Singularity Analysis for the Perspective-Four-Line Problem

International Journal of Computer Vision

Jorge García Fontán, Abhilash Nayak, Sébastien Briot, Mohab Safey El Din

Jorge García Fontán: Sorbonne Université, Laboratoire d'Informatique de Paris 6, LIP6, Equipe PolSys, Paris, France

Abhilash Nayak: Centre national de la Recherche Scientifique, Laboratoire des Sciences du Numérique de Nantes (LS2N), UMR CNRS 6004, Nantes, France

Sébastien Briot: Centre national de la Recherche Scientifique, Laboratoire des Sciences du Numérique de Nantes (LS2N), UMR CNRS 6004, Nantes, France

Mohab Safey El Din: Sorbonne Université, CNRS, INRIA, Laboratoire d'Informatique de Paris 6, LIP6, Equipe PolSys, Paris, France

Corresponding author: Jorge García Fontán, email: Jorge.Garcia-Fontan@lip6.fr

1. Defining the lines, finding the interaction matrix and calssifying its minors into ideals G, H and K

```
> restart;
```

Calling the required libraries

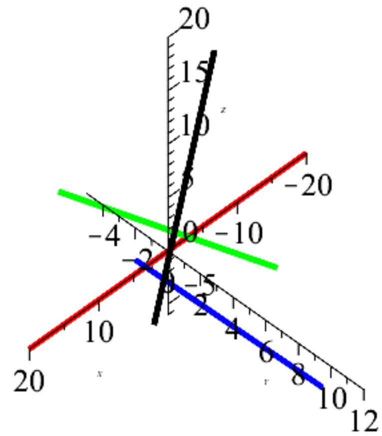
```
> with(LinearAlgebra): with(plots): with(plottools):
```

Defining the 4 lines (direction vector and a point on the line) and the position of the camera in the object frame

```
> OC:=Vector([X,Y,Z]):
> OP1:=Vector([0,0,0]):  CP1:=OP1-OC: u1:=Vector([1,0,0]):
  OP2:=Vector([0,0,d1]): CP2:=OP2-OC: u2:=Vector([s1,s2,0]):
  OP3:=Vector([d2,d3,0]): CP3:=OP3-OC: u3:=Vector([d2,s3,s4]):
  OP4:=Vector([0,d4,d5]): CP4:=OP4-OC: u4:=Vector([s5,d4,s6]):
```

Plotting the observed lines for arbitrary parameters

```
> val:={d1=2,d2=3,d3=5,d4=1,d5=7,d6=-2, s1=4,s2=-5,s3=7,s4=3,s5=-2,
  s6=13}:
> n:=1:
  t1:=spacecurve(eval(OP1,val)+lambda*eval(u1,val),lambda=-20..20,
  color=red,thickness=5,transparency=0):
  t2:=spacecurve(eval(OP2,val)+lambda*eval(u2,val),lambda=-n..n,
  color=green,thickness=5,transparency=0):
  t3:=spacecurve(eval(OP3,val)+lambda*eval(u3,val),lambda=-n..n,
  color=blue,thickness=5,transparency=0):
  t4:=spacecurve(eval(OP4,val)+lambda*eval(u4,val),lambda=-n..n,
  color=black,thickness=5,transparency=0):
> display(t1,t2,t3,t4,labels=[X,Y,Z],axes=normal);
```



The rows of the interaction matrix consisting of an affine line and an ideal line for each observed line

```
> for i from 1 to 4 do
  f||i||1:=CrossProduct(u||i,CP||i):
  m||i||1:=CrossProduct(OP||i,f||i||1):
  f||i||2:=ZeroVector(3):
  m||i||2:=CrossProduct(u||i,f||i||1):
  zf||i:=Vector([Transpose(f||i||1),Transpose(m||i||1)]):
  zm||i:=Vector([Transpose(f||i||2),Transpose(m||i||2)]):
  od:
> zeta:=Transpose(Matrix([zf1,zf2,zf3,zf4,zm1,zm2,zm3,zm4]));
```

$$\zeta := [[0, Z, -Y, 0, 0, 0], \tag{1.1}$$

$[s2\,(d1-Z), -s1\,(d1-Z), X\,s2-Y\,s1, d1\,s1\,(d1-Z), d1\,s2\,(d1-Z), 0],$

$[-s3\,Z-s4\,(d3-Y), d2\,Z+s4\,(d2-X), d2\,(d3-Y)-s3\,(d2-X), d3\,(d2\,(d3-Y)-s3\,(d2-X)), -d2\,(d2\,(d3-Y)-s3\,(d2-X)), d2\,(d2\,Z+s4\,(d2-X))-d3\,(-s3\,Z-s4\,(d3-Y))],$

$[d4\,(d5-Z)-s6\,(d4-Y), -s5\,(d5-Z)-s6\,X, s5\,(d4-Y)+d4\,X, d4\,(s5\,(d4-Y)+d4\,X)-d5\,(-s5\,(d5-Z)-s6\,X), d5\,(d4\,(d5-Z)-s6\,(d4-Y)), -d4\,(d4\,(d5-Z)-s6\,(d4-Y))],$

$[0, 0, 0, 0, Y, Z],$

$[0, 0, 0, s2\,(X\,s2-Y\,s1), -s1\,(X\,s2-Y\,s1), -s1^2\,(d1-Z)-s2^2\,(d1-Z)],$

$[0, 0, 0, s3\,(d2\,(d3-Y)-s3\,(d2-X))-s4\,(d2\,Z+s4\,(d2-X)), -d2\,(d2\,(d3-Y)-s3\,(d2-X))+s4\,(-s3\,Z-s4\,(d3-Y)), d2\,(d2\,Z+s4\,(d2-X))-s3\,(-s3\,Z-s4\,(d3-Y))],$

$[0, 0, 0, d4\,(s5\,(d4-Y)+d4\,X)-s6\,(-s5\,(d5-Z)-s6\,X), -s5\,(s5\,(d4-Y)$

$+ d4\, X) + s6\, (d4\, (d5 - Z) - s6\, (d4 - Y)\,), s5\, (-s5\, (d5 - Z) - s6\, X) - d4\, (d4\, (d5 - Z) - s6\, (d4 - Y)\,)\,]]$

28 minors of the interaction matrix

```
> ch:=combinat[choose](8,2); nops(ch);
```

$ch := [[1, 2], [1, 3], [1, 4], [1, 5], [1, 6], [1, 7], [1, 8], [2, 3], [2, 4], [2, 5], [2, 6], [2, 7], [2, 8], [3, 4], [3, 5], [3, 6], [3, 7], [3, 8], [4, 5], [4, 6], [4, 7], [4, 8], [5, 6], [5, 7], [5, 8], [6, 7], [6, 8], [7, 8]]$

$$28 \tag{1.2}$$

```
> for i from 1 to nops(ch) do
  zt||i:=DeleteRow(zeta,ch[i]):
  D||i:=factor(Determinant(DeleteRow(zeta,ch[i]))):
  od:
> I28:=[seq(D||i,i=1..28)]: nops(%);
```

$$28 \tag{1.3}$$

Determining the ideals G, H and K

```
> zf:=Transpose(Matrix([zf1,zf2,zf3,zf4])):
  zm:=Transpose(Matrix([zm1,zm2,zm3,zm4])):
> k:=1: #splitting the 16 minors into two factors to obtain ideals G and H
  for i from 1 to 4 do
  for j from 1 to 4 do
  zs:=Matrix([[DeleteRow(zf,i)],[DeleteRow(zm,j)]]):
  Ds||k:=factor(Determinant(zs)):
  Gs||k:=factor(Determinant(DeleteRow(zf[1..4,1..3],i))):
  Hs||k:=factor(Determinant(DeleteRow(zm[1..4,4..6],i))):
  k:=k+1:
  od:
  od:
  I16:=[seq(Ds||i,i=1..k-1)]:
  G16t:=[seq(Gs||i,i=1..k-1)]: ListTools[MakeUnique](G16t): G:=[seq(%[i],i=1..nops(%))]: nops(%);
  H16t:=[seq(Hs||i,i=1..k-1)]: ListTools[MakeUnique](H16t): H:=[seq(%[i],i=1..nops(%))]: nops(%);
```

$$4$$

$$4 \tag{1.4}$$

```
> K:=eval([seq(D||i,i=23..28)]): #The remaining 6 minors that cannot be factorized.
```

Substituting parameters as given in Appendix A.

```
> Gv:=eval(G,val);
```

$$Gv := [-1765\, X^2 - 587\, X\, Y - 878\, X\, Z + 660\, Y^2 + 232\, Y\, Z - 122\, Z^2 + 2606\, X + 216\, Y + 598\, Z - 708, -177\, X\, Y - 27\, X\, Z + 75\, Y^2 + 17\, Y\, Z - 8\, Z^2 + 156\, Y + 6\, Z, -130\, X\, Y + 40\, X\, Z - 104\, Y^2 - 62\, Y\, Z + 10\, Z^2 + 188\, Y - 20\, Z, -30\, X\, Y + 145\, X\, Z - 24\, Y^2 + 11\, Y\, Z + 30\, Z^2 + 210\, Y - 60\, Z] \tag{1.5}$$

```
> Hv:=eval(H,val);
```

$$Hv := [-73960\, X^3 - 46428\, X^2\, Y + 320426\, X^2\, Z + 88867\, X\, Y^2 + 163934\, X\, Y\, Z + 184389\, X\, Z^2 + 62940\, Y^3 - 356381\, Y^2\, Z - 32282\, Y\, Z^2 + 27183\, Z^3 + 210018\, X^2 - 721747\, X\, Y - 416981\, X\, Z - 146898\, Y^2 - 118097\, Y\, Z - 116973\, Z^2 + 111106\, X + 377082\, Y \tag{1.6}$$

$+ 153504\, Z - 56580,\ -3038\, X^2\, Y + 3686\, X^2\, Z - 2288\, X\, Y^2 + 16544\, X\, Y\, Z + 3344\, X\, Z^2$
$- 315\, Y^3 - 4111\, Y^2\, Z - 157\, Y\, Z^2 + 663\, Z^3 + 27166\, X\, Y - 4168\, X\, Z + 2769\, Y^2$
$- 13942\, Y\, Z - 1527\, Z^2 - 25806\, Y + 690\, Z,\ -650\, X^2\, Y - 3350\, X^2\, Z - 195\, X\, Y^2$
$+ 8450\, X\, Y\, Z - 845\, X\, Z^2 + 260\, Y^3 + 3410\, Y^2\, Z + 390\, Y\, Z^2 - 13390\, X\, Y + 1630\, X\, Z$
$+ 276\, Y^2 - 8372\, Y\, Z + 15088\, Y,\ 225\, X^2\, Y - 1685\, X^2\, Z + 705\, X\, Y^2 + 450\, X\, Y\, Z - 345\, X\, Z^2$
$+ 420\, Y^3 - 1325\, Y^2\, Z - 645\, Y\, Z^2 - 8056\, X\, Y + 705\, X\, Z - 918\, Y^2 - 1527\, Y\, Z + 5658\, Y]$

```
> Kv:=eval(K,val);
```

$Kv := [3036000\, X^4\, Y - 7529280\, X^4\, Z + 4754460\, X^3\, Y^2 - 21886456\, X^3\, Y\, Z - 6125988\, X^3\, Z^2$ **(1.7)**
$+ 1083858\, X^2\, Y^3 + 2882442\, X^2\, Y^2\, Z - 16238698\, X^2\, Y\, Z^2 - 11096786\, X^2\, Z^3$
$- 1238286\, X\, Y^4 + 11225644\, X\, Y^3\, Z + 1209768\, X\, Y^2\, Z^2 + 11358052\, X\, Y\, Z^3 - 4233002\, X\, Z^4$
$- 493560\, Y^5 - 1136814\, Y^4\, Z - 1771102\, Y^3\, Z^2 + 448850\, Y^2\, Z^3 + 2065542\, Y\, Z^4$
$- 408084\, Z^5 - 55501776\, X^3\, Y + 24113616\, X^3\, Z - 55614846\, X^2\, Y^2 + 94986016\, X^2\, Y\, Z$
$- 8598546\, X^2\, Z^2 + 6469710\, X\, Y^3 + 1278090\, X\, Y^2\, Z + 83999142\, X\, Y\, Z^2 + 5724082\, X\, Z^3$
$+ 12653748\, Y^4 - 34027290\, Y^3\, Z - 6002820\, Y^2\, Z^2 - 3495746\, Y\, Z^3 + 2306412\, Z^4$
$+ 305848404\, X^2\, Y - 34761444\, X^2\, Z + 138271764\, X\, Y^2 - 78335080\, X\, Y\, Z$
$+ 16483548\, X\, Z^2 - 71935092\, Y^3 - 256656\, Y^2\, Z - 52845548\, Y\, Z^2 + 2134032\, Z^3$
$- 498731712\, X\, Y + 19626192\, X\, Z - 38576448\, Y^2 + 18346248\, Y\, Z - 6573024\, Z^2$
$+ 236207952\, Y - 7312032\, Z,\ 6624800\, X^4\, Y + 4456800\, X^4\, Z + 9428380\, X^3\, Y^2$
$- 32123880\, X^3\, Y\, Z + 13889140\, X^3\, Z^2 - 4303208\, X^2\, Y^3 - 45476486\, X^2\, Y^2\, Z$
$- 32385320\, X^2\, Y\, Z^2 + 3087198\, X^2\, Z^3 - 11420032\, X\, Y^4 + 17830726\, X\, Y^3\, Z$
$- 31601150\, X\, Y^2\, Z^2 - 2195978\, X\, Y\, Z^3 - 294190\, X\, Z^4 - 4268160\, Y^5 + 25096600\, Y^4\, Z$
$+ 17577878\, Y^3\, Z^2 - 3261404\, Y^2\, Z^3 - 77010\, Y\, Z^4 - 51168\, Z^5 - 17855680\, X^3\, Y$
$- 7898280\, X^3\, Z + 46018856\, X^2\, Y^2 + 130710864\, X^2\, Y\, Z - 14697912\, X^2\, Z^2$
$+ 68337620\, X\, Y^3 + 121570128\, X\, Y^2\, Z + 60966800\, X\, Y\, Z^2 + 1627348\, X\, Z^3 + 16075920\, Y^4$
$- 50526584\, Y^3\, Z + 12307644\, Y^2\, Z^2 + 295640\, Y\, Z^3 + 1083876\, Z^4 - 116812448\, X^2\, Y$
$+ 16999032\, X^2\, Z - 168966056\, X\, Y^2 - 256357336\, X\, Y\, Z + 3887048\, X\, Z^2 - 329064\, Y^3$
$- 38129632\, Y^2\, Z - 30943256\, Y\, Z^2 - 2549544\, Z^3 + 286261696\, X\, Y - 11929968\, X\, Z$
$+ 53058480\, Y^2 + 133675520\, Y\, Z - 1170960\, Z^2 - 144710976\, Y + 4687776\, Z,$
$1035000\, X^4\, Y - 2566800\, X^4\, Z + 1092450\, X^3\, Y^2 - 9976750\, X^3\, Y\, Z + 26275730\, X^3\, Z^2$
$- 1571280\, X^2\, Y^3 - 3648703\, X^2\, Y^2\, Z + 14898478\, X^2\, Y\, Z^2 + 19447161\, X^2\, Z^3$
$- 2491872\, X\, Y^4 + 10455011\, X\, Y^3\, Z - 24472395\, X\, Y^2\, Z^2 + 6015037\, X\, Y\, Z^3 + 5255985\, X\, Z^4$
$- 852480\, Y^5 + 6642444\, Y^4\, Z - 4430001\, Y^3\, Z^2 - 9241278\, Y^2\, Z^3 - 20959\, Y\, Z^4 + 523734\, Z^5$
$- 1206210\, X^3\, Y + 11993940\, X^3\, Z + 17592132\, X^2\, Y^2 + 5834381\, X^2\, Y\, Z - 39250039\, X^2\, Z^2$
$+ 25674570\, X\, Y^3 - 4269971\, X\, Y^2\, Z - 18022208\, X\, Y\, Z^2 - 19874311\, X\, Z^3 + 8663112\, Y^4$
$- 45928152\, Y^3\, Z + 2752721\, Y^2\, Z^2 - 2587003\, Y\, Z^3 - 2595546\, Z^4 - 60994644\, X^2\, Y$
$+ 9592434\, X^2\, Z - 124995966\, X\, Y^2 - 44707334\, X\, Y\, Z + 25116044\, X\, Z^2 - 16041132\, Y^3$

$$+ 9178620\,Y^2 Z + 2996442\,Y Z^2 + 4959360\,Z^3 + 141802404\,X Y - 12782724\,X Z$$
$$+ 55929780\,Y^2 + 33633948\,Y Z - 5450376\,Z^2 - 58222296\,Y + 3447936\,Z, 245440\,X^3 Y^2$$
$$- 218440\,X^3 Y Z + 118920\,X^3 Z^2 + 388362\,X^2 Y^3 - 1589246\,X^2 Y^2 Z + 4270\,X^2 Y Z^2$$
$$- 15002\,X^2 Z^3 + 182858\,X Y^4 - 767384\,X Y^3 Z - 1087960\,X Y^2 Z^2 + 126912\,X Y Z^3$$
$$- 17690\,X Z^4 + 23400\,Y^5 + 317110\,Y^4 Z + 162978\,Y^3 Z^2 - 62854\,Y^2 Z^3 + 2790\,Y Z^4$$
$$- 1248\,Z^5 - 2531692\,X^2 Y^2 + 2413552\,X^2 Y Z - 275676\,X^2 Z^2 - 2290656\,X Y^3$$
$$+ 2964612\,X Y^2 Z + 875096\,X Y Z^2 + 35908\,X Z^3 - 263724\,Y^4 - 62496\,Y^3 Z$$
$$+ 692408\,Y^2 Z^2 - 92240\,Y Z^3 + 23940\,Z^4 + 5231128\,X Y^2 - 4239208\,X Y Z + 167904\,X Z^2$$
$$+ 2063160\,Y^3 - 820456\,Y^2 Z - 668000\,Y Z^2 - 14304\,Z^3 - 3013536\,Y^2 + 1952976\,Y Z$$
$$- 57168\,Z^2, 51750\,X^3 Y^2 - 249090\,X^3 Y Z + 299460\,X^3 Z^2 + 87810\,X^2 Y^3$$
$$- 622808\,X^2 Y^2 Z + 1256768\,X^2 Y Z^2 + 278606\,X^2 Z^3 + 46578\,X Y^4 - 283939\,X Y^3 Z$$
$$- 122385\,X Y^2 Z^2 + 394487\,X Y Z^3 + 100795\,X Z^4 + 7560\,Y^5 + 62439\,Y^4 Z - 69311\,Y^3 Z^2$$
$$- 105573\,Y^2 Z^3 + 1951\,Y Z^4 + 12774\,Z^5 - 782298\,X^2 Y^2 + 3033480\,X^2 Y Z - 513192\,X^2 Z^2$$
$$- 773088\,X Y^3 + 3006255\,X Y^2 Z - 862598\,X Y Z^2 - 253931\,X Z^3 - 154854\,Y^4$$
$$- 106929\,Y^3 Z + 3408\,Y^2 Z^2 - 264695\,Y Z^3 - 37758\,Z^4 + 3363228\,X Y^2 - 4510470\,X Y Z$$
$$+ 236202\,X Z^2 + 1113174\,Y^3 - 1899918\,Y^2 Z - 148908\,Y Z^2 + 45444\,Z^3 - 2934252\,Y^2$$
$$+ 1551420\,Y Z - 42048\,Z^2, 29250\,X^3 Y^2 - 113050\,X^3 Y Z - 217800\,X^3 Z^2 + 85800\,X^2 Y^3$$
$$- 195265\,X^2 Y^2 Z + 501390\,X^2 Y Z^2 - 119995\,X^2 Z^3 + 81120\,X Y^4 - 252285\,X Y^3 Z$$
$$+ 392990\,X Y^2 Z^2 + 168915\,X Y Z^3 - 16930\,X Z^4 + 24960\,Y^5 - 134740\,Y^4 Z - 46540\,Y^3 Z^2$$
$$+ 68560\,Y^2 Z^3 + 6300\,Y Z^4 - 516080\,X^2 Y^2 - 797290\,X^2 Y Z + 203790\,X^2 Z^2$$
$$- 511894\,X Y^3 + 524534\,X Y^2 Z - 1072472\,X Y Z^2 + 54320\,X Z^3 - 79224\,Y^4$$
$$+ 644008\,Y^3 Z - 151216\,Y^2 Z^2 - 148588\,Y Z^3 - 290756\,X Y^2 + 1353444\,X Y Z$$
$$- 40920\,X Z^2 + 53280\,Y^3 - 545568\,Y^2 Z + 497312\,Y Z^2 + 1054848\,Y^2 - 450672\,Y Z]$$

2. Analysis of the variety V(G) intersection V(K)

Groebner basis of G

```
> indets(G);
```

$$\{X, Y, Z, d1, d2, d3, d4, d5, s1, s2, s3, s4, s5, s6\} \qquad \textbf{(2.1)}$$

```
> GbG:=Groebner[Basis](eval(G),plex(d1, d2, d3, d4, d5, s1, s2, s3,
  s4, s5, s6, X, Y, Z)): nops(%);
```

$$4 \qquad \textbf{(2.2)}$$

The normal form of elements of K w.r.t the Groebner basis.

```
> for i from 1 to 6 do
  Groebner[NormalForm](K[i],GbG,plex(d1, d2, d3, d4, d5, s1, s2,
  s3, s4, s5, s6, X, Y, Z));
  od; # Thus V(G) intersection V(K) = V(G)
```

$$0$$
$$0$$

$$0$$
$$0$$
$$0$$
$$0 \quad \textbf{(2.3)}$$

Groebner basis of G after substituting the parameters d_i and s_i

```
> GbGv:=Groebner[Basis](Gv,plex(X,Y,Z));
```

$$GbGv := [1083\, Y^2 + 583\, Y Z - 25\, Z^2, 1765\, X Z + 329\, Y Z + 360\, Z^2 + 2166\, Y - 720\, Z, 1911495\, X Y + 198073\, Y Z + 8225\, Z^2 - 2042538\, Y + 54150\, Z, 3373788675\, X^2 - 193436537\, Y Z - 143062825\, Z^2 - 4981355970\, X - 1273506198\, Y - 490230780\, Z + 1353338460] \quad \textbf{(2.4)}$$

Analysing GbGv shows that its variety consists of two lines:

```
> allvalues(evala(AFactor(GbGv[1])));
```

$$1083 \left(Y + \left(\frac{1}{2166}\sqrt{448189} + \frac{583}{2166}\right) Z\right) \left(Y + \left(-\frac{1}{2166}\sqrt{448189} + \frac{583}{2166}\right) Z\right), 1083 \left(Y + \left(\frac{1}{2166}\sqrt{448189} + \frac{583}{2166}\right) Z\right) \left(Y + \left(-\frac{1}{2166}\sqrt{448189} + \frac{583}{2166}\right) Z\right) \quad \textbf{(2.5)}$$

```
> map(factor,eval(GbGv,Y=-(((1/2166)*sqrt(448189)+583/2166)*Z)));
```

$$\left[0, -\frac{1}{2166}\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} - 3822990\, X - 587953\, Z + 2822298\right) Z, \frac{1}{4332}\left(\sqrt{448189} + 583\right)\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} - 3822990\, X - 587953\, Z + 2822298\right) Z, -\frac{1}{4332}\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} + 3822990\, X - 587953\, Z - 2822298\right)\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} - 3822990\, X - 587953\, Z + 2822298\right)\right] \quad \textbf{(2.6)}$$

```
> map(factor,eval(GbGv,Y=-((-(1/2166)*sqrt(448189)+583/2166)*Z)));
```

$$\left[0, \frac{1}{2166}\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} + 3822990\, X + 587953\, Z - 2822298\right) Z, \frac{1}{4332}\left(-583 + \sqrt{448189}\right)\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} + 3822990\, X + 587953\, Z - 2822298\right) Z, -\frac{1}{4332}\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} - 3822990\, X + 587953\, Z + 2822298\right)\left(329\, Z\sqrt{448189} + 2166\sqrt{448189} + 3822990\, X + 587953\, Z - 2822298\right)\right] \quad \textbf{(2.7)}$$

The two transversals intersecting the two lines

```
> M:=[Z*sqrt(448189)+2166*Y+583*Z, 329*Z*sqrt(448189)+2166*sqrt
  (448189)-3822990*X-587953*Z+2822298];
```

$$M := \left[Z\sqrt{448189} + 2166\, Y + 583\, Z, 329\, Z\sqrt{448189} + 2166\sqrt{448189} - 3822990\, X - 587953\, Z + 2822298\right] \quad \textbf{(2.8)}$$

```
> N:=[Z*sqrt(448189)-2166*Y-583*Z, 329*Z*sqrt(448189)+2166*sqrt
  (448189)+3822990*X+587953*Z-2822298];
```

$$N := \left[Z\sqrt{448189} - 2166\, Y - 583\, Z, 329\, Z\sqrt{448189} + 2166\sqrt{448189} + 3822990\, X \quad \textbf{(2.9)}\right.$$

$$+587953\,Z-2822298]$$

Alternative approach to find the two transversals

```
> PolynomialIdeals[PrimeDecomposition](PolynomialIdeals[PolynomialIdeal](GbGv),sqrt(448189));
```

$$\langle -Z\sqrt{448189}+2166\,Y+583\,Z,\ 329\,Z\sqrt{448189}+2166\sqrt{448189}+3822990\,X+587953\,Z-2822298\rangle,\ \langle Z\sqrt{448189}+2166\,Y+583\,Z,\ -329\,Z\sqrt{448189}+3822990\,X+587953\,Z-2166\sqrt{448189}-2822298\rangle \quad (2.10)$$

Avoiding these singualrities such that the variety V(G) is null

```
> GbG2:=Groebner[Basis](G,plex(X,Y,Z)): nops(%);
```

$$4 \quad (2.11)$$

```
> cf:=collect(GbG2[1],{Y},factor,distributed);
```

$$cf := -d4\,s2\,(d1\,d2\,d3-d1\,d2\,s3-d1\,s3\,s5-d2\,d3\,d5+d2\,d3\,s6+d2\,d5\,s3-d2\,s3\,s6-d3\,s4\,s5)\,Z^2+(d1\,d2\,d3\,s2\,s6-d1\,d2\,d4\,s2\,s4-d1\,d2\,s2\,s3\,s6+d1\,d3\,d4\,s1\,s4+d1\,d4\,d5\,s1\,s3-d1\,d4\,s1\,s3\,s6-d1\,d4\,s2\,s4\,s5-d1\,d5\,s2\,s3\,s5+d2\,d4\,d5\,s2\,s4-d2\,d4\,s2\,s4\,s6-d3\,d5\,s2\,s4\,s5)\,Y\,Z+d1\,s4\,(d2\,s2\,s6-d3\,s1\,s6-d4\,d5\,s1+d4\,s1\,s6+d5\,s2\,s5)\,Y^2 \quad (2.12)$$

```
> cf2:=coeff(cf,Y,2); cf1:=coeff(cf,Y,1); cf0:=coeff(cf,Y,0);
```

$$cf2 := d1\,s4\,(d2\,s2\,s6-d3\,s1\,s6-d4\,d5\,s1+d4\,s1\,s6+d5\,s2\,s5)$$

$$cf1 := (d1\,d2\,d3\,s2\,s6-d1\,d2\,d4\,s2\,s4-d1\,d2\,s2\,s3\,s6+d1\,d3\,d4\,s1\,s4+d1\,d4\,d5\,s1\,s3-d1\,d4\,s1\,s3\,s6-d1\,d4\,s2\,s4\,s5-d1\,d5\,s2\,s3\,s5+d2\,d4\,d5\,s2\,s4-d2\,d4\,s2\,s4\,s6-d3\,d5\,s2\,s4\,s5)\,Z$$

$$cf0 := -d4\,s2\,(d1\,d2\,d3-d1\,d2\,s3-d1\,s3\,s5-d2\,d3\,d5+d2\,d3\,s6+d2\,d5\,s3-d2\,s3\,s6-d3\,s4\,s5)\,Z^2 \quad (2.13)$$

```
> Disc:=factor(cf1^2-4*cf2*cf0)/Z^2; #We have no one dimensional singularities if Disc<0 (i.e. when the four lines belong to an elliptic congruence).
```

$$Disc := d1^2\,d2^2\,d3^2\,s2^2\,s6^2+2\,d1^2\,d2^2\,d3\,d4\,s2^2\,s4\,s6-2\,d1^2\,d2^2\,d3\,s2^2\,s3\,s6^2+d1^2\,d2^2\,d4^2\,s2^2\,s4^2-2\,d1^2\,d2^2\,d4\,s2^2\,s3\,s4\,s6+d1^2\,d2^2\,s2^2\,s3^2\,s6^2-2\,d1^2\,d2\,d3^2\,d4\,s1\,s2\,s4\,s6-4\,d1^2\,d2\,d3\,d4^2\,d5\,s1\,s2\,s4-2\,d1^2\,d2\,d3\,d4^2\,s1\,s2\,s4^2+4\,d1^2\,d2\,d3\,d4^2\,s1\,s2\,s4\,s6+2\,d1^2\,d2\,d3\,d4\,d5\,s1\,s2\,s3\,s6+4\,d1^2\,d2\,d3\,d4\,d5\,s2^2\,s4\,s5+2\,d1^2\,d2\,d3\,d4\,s1\,s2\,s3\,s4\,s6-2\,d1^2\,d2\,d3\,d4\,s1\,s2\,s3\,s6^2-2\,d1^2\,d2\,d3\,d4\,s2^2\,s4\,s5\,s6-2\,d1^2\,d2\,d3\,d5\,s2^2\,s3\,s5\,s6+2\,d1^2\,d2\,d4^2\,d5\,s1\,s2\,s3\,s4-2\,d1^2\,d2\,d4^2\,s1\,s2\,s3\,s4\,s6+2\,d1^2\,d2\,d4^2\,s2^2\,s4^2\,s5-2\,d1^2\,d2\,d4\,d5\,s1\,s2\,s3^2\,s6-2\,d1^2\,d2\,d4\,d5\,s2^2\,s3\,s4\,s5+2\,d1^2\,d2\,d4\,s1\,s2\,s3^2\,s6^2-2\,d1^2\,d2\,d4\,s2^2\,s3\,s4\,s5\,s6+2\,d1^2\,d2\,d5\,s2^2\,s3^2\,s5\,s6+d1^2\,d3^2\,d4^2\,s1^2\,s4^2+2\,d1^2\,d3\,d4^2\,d5\,s1^2\,s3\,s4-2\,d1^2\,d3\,d4^2\,s1^2\,s3\,s4\,s6-2\,d1^2\,d3\,d4^2\,s1\,s2\,s4^2\,s5-2\,d1^2\,d3\,d4\,d5\,s1\,s2\,s3\,s4\,s5+4\,d1^2\,d3\,d4\,s1\,s2\,s3\,s4\,s5\,s6+d1^2\,d4^2\,d5^2\,s1^2\,s3^2-2\,d1^2\,d4^2\,d5\,s1^2\,s3^2\,s6+2\,d1^2\,d4^2\,d5\,s1\,s2\,s3\,s4\,s5+d1^2\,d4^2\,s1^2\,s3^2\,s6^2-2\,d1^2\,d4^2\,s1\,s2\,s3\,s4\,s5\,s6+d1^2\,d4^2\,s2^2\,s4^2\,s5^2-2\,d1^2\,d4\,d5^2\,s1\,s2\,s3^2\,s5+2\,d1^2\,d4\,d5\,s1\,s2\,s3^2\,s5\,s6-2\,d1^2\,d4\,d5\,s2^2\,s3\,s4\,s5^2 \quad (2.14)$$

$$+ d1^2\, d5^2\, s2^2\, s3^2\, s5^2 - 2\, d1\, d2^2\, d3\, d4\, d5\, s2^2\, s4\, s6 + 2\, d1\, d2^2\, d3\, d4\, s2^2\, s4\, s6^2$$
$$- 2\, d1\, d2^2\, d4^2\, d5\, s2^2\, s4^2 + 2\, d1\, d2^2\, d4^2\, s2^2\, s4^2\, s6 + 2\, d1\, d2^2\, d4\, d5\, s2^2\, s3\, s4\, s6$$
$$- 2\, d1\, d2^2\, d4\, s2^2\, s3\, s4\, s6^2 + 4\, d1\, d2\, d3^2\, d4\, d5\, s1\, s2\, s4\, s6 - 4\, d1\, d2\, d3^2\, d4\, s1\, s2\, s4\, s6^2$$
$$- 2\, d1\, d2\, d3^2\, d5\, s2^2\, s4\, s5\, s6 + 4\, d1\, d2\, d3\, d4^2\, d5^2\, s1\, s2\, s4 + 2\, d1\, d2\, d3\, d4^2\, d5\, s1\, s2\, s4^2$$
$$- 8\, d1\, d2\, d3\, d4^2\, d5\, s1\, s2\, s4\, s6 - 2\, d1\, d2\, d3\, d4^2\, s1\, s2\, s4^2\, s6 + 4\, d1\, d2\, d3\, d4^2\, s1\, s2\, s4\, s6^2$$
$$- 4\, d1\, d2\, d3\, d4\, d5^2\, s2^2\, s4\, s5 - 4\, d1\, d2\, d3\, d4\, d5\, s1\, s2\, s3\, s4\, s6$$
$$+ 2\, d1\, d2\, d3\, d4\, d5\, s2^2\, s4^2\, s5 + 4\, d1\, d2\, d3\, d4\, d5\, s2^2\, s4\, s5\, s6 + 4\, d1\, d2\, d3\, d4\, s1\, s2\, s3\, s4\, s6^2$$
$$- 4\, d1\, d2\, d3\, d4\, s2^2\, s4^2\, s5\, s6 + 2\, d1\, d2\, d3\, d5\, s2^2\, s3\, s4\, s5\, s6 - 2\, d1\, d2\, d4^2\, d5^2\, s1\, s2\, s3\, s4$$
$$+ 4\, d1\, d2\, d4^2\, d5\, s1\, s2\, s3\, s4\, s6 - 2\, d1\, d2\, d4^2\, d5\, s2^2\, s4^2\, s5 - 2\, d1\, d2\, d4^2\, s1\, s2\, s3\, s4\, s6^2$$
$$+ 2\, d1\, d2\, d4^2\, s2^2\, s4^2\, s5\, s6 + 2\, d1\, d2\, d4\, d5^2\, s2^2\, s3\, s4\, s5 - 2\, d1\, d2\, d4\, d5\, s2^2\, s3\, s4\, s5\, s6$$
$$- 2\, d1\, d3^2\, d4\, d5\, s1\, s2\, s4^2\, s5 + 4\, d1\, d3^2\, d4\, s1\, s2\, s4^2\, s5\, s6 + 4\, d1\, d3\, d4^2\, d5\, s1\, s2\, s4^2\, s5$$
$$- 4\, d1\, d3\, d4^2\, s1\, s2\, s4^2\, s5\, s6 - 2\, d1\, d3\, d4\, d5^2\, s1\, s2\, s3\, s4\, s5$$
$$+ 2\, d1\, d3\, d4\, d5\, s1\, s2\, s3\, s4\, s5\, s6 - 2\, d1\, d3\, d4\, d5\, s2^2\, s4^2\, s5^2 + 2\, d1\, d3\, d5^2\, s2^2\, s3\, s4\, s5^2$$
$$+ d2^2\, d4^2\, d5^2\, s2^2\, s4^2 - 2\, d2^2\, d4^2\, d5\, s2^2\, s4^2\, s6 + d2^2\, d4^2\, s2^2\, s4^2\, s6^2$$
$$- 2\, d2\, d3\, d4\, d5^2\, s2^2\, s4^2\, s5 + 2\, d2\, d3\, d4\, d5\, s2^2\, s4^2\, s5\, s6 + d3^2\, d5^2\, s2^2\, s4^2\, s5^2$$

Example

```
> val;
```

$$\{d1 = 2, d2 = 3, d3 = 5, d4 = 1, d5 = 7, d6 = -2, s1 = 4, s2 = -5, s3 = 7, s4 = 3, s5 = -2, s6 = 13\} \quad \textbf{(2.15)}$$

```
> v2:={d1 = 2, d2 = 3, d3 = 5, d4 = 1, d5 = 7, d6 = -2, s1 = 4, s2
  = -5, s3 = 7, s4 = 3, s5 = -2, s6=-9};
```

$$v2 := \{d1 = 2, d2 = 3, d3 = 5, d4 = 1, d5 = 7, d6 = -2, s1 = 4, s2 = -5, s3 = 7, s4 = 3, s5 = -2, s6 = -9\} \quad \textbf{(2.16)}$$

```
> eval(Disc,v2);
```

$$-434304 \quad \textbf{(2.17)}$$

```
> eval(G,v2): Gtemp:=Groebner[Basis](%,plex(X,Y,Z));
```

$$Gtemp := [963\, Y^2 - 1122\, Y Z + 355\, Z^2, 365\, X Z - 29\, Y Z + 100\, Z^2 + 642\, Y - 200\, Z, 351495\, X Y + 63762\, Y Z + 10295\, Z^2 + 527724\, Y - 227910\, Z, 128295675\, X^2 + 4641798\, Y Z - 9331445\, Z^2 + 122320260\, X - 102759804\, Y + 58813620\, Z + 40773420] \quad \textbf{(2.18)}$$

```
> allvalues(evala(AFactor(Gtemp[1]))); #Since the transversals are
  complex in this case, we avoid the one dimensional singularities.
```

$$963 \left(Y + \left(-\frac{2}{321}\, \mathrm{I}\sqrt{754} - \frac{187}{321}\right) Z\right) \left(Y + \left(\frac{2}{321}\, \mathrm{I}\sqrt{754} - \frac{187}{321}\right) Z\right), 963 \left(Y + \left(-\frac{2}{321}\, \mathrm{I}\sqrt{754} - \frac{187}{321}\right) Z\right) \left(Y + \left(\frac{2}{321}\, \mathrm{I}\sqrt{754} - \frac{187}{321}\right) Z\right) \quad \textbf{(2.19)}$$

3. Analsysis of the variety V(H) intersection V(K)

Groebner basis of H after substituting the parameters d_i and s_i

```
> GbHv:=Groebner[Basis](Hv,plex(X, Y, Z)): nops(%);
```

$$5 \quad \textbf{(3.1)}$$

The normal form of elements of K w.r.t the Groebner basis.

```
> for i from 1 to 6 do
  Kn||i:=Groebner[NormalForm](Kv[i],GbHv,plex(X, Y, Z));
  print(nops(Kn||i));
  od: # None of the normal forms are zero.
```

31

31

31

28

28

28 **(3.2)**

V(H) intersection V(K)

```
> HKv:=[op(Hv),op(Kv)]: indets(%);
```

$\{X, Y, Z\}$ **(3.3)**

```
> Groebner[HilbertDimension](HKv);
  Groebner[HilbertSeries](HKv): eval(%,_Z=1);
  #V(H) intersection V(K) is zero-dimensional, meaning that there are isolated points and there can be a maximum of 22 of them in the real domain R[X,Y,Z].
```

0

22 **(3.4)**

```
> Vhk:=RootFinding[Isolate](HKv,[X,Y,Z]); nops(%); # 16 solutions
```

$Vhk := [[X=-9.857670022, Y=-2.473392477, Z=-1.841223571], [X=-0.7203183666, Y=0., Z=0.], [X=-0.3202821945, Y=0.01051953565, Z=0.2205292468], [X=-0.007832845310, Y=0.009791056638, Z=2.000000000], [X=0.05411871357, Y=0.009243449917, Z=1.842242267], [X=0.3286775780, Y=0., Z=0.], [X=0.9179504186, Y=0.1418843101, Z=-2.082049581], [X=0.9387259984, Y=0.5680555079, Z=-2.022514069], [X=0.9719575977, Y=-1.214946997, Z=2.000000000], [X=1.011294049, Y=0.7947132746, Z=-0.8849672794], [X=1.016368185, Y=0.3715257653, Z=-1.983631815], [X=1.218211956, Y=0.3908940221, Z=-0.9183777123], [X=3.231256797, Y=0., Z=0.], [X=3.880385540, Y=7.054232928, Z=0.8803855404], [X=65.09432017, Y=-96.56905971, Z=-0.03639195019], [X=90.31279402, Y=-112.8909925, Z=2.000000000]]$

16 **(3.5)**

To remove the points that lie on the four observed lines or their transversals: Finding the ideal F.

```
> for i from 1 to 4 do
  F||i:=op(PolynomialIdeals[Generators](PolynomialIdeals[Saturate](PolynomialIdeals[PolynomialIdeal]([op(Kv),op(Hv)]),Gv[i]))):
  od:
> F:=[F1,F2,F3,F4]:
> Groebner[HilbertDimension](F);
  Groebner[HilbertSeries](F): eval(%,_Z=1);
  #V(F) is zero-dimensional and there can be a maximum of 10 of them in the real domain R[X,Y,Z].
```

$$0$$
$$10 \qquad (3.6)$$

```
> VF:=RootFinding[Isolate](F,[X,Y,Z]); nops(%); # 6 singular points
```

$VF := [[X=0.9387259984, Y=0.5680555079, Z=-2.022514069], [X=-9.857670022, Y=-2.473392477, Z=-1.841223571], [X=1.011294049, Y=0.7947132746, Z=-0.8849672794], [X=65.09432017, Y=-96.56905971, Z=-0.03639195019], [X=-0.3202821945, Y=0.01051953565, Z=0.2205292468], [X=0.05411871357, Y=0.009243449917, Z=1.842242267]]$

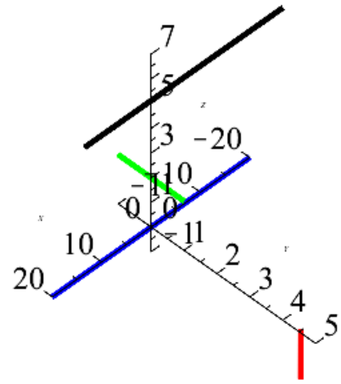
$$6 \qquad (3.7)$$

Results : Singularities in P4P include two lines that are transversal to the four observed lines and up to 10 isolated points. The one dimensional singularities can be avoided by forcing the four observed lines to be in an elliptic congruence there by making sure that their transversals are not real.

4. Singularities in P4L when the lines are subjected to orthogonality and parallelism constraints

Parametrization

```
> OP1:=Vector([0,0,0]): CP1:=OP1-OC: u1:=Vector([1,0,0]):
  OP2:=Vector([0,0,d1]): CP2:=OP2-OC: u2:=Vector([0,1,0]):
  OP3:=Vector([d2,d3,0]): CP3:=OP3-OC: u3:=Vector([0,0,1]):
  OP4:=Vector([0,d4,d5]): CP4:=OP4-OC: u4:=Vector([1,0,0]):
> n:=1:
> t1:=spacecurve(eval(OP1,val)+lambda*eval(u1,val),lambda=-20..20,
  color=blue,thickness=5,transparency=0):
  t2:=spacecurve(eval(OP2,val)+lambda*eval(u2,val),lambda=-n..n,
  color=green,thickness=5,transparency=0):
  t3:=spacecurve(eval(OP3,val)+lambda*eval(u3,val),lambda=-n..n,
  color=red,thickness=5,transparency=0):
  t4:=spacecurve(eval(OP4,val)+lambda*eval(u4,val),lambda=-20..20,
  color=black,thickness=5,transparency=0):
> display(t1,t2,t3,t4,labels=[X,Y,Z],axes=normal);
```



Interaction matrix and deriving ideals G, H and K55

```
> for i from 1 to 4 do
  f||i||1:=CrossProduct(u||i,CP||i):
  m||i||1:=CrossProduct(OP||i,f||i||1):
  f||i||2:=ZeroVector(3):
  m||i||2:=CrossProduct(u||i,f||i||1):
  zf||i:=Vector([Transpose(f||i||1),Transpose(m||i||1)]):
  zm||i:=Vector([Transpose(f||i||2),Transpose(m||i||2)]):
  od:
```

```
> zeta:=Transpose(Matrix([zf1,zf2,zf3,zf4,zm1,zm2,zm3,zm4]));
```

$$\zeta := [[0, Z, -Y, 0, 0, 0], [d1 - Z, 0, X, 0, d1\ (d1 - Z), 0], [-d3 + Y, d2 - X, 0, 0, 0, d2\ (d2 - X) - d3\ (-d3 + Y)], [0, -d5 + Z, d4 - Y, d4\ (d4 - Y) - d5\ (-d5 + Z), 0, 0], [0, 0, 0, 0, Y, Z], [0, 0, 0, X, 0, -d1 + Z], [0, 0, 0, -d2 + X, -d3 + Y, 0], [0, 0, 0, 0, -d4 + Y, -d5 + Z]] \tag{4.1}$$

```
> zf:=Transpose(Matrix([zf1,zf2,zf3,zf4])):
  zm:=Transpose(Matrix([zm1,zm2,zm3,zm4])):
```

28 minors of the interaction matrix

```
> ch:=combinat[choose](8,2); nops(ch);
```

$ch := [[1, 2], [1, 3], [1, 4], [1, 5], [1, 6], [1, 7], [1, 8], [2, 3], [2, 4], [2, 5], [2, 6], [2, 7], [2, 8], [3, 4], [3, 5], [3, 6], [3, 7], [3, 8], [4, 5], [4, 6], [4, 7], [4, 8], [5, 6], [5, 7], [5, 8], [6, 7], [6, 8], [7, 8]]$

$$28 \tag{4.2}$$

```
> for i from 1 to nops(ch) do
  zt||i:=DeleteRow(zeta,ch[i]):
  D||i:=factor(Determinant(DeleteRow(zeta,ch[i]))):
  od:
```

```
> I28:=[seq(D||i,i=1..28)]: nops(%);
```

$$28 \tag{4.3}$$

Determining the ideals G, H and K

```
> zf:=Transpose(Matrix([zf1,zf2,zf3,zf4])):
  zm:=Transpose(Matrix([zm1,zm2,zm3,zm4])):
> k:=1: #splitting the 16 minors into two factors to obtain ideals
  G and H
  for i from 1 to 4 do
  for j from 1 to 4 do
  zs:=Matrix([[DeleteRow(zf,i)],[DeleteRow(zm,j)]]):
  Ds||k:=factor(Determinant(zs)):
  Gs||k:=factor(Determinant(DeleteRow(zf[1..4,1..3],i))):
  Hs||k:=factor(Determinant(DeleteRow(zm[1..4,4..6],i))):
  k:=k+1:
  od:
  od:
  I16:=[seq(Ds||i,i=1..k-1)]:
  G16t:=[seq(Gs||i,i=1..k-1)]: ListTools[MakeUnique](G16t): G:=[seq
  (%[i],i=1..nops(%))]: nops(%);
  H16t:=[seq(Hs||i,i=1..k-1)]: ListTools[MakeUnique](H16t): H:=[seq
  (%[i],i=1..nops(%))]: nops(%);
```

$$4$$

$$4 \tag{4.4}$$

```
> K:=eval([seq(D||i,i=23..28)]): #The remaining 6 minors that
  cannot be factorized.
```

Analysis of the variety V(G)

```
> GbG:=Groebner[Basis](G,tdeg(X,Y,Z)); # No singularities in the
  generic case
```

$$GbG := \left[Y Z d5 - Y d1\, d5 - Z^2 d4 + Z d1\, d4,\ (d1\, d4 - d3\, d5)\, X Z + \left(-d1^2 d4 + d1\, d3\, d5 \right) X + Z^2 d2\, d4 - 2 Z d1\, d2\, d4 + d1^2 d2\, d4,\ Y^2 d5^2 + \left(-d1\, d4\, d5 - d3\, d5^2 \right) Y - Z^2 d4^2 + \left(d1\, d4^2 + d3\, d4\, d5 \right) Z,\ \left(d1\, d4\, d5 - d3\, d5^2 \right) X Y + \left(-d1\, d3\, d4\, d5 + d3^2 d5^2 \right) X + Z^2 d2\, d4^2 + \left(-d1\, d2\, d4^2 - d2\, d3\, d4\, d5 \right) Z + d1\, d2\, d3\, d4\, d5 \right] \tag{4.5}$$

```
> Pd:=PolynomialIdeals[PrimeDecomposition](PolynomialIdeals
  [PolynomialIdeal](GbG,variables={X,Y,Z}));
```

$$Pd := \langle -d1 + Z, -d3 + Y \rangle, \langle (d1\, d4 - d3\, d5)\, X + Z d2\, d4 - d1\, d2\, d4, Y d5 - Z d4 \rangle, \langle X, -d1 + Z, Y d5 - d1\, d4 \rangle \tag{4.6}$$

Two transversals

```
> L:=[op(PolynomialIdeals[Generators](Pd[1]))];
```

$$L := [-d1 + Z, -d3 + Y] \tag{4.7}$$

```
> M:=[op(PolynomialIdeals[Generators](Pd[2]))];
```

$$M := [(d1\, d4 - d3\, d5)\, X + Z d2\, d4 - d1\, d2\, d4, Y d5 - Z d4] \tag{4.8}$$

Avoiding these singualrities such that the variety V(G) is null

```
> cf:=collect(GbG[1],{Z},factor,distributed);
```

$$cf := -Z^2 d4 + (Y d5 + d1\, d4)\, Z - Y d1\, d5 \tag{4.9}$$

```
> cf2:=coeff(cf,Z,2); cf1:=coeff(cf,Z,1); cf0:=coeff(cf,Z,0);
```

$$cf2 := -d4$$

$$cf1 := Y d5 + d1\, d4$$

(4.10)

$$cf0 := -Y\,d1\,d5 \tag{4.10}$$

```
> Disc:=factor(cf1^2-4*cf2*cf0); #We cannot avoid one dimensional singularities since Disc>=0
```

$$Disc := (Y\,d5 - d1\,d4)^2 \tag{4.11}$$

Analysis of the variety V(H) intersection V(K)

```
> HK:=[op(H),op(K)]: indets(%);
```

$$\{X, Y, Z, d1, d2, d3, d4, d5\} \tag{4.12}$$

```
> Groebner[HilbertDimension](HK,plex(X,Y,Z));
```

$$0 \tag{4.13}$$

```
> Ghk:=map(factor,Groebner[Basis](HK,plex(X,Y,Z)));
```

$$Ghk := \big[Z\,(-d5+Z)\,(Z\,d4 - d3\,d5)\,(-d1+Z),\ Y\,d5 - Z\,d4,\ X\,d1^3\,d4^3 + X\,d1^2\,d3\,d4^2\,d5 - 2\,X\,d1^2\,d4^3\,d5 - X\,d1\,d3^2\,d4\,d5^2 - X\,d3^3\,d5^3 + 2\,X\,d3^2\,d4\,d5^3 + 4\,Z^3\,d2\,d4^3 - 2\,Z^2\,d1\,d2\,d4^3 - 2\,Z^2\,d2\,d3\,d4^2\,d5 - 4\,Z^2\,d2\,d4^3\,d5 - Z\,d1^2\,d2\,d4^3 + 2\,Z\,d1\,d2\,d3\,d4^2\,d5 + 2\,Z\,d1\,d2\,d4^3\,d5 - Z\,d2\,d3^2\,d4\,d5^2 + 2\,Z\,d2\,d3\,d4^2\,d5^2 - d1^3\,d2\,d4^3 + 2\,d1^2\,d2\,d4^3\,d5 + d1\,d2\,d3^2\,d4\,d5^2 - 2\,d1\,d2\,d3\,d4^2\,d5^2\big] \tag{4.14}$$

```
> solve(Ghk,{X,Y,Z}); # The variety consists of 4 points
```

$$\left\{X = \frac{d1\,d2\,d4}{d1\,d4 + d3\,d5},\ Y=0,\ Z=0\right\},\ \left\{X = \frac{(d1-d5)\,d2\,d4}{d1\,d4 + d3\,d5 - 2\,d4\,d5},\ Y=d4,\ Z=d5\right\},\ \left\{X=0,\ Y = \frac{d1\,d4}{d5},\ Z=d1\right\},\ \left\{X=d2,\ Y=d3,\ Z=\frac{d5\,d3}{d4}\right\} \tag{4.15}$$

To remove the points that lie on the four observed lines or their transversals: Finding the ideal F

```
> for i from 1 to 4 do
  F||i:=op(PolynomialIdeals[Generators](PolynomialIdeals[Saturate](PolynomialIdeals[PolynomialIdeal]([op(K),op(H)]),G[i]))):
  od:
> F:=[F1,F2,F3,F4];
```

$$F := \big[Y\,d4^2 + Y\,d5^2,\ -Y\,d5 + Z\,d4,\ Y\,d4 + Z\,d5,\ -2\,X\,Y\,Z + X\,Y\,d1 + X\,Z\,d3 + Y\,Z\,d2 - Y\,d1\,d2,\ -2\,X\,Y\,d5 + X\,d1\,d4 + X\,d3\,d5 + Y\,d2\,d5 - d1\,d2\,d4,\ 2\,X\,Y^2\,d4 + X\,Y\,d1\,d5 - X\,Y\,d3\,d4 - Y^2\,d2\,d4 - Y\,d1\,d2\,d5,\ 4\,X^2\,Y^3\,d4 - 4\,X^2\,Y^2\,d3\,d4 + X^2\,Y\,d1^2\,d4 + X^2\,Y\,d3^2\,d4 - 4\,X\,Y^3\,d2\,d4 + 2\,X\,Y^2\,d2\,d3\,d4 - 2\,X\,Y\,d1^2\,d2\,d4 + Y^3\,d2^2\,d4 + Y\,d1^2\,d2^2\,d4,\ 4\,X\,Y^3\,d4 - 4\,X\,Y^2\,d3\,d4 + X\,Y\,d1^2\,d4 + X\,Y\,d3^2\,d4 - 2\,Y^3\,d2\,d4 - Y^2\,d1\,d2\,d5 + Y^2\,d2\,d3\,d4 - Y\,d1^2\,d2\,d4 + Y\,d1\,d2\,d3\,d5,\ X,\ d2,\ X,\ d2,\ -Y\,d5 + Z\,d4,\ -Y\,d4 - Z\,d5 + d4^2 + d5^2,\ -2\,X\,Y\,Z + X\,Y\,d1 + X\,Z\,d3 + Y\,Z\,d2 - Y\,d1\,d2,\ -2\,X\,Y\,d5 + X\,d1\,d4 + X\,d3\,d5 + Y\,d2\,d5 - d1\,d2\,d4,\ -2\,X\,Y^2\,d4 - X\,Y\,d1\,d5 + X\,Y\,d3\,d4 + 2\,X\,Y\,d4^2 + X\,d1\,d4\,d5 - X\,d3\,d4^2 + Y^2\,d2\,d4 + Y\,d1\,d2\,d5 - Y\,d2\,d4^2 - d1\,d2\,d4\,d5,\ -4\,X^2\,Y^3\,d4 + 4\,X^2\,Y^2\,d3\,d4 + 4\,X^2\,Y^2\,d4^2 - X^2\,Y\,d1^2\,d4 - X^2\,Y\,d3^2\,d4 - 4\,X^2\,Y\,d3\,d4^2 + X^2\,d1^2\,d4^2 + X^2\,d3^2\,d4^2 + 4\,X\,Y^3\,d2\,d4 - 2\,X\,Y^2\,d2\,d3\,d4 - 4\,X\,Y^2\,d2\,d4^2 + 2\,X\,Y\,d1^2\,d2\,d4 + 2\,X\,Y\,d2\,d3\,d4^2 - 2\,X\,d1^2\,d2\,d4^2 - Y^3\,d2^2\,d4 + Y^2\,d2^2\,d4^2 - Y\,d1^2\,d2^2\,d4 + d1^2\,d2^2\,d4^2,\ -4\,X\,Y^3\,d4 + 4\,X\,Y^2\,d3\,d4 + 4\,X\,Y^2\,d4^2 - X\,Y\,d1^2\,d4 \tag{4.16}$$

$$- X\,Y\,d3^2\,d4 - 4\,X\,Y\,d3\,d4^2 + X\,d1^2\,d4^2 + X\,d3^2\,d4^2 + 2\,Y^3\,d2\,d4 + Y^2\,d1\,d2\,d5$$
$$- Y^2\,d2\,d3\,d4 - 2\,Y^2\,d2\,d4^2 + Y\,d1^2\,d2\,d4 - Y\,d1\,d2\,d3\,d5 - Y\,d1\,d2\,d4\,d5 + Y\,d2\,d3\,d4^2$$
$$- d1^2\,d2\,d4^2 + d1\,d2\,d3\,d4\,d5\,]$$

```
> Groebner[Basis](F,plex(X,Y,Z)); # The four points in the variety V(H) intersection V(K) lie on the four observed lines or their transversals which we already know to be singularities. Thus, there are no isolated singularities in this case.
```

$$[1] \tag{4.17}$$

Results: For the special case of P4L with orthogonality and parallelism constraints, generically, the only singularities are when the camera center lies on the four observed lines or their two transversal lines.