Point cloud registration: matching a maximal common subset on pointclouds with noise (with 2D implementation)

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Motivation

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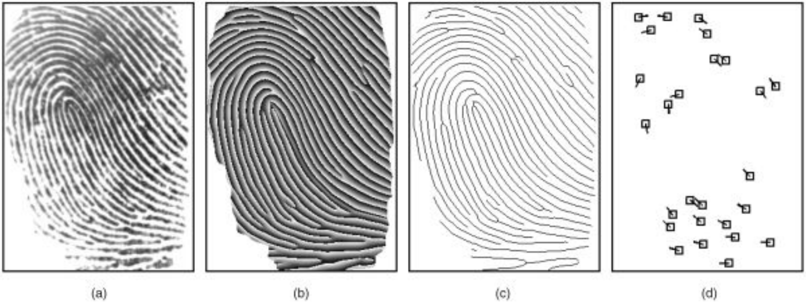

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- Question: can we adapt a conventional method to deal with this?

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where the i_k are all different from each other, as well as the j_k. (This ensures no point is matched to other points twice)

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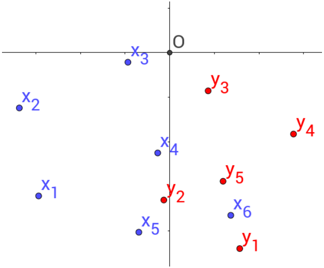

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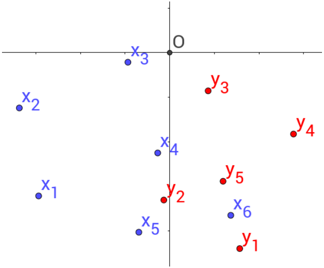

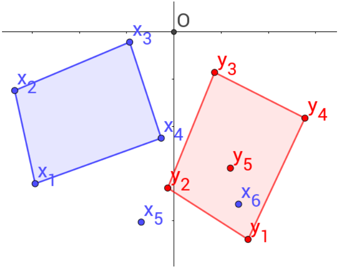

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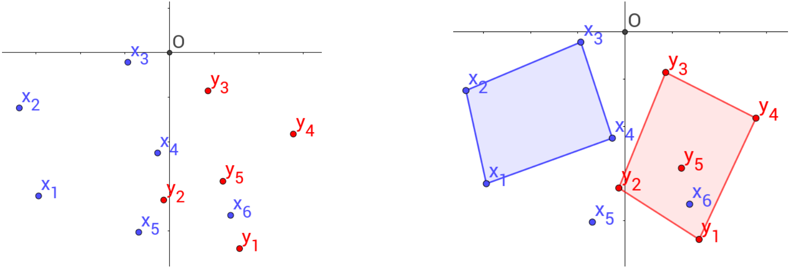

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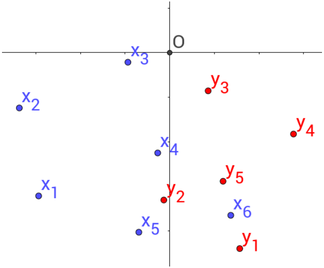

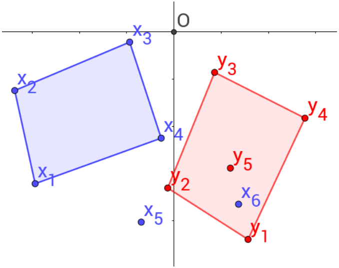

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Finally, we compare the rotations found for each pair of pivots.

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It is therefore necessary to find a different ϕ.

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3. $\phi(r)$ should be small for small distances r, in order to reward close points as correspondent.

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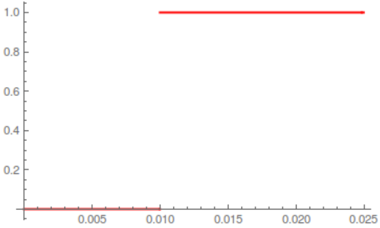
$$\phi(r) = \begin{Bmatrix} 0 & \text{for } 0 \leq r < \delta \\ 1 & \text{for } \delta \leq r \end{Bmatrix}$$



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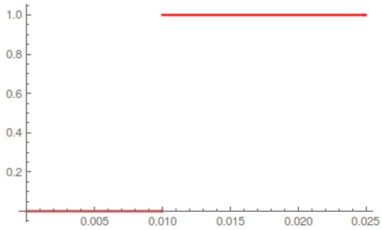

Any pair of points from X and Y that are closer than δ are seen as correspondent to each other.

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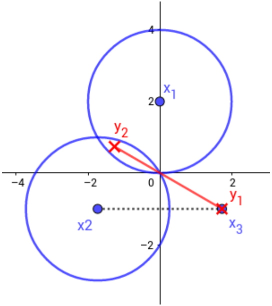

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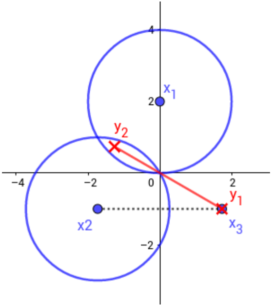

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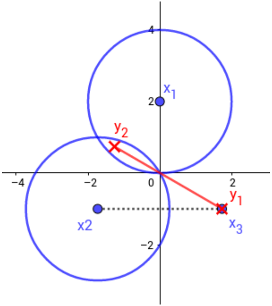

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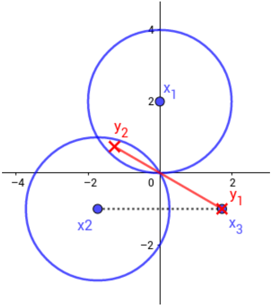

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Therefore, minimizing energy $\Rightarrow$ maximizing common subset!

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Insight on the structure of $E_{p,q}$ is needed to optimize it.

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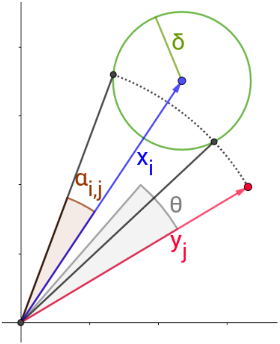

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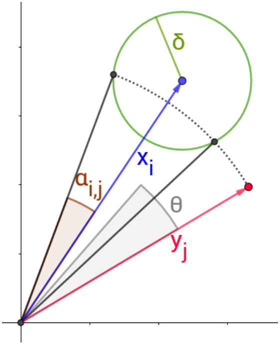

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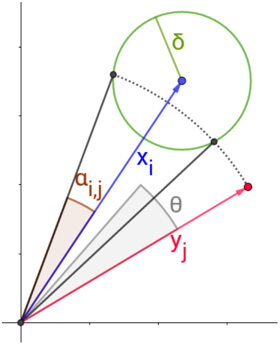

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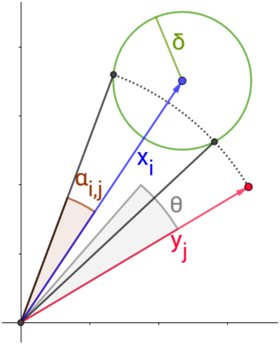

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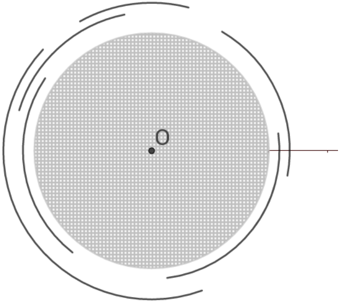
Next: find where the maximum number of these intervals overlap



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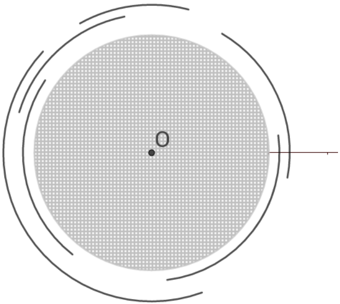

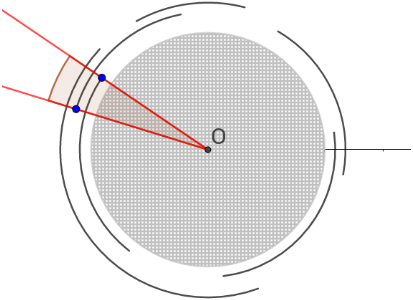

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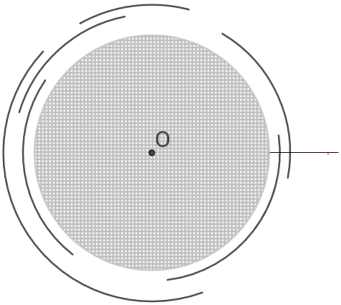

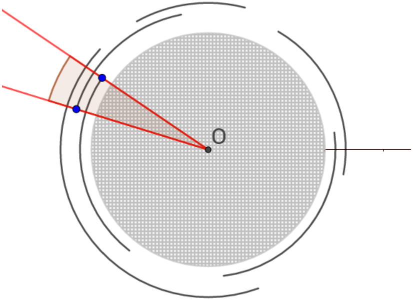

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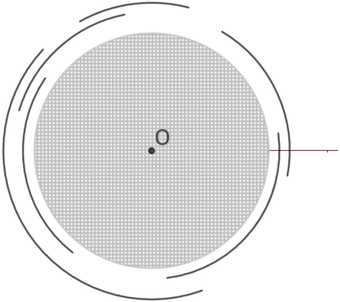

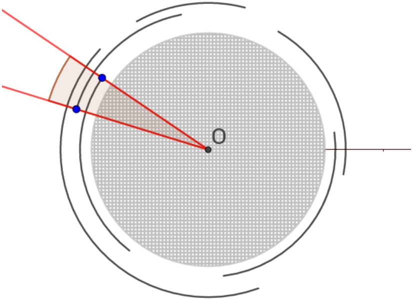

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- Periodic variation: simply consider number of guests at $\theta = 0$ as initial condition.

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        I_ij ← [∠x_i − ∠y_j − α_{i,j} , ∠x_i − ∠y_j + α_{i,j}], ∀ valid i, j
        θ_{p,q} ← angle of maximum number of intersections  mod 2π
    end for
end for
θ ← max_{p,q} θ_{p,q}
(p, q) ← arg max_{p,q} θ_{p,q}
```

$$\alpha_{ij} \leftarrow \arccos\left(\frac{||x_i||^2+||y_j||^2-\delta^2}{2||x_i||\cdot||y_j||}\right), \forall \text{ valid } i,j$$

$$I_{ij} \leftarrow [\angle x_i - \angle y_j - \alpha_{i,j}\ , \angle x_i - \angle y_j + \alpha_{i,j}], \forall \text{ valid } i,j$$

First experiments

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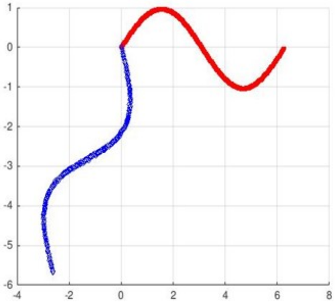

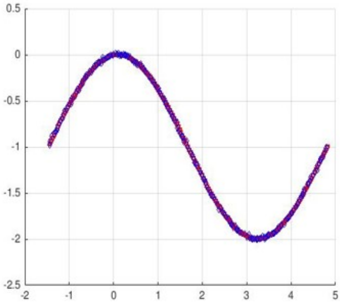

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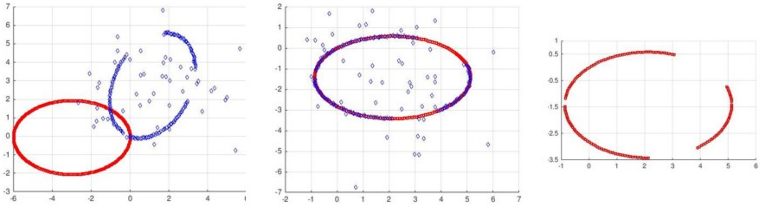

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- The procedure was repeated 50 times. Successful every time. Difference in rotation angles: 0.0028 degrees mean, 0.0023 standard deviation, 0.0080 maximum

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- We study values of k for $3 \leq k \leq 150$, 20 iterations for each.

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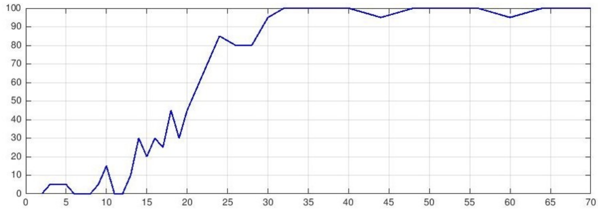

(For $k > 70$: successful matches every single time.)

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Thanks for your attention!