Homework 2: set identification

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Question 1

We know from the model

$$\begin{aligned} P^0_{01} &= -\alpha_1(1+\alpha_2) + u\,\alpha_1\alpha_2, \\ P^0_{10} &= -\alpha_2(1+\alpha_1) + (1-u)\,\alpha_1\alpha_2, \qquad (\alpha_1,\alpha_2) \in [-1,0]^2,\ u \in [0,1]. \\ P^0_{11} &= (1+\alpha_1)(1+\alpha_2), \end{aligned}$$

The third equation does not depend on u, so any feasible (α_1, α_2) must satisfy

$$(1+\alpha_1)(1+\alpha_2) = P^0_{11}.$$

This immediately implies $1+\alpha_i > 0$, so in fact $\alpha_i > -1$ and the boundary $\alpha_1 = -1$ is not allowed. Solving explicitly,

$$\alpha_2 = -1 + \frac{P^0_{11}}{1+\alpha_1}, \qquad \alpha_1 \in (-1, 0].$$

So all admissible parameters lie on this one-dimensional curve.
Now I need to check feasibility of $u \in [0,1]$. From the first equation,

$$u(\alpha_1,\alpha_2) = \frac{P^0_{01} + \alpha_1(1+\alpha_2)}{\alpha_1\alpha_2}, \qquad \text{so I need } 0 \le u(\alpha_1,\alpha_2) \le 1.$$

Equivalently,

$$-\alpha_1(1+\alpha_2) \ \le\ P^0_{01} \ \le\ -\alpha_1(1+\alpha_2) + \alpha_1\alpha_2.$$

Final Identified Set

$$\Theta_I = \left\{ (\alpha_1,\alpha_2) \in (-1,0]^2 : (1+\alpha_1)(1+\alpha_2) = P^0_{11},\ \frac{P^0_{01} + \alpha_1(1+\alpha_2)}{\alpha_1\alpha_2} \in [0,1] \right\}.$$

As expected, it is a curve in $\mathbb{R}^2$, not an area.

Question 2

a)

We can create boxunds on the observed probabilities and then define moment inequalities. For $y = (1,0)$ we have

$$\hat{P}_{01} = -\alpha_1(1+\alpha_2) + u\,(\alpha_1\alpha_2), \qquad u \in [0,1].$$

Since $\alpha_1, \alpha_2 \in [-1,0]$, we have $\alpha_1\alpha_2 \ge 0$, so $\hat{P}_{01}$ is (weakly) increasing in u. Hence

$$-\alpha_1(1+\alpha_2) \ \le\ \hat{P}_{01} \ \le\ -\alpha_1(1+\alpha_2) + \alpha_1\alpha_2,$$

which yields the two moment inequalities

$$-\alpha_1(1+\alpha_2) - \hat{P}_{01} \ \le\ 0, \qquad \hat{P}_{01} + \alpha_1 \ \le\ 0.$$

Analogously, for $y = (0,1)$:

$$\hat{P}_{10} = -\alpha_2(1+\alpha_1) + (1-u)\,(\alpha_1\alpha_2),$$

which is (weakly) decreasing in u, giving

$$-\alpha_2(1+\alpha_1) \ \leq\ \hat{P}_{10} \ \leq\ -\alpha_2(1+\alpha_1)+\alpha_1\alpha_2,$$

and the two inequalities

$$-\alpha_2(1+\alpha_1)-\hat{P}_{10} \ \leq\ 0, \qquad \hat{P}_{10}+\alpha_2 \ \leq\ 0.$$

Finally, $P_{11}=(1+\alpha_1)(1+\alpha_2)$ provides the equality

$$(1+\alpha_1)(1+\alpha_2)-\hat{P}_{11}=0.$$

Thus, in principle we obtain five (in)equalities. However, in the specific data configuration $\hat{P}_{01}=0.35$, $\hat{P}_{10}=0.15$, and $\hat{P}_{11}=0.5$, the system reduces to the simpler set

$$\hat{P}_{01}\leq-\alpha_1, \qquad \hat{P}_{10}\leq-\alpha_2, \qquad (1+\alpha_1)(1+\alpha_2)=\hat{P}_{11},$$

since the lower bounds are automatically satisfied once the upper bounds and the equality are imposed.

b)

I propose the following test:

$$Tn(\theta)=max_j\sqrt{n}\frac{\bar{m_j}}{\sigma_j}$$

c)

I will compute the critical value using GMS. The idea is to simulate a multivariate normal, that has the variance-covariance matrix of my observed data. Then I define

$$\xi_j(\theta)=\sqrt{n}\frac{\bar{m_j}}{\sigma_j}\frac{1}{\kappa}$$

Where $\kappa=\sqrt{2\log(\log(n))}$. For every value of θ in the grid (or potential combination) I then compute this variable and add the initially simulated. Then I take the row maximum to have an asymptotic distribution of the test statistic and then I take the 95th percentile to define the critical value.

d)

My bounds are the following:

Table 1: GMS 95% confidence bounds for (α_1,α_2)

Parameter	Lower	Upper
α_1	-0.470	-0.315
α_2	-0.325	-0.125

The total number of points are 504.

e)

For each parameter value $\theta=(\alpha_1,\alpha_2)$ I do the following:

1. Compute the sample moment vector

$$m_n(\theta)=\begin{pmatrix}\hat{P}_{01}+\alpha_1\\ \hat{P}_{10}+\alpha_2\\ \hat{P}_{11}-(1+\alpha_1)(1+\alpha_2)\end{pmatrix},$$

   where the first two are inequality moments (≤ 0) and the last one is an equality ($=0$).

2. Estimate the covariance matrix $\hat{\Sigma}$ of $(\hat{P}_{01}, \hat{P}_{10}, \hat{P}_{11})$ using the usual multinomial formula

$$\hat{\Sigma} = \mathrm{diag}(\hat{p}) - \hat{p}\,\hat{p}', \qquad \hat{p} = (\hat{P}_{01}, \hat{P}_{10}, \hat{P}_{11})'.$$

   This matrix is singular, so I use the Moore–Penrose inverse $\hat{\Sigma}^{+}$.

3. Project $m_n(\theta)$ onto the feasible set

$$\mathcal{M} = \{\mu \in \mathbb{R}^3 : \mu_1 \leq 0,\ \mu_2 \leq 0,\ \mu_3 = 0\}.$$

   In other words, I solve numerically

$$\hat{\mu}(\theta) = \arg\min_{\mu \in \mathcal{M}}\ n\,(m_n(\theta) - \mu)'\,\hat{\Sigma}^{+}\,(m_n(\theta) - \mu).$$

   (In code this is just an `optim()` with box constraints.)

4. Count how many moments are "close to" binding at the solution $\hat{\mu}(\theta)$. The equality is always binding, and I add one more for each inequality with $\hat{\mu}_j \approx 0$.

5. Compute the critical value as $\chi^2_{r,\,1-\alpha}$ where r is the number of binding moments.

6. Keep θ if

$$T_n(\theta) := n\,(m_n(\theta) - \hat{\mu}(\theta))'\,\hat{\Sigma}^{+}\,(m_n(\theta) - \hat{\mu}(\theta))\ \leq\ \chi^2_{r,\,1-\alpha}.$$

The confidence set is just all grid points that survive this test. However, there seems to be an error in my code, as I am not able to generate correct bounds.