backward induction proposes that we start from the end—that is, with the final actions. In the tree, actions are represented by rectangles, so the last actions we make are whether to merge or not, in the case of initial success.

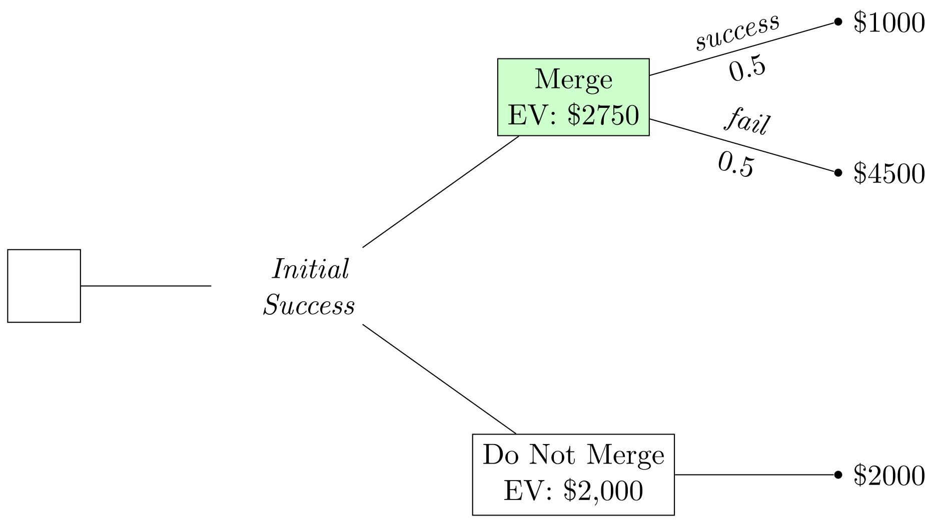
Let's start with the physical store. In the event of initial success, if we merge, we get \$4,500 with probability 0.5 and \$1,000 with probability 0.5. The expected value of merging is then:

$EV_{merge} = 4500 \times 0.5 + 1000 \times 0.5 = \$2,750.$

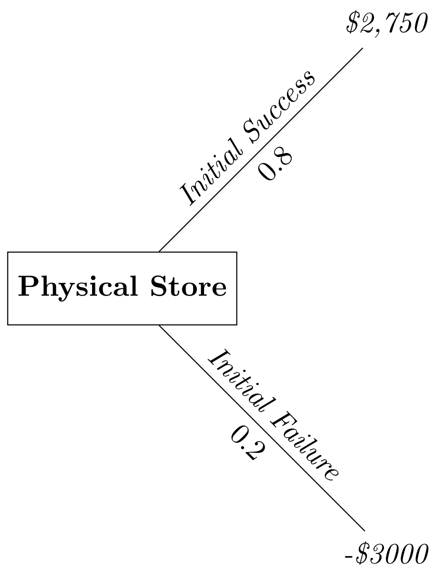
If we do not merge, we receive \$2,000 for sure.

Because $2,750 > 2,000$, if we succeed initially with the physical store, we would choose to merge, for an expected income of \$2,750.

Representing this in the sub-tree:



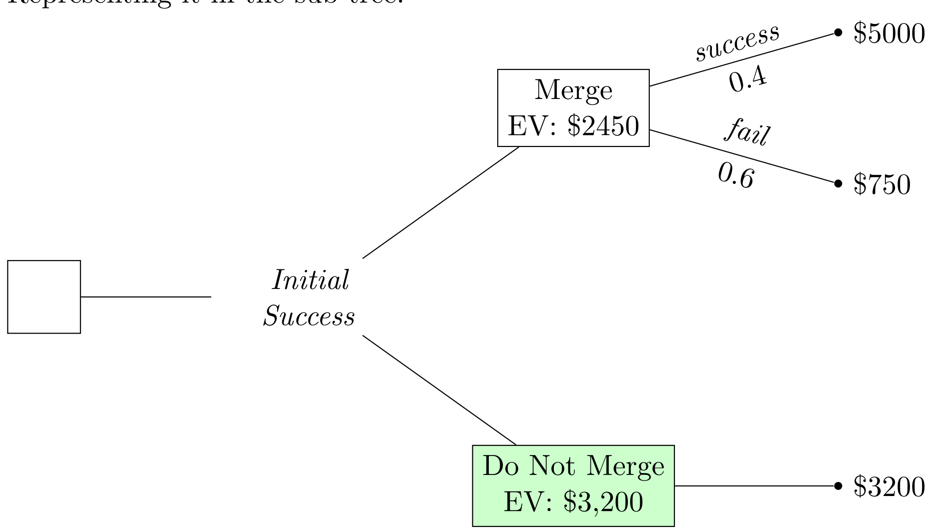
But this means that the expected value, if we get an initial success in the physical store, is precisely $\$2,750$. So, we can **trim** all the branches that follow "initial success" and add the EV of $\$2,750$, that is:



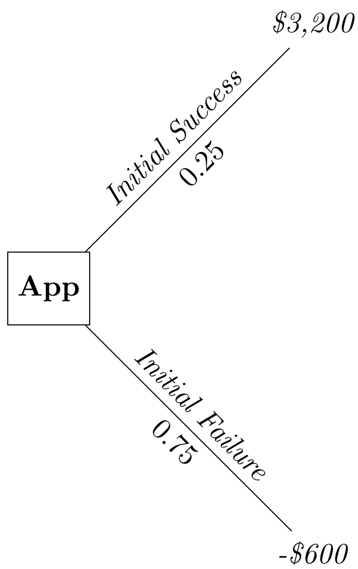
Let's do the same with the case when we choose the app. In the event that we have an initial success, if we merge we succeed with probability 0.4 and get $5,000, and fail with probability 0.6 and get only $750. The expected value of merging is then $EV = 0.4 * 5000 + 0.6 * 750 = \$2,450$. On the other hand, if we don't merge, we get $3,200 for sure.

In this case because $\$2,450 < 3,200$, we will not merge if the app turns out to succeed.

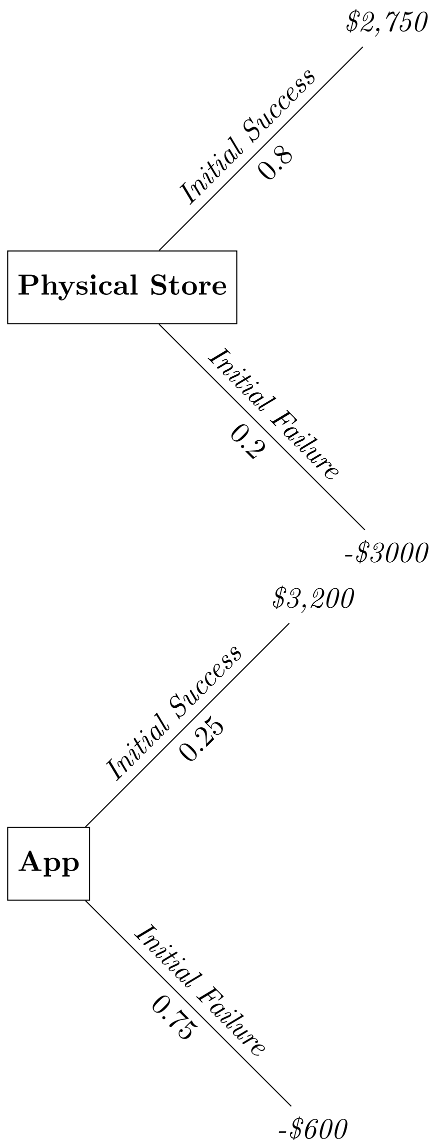
Representing it in the sub-tree:



Again, this means that the expected value if we get an initial success with the app is $\$3,200$, so we can trim all the branches after "initial success" and add the EV of $\$3,200$, that is:


$3,200
Initial Success
0.25
App
Initial Failure
0.75
-$600

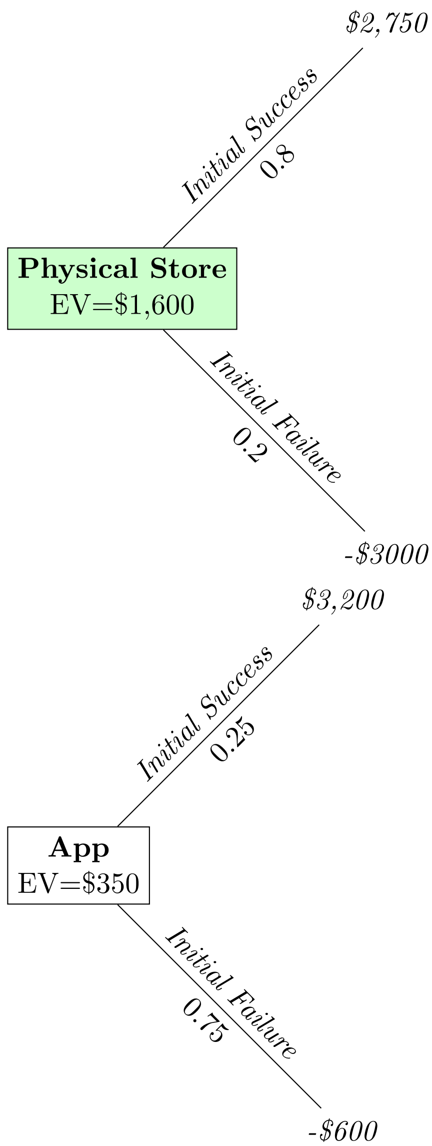
So our tree now looks like this:



This tree is simple, we only need to compute the expected value for each option.

For the Physical store, we have an initial success with prob. 0.8 and get $2,750. We have an initial fail with prob. 0.2 and get -$3000. This results in an expected value of $EV = 0.8 * 2750 + 0.2 * (-3000) = 1600$.

For the App, we have an initial success with prob. 0.25 and get $3,200. We have an initial fail with prob. 0.75 and get -$600. This results in an expected value of $EV = 0.25*3200+0.75*(-600) = 350$.



Because $1,600 > 350$, we (or the young entrepreneur) will choose to invest in the physical store.

Solution: Under the expected value criterion, choose the physical store.

Recap. When we face problems of choice under uncertainty with multiple decision layers, it can be complicated to solve the entire problem at once. The **backward induction** technique suggests starting from the last decision we need to make, recording our choice (and its corresponding expected value), and then moving backward to the previous level (as if we were *"trimming the tree"*). In this way, we focus only on the scenarios that could actually occur, ignoring states that will never be chosen (for example, an initial success but not merging for the physical store).

Important. This trimming procedure is not feasible with payoff matrices. There, one would need to compare every possible combination of actions/states, which can be lengthy and confusing.

This is why decision trees are especially helpful in such cases.

3.4 Problem Set 2

Problem 1: Case

A mining company is considering exploring for gold in a particular area.

- If they **do not** start the expedition, the payoff is \$0.
- If they start the **first expedition**:
  - With probability 45%, they find gold and the final payoff is \$30M.
  - With probability 55%, they do not find gold. They may either:
    * Stop, with a payoff of −\$1M (cost of the first expedition), or
    * Start a **second expedition**.
- If they start the **second expedition**:
  - With probability 30%, they find gold and the final payoff is \$29M.
  - With probability 70%, they do not find gold. They may either:
    * Stop, with a payoff of −\$3M (cost of the first two expeditions), or
    * Start a **third expedition**.
- If they start the **third expedition**:
  - With probability 10%, they find gold and the final payoff is \$27M.
  - With probability 90%, they do not find gold. They may either:
    * Stop, with a payoff of −\$5M (cost of the first three expeditions), or
    * Start a **fourth (and last) expedition**.
- If they start the **fourth expedition**:
  - With probability 1%, they find gold and the final payoff is \$25M.
  - With probability 99%, they do not find gold and the final payoff is −\$8M (cost of all four expeditions).

Here, M denotes millions of dollars. Reported "final payoffs" already account for previous costs.

With the above information, and assuming that the mining company cares about maximizing the expected value of its activities:

(a) Represent this problem as a decision tree. Include all relevant elements: actions, outcomes, states, and probabilities.

(b) Should they engage in the first expedition? Justify your answer using the expected value (EV) criterion.

(c) If your answer to (b) is "yes", what is the maximum number of expeditions they should be willing to undertake?

(d) Is it theoretically possible to engage in exploration after the first expedition, even though the expected benefits are negative? why or why not?

Problem 2: Case

A couple is deciding between buying a house in an urban city or a rural city. The properties have the same price. Both cities are somewhat underdeveloped, so major changes are expected over the next 10 years.

Urban city. With probability 0.25 the area becomes industrial (property value decreases by \$200K); with probability 0.45 it becomes a college area (value increases by \$500K); and with probability 0.30 it stays the same (no change).

Rural city. With probability 0.20 the area becomes more touristic (value increases by \$900K); with probability 0.80 it develops toward farming (value decreases by \$75K).

The changes in each city are independent.

Here, K denotes thousands of dollars. Changes are net changes in value relative to today's price (the current purchase prices are equal).

Based on the previous case:

(a) What are all the possible states of the world?

(b) Write the regret matrix.

(c) If the couple minimizes the expected regret, what should they choose? Item If the couple minimizes max-regret, what should they choose?

Problem 3: Case

A presidential candidate must choose between promoting a *moderate* agenda or an *extremist* agenda.

- If they choose the moderate agenda, they win the election with probability 0.30.
- If they choose the extremist agenda, they win the election with probability 0.70.
- If they do not win the election, the payoff is -25.
- If they win the election, the probability of obtaining a congressional majority for the agenda depends on the agenda:
  - Moderate agenda: majority with probability 0.55.
  - Extremist agenda: majority with probability 0.40.

  These two events are independent from each other.
- If the agenda is passed, the payoff is 200. If the agenda fails, the payoff depends on agenda type:
  - Fail as moderate: payoff -50.
  - Fail as extremist: payoff -100.

Based on the previous case:

(a) What are all the possible states of the world?

(b) Represent this problem as a decision tree.

(c) What type of agenda would a candidate that maximizes expected value choose?

Problem 4: *Challenging*

Consider the following lottery game.

The lottery ticket costs \$70.

Five fair coins (each with probability 1/2 of heads or tails) are flipped. If *all five* coins land heads, the payoff is \$0. If at least one coin lands tails, the game proceeds to a second phase.

In the second phase four fair coins are flipped. If *all four* of those coins land tails, the prize is \$200; otherwise the prize is \$50.

Based on the previous case:

(a) What would someone following the EV criterion do? Play the game or not?

4 Chapter 4: Expected Utility

Let's start this chapter with a question: Why do people play the national lottery, even though the odds are extremely low and the expected monetary return is usually less than the cost of a ticket?

The expected value suggests not buying a national lottery ticket.

Now let's think about this other fictitious lottery: You can win 20 million dollars with chances $\frac{1}{10000}$ (or 0.0001) or nothing with chances $\frac{9999}{10000}$ (or 0.9999).

Before continuing, ask yourself this question: What is the maximum amount you would be willing to pay to play this lottery?

Imagine the ticket to play this lottery costs $\$1,700$; would you buy the ticket?

The expected value criterion says you should. Indeed, you should buy any ticket below $2,000, which is the expected value of this lottery.

Most people probably won't buy the ticket, even if they have more than $2,000 available. Why?

Some of the other criteria could explain some of these choices, but so far, we haven't studied why people would use one criterion over another, or why we need different criteria to explain choices in different situations.

John von Neumann and Oskar Morgenstern proposed in 1947 the expected utility theory, which incorporates utility functions in the domain of decisions under uncertainty. The main point of this theory is that the valuation of each possible payoff may not be linear, as it is when we take the expected value.

The precursor of this idea was Nicolaus Bernoulli in the 18th century, when describing the St. Petersburg Paradox (we will talk about this paradox later in this chapter).

Let's have a closer look at expected utility theory. It proposes that people don't evaluate lotteries based purely on expected monetary value. Instead, they consider how much *utility* or satisfaction they get from the lottery. For example, the jump in utility from having $0 to having $100 may be different (greater or smaller) than the jump in utility from having $100 to having $200.

So, if we take an average over the utilities of the different payoffs of the lotteries, rather than the payoffs themselves, the ranking of the payoffs may be very different. Of course, we need the utility to be non-linear to have these differences. If the utility is linear, taking an average over the utilities of payoffs or over the payoffs themselves will give the same ranking.

For example, consider a decision maker with utility function $u(x) = x^{0.5}$, where x represents money. This person faces a lottery that pays $100 with probability 50% or $0 with probability 50%. The lottery ticket costs $49.

Clearly, $49 < EV_{lottery} = 100 * 0.5 = 50$, so by expected value, the person should buy the ticket.

Now, let's look at the utility-level analysis.

If the person wins the lottery, they get \$100, which means a utility of $u(100) = 100^{0.5} = 10$. If the person does not win, they get \$0, for a utility $u(0) = 0^{0.5} = 0$. Hence, they get utility 10 with a chance of 0.5 and utility 0 with a chance of 0.5. The *Expected Utility* is then $EU_{lottery} = 0.5 * 10 + 0.5 * 0 = 5$.

If the person keeps the \$49 for themselves, the utility is $u(49) = 49^{0.5} = 7$.

Since $7 > 5$, playing the lottery will result in an expected utility loss of 2 (as $5 - 7 = -2$). If the decision maker is a utility-maximizer, as we usually assume, they will not buy the ticket.

Bonus: The St. Petersburg Paradox

Another famous puzzle illustrates the limitations of expected value. Imagine a game where you flip a coin repeatedly until it lands heads, and the payoff doubles with each additional flip:

- Heads on the 1st flip: win \$2
- Heads on the 2nd flip: win \$4
- Heads on the 3rd flip: win \$8
- ...and so on.

The expected monetary value of this game is infinite because the payoff grows exponentially with each flip.

Yet, most people would only pay a modest amount to play. Expected utility theory explains this by allowing utility to be nonlinear — extremely high payoffs contribute less to utility than expected value would suggest, tempering the paradox.

4.1 What is Expected Utility?

The expected utility criterion says that when making choices involving uncertain outcomes, people maximize the average *utility* they expect to get, not just the average monetary amount.

Mathematically, for a lottery p with outcomes $x_1, x_2, \ldots, x_n$ and probabilities $p_1, p_2, \ldots, p_n$, the expected utility is:

$$EU(p) = \sum_{i=1}^{n} p_i \cdot u(x_i)$$

where $u(\cdot)$ is a **utility function** that transforms the monetary outcome x_i into a subjective value.

The expected utility criterion chooses lottery p over lottery q if $EU(p) \geq EU(q)$.

Notice that the expected utility criterion assumes that the utility function exists and that it is increasing in outcomes.

As we mentioned earlier, u may be nonlinear in outcomes. It is this possibility that makes EU different from EV.

It also assumes that EU is increasing in outcomes (inherited from u being increasing in outcomes) and linear in probabilities (as any expectation is).

4.2 Expected Utility Theory: Axioms and Representation

Let's start with some notation and definitions.

Let X be a set (a collection of elements). To represent that an element x belongs to X, we write $x \in X$, where "$\in$" should be read as "belongs to".

The outcome set

We use Z to denote the set of outcomes, i.e.,

$$Z = \{z_1, z_2, z_3, \ldots, z_N\},$$

where each z_i is a possible outcome. Every possible outcome is included in Z.

A lottery

A lottery p is a rule

$$p : Z \to [0, 1],$$

that assigns to every possible outcome z_i a probability p_i. Naturally, $0 \leq p_i \leq 1$ and

$$\sum_{i=1}^{N} p_i = 1.$$

The set of all possible lotteries over Z

The set of all lotteries over Z consists of all probability rules p that satisfy $0 \leq p_i \leq 1$ and $\sum_{i=1}^{N} p_i = 1$. We denote this set as

$$\Delta(Z) := \left\{ p : z_i \mapsto p_i \;\middle|\; 0 \leq p_i \leq 1 \text{ and } \sum_{i=1}^{N} p_i = 1 \right\}.$$

When we write $p, q \in \Delta(Z)$, we mean that both p and q are lotteries over Z.

Lottery mixtures

If $p, q \in \Delta(Z)$ and $\alpha \in (0, 1)$, the *lottery mixture* $\alpha p + (1 - \alpha)q$ is defined as a lottery where, with probability α, the outcome is determined by p, and with probability $1 - \alpha$, it is determined by q.

Property of lottery mixtures

For every $\alpha \in (0, 1)$ and $p, q \in \Delta(Z)$, the mixture is also a lottery:

$$\alpha p + (1 - \alpha)q \in \Delta(Z).$$

Indeed, in the mixture, outcome z_i occurs with probability $\alpha p_i + (1-\alpha)q_i$. Since $0 \leq p_i, q_i \leq 1$ and $\alpha \in (0,1)$, we have

$$0 \leq \alpha p_i + (1-\alpha)q_i \leq 1,$$

and

$$\sum_{i=1}^{N}(\alpha p_i + (1-\alpha)q_i) = \alpha \sum_{i=1}^{N} p_i + (1-\alpha)\sum_{i=1}^{N} q_i = \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1.$$

Preference relation over lotteries

We define a binary preference relation $\succsim$ over $\Delta(Z)$ as follows:

Given $p, q \in \Delta(Z)$ (that is, given two lotteries),

$p \succsim q$ denotes that lottery p is weakly preferred over lottery q,

$p \precsim q$ denotes that lottery q is weakly preferred over lottery p,

$p \succ q$ denotes that lottery p is strongly preferred over lottery q,

$p \prec q$ denotes that lottery q is strongly preferred over lottery p,

$p \sim q$ denotes that lottery p is indifferent with respect to lottery q.

Axioms

We now define a set of properties that a preference relation ($\succsim$) over lotteries ($\Delta(Z)$) could satisfy.

- **Completeness**: For any p and $q \in \Delta(Z)$, either $p \succeq q$ or $q \succeq p$.

  In other words, the decision-maker can always compare any two lotteries and decide whether one is preferred over the other or whether they are indifferent.

- **Transitivity**: For any p, q and $r \in \Delta(Z)$, if $p \succeq q$ and $q \succeq r$, then $p \succeq r$.

  That is, if lottery p is weakly preferred over lottery q, and lottery q is weakly preferred over lottery r, then it must be the case that lottery p is weakly preferred over lottery r.

- **Independence**: For any p, q and $r \in \Delta(Z)$ and any $\alpha \in (0,1)$, $p \succsim q$ if and only if $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$.

  In words, the decision-maker prefers p to q if and only if they prefer the mixture $\alpha p + (1-\alpha)r$ to $\alpha q + (1-\alpha)r$.

  Intuitively, the preference ranking between p and q does not change when mixing each of them with another lottery (r) in the same proportion (α).

- **Mixture Continuity**: For any p, q and $r \in \Delta(Z)$, if $p \succ q \succ r$, there exist $\alpha, \beta \in (0,1)$ such that:

  - $\alpha p + (1-\alpha)r \succ q$
  - $q \succ \beta p + (1-\beta)r$

In words, if $p \succ q \succ r$, there exist probabilities $\alpha, \beta \in (0,1)$ such that the mixture $\alpha p+(1-\alpha)r$ is preferred to q, and q is preferred to the mixture $\beta p + (1-\beta)r$.

There is not a lot of intuition for this property, beyond ensuring that preferences vary smoothly with probabilistic mixtures.

Expected Utility Representation Theorem

One of the main contributions of von Neumann and Morgenstern was to show that if the preferences over lotteries satisfy a set of properties (axioms), then they can be represented by an expected utility function like the one we just described.

The celebrated von Neumann-Morgenstern theorem states:

If a decision-maker's preferences ($\succsim$) over lotteries ($\Delta(Z)$) satisfy completeness, transitivity, independence, and mixture continuity, then there exists a utility function $u(\cdot)$ such that for any p and $q \in \Delta(Z)$,

$$p \succeq q \quad \text{if and only if} \quad EU(p) \geq EU(q),$$

where

$$EU(p) = \sum_i p_i u(x_i)$$

is the expected utility of lottery p.

Moreover, the utility function $u(\cdot)$ is unique up to positive affine transformations. That is, if we define another utility function $\hat{u}(\cdot) = a + bu(\cdot)$, for $b > 0$ and $a \in \mathbb{R}$, then $\hat{u}(\cdot)$ will also represent the preferences that $u(\cdot)$ represents.

This theorem grounds the expected utility model: under reasonable axioms, preferences can be represented as maximizing expected utility.

5 Chapter 5: Risk Preferences

In the previous chapter, we introduced the idea that people sometimes do not follow the expected value criterion but rather opt into lotteries with an expected net loss or avoid lotteries with an expected net gain.

This idea suggests that people have attitudes (or preferences) toward risky prospects (or lotteries). We will now provide a classification of those risk attitudes and then study how Expected Utility Theory explains them.

Risk Preferences

We will use some of the notation we defined in the previous chapter. Let $Z = \{z_1, z_2, ..., z_N\}$ denote the set of possible outcomes, $p : Z \to [0, 1]$ a lottery, such that $p_i \in [0, 1]$ for all $i = 1, ..., N$ and $\sum_{i=1}^{N} p_i = 1$. Let $\Delta(Z)$ denote the set of all possible lotteries, and $\succsim$ a preference relation defined over the elements of $\Delta(Z)$.

A decision maker is:

- **Risk Averse**, if for every $p \in \Delta(Z)$, $p \precsim EV(p)$. That is, a risk-averse person **weakly** prefers receiving the expected value of a lottery for sure over playing the lottery.

- **Risk Loving**, if for every $p \in \Delta(Z)$, $p \succsim EV(p)$. That is, a risk-loving person **weakly** prefers playing the lottery over receiving its expected value for sure.

- **Risk Neutral**, if for every $p \in \Delta(Z)$, $p \sim EV(p)$. That is, a risk-neutral person is indifferent between playing the lottery and receiving its expected value for sure. *The risk-neutral person evaluates each lottery based on its expected value, so the lottery and its expected value are equally good.*

We know that under Expected Utility Theory, it is the utility of each possible payoff, rather than the payoff itself, that drives decisions over lotteries.

Since risk aversion and risk loving are found in almost all real-life decisions, it is important to study how Expected Utility Theory makes sense of these risk attitudes.

As we mentioned earlier, the key lies in the non-linearity of $u(x)$, so we need to pause and review the mathematical properties of some linear and non-linear functions.

5.1 Convex, Concave, and Linear Functions

In economics, and especially in decision theory, the curvature of the utility function tells us a lot about the decision maker's attitude toward risk.

Definition of convexity and concavity: A function $f(x)$ is *convex* if for any two points x_1 and x_2, and any $\lambda \in [0,1]$:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

It is *concave* if:

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$

In words:

- A **convex** function lies *below* the straight line connecting any two points on the curve.
- A **concave** function lies *above* that line.
- A **linear** function satisfies equality—the function is exactly equal to the straight line connecting any two points on the curve.

Notice that when the inequalities are strict, we talk about *strictly* convex and *strictly* concave. In the case of equality, we call the function **linear**.

The second derivative test:
For functions that are twice differentiable, one way to check concavity is by the second derivative:

- If $f''(x) > 0$, the function is convex at x.
- If $f''(x) < 0$, the function is concave at x.
- If $f''(x) = 0$, the function is linear at that point.

Examples:

- **Convex function:** $u(x) = x^2$

$$u'(x) = 2x, \quad u''(x) = 2 > 0 \quad \text{for all } x$$

- **Concave function:** $u(x) = \sqrt{x}$ for $x \geq 0$

$$u'(x) = \frac{1}{2\sqrt{x}}, \quad u''(x) = -\frac{1}{4x^{3/2}} < 0 \quad \text{for } x > 0$$

- **Linear function:** $u(x) = 3x + 2$

$$u'(x) = 3, \quad u''(x) = 0$$

5.2 Risk Preferences and Utility Functions

Let's now link risk preferences with our expected utility framework.

We defined risk aversion as: for every $p \in \Delta(Z)$,

$$p \precsim EV(p).$$

If preferences are represented by a VoM[3] expected utility function, this means that $EU(p) \leq EU(EV_p)$, which is the same as

$$\sum_i p_i u(z_i) \leq u\left(\sum_i p_i z_i\right).$$

From Jensen's inequality (in probability), we know that a function f is concave if and only if $E(f(X)) \geq f(E(X))$. If we plug u as the function into the inequality, it says that $E(u(p)) \geq u(E(p))$, which is the same as $EU(p) \leq u(EV(p))$.

So, risk-averse preferences are represented by concave utility functions.

If you are not familiar with Jensen's inequality, we can look directly at the definition of concavity. Consider a lottery p that pays z_1 with probability p_1 and z_2 with probability $1 - p_1$.
Risk aversion implies that

$$p_1 u(z_1) + (1 - p_1)u(z_2) \leq u(p_1 z_1 + (1 - p_1)z_2),$$

but this is identical to the formal definition of a concave function, hence u is concave.

In summary, risk-averse preferences are represented by concave utility functions.

In a similar fashion, we can say that:

- **Risk-neutral** preferences are represented by a *linear* utility function.
- **Risk-loving** preferences are represented by a *convex* utility function.

Which is the same as:

- $u(x)$ is convex $\Rightarrow EU(p) > EU(EV_p)$, that is, $\sum_i p_i u(x_i) > u(\sum_i p_i x_i)$.
- $u(x)$ is linear $\Rightarrow EU(p) = EU(EV_p)$, that is, $\sum_i p_i u(x_i) = u(\sum_i p_i x_i)$.

[3]VoM refers to von Neumann and Morgenstern

5.3 Certainty Equivalent and Risk Aversion

We know that for risk-averse people $\Rightarrow EU(p) \leq EU(EV_p)$. If the inequality is strict and $u(x)$ is a continuous function, as we assume it is, then we may find some small number $\epsilon > 0$ such that $EU(p) < EU(EV_p - \epsilon)$. That is, between the lottery and a certain amount that is a tiny bit smaller than the expected value, the risk-averse person will still prefer the certain amount.

But not every risk-averse person is the same; some of them will still prefer the risky option even if it is much lower than the expected value. For others, if a certain amount is not close enough to the expected value, they will switch to the lottery.

In other words, among risk-averse decision makers, some are more risk-averse than others.

How do we compare two decision makers' risk aversion? And how do we know how small the certain outcomes are that they will prefer over the lottery?

The **certainty equivalent (CE)** of a lottery is the **guaranteed** amount of money that the person considers equally desirable as the lottery.

In other words, the CE of a lottery p is the certain amount that satisfies $EU(p) = EU(CE)$ or $p \sim CE$.

For example, if a lottery pays \$0 with 50% chance and \$100 with 50% chance, its expected value is \$50. Suppose that a risk-averse person chooses according to the following rule: "If they offer me any sure payoff of \$40 or more, I will take that payoff over the lottery. But if they offer me a sure payoff of less than \$40, I'd rather play the lottery."

For this person, the CE of the lottery is \$40. We know that for any guaranteed payoff of \$40 or more, they will take it over the lottery, but not below that.

Now imagine that another person, say Person 2, says: "If they offer me any sure payoff of \$20 or more, I will take that payoff over the lottery. But if they offer me a sure payoff of less than \$20, I'd rather play the lottery."

For Person 2, the CE of the same lottery is \$20, which is lower than the CE of the first person.

Who is more risk-averse? If the sure payoff is \$35, the first person will rather play the lottery (because \$35 is below their CE), but Person 2 will take the sure payoff (because it is above their CE). So the first person, although risk-averse, is willing to take more risk than Person 2.

From this example, we can define the following classification rule:

If:

- Person 1 has certainty equivalent CE_1 for a given lottery.
- Person 2 has certainty equivalent CE_2 for the same lottery.
- And $CE_1 > CE_2$.

Then Person 2 is MORE risk-averse than Person 1.

If we know the form of the utility function u, we can compute the CE of a lottery p by solving the equation $EU(p) = u(CE)$.

Example. Consider the utility function $u(z) = z^{1/2}$ and the lottery p that pays 100 with probability 0.5 and 0 with probability 0.5.

In this case we have: $(CE)^{1/2} = 0.5(100)^{1/2} + (0)^{1/2} = 5 \Rightarrow (CE)^{1/2} = 5$.

Taking both sides to the power of 2:

$CE = 25$.

This means that someone with preferences represented by $u(z) = z^{1/2}$ will prefer any sure amount of 25 or more over the lottery, but will prefer the lottery over any sure amount smaller than 25.

Because $EV(p) = 50$, we can also conclude that $u^{1/2}$ represents risk-averse preferences, even without looking at the curvature of u.

Certainty Equivalent and Risk Preferences

Since the CE of a lottery p is the amount that the decision maker considers equivalent to p, we can restate risk preference categories as follows:

For every $p \in \Delta(Z)$:

- Risk-averse: $CE_p \leq EV(p)$
- Risk-loving: $CE_p \geq EV(p)$
- Risk-neutral: $CE_p = EV(p)$

Risk Premium

Another concept related to risk preferences and certainty equivalents is the risk premium.

The **risk premium** of a lottery p is the difference between the expected value and the certainty equivalent:

$$\text{Risk Premium} = EV(p) - CE_p$$

The risk premium provides a simple measure of how much money a risk-averse individual is willing to give up (in expectation) to avoid risk (avoid playing the lottery).

If the risk premium is large (small CE compared to EV), the decision maker is willing to give up a lot of money to avoid the risky situation.

The more risk-averse someone is, the higher the risk premium (the more they are willing to give up to avoid the risk).

Visual intuition: For concave utility functions, the expected utility of the lottery lies below the utility of the expected value. This gap is what drives the risk premium.

For risk-neutral decision makers, the risk premium is zero, as $CE = EV$.

For risk-loving decision makers, the risk premium is negative, as $CE > EV$. The interpretation in this case is the opposite of risk aversion; the risk premium tells us how much money the risk-loving decision maker is willing to give up (in expectation) to have the chance of playing the lottery.

Demand for Risky Assets

An alternative way to look at risk aversion is by the demand for risky assets.

Assume you have two assets a_0 (safe) and a_1 (risky). The returns are r_0 and r_1 respectively, with $E[r_1] > r_0$, so a_1 is more attractive in expectation. Also assume that $P(r_1 > r_0) > 0$, so a_1 is riskier than a_0 (there will be states in which a_1 returns less than a_0).

Now consider a person with wealth W who decides how much to invest in a_1 and a_0. Call $\alpha \in [0, 1]$ the proportion invested in a_1, so the expected return for the agent is:

$$\alpha W E[r_1] + (1 - \alpha) r_0$$

If the optimal investment in a_1, αW, is smaller than W, we say that the person is risk-averse.

If we consider two individuals with the same wealth W, but distinct investment levels $\alpha_1 W$ and $\alpha_2 W$ with $\alpha_1 \geq \alpha_2$,

then we say that the individual investing $\alpha_1 W$ is less risk-averse than the one investing $\alpha_2 W$.

5.4 Absolute Risk Aversion and Relative Risk Aversion

Additive risks.

Consider the following lottery: Gain \$1,000 with probability 20% or \$0 with probability 80%.

This lottery is the same whether you originally have \$500 in your bank account (so the lottery is really between having \$500 + 1000 with chance 20% and \$500 + 0 with chance 80%), or \$3000 in your bank account (so the lottery is really between having \$3000 + 1000 with chance 20% and \$3000 + 0 with chance 80%). This type of risk that is independent of the level of wealth is called **additive risk.**

While the risk does not depend on the initial level of wealth w, our attitudes towards this risk may.

The concept of Absolute Risk Aversion measures how much risk someone is willing to bear at wealth level w.

It classifies decision makers into three groups:

Decreasing Absolute Risk Aversion (DARA) means the person becomes less risk-averse as they get wealthier.

Example: A person with DARA may avoid risky investments when they are poor, but as they get richer, they put their money into riskier projects.

Increasing Absolute Risk Aversion (IARA) means the person becomes more risk-averse as they get wealthier.

Constant Absolute Risk Aversion (CARA) means the person's risk aversion remains constant at any wealth level.

Another way to see the different types of absolute risk aversion is through the demand for risky assets.

If a person invests \$20 in a risky asset when their wealth is \$100, then:

- If they have CARA utility, they will still invest \$20 in the risky asset when their wealth is \$500.
- If they have DARA utility, they will invest more than \$20 in the risky asset when their wealth is \$500.
- If they have IARA utility, they will invest less than \$20 in the risky asset when their wealth is \$500.

Empirical studies like Guiso and Paiella (2008) and Levy (1994), among others, show that DARA is more prevalent than CARA or IARA.

Mathematically, absolute risk aversion is defined by Pratt's measure:

$$A(w) = -\frac{u''(w)}{u'(w)}$$

For a risk-averse person, $A(w) \geq 0$ for every $w \geq 0$.

DARA means $A(w)$ is decreasing in w, IARA means $A(w)$ is increasing in w, and CARA means $A(w)$ is constant in w.

Multiplicative risks.

Now consider the following lottery: If you bet $\$M$ on this lottery, you may lose 50% of your bet with probability 0.20 or win 10% with probability 0.80.

The stakes of this lottery change with your level of wealth ($\$M$). If you bet \$500, the lottery is really between having \$250 with chance 20% and \$550 with chance 80%. If you bet \$3000, the lottery is really between having \$1500 with chance 20% and \$3300 with chance 80%.

This type of risk that depends on the original level of wealth is called **multiplicative risk.** They are common in financial markets, where gains (and losses) are typically expressed as proportions of investments.

To understand how people respond to **multiplicative** risks, we use the concept of **Relative Risk Aversion**, which measures risk attitudes *relative to the stakes*.

Mathematically, Relative Risk Aversion (RRA) is defined by:

$$R(w) = -\frac{w \cdot u''(w)}{u'(w)}$$

It classifies decision-makers into three groups:

- **Decreasing Relative Risk Aversion (DRRA)**: Risk aversion decreases as the stakes increase.
- **Increasing Relative Risk Aversion (IRRA)**: Risk aversion increases with the stakes.
- **Constant Relative Risk Aversion (CRRA)**: Risk aversion stays constant with the stakes.

Going back to the interpretation using the demand for risky assets:

If a person invests \$20 of a risky asset when their wealth is \$100 (investment is 20%), then:

- If they have CRRA utility: when their wealth is \$500, they invest \$100 (20%).
- If they have DRRA utility: when their wealth is \$500, they invest more than \$100 (20%).
- If they have IRRA utility: when their wealth is \$500, they invest less than \$100 (20%).

Who is more risk-averse?

We previously mentioned that the risk premium and certainty equivalent can be used to compare the risk aversion between two decision makers.

Here are other criteria that also help compare levels of risk aversion:

A decision maker is **more** risk-averse than another person if they:

- Have a lower CE (higher risk premium).
- Have a **more** concave utility function: their utility function is an increasing concave transformation of the utility function of the other person.
- Have a higher Arrow-Pratt risk aversion coefficient.
- Have a lower demand for risky assets, given the same level of wealth.

References

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5.5 Preferences and Data

Even though we defined risk preference categories—risk aversion, risk loving, and risk neutrality—as requiring conditions for every lottery, which is impossible to verify, we can still learn about (local) risk preferences by looking at the choices people make.

If we make some assumptions about their preferences, we can then infer more general properties.

Example 1

We have a lottery described by $(1000, 0.2; 0, 0.8)$, that is, it pays \$1000 with probability 0.2 and \$0 with probability 0.8. For lottery 1, the ticket cost is \$190. For lottery 2, with the same prospects, the ticket cost is \$120.

The last three columns of the table show the choices of three individuals for each lottery. For example, Ann doesn't buy the lottery ticket at the price of \$190, but she buys it when the ticket price is \$120.

			Choice		
	Prospects	Ticket Price	Ann	Bob	Charlie
Lottery 1	(1000, 0.2; 0, 0.8)	190	Don't buy ticket	Don't buy ticket	buy ticket
Lottery 2	(1000, 0.2; 0, 0.8)	120	buy ticket	Don't buy ticket	buy ticket

What can we say about these individuals?

Since $EV_{lottery} = 200$, we can say:

- Ann's choices are consistent with risk aversion, since she didn't buy the lottery ticket at 190. We shouldn't say that Ann is risk-averse because we don't have enough observations about other choices.

- Bob's choices are consistent with risk aversion, since he didn't buy the lottery ticket at any of the prices.
- Bob's choices are consistent with him being more risk-averse than Ann, since Ann was willing to take the risk when the ticket price was 120 but Bob was not.
- Charles is more willing to take risks than both Bob and Ann, since he played the lottery at both prices.

Can we say that Charles's choices are consistent with risk-loving or risk-neutrality? Not really. Remember that for risk-loving, we need the lottery to be strictly preferred over its expected value (200), and for risk neutrality, we need indifference between the two.

The current information is not enough to conclude further about Charles's preferences. We don't know what would happen if the ticket cost was \$195, or \$196, or \$199, or \$200; hence the only thing we can say for sure with this information is that Charles is more willing to take risks than both Bob and Ann, since he played the lottery at both prices.

During the lecture, we will also go over how experimental economists measure risk preferences.

5.6 Riskier lotteries

This subsection was taken from the lecture slides ("Chapter 7").

Risk. We never really defined what risk is, or more importantly, when one lottery is riskier than another. We mentioned that, in finance, variance is commonly used as a measure of risk, since it captures the variability of outcomes.

In this section, we provide a criterion for classifying a lottery as riskier or less risky than another lottery by looking at how a risk-averse decision maker would rank such lotteries.

> A lottery q is riskier than a lottery p if every risk-averse person prefers p over q.

This definition connects risk to preferences under expected utility, rather than relying only on variance.

Revisiting the Mean-Variance Criterion

Suppose preferences can be represented with quadratic utility:

$$u(z) = az - bz^2, \quad b \geq 0.$$

This utility function is concave, since

$$u''(z) = -2b < 0,$$

so it exhibits risk aversion.

If X is a random variable representing a lottery, expected utility is

$$EU(X) = aE[X] + bE[X^2].$$

Using the identity $Var(X) = E[X^2] - (E[X])^2$, we can rewrite this as

$$EU(X) = aE[X] - b(E[X])^2 - bVar(X).$$

Thus, with quadratic utility, more variance always reduces expected utility for risk-averse people.

Two Criteria to Evaluate Riskiness

To compare lotteries, we will focus only on those with the same expected value (a partial ordering). We then look for criteria that take into account the entire distribution, not just the mean and variance. The most important are the **Mean-Preserving Spread** and **Second Order Stochastic Dominance (SOSD)**.

Mean-Preserving Spreads

Given a lottery p, we say that q is a mean-preserving spread of p if

$$q = p + \epsilon, \quad \text{with } E[\epsilon] = 0.$$

In other words, q has the same mean as p, but adds some noise that increases dispersion.

Example 1. If $p = (10, 1)$ and $\epsilon = (-5, 0.5; +5, 0.5)$, then

$$q = (5, 0.5; 15, 0.5)$$

is a mean-preserving spread of p. Notice that $E[q] = E[p] = 10$, but q shifts probability mass toward the extremes.

Example 2. If $p = (10, 0.4; 20, 0.6)$, then

$$q = (0, 0.2; 10, 0.3; 20, 0.2; 30, 0.3)$$

is a mean-preserving spread of p, generated by $\epsilon = (-10, 0.5; +10, 0.5)$.

In general, if q is a mean-preserving spread of p, then every risk-averse decision maker prefers p to q. Hence, q is considered *riskier* than p.

Second Order Stochastic Dominance (SOSD)

Sometimes it is hard to directly construct the noise ϵ that turns p into q.

Another way to check whether q is a mean-preserving spread of p is through **Second Order Stochastic Dominance (SOSD)**.

Definition. Let p and q be lotteries with cumulative distribution functions F_p and F_q. Then p second-order stochastically dominates q, written $p \succeq_{SOSD} q$, if

$$\sum_{i=1}^{k} F_p(x_i) \leq \sum_{i=1}^{k} F_q(x_i), \quad \text{for all } k.$$

Example: Second-Order Stochastic Dominance

Consider two lotteries p and q over outcomes $\{0, 1, 2, 3\}$:

$$p = (0, 0.25;\ 1, 0.25;\ 2, 0.25;\ 3, 0.25), \qquad q = (0, 0.4;\ 1, 0.1;\ 2, 0.1;\ 3, 0.4).$$

That is, lottery p assigns equal probability 0.25 to each outcome, while lottery q places more probability mass on the extremes (0 and 3).

The cumulative distribution functions (CDFs) are:

$$F_p(z) = \begin{cases} 0.25 & 0 \le z < 1, \\ 0.50 & 1 \le z < 2, \\ 0.75 & 2 \le z < 3, \\ 1 & 3 \le z, \end{cases} \qquad F_q(z) = \begin{cases} 0.40 & 0 \le z < 1, \\ 0.50 & 1 \le z < 2, \\ 0.60 & 2 \le z < 3, \\ 1 & 3 \le z. \end{cases}$$

To test for **second-order stochastic dominance (SOSD)**, we compare the cumulative sums of these probabilities up to each outcome:

$$\begin{aligned} \text{At } z = 0: &\quad F_p(0) = 0.25 \ < \ 0.40 = F_q(0), \\ \text{Up to } z = 1: &\quad F_p(0) + F_p(1) = 0.25 + 0.50 = 0.75 \ < \ 0.40 + 0.50 = 0.90, \\ \text{Up to } z = 2: &\quad 0.25 + 0.50 + 0.75 = 1.50 \ = \ 1.50 = 0.40 + 0.50 + 0.60, \\ \text{Up to } z = 3: &\quad 0.25 + 0.50 + 0.75 + 1 = 2.50 \ = \ 2.50. \end{aligned}$$

Since the cumulative sums for p are never larger than those for q (and strictly smaller at some points), we conclude:

$$p \succeq_{SOSD} q.$$

Interpretation. Neither lottery first-order stochastically dominates the other, because their CDFs cross. However, p second-order stochastically dominates q, meaning that *every risk-averse decision maker* would weakly prefer p.

According to our definition, we conclude that q is riskier than p.

Some necessary and sufficient conditions for SOSD.

SOSD relates to other criteria as follows:

- $FOSD \Rightarrow SOSD$, but the opposite is not true!
- If $p \succsim_{SOSD} q$ then $EV(p) \geq EV(q)$.
- If $p \succsim_{SOSD} q$ then $min(p) \geq min(q)$.

5.7 Problem Set 3

Problem 1

Lotteries.

$$\begin{aligned} A &= (-\$10,\ 0.2;\ \ \$10,\ 0.8), \\ B &= (\ \$0,\ 0.4;\ \ \$10,\ 0.6), \\ C &= (\ \$10,\ 0.8;\ \ \$20,\ 0.2), \\ D &= (-\$10,\ 0.1;\ \ \$10,\ 0.8;\ \ \$20,\ 0.1), \\ E &= (\ \$0,\ 0.2;\ \ \$10,\ 0.7;\ \ \$20,\ 0.1). \end{aligned}$$

a) Consider a mixture of Lotteries A and C, $xA + (1-x)C$. For what value of $x \in (0,1)$ is this mixture equivalent to Lottery D? (Note: equivalent means that the resulting lottery from the mixture is identical to Lottery D in payoffs and probabilities).

b) Consider a mixture of Lotteries B and C, $yB + (1-y)C$. For what value of y is this mixture equivalent to Lottery E?

c) Suppose an individual has preferences $C \succ E \succ D \succ A \succ B$. Which of the four axioms is violated by these preferences? Explain.

d) Are the preferences in part (c) consistent with expected utility maximization? Explain.

Problem 2

Consider the lottery: $L = (-\$100,\ 0.2;\ \ \$70,\ 0.6;\ \ \$110,\ 0.2)$,

The choices of different people between L and *sure* amounts is are as follows (*sure* means the prefer the sure amount over the lottery, and L means that they prefer the lottery):

	Choices per person			
Safe payoff	Person 1	Person 2	Person 3	Person 4
\$10	L	L	L	L
\$20	*safe*	L	L	L
\$30	*safe*	L	L	*safe*
\$40	*safe*	L	*safe*	L
\$50	*safe*	*safe*	*safe*	*safe*
\$60	*safe*	*safe*	*safe*	safe

a) Can Person 4's preferences be represented by EU (with an increasing utility function)? If no, which axiom is violated?

b) Which people's choices are consistent with risk aversion? Which people's choices are consistent with risk-loving? (exclude person 4).

c) Provide the best estimate for the CE for each person (from 1 to 3). *Note: the estimate should be of the form "the CE is between this and this number".*

Problem 3

Three decision makers invest in a risky asset. When their income is \$2400, they invest \$600 each.

When their income is \$4800, the investment is as follows:

- Person 1 invests \$900 in the risky asset.
- Person 2 invests \$1200 in the risky asset.
- Person 3 invests \$600 in the risky asset.

a) Classify each person according to their absolute and relative risk aversion behavior.

Extra Questions for people interested in academia:

Problem 4

Ralph's Preferences. Ralph's preferences are consistent with expected utility maximization with the utility function $u(x) = -e^{-2x}$.

a) Calculate Ralph's coefficient of absolute risk aversion. Is Ralph risk averse? Classify Ralph's risk preferences according to absolute risk aversion.

b) Calculate Ralph's coefficient of relative risk aversion. Classify Ralph's risk preferences according to relative risk aversion.

c) Find Ralph's risk premia for the lotteries:

$$p = (\$0, 0.5;\ \$25, 0.5), \quad q = (\$5000, 0.5;\ \$5025, 0.5).$$

d) Repeat parts a), b), and c) for $u(x) = -e^{-2\sqrt{x}}$ and $u(x) = x^{\beta-1}$ with $\beta \in (0, 1)$.

6 Chapter 6: Insurance

This chapter explores the concept of insurance through the lens of expected utility and risk preferences. We discuss how individuals evaluate risky prospects, when and why they demand insurance, and how insurance markets operate, including challenges like adverse selection and moral hazard.

6.1 Motivation: Why Do We Need Insurance?

Imagine you recently bought a new car with a two-year credit. One day, you have an accident and, though nothing happens to you, your car is seriously damaged and needs a very costly repair. You don't have the money to pay for the repair out of pocket; you haven't even finished paying for your (now damaged) car.

A situation like this would seriously hurt many people's finances. Since accidents are not uncommon, the fear of facing a financial issue like the one we just described could deter many people from making large expenses/investments: cars, houses, expensive technological devices, or even planning vacations.

One way in which the market tries to alleviate this potential financial catastrophe is through insurance.

Insurance is a contract in which the insurer commits to pay a percentage of the insured's loss if a particular incident happens, and in exchange, the insured pays the insurer a fee, whether the incident happens or not.

The social value of insurance

Imagine a person with utility function $u(z) = z^{0.5}$. They are deciding whether to buy a car or not. If they buy the car, with probability 4%, the car will be involved in an accident that will cost \$9000 in repairs.

Assuming the person has 10000, with probability 0.96, they will have a utility of $10000^{0.5} = 100$, and with probability 0.04, the utility is $(10000 - 9000)^{0.5} = 31.6$, so the expected utility loss is equal to 97.26.

If the person does not get the car, they would have to pay \$500 for public transportation, for a utility of $(10000 - 500)^{0.5} = 97.47$. In this case, the person will choose not to buy a car, since $97.47 > 97.26$ (greater utility).

Now imagine an insurance policy exists as follows: The person buys the insurance and pays \$360. In exchange, if an accident occurs, the insurance company will cover 100% of the repair. That is, regardless of whether the accident occurs or not, the person will always have a utility of $(10000 - 360)^{0.5} \sim 98.18$ (their only payment is the insurance fee). Again, if the person does not get the car and uses public transportation, the expected utility is ~ 97.46.

In this case, the person would rather get the car and buy insurance against car accidents, since this gives more utility than not buying the car.

The insurance helps the person bring "welfare" from luckier states of the world to unluckier states of the world, thereby smoothing or partially mitigating the risks. The insurance allows for mutually beneficial transactions that would have been impossible without it.

Some markets do not offer the possibility to buy contracts that specify obligations and benefits for every "contingency" (every possible state of the world). These markets are called **incomplete markets**.

For example, suppose you have the option to take a job in a foreign country (say England). Your payoff is in local currency: pound sterling. You are offered £67,000 (around \$90K as of the July 2025 exchange rate).

There is a state of the world in which the pound sterling undergoes a major depreciation, so your £67,000 becomes only around \$60K. This is not a good state of the world, but you can't insure against it.

If someone believes that a major depreciation of the pound is "likely enough" and this person is sufficiently risk-averse, they will not take the job. If a mechanism of insurance against fluctuations of the sterling pound were available, the person would probably have taken the job.

Incomplete markets are **inefficient** in the sense that mutually beneficial agreements/trade will not happen due to the impossibility of smoothing the risk across states of the world.

Insurance reduces or even fully eliminates this source of inefficiency by allowing people to transfer welfare across states. For a deeper discussion of inefficiencies in the market, I recommend taking Econ 100B (formerly Econ 100C).

6.2 Demand for Insurance

When Does a Person Buy Insurance?

A decision-maker with initial wealth W faces a lottery $L = (W - X, p; W, 1 - p)$ — that is, they lose X with probability p, no loss otherwise.

Let's consider an insurance contract that covers 100% of the loss and charges a **premium** (the insurance fee) of $\$\pi$.

Should the decision-maker buy the insurance?

Let's see:

- **Without insurance**, final wealth is a lottery:

$$W_L = \begin{cases} W - X & \text{with probability } p, \\ W & \text{with probability } 1 - p. \end{cases}$$

- **With insurance**, they pay a premium π and receive full coverage of the loss:

$$W_I = W - \pi \quad \text{in both states (certainty).}$$

Notice that the insurance cost $-\pi$ is paid in every state (regardless of whether there was a loss or not), so it is a certain cost.

The person buys insurance if:

$$W_I \succsim W_L$$

or, in terms of expected utility, if

$$U(W-\pi) > \mathbb{E}[U(W_L)].$$

Think about this question: Who are the customers of insurance? Risk-averse people, risk-neutral people, risk-loving people, all of them, or does it depend?

Will a risk-loving person buy the insurance?

Remember that a risk-loving person strictly prefers any lottery over its expected value with certainty. For this example, they will prefer W_L over having $W - pX$ with certainty (make sure you can verify why $W - pX$ is the expected value of the lottery W_L).

Then, if $\pi \geq pX \Rightarrow W - \pi < W - pX$, the risk-loving person will not buy the insurance.

In other words, the insurance premium (fee) needs to be **lower** than the expected loss for a risk-loving person to consider buying it. In real life, an insurer will never charge a premium below the expected loss because they would incur expected losses (we will see more about the supply of insurance in the next section).

Similarly, a risk-neutral person will only buy insurance if the insurance premium is below the expected loss and will be indifferent if they are equal.

It seems that only risk-averse people would be willing to buy insurance. Remember that they prefer the expected value for sure ($W - pX$) over the lottery W_L. Hence, if the insurance premium is $\pi = pX$, every risk-averse person will prefer to buy insurance.

On the other hand, remember that risk-averse people may prefer $W - \pi$ for sure over the lottery W_L (even if $W - \pi < W - pX$). This depends on how risk-averse they are and how much greater π is with respect to pX.

Let's look at it with an example:

Consider $W = 1000$, $p = 0.1$, $X = 500$, and $u(x) = 100 \cdot x^{0.25}$.

$$\mathbb{E}[U(W_L)] = 0.9 \cdot 100 \cdot (1000)^{0.25} + 0.1 \cdot 100 \cdot (500)^{0.25} = 553.4$$

With the insurance, if $\pi = 55$, the utility is:

$$U(1000 - 55) = 945^{0.25} = 554.4 > 553.4$$

So this person is better off buying the insurance than exposing themselves to the lottery, even if $55 > 50 = 0.1 \cdot 500$. Notice, however, that if the insurance premium was a bit higher, say \$80, their utility with insurance would be around 550.7, so they would not buy the insurance despite being risk-averse.

Similarly, consider another person with utility $u(x) = x^{0.9}$; this person is also risk-averse (check that the second derivative is negative), but:

$$\mathbb{E}[U(W_L)] = 0.9 \cdot (1000)^{0.9} + 0.1 \cdot (500)^{0.9} = 477.9$$

and

$$U(1000 - 55) = 945^{0.9} = 476.3$$

So this other person, who is also risk-averse, is not risk-averse enough to buy the insurance at the fee of \$55.

This example shows us that when the insurance premium is above the expected loss, different decision-makers will or will not buy it depending on their risk preferences—specifically, how risk-averse they are.

Relation to the Certainty Equivalent and Risk Premium

Is there a way to know the maximum a particular individual is willing to pay for insurance?

In the previous chapter, we studied two concepts related to risk preferences: the **Certainty Equivalent (CE)** and the **Risk Premium**. Let's review them again:

- The **certainty equivalent (CE)** of a lottery p is the guaranteed amount CE_p such that:

$$U(CE_p) = \mathbb{E}[U(p)].$$

That is, the CE is the sure amount of money that makes the decision-maker indifferent between that amount and the lottery.

Because we assume preferences are monotonic (the more, the better), the decision-maker will strictly prefer the lottery over any sure amount below the CE and will strictly prefer the sure amount over the lottery whenever it is greater than the CE.

- The **risk premium** (RP_L):

$$RP_L = EV(p) - CE_p,$$

is the maximum amount the individual is willing to give up to avoid the risk.

In terms of insurance, given some initial wealth W and a probability p of having a loss of X, remember that the individual pays an insurance premium π and in exchange they have full coverage; that is, they receive for sure:

$$W - \pi$$

The alternative is not buying the insurance and facing the lottery, receiving $\mathbb{E}[L]$.

Because $W - \pi$ is a sure amount, we must compare it with the CE. If $W - \pi \geq CE_L$, the individual will buy the insurance. Otherwise, they will rather face the lottery.

If we isolate CE_L from $RP_L = EV(L) - CE_L$, that is $CE_L = EV(L) - RP_L$, and plug it into $W - \pi \geq CE_L$, that is $W - \pi \geq EV(L) - RP_L$, we get $W - EV(L) + RP_L \geq \pi$.

Further substituting $EV(L) = W - pX$ into the above expression, we get $W - (W - pX) + RP_L \geq \pi \Rightarrow pX + RP_L \geq \pi$.

This last expression defines an upper limit for the insurance premium π:

$$\boxed{pX + RP_L \geq \pi}$$

In words, the risk premium cannot exceed the expected loss (pX) of the lottery by more than the risk premium of the lottery. So in the insurance problem, RP is literally the **maximum the person is willing to pay in order to avoid the gamble!**

Example: Let initial wealth $W = 10000$, loss $X = 5000$, with $p = 0.2$. Assume utility $U(x) = \sqrt{x}$.

Calculate:

$$\mathbb{E}[U(W_L)] = 0.2 \cdot \sqrt{10000 - 5000} + 0.8 \cdot \sqrt{10000} = 0.2 \cdot 70.71 + 0.8 \cdot 100 = 14.14 + 80 = 94.14.$$

Find CE such that $\sqrt{CE} = 94.14 \Rightarrow CE \approx 8861$.
Expected wealth without insurance:

$$\mathbb{E}[W_L] = 0.2 \cdot 5000 + 0.8 \cdot 10000 = 9000.$$

Risk premium:

$$RP_L = 9000 - 8861 = 139.$$

The person is willing to pay up to \$139 above the expected loss to avoid risk.

Partial Coverage

Oftentimes, insurance policies do not cover 100% of the loss but rather a fraction of it (or up to a maximum amount). In these cases, the individual pays a percentage $\alpha \in (0, 1)$ of the insurance premium, and if the bad state occurs, the insurance covers $\alpha \times X$ (a proportion α of the loss X).

In this case, the optimization problem is to find the proportion of coverage α that maximizes expected utility.

That is,

Find α to maximize

$$EU(\alpha) = p\,u(W - X - \alpha\pi + \alpha X) + (1-p)u(W - \alpha\pi).$$

This has the F.O.C.:

$$\frac{\partial EU(\alpha)}{\partial \alpha} = pu^{'}(W - X - \alpha\pi + \alpha X)(X - \pi) - (1-p)u^{'}(W - \alpha\pi)\pi = 0,$$

so

$$\frac{p(X-\pi)}{(1-p)\pi} = \frac{u^{'}(W-\alpha\pi)}{u^{'}(W - X - \alpha\pi + \alpha X)}.$$

Notice that we can rearrange the F.O.C. into

$$\frac{u^{'}(W - X - \alpha\pi + \alpha X)}{(1-p)\pi} = \frac{u^{'}(W-\alpha\pi)}{p(X-\pi)},$$

where the LHS is the marginal utility of an extra dollar in the "loss" state over the price of transferring money to that state from the "no loss" state, and the RHS is the marginal utility of an extra dollar in the "no loss" state over the price of transferring money from that state to the "loss" state.

Hence, the F.O.C. of the insurance problem is nothing other than an optimal arrangement of wealth across different states of the world, or an optimal transfer of resources from the good-luck state to the bad-luck state.

To see an example with utility functions, let's consider $u(x) = x^{0.5}$, with parameters $W = 1000$, $p = 0.1$, and $X = 500$, and an insurance premium π.

The F.O.C. becomes:

$$\frac{0.1(500-\pi)}{0.9\pi} = \frac{0.5(1000-\alpha\pi)^{-0.5}}{0.5(500 - \alpha\pi + \alpha 500)^{-0.5}},$$

which further simplifies to

$$\frac{0.1(500-\pi)}{0.9\pi} = \frac{(500 - \alpha\pi + \alpha 500)^{0.5}}{(1000-\alpha\pi)^{0.5}}.$$

We can square both sides to get

$$\frac{0.1^2(500-\pi)^2}{(0.9\pi)^2} = \frac{500 - \alpha\pi + \alpha 500}{1000-\alpha\pi}.$$

After some algebra, we can isolate α^*:

$$\boxed{\alpha^* = \frac{(0.9\pi)^2 \cdot 500 - 0.1^2(500-\pi)^2 \cdot 1000}{-0.1^2\pi(500-\pi)^2 - (0.9\pi)^2(500-\pi)}}$$

We now plug some values for the risk premium π and see how the optimal coverage varies with the size of the insurance premium.

First, let's consider $\pi = \$58$, then we have:

$$\alpha^* = \frac{(0.9 \cdot 58)^2 \cdot 500 - 0.1^2(500 - 58)^2 \cdot 1000}{-0.1^2 \cdot 58 \cdot (500 - \pi)^2 - (0.9 \cdot 58)^2(500 - 58)} \sim 0.449.$$

If we consider $\pi = \$63$:

$$\alpha^* \sim 0.198.$$

And if we consider $\pi = \$70$:

$$\alpha^* \sim -0.075.$$

But since $\alpha \in [0, 1]$, a negative α just means no insurance at all.

We conclude that the further away π is from the expected loss, the lower the optimal percentage of coverage.

Exercise. Show that if $u(z) = z^{1/2}$, for any given loss X, wealth W and probability of loss $p \in (0, 1)$, if the risk premium $\pi = pX$, then the optimal percentage of coverage is full coverage $(\alpha^* = 1$.

Heterogeneity in Preferences

In the previous chapter, we used the CE and risk premium to classify individuals as more or less risk-averse. We said that the person with the lower CE (higher risk premium) is more risk-averse.

This classification applied to the demand for insurance means that more risk-averse people are willing to pay higher insurance premiums to avoid risk.

Hence, another way to identify and classify decision-makers' risk preferences is by looking at their demand for insurance.

6.3 Supply of Insurance

What is the business of insurance?

Imagine a person or company that tells others: "Give me a small amount of money, and if you happen to suffer a big financial loss at some moment, I've got you covered."

This person is not selling a physical product or a tangible service; they are selling a commitment to cover someone's losses if a bad event occurs.

Even though we are used to having insurance in several aspects of our lives, this is quite a sophisticated product. How is this business sustainable? How does perfect competition or market concentration look? What are the potential issues for the good functioning of these markets?

Perfectly Competitive Insurance Markets

Consider a lottery in which an individual loses \$$X$ with probability p and nothing with probability $1-p$.

We start the study of the supply of insurance with the concept of the **actuarially fair premium**. An actuarially fair insurance premium is defined as

$$\pi^* = pX.$$

That is, the risk premium is equal to the expected loss.

Selling at this price yields zero expected profit for the insurance company. How so?

For any premium π, the expected profit of the insurance company is computed as follows:

- With probability $1-p$, nothing happens, and the insurance company receives π.
- With probability p, the insured suffers a loss of \$$X$, and the insurance company must fully cover that loss, incurring a net revenue of $\pi - X$.

Then the expected profit is:

$$\mathbb{E}[\text{Profits}] = (1-p)\pi + p(\pi - X) = \pi - pX.$$

Expected profits are positive if $\pi > pX$ and zero if $\pi = pX$.

So, in perfectly competitive insurance markets, where expected profits are zero, the premium is $\pi^* = pX$, also called the actuarially fair premium.

Sustainability: Law of Large Numbers

Imagine a home insurance company that only has one client. With probability 1%, the house suffers damages from a natural disaster amounting to \$100,000. The actuarially fair premium is \$1,000.

The company receives \$1,000 and incurs no expenses with probability 99%, but with probability 1%, it incurs a loss of \$99,000 ($100{,}000 - 1{,}000$). Unless the company has a lot of capital from other sources, this event would ruin it financially. Indeed, the first reason why the client seeks insurance is that they do not want to bear that scenario themselves, so why would the company want to?

You may argue that the company could be risk-loving or could charge more than the actuarially fair premium (say, \$2,000). But even then, there is still a 1% chance the company will default.

The business is not sustainable unless risk is pooled.

Law of Large Numbers (LLN)

In statistics, the Law of Large Numbers (LLN) states that if a random variable Z has mean μ_Z, and we take a sample of size N from Z (that is, we draw from Z independently N times), then as N grows large, the sample average converges to μ_Z.

In other words, the average of a large sample will be close to the true mean. This is not necessarily true for small samples. If you only have one client, even though the probability of a disaster is 1%, it may still occur—and you (the insurer) will have to pay, likely defaulting.

So how does LLN apply to insurance?

Define the random variable $Z = 1$ if an accident occurs (you must pay the client), and $Z = 0$ otherwise. Since the accident occurs with probability $p = 0.01$, the expected value is:

$$\mathbb{E}[Z] = 0.01 \cdot 1 + 0.99 \cdot 0 = 0.01.$$

By the LLN, if the insurance company has many clients, the proportion of clients experiencing a loss converges to 1%. This allows the company to use premiums collected from all clients to cover the few who suffer a loss.

Example:

Continuing with the house insurance example, imagine the firm has 2,000 clients. If each client pays the actuarially fair premium of \$1,000, the firm collects:

$$1{,}000 \times 2{,}000 = \$2{,}000{,}000.$$

Now the firm has a large pool of funds to pay out claims. By the LLN, about 1% of the clients—i.e., 20 clients—will suffer damage. The firm must then pay:

$$100{,}000 \times 20 = \$2{,}000{,}000.$$

This matches the amount collected, so on average, the insurance company breaks even: it does not default, but it also does not make a profit.

Administrative Costs

In real life, insurance companies face other costs, such as administrative expenses. To achieve true zero expected profit, the insurance premium must cover these additional costs as well. That is, the zero-profit premium becomes:

$$\pi = \pi^* + \text{ad},$$

where `ad` represents administrative costs and π^* the actuarially fair premium. Hence, the real-world premium is usually slightly higher than the actuarially fair premium.

6.4 Real-World Insurance Contracts

In the real world, insurance contracts are not as simple as paying a premium and receiving full compensation in the event of a loss. Instead, they include several important features that shape both the demand for insurance and the risk borne by insurers. Below, we outline the most common features and their implications.

1. Coverage Limit

The **coverage limit** is the maximum amount the insurance company will pay if a covered loss occurs. Any loss amount beyond this limit must be borne by the insured.

Example: Suppose you purchase car insurance with a coverage limit of $20,000. If your car is totaled in an accident and the damage is assessed at $25,000, your insurer will only pay up to the $20,000 limit. You will have to cover the remaining $5,000 yourself.

2. Deductible

The **deductible** is the amount you must pay out of pocket before the insurance coverage begins. Deductibles reduce premiums and help mitigate moral hazard (we will study this concept in the next section) by making insured parties share in the cost of their claims.

Example: If your deductible is $1,000 and you file a claim for $700, you receive no reimbursement. If you file a second claim for $1,500, the insurer will reimburse only $1,200, as the first $300 of the second claim completes your total deductible of $1,000.

3. Premium

The **premium** is the price paid (monthly, quarterly, or annually) for insurance coverage. Premiums are influenced by the expected loss, administrative costs, and contract features such as limits, deductibles, and co-insurance.

4. Co-insurance

Co-insurance refers to the percentage of covered costs the insured must pay even after the deductible is met. It further shares risk between the insurer and the insured.

Example: If your policy includes 20% co-insurance and you incur a loss of $10,000 after the deductible is paid, you would pay $2,000 (20%), and the insurer would cover the remaining $8,000.

These contract features serve several purposes: they discourage over-insurance and risky behavior, reduce the frequency of small claims, and help insurers avoid catastrophic losses. However, they also introduce complexity that can influence individuals' willingness to purchase insurance.

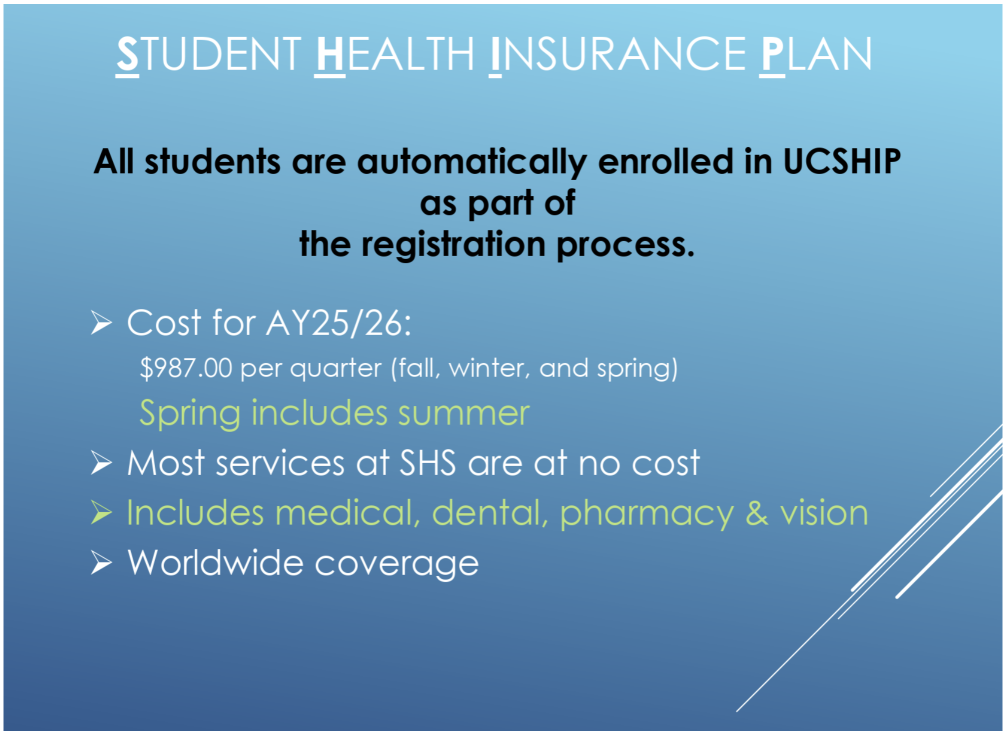
Example: UC SHIP Health Insurance Plan

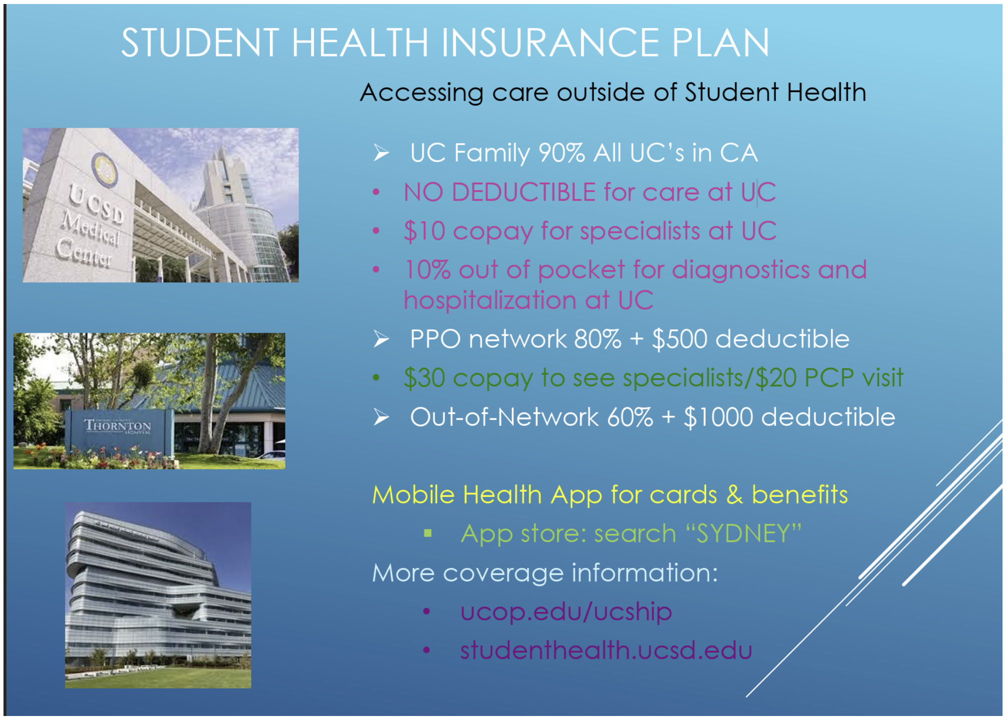
Let's take a closer look at a real-world example: the UC San Diego Student Health Insurance Plan (UC SHIP).

UC SHIP includes multiple tiers of coverage:

- **UC Family**: Services within the UC health system.
- **PPO Network**: Approved providers outside the UC system.
- **Out-of-Network**: Any provider not in the approved network.

Each tier comes with different deductibles, co-insurance rates, and co-pays. Understanding these features is essential for comparing coverage levels and making informed healthcare decisions.





Important Questions	Answers	Why This Matters:
What is the overall deductible?	There is no deductible for UC Family providers. For network providers: $500/ person or $1000/family; Out-of-network provider: $1000/person or $2000/family.	Generally, you must pay all the costs from providers up to the deductible amount before this plan begins to pay. If you have other family members on the plan, each family member must meet their own individual deductible until the total amount of deductible expenses paid by all family members meets the overall family deductible.
Are there services covered before you meet your deductible?	Yes, network preventive services, emergency room, urgent care, acupuncture, chiropractic, physician office visits, family planning, medical evacuation, repatriation and prescription drugs.	This plan covers some items and services even if you haven't yet met the deductible amount. But a copayment or coinsurance may apply. For example, this plan covers certain preventive services without cost-sharing and before you meet your deductible. See a list of covered preventive services at https://www.healthcare.gov/coverage/preventive-care-benefits
Are there other deductibles for specific services?	Yes. Pediatric dental: $60/person or $120/family. There are no other specific deductibles.	You must pay all of the costs for these services up to the specific deductible amount before this plan begins to pay for these services.
What is the out-of-pocket limit for this plan?	For UC family providers/network providers: $4500/person or $9000/family. For out-of-network providers: $9000/person or $18000/family.	The out-of-pocket limit is the most you could pay in a year for covered services. If you have other family members in this plan, they have to meet their own out-of-pocket limits until the overall family out-of-pocket limit has been met.
What is not included in the out-of-pocket limit?	Premiums, balance-billed charges and health care this plan doesn't cover.	Even though you pay these expenses, they don't count toward the out-of-pocket limit.

Common Medical Event	Services You May Need	UC Family Provider (You will pay the least)	Network Provider (You will pay the least)	Out-of-Network Provider (You will pay the most)	Limitations, Exceptions, & Other Important Information
If you visit a health care provider's office or clinic	Primary care visit to treat an injury or illness	No charge Student Health Services (SHS); $5 copayment/visit (UC Family). Deductible does not apply.	$20 copayment/visit. Deductible does not apply.	40% coinsurance	——————none——————
	Specialist visit	No charge (SHS); $10 copayment/visit (UC Family). Deductible does not apply.	$30 copayment/visit. Deductible does not apply.	40% coinsurance	——————none——————
	Preventive care/screening/ immunization	No charge. Deductible does not apply.	No charge. Deductible does not apply.	Not covered	You may have to pay for services that are not preventive. Ask your provider if the services needed are preventive. Then check what your plan will pay for.
If you have a test	Diagnostic test (x-ray, blood work)	10% coinsurance. Deductible does not apply.	20% coinsurance	40% coinsurance	——————none——————
	Imaging (CT/PET scans, MRIs)	10% coinsurance. Deductible does not apply.	20% coinsurance	40% coinsurance	You should refer to your policy or plan document for details (*see pages 30, 33, 37, 39, & 67).
If you need drugs to treat your illness or	Generic drugs	$5 copayment (SHS), $10 copayment (UC Family)/prescription.	$10 copayment/ prescription at retail pharmacies. Deductible	$10 plus any amount over the allowed amount/ prescription. Deductible	Covers up to a 30-day supply of medications and up to 180-days for oral contraceptives at retail or

6.5 Big Picture: How Should We Think About Insurance?

Now that we understand both the demand and the supply sides of insurance, and have a clear initial idea about how contracts work in real life, let's step back and discuss broader questions.

Are Insurance Companies Villains or Saviors?

As we already mentioned, insurance increases the efficiency of markets by reducing or eliminating part of the "incomplete markets" problem. It is also true that this improvement typically assumes that (1) insurance markets are perfectly competitive and (2) insurance markets work without any other frictions.

To the extent that these assumptions are not true, the benefits of insurance could be questioned.

Recently, a health insurance company executive was murdered by a young man, who wrote a manifesto against health insurance companies because of what he perceived to be immoral practices. Social media was polarized about how people felt about this act, but many supported the young man and his view.

We will study some aspects that make insurance markets complex and give rise to these polemics.

Real-World Insurance Markets: Are They Competitive?

Several reports indicate that health insurance markets are highly concentrated in the U.S. That is, a few companies capture most of the enrollments. When consumers don't have many options, they have to accept what is offered to them, especially if they are risk-averse—and in matters of health, most of us are.

If consumers don't have many options but are still willing to buy, insurance companies can raise the insurance premium above the competitive level, reduce the benefits, or both.

According to this short article by the American Medical Association, both in 2013 and 2024, most of the health insurance markets were highly concentrated. For example:

"In 89% (339) of MSAs (metropolitan statistical areas), at least one insurer held a commercial market share of 30% or greater, and in 47% (181) of MSAs, one insurer's share was at least 50%."

Another data point, according to the U.S. Health Insurance Industry Analysis Report for 2024 (mid-year report) by the National Association of Insurance Commissioners: the **net** income for the health insurance industry for that period was **$16 billion**, a decrease of 14% with respect to the same period in the previous year. That represented a 2.7% profit margin.

Adverse Selection: When Buyers Know More Than Sellers

Adverse selection arises when individuals know more about their risk level than insurance companies do—and those with higher risk are more likely to buy insurance. This leads to rising premiums, which in turn may drive lower-risk individuals out of the market.

Example: Consider a health insurance plan. People who expect high medical expenses are more likely to buy it. Healthier individuals opt out, making the insured pool riskier, which increases costs for the insurer. To stay in business, the company raises premiums, further driving out low-risk individuals. This spiral is sometimes referred to as a *death spiral*.

To address this, insurers often use risk-based pricing, require medical exams, or deny coverage/raise prices based on pre-existing conditions. These practices are no longer allowed in the U.S. for health insurance after the Affordable Care Act (ACA) of 2010.

Moral Hazard: When People Take More Risks Because They're Insured

Once individuals are insured, they may have less incentive to avoid risky behavior because they're protected from the consequences. This is called moral hazard.

Example: A person with full auto insurance might drive less carefully, knowing that "the insurance company will pay".

This is an issue for insurance companies because it means that the probability of a bad state occurring increases above its natural level, making the actuarially fair premium insufficient to cover the new expected expenditure.

To combat moral hazard, insurers design contracts with deductibles, co-insurance, and monitoring mechanisms (e.g., requiring car maintenance records).

Social Insurance

When market frictions are too high and there is no clear way to mitigate them, an alternative to private insurance is social insurance.

The pillar of social insurance is solidarity. For example, in some countries, health insurance is social.

One scheme may be as follows: every worker in the country pays a percentage of their salary to social insurance. Every employer also pays a proportion of their employees' salaries to social insurance, and the government also contributes.

Every worker and their dependents are covered by this insurance, regardless of their income, type of disease, or pre-existing conditions.

There is no opt-out from this insurance. Young, healthy, and wealthy people must contribute, thereby subsidizing the medical coverage of those who are older, sicker, and poorer.

A problem with social insurance systems is that they are typically overcrowded and overused. They also often face criticism for limiting individuals' freedom by forcing them to contribute when they don't want to.

In summary, insurance markets are complicated. Despite their social benefit of increasing efficiency in the presence of uncertainty, they are vulnerable to market concentration (which increases inefficiency), and their healthy functioning may be threatened by the consumers of insurance themselves, through adverse selection and moral hazard. Social security emerges as a solidarity-based alternative, but it is not free of challenges, restrictions, and criticism.

6.6 Behavioral Economics of Insurance: Are people good at choosing insurance plans?

In this section, we briefly review a working paper by Anya Samek and Justin Sydnor: *"Impact of Consequence Information on Insurance Choice" (2020).*

Motivation: They motivate their paper by stating that recent research highlights that consumers often make suboptimal insurance choices, deviating from predictions of standard economic theory (e.g., Johnson et al., 2013; Bhargava et al., 2017b; Handel et al., 2020). A likely reason is the cognitive difficulty of understanding complex insurance plan features—premiums, deductibles, coinsurance, etc.—and mapping them into meaningful financial outcomes.

References

Johnson, E. J., Hassin, R., Baker, T., Bajger, A. T., and Treuer, G. (2013). Can consumers make affordable care affordable? The value of choice architecture. PloS one, 8(12).

Bhargava, S., Loewenstein, G., and Sydnor, J. (2017b). Choose to lose: Health plan choices from a menu with dominated option. The Quarterly Journal of Economics, 132(3):13191372.

Handel, B., Kolstad, J., Minten, T., and Spinnewijn, J. (2020). The social determinants of choice quality: Evidence from health insurance in the Netherlands. National Bureau of Economic Research, Working Paper No. 27785.

What they test: The paper tests predictions of **expected utility theory** in insurance choice: individuals should evaluate plans by comparing the *distribution of final wealth* under each plan and select the one that maximizes *expected utility*, based on their beliefs and risk preferences.

However, this mapping is cognitively demanding in real life, suggesting behavioral deviations from rational choice.

How they tested it: They developed a **"consequence graph"**, a visual aid mapping insurance plan features into a distribution of total costs (premiums + out-of-pocket) across quantiles of expected medical spending. The idea is to simplify decisions without adding new information.

They run two experiments:

1. **Incentivized lab experiment** with university students:
    - Controlled choice for a fictitious person
    - Financially incentivized
2. **Survey experiment** with a nationally representative sample:
    - More realistic menus
    - Personalized spending distributions

Participants are randomly assigned to:

- **Feature-only group**: standard plan comparison tables
- **Consequence-information group**: table + visual consequence graph.

What they found:

1. **Improved Decision Quality**:
    - Dominated plan choices fell sharply:
        - Lab: from 40–60% to 10–20%
        - Survey: from 50% to 24%
    - Effects comparable to raising numeracy by 2.5 standard deviations
2. **Higher Uptake of High-Deductible Plans**:
    - When high-deductible plans were efficient, uptake rose by 19 percentage points
    - Matches prior findings that informed consumers prefer such plans
3. **Unexpected Risk-Taking Behavior**:
    - In menus where low-deductible plans were dominant, some chose higher-deductible plans under the consequence graph
    - Suggests bias toward low-premium options when tradeoffs are visualized
4. **Shift in Decision Models**:

- Feature-only group relied on simple heuristics (e.g., lowest deductible)
- Graph group aligned more with utility-maximization
- Heuristic users moved toward "lowest premium" rule

5. **Risk Preferences Only Matter with Better Information**:
   - With only plan features, risk preferences did not predict choices
   - With consequence graphs, weak correlation between risk preferences and plan selection

6.7 Problem Set 4

Problem 1

Every citizen in a town has some probability of developing a disease. Someone developing the disease will spend on average \$15,000. Every person in this city has the same wealth level W.

A town has three types of citizens. Type 1 has a 4% probability of developing the disease and a risk premium of \$700 for the induced lottery $(W - 15k, 0.05; W, 0.95)$. Type 2 has a probability of 0.09% of developing the disease and a risk premium of \$1400, and type 3 has a probability of 12% and a risk premium of \$2000.

The last census counted 200 type 1 citizens, 400 type 2, and 400 type 3 citizens.

A company is considering offering full insurance to cover disease-related expenses.

a) Compute the (aggregated) actuarially fair risk premium.

b) Show that type 1 citizens will not buy the insurance at the premium you found in (1).

c) Show that with the risk premium you find in (1), the firm expects negative profits.

d) Compute the risk premium that achieves zero expected profits (given that type 1 will not participate).

e) Considering the new risk premium you found in (4), will type 2 citizens not buy the insurance?

f) If your answer to (e) is no, compute the new risk premium that achieves zero profits.

g) What type of issue is described in points (a) to (f). How is this particular process known as?

Problem 2

consider the lottery $a = (\$10, 0.2; \$20, 0.3; \$36, 0.5)$
Consider the random variable:

$$\epsilon = \begin{cases} (-\$10,\ p;\ +\$30,\ 1-p), & \text{if } a = \$10, \\ (-\$36,\ q;\ +\$4,\ 1-q), & \text{if } a = \$36. \end{cases}$$

a) For which values of p and $q \in [0,1]$ is $E[\epsilon|a] = 0$? (*Note: $E[\epsilon|a]$ means the conditional expectation. To compute it, fix the level of a, and compute $E[\epsilon]$ for each level of a.*)

b) Compute the (reduced) form of $b = a + \epsilon$.

c) Is a or b riskier? How do you know?

d) Without making **any** calculation, can you argue that $a \succsim_{SOSD} b$?

Problem 3

Consider the following two health insurance plans:

Plan 1

- Premium: \$500
- co-insurance 5%
- out-of-pocket limit \$6000

Plan 2

- Premium: \$200
- co-insurance 15%
- out-of-pocket limit \$6000

Assume that Steve, expects his health expenses in the upcoming year to be as follows:

- \$0 20%
- \$500 20%
- \$4000 20%
- \$5000 20%
- \$8000 20%

If Steve is risk-averse, which plan should he choose?

Extra Questions for people interested in academia:

Problem 4

Consider a risk-averse agent with wealth W who faces this risk of losing X with probability $p \in (0, 1)$.

a) Write the underlying lottery L.

b) Show that at the risk premium, $\pi = pX$, the agent always buy full insurance.

c) Show that the agent buys full insurance as long as $\pi \leq pX + Rp_L$

d) Show using the optimality conditions of the partial insurance problem that if $u(z) = z^{0.5}$, and $\pi = pX$, the optimal percentage of insurance α^* is 1.

e) What is the relation between parts (b) and (d)?