SPREADING OUT THE HODGE FILTRATION IN RIGID ANALYTIC GEOMETRY

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ABSTRACT.

CONTENTS

1. INTRODUCTION

In this paper, we will provide a rigid analytic construction of the deformation to the normal cone, studied in [6]. Our goal is to use this geometric construction to deduce certain important results concerning both *rigid analytic* and *over-convergent* (Hodge complete) *derived de Rham cohomology* of rigid analytic spaces over a non-archimedean field of characteristic zero. We will then exploit this ideas to come up with analogues concerning *derived rigid cohomology* of finite type schemes over a perfect field in characteristic zero. In particular, our main goal is to extrapolate the main result of [4] to the setting of derived rigid cohomology.

1.1. **Notations and Conventions.** We shall denote the *analytic mapping stack* as $\mathbf{Map}_{\mathrm{dAnSt}_k}(X, Y)$.

1.2. **Preliminaries.** Let $\mathcal{X}$ be an ∞-topos. The notion of a *local* $\mathcal{T}_{\mathrm{an}}(k)$*-structure on* $\mathcal{X}$ was first introduced in [14, Definition 2.4], see also [1, §2].

Let $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ be a local $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$. Since the pregeometry $\mathcal{T}_{\mathrm{an}}(k)$ is compatible with n-truncations, cf. [14, Theorem 3.23], it follows that $\pi_0(\mathcal{O}) \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, as well.

Denote by $\mathcal{J} \subseteq \pi_0(\mathcal{O})$, the *Jacobson ideal* of $\pi_0(\mathcal{O}^{\mathrm{alg}})$, which can be naturally regarded as an object in the ∞-category

$$\mathrm{Mod}_{\pi_0(\mathcal{O}^{\mathrm{alg}})} \simeq \mathrm{Mod}_{\pi_0(\mathcal{O})},$$

for a justification of the latter equivalence, see for instance [12, Theorem 4.5] . Since the ∞-category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ is a presentable ∞-category we can consider the quotient

$$\pi_0(\mathcal{O})_{\mathrm{red}} \coloneqq \pi_0(\mathcal{O})/\mathcal{J} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}),$$

which we refer to the *reduced* $\mathcal{T}_{\mathrm{an}}(k)$*-structure on* $\mathcal{X}$ *associated to* $\pi_0(\mathcal{O})$. Moreover, the corresponding *underlying algebra* satisfies

$$(\pi_0(\mathcal{O})_{\mathrm{red}})^{\mathrm{alg}} \simeq \pi_0(\mathcal{O})^{\mathrm{alg}}/\mathcal{J} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

One can further prove that $\pi_0(\mathcal{O})_{\mathrm{red}} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ actually lies in the full subcategory $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$.

Definition 1.1. Let $Z = (\mathcal{Z}, \mathcal{O}_Z) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote a $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos. We define the *reduced* $\mathcal{T}_{\mathrm{an}}(k)$*-structure* ∞*-topos* as

$$Z_{\mathrm{red}} \coloneqq (\mathcal{Z}, \pi_0(\mathcal{O}_Z)_{\mathrm{red}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

We shall denote by $\mathrm{Afd}_k^{\mathrm{red}}$ (resp., $\mathrm{An}_k^{\mathrm{red}}$) the full subcategory of dAfd_k (resp., dAn_k) spanned by reduced k-affinoid (resp., k-analytic spaces).

Notation 1.2. Let $(-)^{\mathrm{red}} \colon \mathrm{dAn}_k \to \mathrm{An}_k^{\mathrm{red}}$ denote the functor obtained by the formula

$$Z = (\mathcal{Z}, \mathcal{O}_Z) \in \mathrm{dAn}_k \mapsto Z_{\mathrm{red}} = (\mathcal{Z}, \pi_0(\mathcal{Z})_{\mathrm{red}}) \in \mathrm{An}_k^{\mathrm{red}}.$$

We shall refer to it as the *underlying reduced* k*-analytic space.*

Definition 1.3. Let $f \colon X \to Y$ be a morphism in dAn_k. We shall say that f is an *admissible open immersion* if the induced morphism on 0-th truncations

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y),$$

is an admissible open immersion in the sense of [3, §1.3].

Lemma 1.4. *Let* $f \colon X \to Y$ *be an admissible open immersion of derived* k*-analytic spaces. Then* $f^{\mathrm{red}} \colon X^{\mathrm{red}} \to Y^{\mathrm{red}}$ *is also an admissible open immersion.*

Proof. By the definitions, it is clear that the truncation

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y),$$

is an admissible open immersion of ordinary k-analytic spaces. In the case of ordinary k-analytic spaces it is clear from the construction that the reduction of Zariski open immersions is again a Zariski open immersion. $\square$

Definition 1.5. In [12, Definition 5.41] the authors introduced the notion of a square-zero extension between $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi. In particular, given a morphism $f \colon Z \to Z'$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, we shall say that f *has the structure of a square-zero extension* if f exhibits Z' as a square-zero extension of Z.

Recall the definition of the ∞-categories of derived k-affinoid and derived k-analytic spaces given in [14, Definition 7.3 and Definition 2.5.], respectively.

Remark 1.6. Let $X \in \mathrm{An}_k$. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be an ideal satisfying $\mathcal{J}^2 = 0$. Consider the fiber sequence

$$\mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J},$$

in the ∞-category $\mathrm{Coh}^+(X)$. We have a fiber sequence of the form

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{J}} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{J}/\mathcal{O}_X},$$

and we have a further identification $\tau_{\leq 1}(\mathbb{L}_{\mathcal{O}_X/\mathcal{J}/\mathcal{O}_X}) \simeq \mathcal{J}[1]$. For this reason, we obtain a well defined morphism

$$d\colon \mathbb{L}_{\mathcal{O}_X/\mathcal{J}} \to \mathcal{J}[1],$$

in the derived ∞-category $\mathrm{Mod}_{\mathcal{O}_X/\mathcal{J}}$. This derivation classifies a square-zero extension of $\mathcal{O}_X/\mathcal{J}$ by $\mathcal{J}[1]$ which can be identified with the object $\mathcal{O}_X$ itself. In particular, we deduce that X is a square-zero extension of X_{red}.

Lemma 1.7. *Let $Z \coloneqq (\mathcal{Z}, \mathcal{O}_Z) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote a $\mathcal{T}_{\mathrm{an}}(k)$-structure ∞-topos such that $\pi_0(\mathcal{O}_Z^{\mathrm{alg}})$ is Noetherian derived k-algebra on $\mathcal{Z}$. Suppose that the reduction Z_{red} is equivalent to a derived k-affinoid space. Then the truncation $\mathrm{t}_0(Z)$ is isomorphic to an ordinary k-affinoid space. If we assume further that for every $i > 0$, the homotopy sheaves $\pi_i(\mathcal{O}_Z)$ are coherent $\pi_0(\mathcal{O}_Z)$-modules, then Z itself is equivalent to a derived k-affinoid space.*

Proof. The second claim of the Lemma follows readily from the first assertion together with the definitions. We are thus reduced to prove that $\mathrm{t}_0(Z)$ is isomorphic to an ordinary k-affinoid space. Let $\mathcal{J} \subseteq \pi_0(\mathcal{O}_Z)$, denote the coherent ideal sheaf associated to the closed immersion $Z_{\mathrm{red}} \hookrightarrow Z$. Notice that the ideal $\mathcal{J}$ agrees with the Jacobson ideal of $\pi_0(\mathcal{O}_Z)$. Thanks to our assumption that $\pi_0(\mathcal{O}_Z)$ is a Noetherian derived k-algebra on $\mathcal{Z}$, it follows that there exists a sufficiently large integer $n \geq 2$ such that

$$\mathcal{J}^n = 0.$$

Arguing by induction, we can suppose that $n = 2$, that is to say that

$$\mathcal{J}^2 = 0.$$

In particular, Remark 1.6 implies that the the natural morphism $Z_{\mathrm{red}} \to Z$ has the structure of a square zero extension. The assertion now follows from [12, Proposition 6.1] and its proof. □

Remark 1.8. We observe that the converse of Lemma 1.7 holds true. Indeed, the natural morphism $Z_{\mathrm{red}} \to Z$ is a closed immersion. In particular, if $Z \in \mathrm{dAfd}_k$ we deduce readily from that $Z_{\mathrm{red}} \in \mathrm{dAfd}_k$, as well.

Definition 1.9. Let $f\colon X \to Y$ be a morphism in the ∞-category dAn_k. We shall say that f is an *affine morphism* if for every morphism $Z \to Y$ in dAn_k such that Z is equivalent to a derived k-affinoid space, the pullback

$$Z' \coloneqq Z \times_Y X \in \mathrm{dAn}_k,$$

is also equivalent to a derived k-affinoid space.

Lemma 1.10. *Let $f\colon X \to Y$ be an affine morphism in dAn_k. Suppose we are given an admissible open covering*

$$g\colon \coprod_{j \in J} U_j \to Y,$$

where for each $j \in J$, $U_j \in \mathrm{dAfd}_k$. For each $j \in J$, let

$$V_j \coloneqq U_j \times_X Y \in \mathrm{dAfd}_k,$$

then $\coprod_{j \in J} V_j \to Y$ is an admissible open covering.

Proof. It is clear from our assumption that f is an affine morphism that for every index $j \in J$, the objects $V_j \in \mathrm{dAfd}_k$. The claim of the Lemma follows immediately from the observation that both the classes of effective epimorphisms of ∞-topoiand admissible open immersions of derived k-analytic spaces are stable under pullbacks, cf. [9, Proposition 6.2.3.15] and [12, Corollary 5.11, Proposition 5.12], respectively. □

2. Non-archimedean differential geometry

2.1. **Analytic formal moduli problems under a base.** In this §, we will study the notion of *analytic formal moduli problems under* a fixed derived k-analytic space. The results presented here will prove to be crucial for the study of the deformation to the normal cone in the k-analytic setting, presented in the next section. We start with the following definition:

Definition 2.1. (1) Let $f \colon X \to Y$ be a morphism in dAn_k. We say that f is a *nil-isomorphism* if $f_{\mathrm{red}} \colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is an isomorphism of k-analytic spaces.

(2) Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived k-analytic space. We shall say that a nil-isomorphism $f \colon X \to Y$ is *of finite type* if for every

$$\mathcal{O}_Y \to \mathcal{O}_X,$$

induces an equivalence $\tau_{\geq m}(\mathcal{O}_Y) \to \tau_{\geq m}(\mathcal{O}_X)$ in the ∞-category $\mathcal{D}_{\mathrm{Ab}}(\mathcal{X})$.

(3) We will denote by $\mathrm{AnNil}_{/X}$ the full subcategory of $(\mathrm{dAn}_k)_{X/}$ spanned by nil-isomorphisms $X \to Y$ of finite type.

Lemma 2.2. *Let $f \colon X \to Y$ be a nil-isomorphism of derived k-analytic spaces. Then:*

(1) Given any morphism $Z \to Y$ in dAn_k, the induced morphism

$$Z \times_X Y \to Z,$$

is again an nil-isomoprhism.

(2) f is an affine morphism.

(3) f is a finite morphism.

Proof. To prove (i), it suffices to prove that the functor $(\text{-})^{\mathrm{red}} \colon \mathrm{dAn}_k \to \mathrm{An}_k^{\mathrm{red}}$ commutes with finite limits. The truncation functor

$$\mathrm{t}_0 \colon \mathrm{dAn}_k \to \mathrm{An}_k,$$

commutes with finite limits, c.f. [14, Proposition 6.2 (v)]. It suffices then to prove that the usual underlying reduced functor

$$(-)^{\mathrm{red}} \colon \mathrm{An}_k \to \mathrm{An}_k^{\mathrm{red}},$$

commutes with finite limits. By construction, the latter assertion is equivalent to the claim that the usual complete tensor product of ordinary k-affinoid algebras commutes with the operation of taking the quotient by the Jacobson radical, which is immediate.

We now prove (ii). Let $Z \to Y$ be an admissible open immersion such that Z is a derived k-affinoid space. We claim that the pullback $Z \times_X Y$ is again a derived k-affinoid space. Thanks to Lemma 1.7 we reduced to prove that $(Z \times_X Y)_{\mathrm{red}}$ is equivalent to an ordinary k-affinoid space. Thanks to (i), we deduce that the induced morphism

$$(Z \times_X Y)_{\mathrm{red}} \to Z_{\mathrm{red}},$$

is an isomorphism of ordinary k-analytic spaces. In particular, $(Z \times_X Y)_{\mathrm{red}}$ is a k-affinoid space. The result now follows from Lemma 1.7.

To prove (iii), we shall show that the induced morphism on the 0-th truncations $\mathrm{t}_0(X) \to \mathrm{t}_0(Y)$ is a finite morphism of ordinary k-affinoid spaces. But this follows immeaditely from the fact that both $\mathrm{t}_0(X)$ and $\mathrm{t}_0(Y)$ can be obtained from the reduced X_{red} by means of a finite sequence of square-zero extensions as in Remark 1.6. □

Definition 2.3. A morphism $X \to Y$ be a morphism in dAn_k is called a *nil-embedding* if the induced morphism of ordinary k-analytic spaces $\mathrm{t}_0(X) \to \mathrm{t}_0(Y)$ is a closed immersion whose ideal of definition assumed to be nilpotent.

Proposition 2.4. *Let $f \colon X \to Y$ be a nil-embedding of derived k-analytic spaces. Then there exists a sequence of morphisms*

$$X = X_0^0 \hookrightarrow X_0^1 \hookrightarrow \cdots \hookrightarrow X_0^n = X_0 \hookrightarrow X_1 \dots X_n \hookrightarrow \cdots \hookrightarrow Y,$$

such that for each $0 \leq i \leq n$ and $j \geq 0$ the morphisms $X_0^i \hookrightarrow X_0^{i+1}$ and $X_j \to X_{j+1}$ have the structure of square-zero extensions. Furthermore, the induced morphisms $\mathrm{t}_{\leq j}(X_j) \to \mathrm{t}_{\leq j}(Y)$ are equivalences of derived k-analytic spaces, for every $j \geq 0$.

Proof. The proof follows the same scheme of reasoning as of [6, Proposition 5.5.3]. For the sake of completeness we present the complete argument here. Consider the induced morphism on the underlying truncations

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y).$$

By construction, there exists a sufficiently large integer $n \geq 0$ such that

$$\mathcal{J}^{n+1} = 0,$$

where $\mathcal{J} \subseteq \pi_0(\mathcal{O}_Y)$ denotes the ideal associated to the nil-embedding $\mathrm{t}_0(f)$. Therefore, we can factor the latter as a finite sequence of square-zero extensions of ordinary k-analytic spaces

$$\mathrm{t}_0(X) \hookrightarrow X_0^{\mathrm{ord},0} \hookrightarrow \cdots \hookrightarrow X_0^{\mathrm{ord},n} = \mathrm{t}_0(Y),$$

as in the proof of Lemma 1.7. For each $0 \leq i \leq n$, we set

$$X_0^i \coloneqq X \coprod_{\mathrm{t}_0(X)} X_0^{\mathrm{ord},i}.$$

By construction, we have that the natural morphism $\mathrm{t}_0(X_0^n) \to \mathrm{t}_0(Y)$ is an isomorphism of ordinary k-analytic spaces. We now argue by induction on the Postnikov tower associated to the morphism $f \colon X \to Y$. Suppose that for a certain integer $i \geq 0$, we have constructed a derived k-analytic space X_i together with morphisms $g_i \colon X \to X_i$ and $h_i \colon X_i \to Y$ such that $f \simeq h_i \circ g_i$ and the induced morphism

$$\mathrm{t}_{\leq i}(X_i) \to \mathrm{t}_{\leq i}(Y)$$

is an equivalence of derived k-analytic spaces. We shall proceed as follows: by the assumption that h_i is $(i+1)$-connective, we deduce from [12, Proposition 5.34] the existence of a natural equivalence

$$\tau_{\leq i}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}) \simeq 0,$$

in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. Consider the natural fiber sequence

$$h_i^* \mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_{X_i} \to \mathbb{L}^{\mathrm{an}}_{X_i/Y},$$

in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. The natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X_i/Y} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1],$$

induces a morphism $\mathbb{L}^{\mathrm{an}}_{X_i} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1]$, such that the composite

$$h_i^* \mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_{X_i} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1], \tag{2.1}$$

is null-homotopic, in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. By the universal property of the relative analytic cotangent complex, (2.1) produces a square-zero extension

$$X_i \to X_{i+1},$$

together with a morphism $h_{i+1}\colon X_{i+1} \to Y$, factoring $h_i\colon X_i \to Y$. We are reduced to show that the morphism

$$\mathcal{O}_Y \to h_{i+1,*}(\mathcal{O}_{X_{i+1}}),$$

is $(i+2)$-connective. Consider the commutative diagram

$$\begin{tikzcd} h_{i,*}(\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}))[i] \arrow[r] & h_{i+1,*}(\mathcal{O}_{X_{i+1}}) \arrow[r] & h_{i,*}(\mathcal{O}_{X_i}) \\ \mathcal{I} \arrow[u, "s_i"] \arrow[r] & \mathcal{O}_Y \arrow[u] \arrow[r] & h_*(\mathcal{O}_{X_i}) \arrow[u] \\ \mathcal{J} \arrow[u] \arrow[r, "="] & \mathcal{J} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}, \tag{2.2}$$

in $\mathrm{Mod}_{\mathcal{O}_Y}$, where both the vertical and horizontal composites are fiber sequences. By our inductive hypothesis, $\mathcal{J}$ is $(i+1)$-connective. Moreover, thanks to [12, Proposition 5.34] we can identify the natural morphism

$$s_i\colon \mathcal{I} \to h_{i,*}(\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}))[i]$$

with the natural morphism $\mathcal{I} \to \tau_{\leq i}(\mathcal{I})$. We deduce that the fiber of the morphism s_i must be necessarily $(i+2)$-connective. The latter observation combined with the structure of (2.2) implies that $h_{i+1}\colon X_{i+1} \to Y$ induces an equivalence of derived k-analytic spaces

$$\mathrm{t}_{\leq i+1}(X_{i+1}) \to \mathrm{t}_{\leq i+1}(Y),$$

as desired. □

Corollary 2.5. *The following assertions hold:*

(1) Let $X \in \mathrm{dAn}_k$. Then the natural morphism

$$X_{\mathrm{red}} \to X,$$

in dAn_k, can be approximated *by successive square zero extensions.*

(2) If $f\colon X \to Y$ is a nil-embedding of finite type then it can be approximated by a finite sequence of square-zero extensions.

Proof. Both the assertions of the Corollary follow readily from Proposition 2.4 by observing that the canonical morphism $X_{\mathrm{red}} \to X$ has the structure of a nil-embedding and that the finiteness assumption on f forces the finiteness of the approximation sequence. □

Lemma 2.6. *Let $f\colon X \to Y$ be a finite morphism of derived k-affinoid spaces. Let $Z \to Y$ be an admissible open immersion and denote by*

$$A \coloneqq \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X), \quad B \coloneqq \Gamma(Y, \mathcal{O}^{\mathrm{alg}}_Y), \quad C \coloneqq \Gamma(Z, \mathcal{O}^{\mathrm{alg}}_Z),$$

the corresponding derived k-algebras of derived global sections. Denote by

$$Z' \coloneqq Z \times_Y X \in \mathrm{dAfd}_k,$$

then one has a natural equivalence

$$\Gamma(Z', \mathcal{O}^{\mathrm{alg}}_{Z'}) \simeq A \otimes_B C,$$

in the ∞-category CAlg_k.

Proof. Consider the natural morphism of derived k-algebras

$$\theta \colon A \otimes_B \to C \to \Gamma(Z', \mathcal{O}_{Z'}^{\mathrm{alg}}).$$

Our goal is to show that θ is an equivalence. We start by observing that since the morphism $f \colon X \to Y$ is finite, the induced morphism $B \to A$ in CAlg_k is finite as well. In particular, the (ordinary) complete tensor product coincides with the (ordinary) algebraic tensor product, i.e., the natural morphism

$$\pi_0(A) \otimes_{\pi_0(B)} \pi_0(C) \to \pi_0(A) \widehat{\otimes}_{\pi_0(B)} \pi_0(C),$$

is an equivalence of ordinary rings. In particular, thanks to [14, Proposition 6.2 (v) and Theorem 6.5] we deduce that

$$\pi_0(A \otimes_B C) \to \pi_0(\Gamma(Z', \mathcal{O}_{Z'}^{\mathrm{alg}})),$$

is an equivalence of derived rings, as well. In order to conclude, we shall prove that for every $i > 0$, the natural morphism

$$\pi_i(A \otimes_B C) \to \pi_i(\Gamma(Z', \mathcal{O}_{Z'})),$$

is an equivalence as well. Since the morphism $f \colon X \to Y$ is a finite morphism we deduce that $\pi_i(\mathcal{O}_X)$ is a coherent $\pi_0(\mathcal{O}_Y)$-module, for every $i \geq 0$. By unwinding the definitions, we deduce then that we have a natural identification

$$\pi_i(\mathcal{O}_{Z'}) \simeq \pi_i 0(\mathcal{O}_Z) \otimes_{\pi_0(\mathcal{O}_Y)} \pi_i(\mathcal{O}_X),$$

in the ∞-category $\mathrm{Mod}_{\pi_0(\mathcal{O}'_Z)}$. The result now follows by taking global sections and analyzing the spectral sequence

$$\pi_i(\Gamma(Z', \pi_j(\mathcal{O}_{Z'})^{\mathrm{alg}})) \Rightarrow \pi_{i+j}(\Gamma(Z', \mathcal{O}_{Z'}^{\mathrm{alg}})),$$

whose existence is guaranteed by [10, 1.2.2.14]. □

Lemma 2.7. *Let $f \colon X \to Y$ be a finite morphism of derived k-affinoid spaces and $g \colon Z \to Y$ an admissible open immersion in dAfd_k. Form the pullback diagram*

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

in the ∞-category dAfd_k. Then the commutative diagram

$$\begin{tikzcd} \mathrm{Coh}^+(Y) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{Coh}^+(X) \arrow[d, "(g')^*"] \\ \mathrm{Coh}^+(Z) \arrow[r, "(f')^*"] & \mathrm{Coh}^+(Z'), \end{tikzcd}$$

is right adjointable. In other words, the Beck-Chevalley natural transformation

$$\alpha \colon g^* \circ f_* \to (f')_* \circ g'_*$$

is an equivalence of functors.

Proof. Since f is assumed to be a finite morphism of derived k-affinoid spaces, it follows from the derive Tate aciclicity theorem, c.f. [13, Theorem 3.1] that the right adjoint $f_* \colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(Y)$ is well defined. The assertion of the Lemma is now an immediate consequence of Lemma 2.6 together with [11, Proposition 2.5.4.5] and the derived Tate aciclicity theorem. □

Proposition 2.8. *Let $f\colon S \to S'$ be a nil-isomorphism between derived k-analytic spaces. Then the pullback functor*

$$f^*\colon \mathrm{Coh}^+(S') \to \mathrm{Coh}^+(S),$$

admits a well defined right adjoint, f_*.

Proof. Since $f\colon S \to S'$ is a nil-isomorphism, we conclude from Lemma 2.2 that f is an affine morphism between derived k-analytic spaces. By admissible descent of Coh^+, cf. [2, Theorem 3.7], together with Lemma 1.10 and Lemma 2.7 we reduce the statement of the Lemma to the case where both S and S' are equivalent to derived k-affinoid spaces. In this case, the result follows by our assumptions on f and Lemma 2.2. □

Corollary 2.9. *Let $f\colon S \to S'$ be a square-zero extension and $g\colon S \to T$ a nil-isomorphism in dAn_k. Suppose we are given a pushout diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd},$$

in dAn_k. Then the induced morphism $T \to T'$ is a square-zero extension.

Proof. Since g is a nil-isomorphism of derived k-analytic spaces, Proposition 2.8 implies that the pullback functor $g^*\colon \mathrm{Coh}^+(T) \to \mathrm{Coh}^+(S)$ admits a well defined right adjoint

$$g_*\colon \mathrm{Coh}^+(S) \to \mathrm{Coh}^+(T).$$

Let $\mathcal{F} \in \mathrm{Coh}^+(S)^{\geq 0}$ and $d\colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}[1]$ be a derivation associated with the morphism $f\colon S \to S'$. Consider now the natural composite

$$d'\colon \mathbb{L}^{\mathrm{an}}_T \to g_*(\mathbb{L}^{\mathrm{an}}_S) \xrightarrow{g_*(d)} g_*(\mathcal{F})[1],$$

in the ∞-category $\mathrm{Coh}^+(T)$. By the universal property of the relative analytic cotangent complex, we deduce the existence of a square-zero extension

$$T \to T',$$

in the ∞-category dAn_k. Let $X \in \mathrm{dAn}_k$ together with morphisms $S' \to X$ and $T \to X$ compatible with both f and g. By the universal property of the relative analytic cotangent complex, the morphism $S' \to X$ induces a uniquely defined (up to a contractible indeterminacy space) morphism

$$\mathbb{L}^{\mathrm{an}}_{S/X} \to \mathcal{F}[1],$$

in $\mathrm{Coh}^+(S)$, such that the compositve $\mathbb{L}^{\mathrm{an}}_S \to \mathbb{L}^{\mathrm{an}}_{S/X} \to \mathcal{F}[1]$ agrees with d. By applying the right adjoint g_* above we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{L}^{\mathrm{an}}_T \arrow[r, "\mathrm{can}"] \arrow[d] & \mathbb{L}^{\mathrm{an}}_{T/X} \arrow[d] \arrow[dr, "d''"] & \\ g_*(\mathbb{L}^{\mathrm{an}}_S) \arrow[r, "g_*(\mathrm{can})"] & g_*(\mathbb{L}^{\mathrm{an}}_{S/X}) \arrow[r] & g_*(\mathcal{F})[1], \end{tikzcd}$$

in the ∞-category $\mathrm{Coh}^+(T)$. From this, we conclude again by the universal property of the relative analytic cotangent complex the existence of a uniquely defined natural morphism $T' \to X$ extending both $T \to X$

and $S' \to X$ and compatible with the restriction to S. The latter assertion is equivalent to state that the commutative square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T', \end{tikzcd}$$

is a pushout diagram in dAn_k. The proof is thus concluded. □

Proposition 2.10. *Let $f\colon X \to Y$ be a nil-embedding of derived k-analytic spaces. Let $g\colon X \to Z$ be a finite morphism in dAn_k. The the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z & \end{tikzcd}$$

admits a pushout in dAn_k, denoted Z'. Moreover, the natural morphism $Z \to Z'$ is also a nil-embedding.

Proof. The ∞-category of $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ is a presentable ∞-category. Consider the pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d] \\ Z \arrow[r] & Z', \end{tikzcd}$$

in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. By construction, the underlying ∞-topos of Z' can be computed as the pushout in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ of the induced diagram on the underlying ∞-topoi of X, Z and Y. Moreover, since g is a nil-isomorphism it induces an equivalence on underlying ∞-topoi of both X and Y. It follows that the induced morphism $Z \to Z'$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ induces an equivalence on the underlying ∞-topoi. Moreover, it follows essentially by construction that we have a natural equivalence

$$\begin{aligned} \mathcal{O}_{Z'} &\simeq g_*(\mathcal{O}_Y) \times_{g_*(\mathcal{O}_Y)} \mathcal{O}_Z \\ &\in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(Z). \end{aligned}$$

As the morphism $g_*(\mathcal{O}_Y) \to g_*(\mathcal{O}_X)$ is an effective epimorphism and the latter are preserved under fiber products in an ∞-topos, c.f. [9, Proposition 6.2.3.15], it follows that the natural morphism

$$\mathcal{O}_{Z'} \to \mathcal{O}_Z,$$

is an effective epimorphism, as well. Consider now the commutative diagram of fiber sequences

$$\begin{tikzcd} \mathcal{J}' \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[d] \\ \mathcal{J} \arrow[r] & g_*(\mathcal{O}_Y) \arrow[r] & g_*(\mathcal{O}_X), \end{tikzcd}$$

in the stable ∞-category $\mathrm{Mod}_{\mathcal{O}'_Z}$. Since the right commutative square is a pullback square it follows that the morphism

$$\mathcal{J}' \to \mathcal{J},$$

is an equivalence. In particular, $\pi_0(\mathcal{J}')$ is a finitely generated nilpotent ideal of $\pi_0(\mathcal{O}^{\mathrm{alg}}_{\mathcal{J}'})$. Indeed, finitely generation follows from our assumption that g is a finite morphism. Thanks to Lemma 1.7, it follows that $\mathrm{t}_0(Z')$ is an ordinary k-analytic space and the morphism $\mathrm{t}_0(Z') \to \mathrm{t}_0(Z)$ is a nil-embedding. We are thus reduced to show that for every $i > 0$, the homotopy sheaf $\pi_i(\mathcal{O}_{Z'}) \in \mathrm{Coh}^+(\mathrm{t}_0(Z'))$. But this follows immediately from the existence of a fiber sequence

$$\mathcal{O}_{Z'} \to g_*(\mathcal{O}_Y) \oplus \mathcal{O}_Z \to g_*(\mathcal{O}_X),$$

in the ∞-category $\mathrm{Mod}_{\mathcal{O}_{Z'}}$ together with the fact that $g_*(\mathcal{O}_Y)$ and $g_*(\mathcal{O}_Z)$ have coherent homotopy sheaves, by our assumption that g is a finite morphism combined with Lemma 2.2. $\square$

Definition 2.11. An *analytic formal moduli problem under X* corresponds to the datum of a functor

$$F \colon (\mathrm{AnNil}_{X/})^{\mathrm{op}} \to \mathcal{S},$$

satisfying the following two conditions:

(1) $F(X) \simeq *$ in $\mathcal{S}$;
(2) Given any pushout diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T', \end{tikzcd}$$

in the ∞-category $\mathrm{AnNil}_{X/}$, such that f is has the structure of a square zero extension, the induced morphism

$$F(T') \to F(T) \times_{F(S)} F(S),$$

is an equivalence in $\mathcal{S}$.
(3) For each $n \geq 0$, consider the well defined functor

$$\mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/} \to \mathrm{AnNil}_{X/},$$

given by the formula

$$(\mathrm{t}_{\leq n}(X) \to S) \in \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/} \mapsto (X \to S \bigsqcup_{\mathrm{t}_{\leq n}(X)} X).$$

Define $F^{\leq n} \colon \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/} \to \mathcal{S}$ as the functor given on objects by the association

$$(rt_{\leq n}(X) \to S) \in \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/} \mapsto F^{\leq n}(S \bigsqcup_{\mathrm{t}_{\leq n}(X)} X) \in \mathcal{S}$$

Given any $S \in \mathrm{AnNil}^{\mathrm{op}}_{X/}$, we then require that the natural morphism

$$F(S) \to \lim_{n \geq 0} F^{\leq n}(\mathrm{t}_{\leq n}(S)),$$

to be an equivalence in $\mathcal{S}$

We shall denote by $\mathrm{AnFMP}_{X/}$ the full subcategory of $\mathrm{Fun}((\mathrm{AnNil}_{X/})^{\mathrm{op}}, \mathcal{S})$ spanned by analytic formal moduli problems under X.

Remark 2.12. In the previous definition we remark that the functor

$$F^{\leq n} \colon \mathrm{AnNil}^{\mathrm{op}}_{\mathrm{t}_{\leq n}(X)/} \to \mathcal{S},$$

is itself an analytic formal moduli problem under $\mathrm{t}_{\leq n}(X)$.

We shall give some important examples of formal moduli problems under X:

Example 2.13. (1) Let $X \in \mathrm{dAn}_k$. As in the algebraic case, we can consider the *de Rham pre-stack associated to* X, $X_{\mathrm{dR}} \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$, determined by the formula

$$X_{\mathrm{dR}}(Z) \coloneqq X(Z_{\mathrm{red}}), \quad Z \in \mathrm{dAfd}_k.$$

We have a natural morphism $X \to X_{\mathrm{dR}}$ induced from the natural morphism $Z_{\mathrm{red}} \to Z$. We claim that $h_*(X_{\mathrm{dR}}) \in \mathrm{Fun}(\mathrm{AnNil}_{X/}^{\mathrm{op}}, \mathcal{S})$ belongs to the full subcategory $\mathrm{AnFMP}_{X/}$. Indeed, in this case it is clear that $h_*(X_{\mathrm{red}})$ is the final object in $\mathrm{AnFMP}_{X/}$ which clearly satisfies conditions i) and ii) in Definition 2.11.

(2) Let $f \colon X \to Y$ be a morphism in the ∞-category dAn_k. We define the *formal completion of* X *in* Y *along* f as the pullback

$$\begin{aligned} Y^\wedge_X &\coloneqq Y \times_{Y_{\mathrm{dR}}} X_{\mathrm{dR}} \\ &\in \mathrm{dAnSt}_k. \end{aligned}$$

By construction we have a natural factorization $X \to Y^\wedge_X \to Y$ in dAnSt_k, and moreover the restriction of $X \to Y^\wedge_X$ to the ∞-category $\mathrm{AnNil}_{X/}^{\mathrm{op}}$ along the natural functor

$$\mathrm{AnNil}_{X/}^{\mathrm{op}} \to (\mathrm{dAnSt}_k)_{X/}^{\mathrm{op}},$$

exhibits $Y^\wedge_X$ as a formal moduli problem under X.

(3) Let $f \colon X \to Y$ be a closed immersion in the ∞-category dAn_k. Consider the shifted tangent bundle associated to f together with the zero section

$$\begin{tikzcd} X \arrow[r, "s_0"] \arrow[dr, "="'] & \mathbf{T}^{\mathrm{an}}_{X/Y}[-1] \arrow[d, "p"] \\ & X. \end{tikzcd}$$

The completion $\mathbf{T}^{\mathrm{an}}_{X/Y}[-1]^\wedge_X$ will play an important role in what follows.

Notation 2.14. We set $\mathrm{AnNil}^{\mathrm{cl}}_{X/} \subseteq \mathrm{AnNil}_{X/}$ to be the full subcategory spanned by those objects corresponding to nil-embeddings of the form

$$X \to S,$$

in dAn_k.

Proposition 2.15. *Let* $Y \in \mathrm{AnNil}_{X/}$. *The following assertions hold:*

(1) Then the inclusion functor

$$\mathrm{AnNil}^{\mathrm{cl}}_{X//Y} \hookrightarrow \mathrm{AnNil}_{X//Y},$$

is cofinal.

(2) The natural morphism

$$\operatorname*{colim}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} Z \to Y,$$

is an equivalence in $\mathrm{Fun}((\mathrm{AnNil}_{X//Y})^{\mathrm{op}}, \mathcal{S})$.

(3) The ∞*-category* $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ *is filtered.*

Proof. We start by proving claim (i). Let $n \geq 0$, and consider the usual restriction along the natural morphism $X_{\mathrm{red}} \to \mathrm{t}_{\leq n}(X)$ functor

$$\mathbf{res}^{\leq n} \colon \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/} \to \mathrm{AnNil}_{X_{\mathrm{red}}/}.$$

Such functor admits a well defined left adjoint

$$\mathbf{push}^{\leq n} \colon \mathrm{AnNil}_{X_{\mathrm{red}}/} \to \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/},$$

which is determined by the formula

$$(X_{\mathrm{red}} \to T) \in \mathrm{AnNil}_{X_{\mathrm{red}}/} \mapsto (\mathrm{t}_{\leq n}(X) \to T') \in \mathrm{AnNil}_{X/},$$

where we set

$$T' \coloneqq \mathrm{t}_{\leq n}(X) \bigsqcup_{X_{\mathrm{red}}} T \in \mathrm{AnNil}_{X/}. \tag{2.3}$$

We claim that $T' \in \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/}$ belongs to the full subcategory $\mathrm{AnNil}^{\mathrm{cl}}_{\mathrm{t}_{\leq n}(X)/} \subseteq \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)/}$. Indeed, since the structural morphism $X_{\mathrm{red}} \to T$, is necessarily a nil-embedding we deduce the claim from Proposition 2.10. We shall denote by

$$\mathbf{res}_!^{\leq n}(Y) \colon \mathrm{AnNil}^{\mathrm{op}}_{X_{\mathrm{red}}/} \to \mathcal{S},$$

the left Kan extension of Y along the functor $\mathbf{res}^{\leq n}$ above. By the colimit formula for left Kan extensions, c.f. [9, Lemma 4.3.2.13], it follows that $\mathbf{res}_!^{\leq n}(Y)$ is given by the formula

$$(X_{\mathrm{red}} \to T) \in \mathrm{AnNil}_{X_{\mathrm{red}}/} \mapsto Y^{\leq n}(T') \in \mathcal{S},$$

where T' is as in (2.3). We thus have a diagram of functors

$$\mathbf{res}^{\leq n} \colon \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)//Y^{\leq n}} \rightleftarrows \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!^{\leq n}(Y)} \colon \mathbf{push}^{\leq n},$$

where $\mathbf{res}^{\leq n}$ is given on objects by the formula

$$(\mathrm{t}_{\leq n}(X) \to S \to Y^{\leq n}) \in \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)//Y^{\leq n}} \mapsto (X_{\mathrm{red}} \to S \to \mathbf{res}_!^{\leq n}(Y))$$

and the functor $\mathbf{push}^{\leq n}$ is given by the association

$$(X_{\mathrm{red}} \to T \to Y^{\leq n}) \in \mathrm{AnNil}_{X_{\mathrm{red}}//Y^{\leq n}} \mapsto (\mathrm{t}_{\leq n}(X) \to T \sqcup_{X_{\mathrm{red}}} \mathrm{t}_{\leq n}(X) \to Y^{\leq n}) \in \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)//Y^{\leq n}}.$$

We claim that the pair $(\mathbf{res}^{\leq n}, \mathbf{push}^{\leq n})$ form an adjunction. Indeed, a morphism

$$(X_{\mathrm{red}} \to S \to \mathbf{res}_!^{\leq n}(Y)) \to \mathbf{res}_!^{\leq n}(\mathrm{t}_{\leq n}(X) \to T \to Y^{\leq n}),$$

corresponds to a commutative diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r] \arrow[d, "="] & S \arrow[r] \arrow[d] & \mathbf{res}_!^{\leq n}(Y) \arrow[d, "="] \\
X_{\mathrm{red}} \arrow[r] & T \arrow[r] & \mathbf{res}_!^{\leq n}(Y),
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\mathrm{AnNil}^{\mathrm{op}}_{X_{\mathrm{red}}/}, \mathcal{S})$. The latter datum is equivalent to the datum of a commutative diagram

$$\begin{tikzcd}
& & \mathrm{t}_{\leq n}(X) \arrow[d] \arrow[dr] & \\
X_{\mathrm{red}} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & S' \arrow[r] \arrow[d] & Y^{\leq n} \arrow[d, "="] \\
X_{\mathrm{red}} \arrow[r] & T \arrow[r] & T' \arrow[r] & Y^{\leq n}
\end{tikzcd} \quad . \tag{2.4}$$

Since the morphism $\mathrm{t}_{\leq n}(X) \to T'$ factors through the structural map

$$\mathrm{t}_{\leq n}(X) \to T,$$

we deduce that the datum of (2.4) is equivalent to the datum of a commutative diagram

$$\begin{tikzcd} \mathrm{t}_{\leq n}(X) \arrow[r] \arrow[d] & S' \arrow[r] \arrow[d] & Y^{\leq n} \arrow[d, "="] \\ \mathrm{t}_{\leq n}(X) \arrow[r] & T \arrow[r] & Y^{\leq n}, \end{tikzcd}$$

which corresponds to a uniquelly well defined morphism

$$\mathbf{push}^{\leq n}(X_{\mathrm{red}} \to S \to \mathbf{res}_!^{\leq n}(Y)) \to (\mathrm{t}_{\leq n}(Y) \to T \to Y^{\leq n}),$$

in the ∞-category $\mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)//Y^{\leq n}}$. We further observe that for every $n \geq m \geq 0$, the objects

$$\mathbf{res}_!^{\leq n}(Y) \quad \text{and} \quad \mathbf{res}_!^{\leq m}(Y),$$

are equivalent as functors $\mathrm{AnNil}^{\mathrm{op}}_{X_{\mathrm{red}}/} \to \mathcal{S}$, we shall denote this functor simply by $\mathbf{res}_!(Y)$.

Passing to the limit over $n \geq 0$ we obtain a commutative diagram of the form

$$\begin{tikzcd} \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \arrow[rr] \arrow[dr] & & \lim_{n\geq 0} \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)//Y^{\leq n}} \\ & \mathrm{AnNil}_{X//Y}. \arrow[ur] & \end{tikzcd}$$

The horizontal morphism is cofinal since it fits into an adjunction, by our previous considerations. Thanks to [9,] in order to show that the natural morphism

$$\mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \to \mathrm{AnNil}_{X//Y},$$

is cofinal, it suffices to prove that $\mathrm{AnNil}_{X//Y} \to \lim_{n\geq 0} \mathrm{AnNil}_{\mathrm{t}_{\leq n}(X)//Y^{\leq n}}$ is itself cofinal. But the latter is an immediate consequence of the fact that derived k-analytic spaces are nilcomplete, c.f. [12, Lemma 7.7], together with our assumption (iii) in Definition 2.11. Assertion (i) of the Proposition now follows from the observation that the functor

$$\mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \to \mathrm{AnNil}_{X//Y},$$

factors through the full subcategory $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y} \subseteq \mathrm{AnNil}_{X//Y}$.

Claim (ii) follows immediately from (i) combined with Yoneda Lemma. To prove (iii) we shall make use of [9, Lemma 5.3.1.12]. Let

$$F \colon \partial \Delta^n \to \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}.$$

For each $[m] \in \Delta^n$, denote by $S_m := F([m])$ in $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$. We then have that the pushout

$$S_n \bigsqcup_X S_{n-1},$$

exists in $\mathrm{AnNil}^{\mathrm{cl}}_{X/}$. We wish to show that $S_n \bigsqcup_X S_{n-1}$ admits a morphism

$$S_n \bigsqcup_X S_{n-1} \to Y,$$

compatible with the diagram F. In order to prove the latter assertion, we observe that Proposition 2.4 can filter the diagram F by diagrams $F_i \to F$ such that $X \to F_0$ is formed by square-zero extensions and so are each $F_i \to F_{i+1}$. Moreover, by the fact that Y satisfies condition (ii) in Definition 2.11 it follows that we can find a well defined morphism

$$S_n \bigsqcup_X S_{n-1} \to Y,$$

which is compatible with F, as desired. $\square$

Definition 2.16. Let $Y \in \mathrm{AnFMP}_{X/}$ denote an analytic formal moduli problem under X. The *relative pro-analytic cotangent complex of* Y *under* X is defined as the pro-object

$$\begin{aligned} \mathbb{L}^{\mathrm{an}}_{X/Y} &:= \{\mathbb{L}^{\mathrm{an}}_{X/Z}\}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \\ &\in \mathrm{Pro}(\mathrm{Coh}^+(X)), \end{aligned}$$

where, for each $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$

$$\mathbb{L}^{\mathrm{an}}_{X/Z} \in \mathrm{Coh}^+(X),$$

denotes the usual analytic relative cotangent complex associated to the structural morphism $X \to Z$ in $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$.

Remark 2.17. Let $Y \in \mathrm{AnFMP}_{X/}$. For a general $Z \in \mathrm{dAn}_k$, there exists a natural morphism

$$\mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/Z},$$

in the infcat $\mathrm{Coh}^+(X)$. Passing to the limit over $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$, we obtain a natural map

$$\mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/Y},$$

in $\mathrm{Pro}(\mathrm{Coh}^+(X))$, as well.

The following result provides justifies our choice of terminology for the object $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$:

Lemma 2.18. *Let $Y \in \mathrm{AnFMP}_{X/}$. Let $X \hookrightarrow S$ be a square zero extension associated to an analytic derivation*

$$d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}[1],$$

where $\mathcal{F} \in \mathrm{Coh}^+(X)^{\geq 0}$. Then there exists a natural morphism

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \to \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}, \mathcal{F})} \{d\}$$

which is furthermore an equivalence in the ∞-category $\mathcal{S}$.

Proof. Thanks to Proposition 2.15 combined with the Yoneda Lemma we can identify the space of liftings of the map $X \to Y$ along $X \to S$ with the mapping space

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \simeq \operatorname*{colim}_{Z \in \mathrm{AnNil}_{X//Y}} \mathrm{Map}_{\mathrm{AnNil}_{X/}}(S, Z).$$

Fix $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$. Then we have a natural identification of mapping spaces

$$\mathrm{Map}_{\mathrm{AnNil}_{X/}}(S, Z) \simeq \mathrm{Map}_{(\mathrm{dAn}_k)_{X/}}(S, Z) \tag{2.5}$$

$$\simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Z}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_X, \mathcal{F})} \{d\}, \tag{2.6}$$

see [12, §5.4] for a justification of the latter assertion. Passing to the colimit over $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$, we conclude thanks to the formula for mapping spaces in pro-∞-categories that we have a natural equivalence

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}, \mathcal{F})} \{d\},$$

as desired. □

Construction 2.19. Let $f \colon Y \to Z$ denote a morphism in $\mathrm{AnFMP}_{X/}$. Then, for every $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ the induced morphism

$$S \to Z,$$

in $\mathrm{AnFMP}_{X/}$ factors necessarily through some $S' \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$. For this reason, we obtain a natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X/S'} \to \mathbb{L}^{\mathrm{an}}_{X/S},$$

in the ∞-category $\mathrm{Coh}^+(X)$. Passing to the limit over $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ we obtain a canonically defined morphism

$$\theta(f) \colon \mathbb{L}^{\mathrm{an}}_{X/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Y},$$

in $\mathrm{Pro}(\mathrm{Coh}^+(X))$. Moreover, this association is functorial and thus we obtain a well defined functor

$$\mathbb{L}^{\mathrm{an}}_{X/\bullet} \colon \mathrm{AnFMP}_{X/} \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

given by the formula

$$(X \to Y) \in \mathrm{AnFMP}_{X/} \mapsto \mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(X)).$$

Proposition 2.20. *Let $X \in \mathrm{dAn}_k$ be a derived k-analytic space. Then the functor*

$$\mathbb{L}^{\mathrm{an}}_{X/\bullet} \colon \mathrm{AnFMP}_{X/} \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

obtained via Construction 2.19, is conservative.

Proof. Let $f \colon Y \to Z$ be a morphism in $\mathrm{AnFMP}_{X/}$. Thanks to Proposition 2.15 we are reduced to show that given any

$$S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z},$$

the structural morphism $g_S \colon X \to S$ admits a unique extension $S \to Y$ which factors the structural morphism $X \to Y$. Thanks to Proposition 2.4 we can reduce ourselves to the case where $X \to S$ has the structure of a square zero extension. In this case, the result follows from Lemma 2.18 combined with our hypothesis. ☐

Our goal now is to give an alternative description of analytic formal moduli problems under $X \in \mathrm{dAn}_k$, in terms of derived k-analytic stacks:

Construction 2.21. Consider the ∞-category of derived k-analytic stacks, dAnSt_k. We have a natural functor

$$h \colon \mathrm{AnNil}_{X/} \to \mathrm{dAn}_k \hookrightarrow \mathrm{dAnSt}_k.$$

Therefore, given any derived k-analytic stack Y equipped with a morphism $X \to Y$, one can consider its restriction to the ∞-category $\mathrm{AnNil}_{X/}$:

$$Y \circ h \colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathcal{S}.$$

We have thus a natural restriction functor

$$h_* \colon \mathrm{dAnSt}_k \to \mathrm{Fun}(\mathrm{AnNil}^{\mathrm{op}}_{X/}, \mathcal{S}).$$

On the other hand, Proposition 2.15 allows us to define a natural functor

$$F \colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathrm{dAnSt}_k$$

via the formula

$$(X \to Y) \in \mathrm{AnFMP}_{X/} \mapsto \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} S,$$

the colimit being computed in the ∞-category dAnSt_k. The latter agrees with the left Kan extension of the functor

$$h \colon \mathrm{AnNil}_{X/} \to \mathrm{dAnSt}_k,$$

along the natural inclusion functor $\mathrm{AnNil}_{X/} \hookrightarrow \mathrm{AnFMP}_{X/}$. In particular, any analytic formal moduli under X when regarded as a derived k-analytic stack can be realized as an *ind-inf*-object, i.e. it can be written as a filtered colimit of nil-embeddings $X \to Z$. We refer the reader to [6, §1] for a precise meaning of the latter notion in the algebraic setting.

Definition 2.22. Let $Y \in \mathrm{dAnSt}_k$. We shall say that Y has a deformation theory if it satisfies the following conditions:

(1) Y is *nilcomplete*, c.f. [12, Definition 7.4];
(2) Y is *infinitesimally cartesian* if it satisfies [12, Definition 7.3];
(3) Y admits a *pro-cotangent complex*, i.e., if it satisfies [12, Definition 7.6] under the weaker assumption that the corresponding derivation functor is pro-corepresentable.

Proposition 2.23. *Let $Y \in (\mathrm{dAnSt}_k)_{X/}$. Assume further that Y admits a deformation theory. Then Y is equivalent to an analytic formal moduli problem under X.*

Proof. We must prove that given a pushout diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"] & S' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

in the ∞-category $\mathrm{AnNil}_{X/}$, where f has the structure of a square-zero extension, then the natural morphism

$$Y(T') \to Y(T) \times_{Y(S)} Y(S'),$$

is an equivalence in the ∞-category $\mathcal{S}$. Suppose further that $S \hookrightarrow S'$ is associated to some analytic derivation

$$d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}[1],$$

for some $\mathcal{F} \in \mathrm{Coh}^+(S)^{\geq 0}$. Thanks to Corollary 2.9 we deduce that the induced morphism $T \to T'$ admits a structure of a square-zero extension, as well. Then, by our assumptions of Y being infinitesimally cartesian and admitting a relative pro-cotangent complex, we have a chain of natural equivalences of the form.

$$\begin{aligned} Y(T') &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{T/}(T', Y) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(T))_{\mathbb{L}^{\mathrm{an}}_T/}}(\mathbb{L}^{\mathrm{an}}_{T/Y}, g_*(\mathcal{F})[1]) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))_{g^*\mathbb{L}^{\mathrm{an}}_T/}}(g^*\mathbb{L}^{\mathrm{an}}_{T/Y}, \mathcal{F}[1]) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))_{\mathbb{L}^{\mathrm{an}}_S/}}(\mathbb{L}^{\mathrm{an}}_{S/Y}, \mathcal{F}[1]) \\ &\simeq \coprod_{f \colon t \to Y} \mathrm{Map}_{S/}(S', Y) \\ &\simeq Y(T) \times_{Y(S)} Y(S'), \end{aligned}$$

where the third equivalence follows from the existence of a commutative diagram between fiber sequences

$$\begin{tikzcd} g^* f^* \mathbb{L}^{\mathrm{an}}_Y \arrow[r] \arrow[d, "="] & g^* \mathbb{L}^{\mathrm{an}}_T \arrow[r] \arrow[d] & g^* \mathbb{L}^{\mathrm{an}}_{T/Y} \arrow[d] \\ (f \circ g)^* \mathbb{L}^{\mathrm{an}}_Y \arrow[r] & \mathbb{L}^{\mathrm{an}}_S \arrow[r] & \mathbb{L}^{\mathrm{an}}_{S/Y}, \end{tikzcd}$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(S))$ combined with the fact that the derivation $d_T \colon \mathbb{L}^{\mathrm{an}}_T \to g_*(\mathcal{F})[1]$ is induced from

$$d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}[1],$$

as in the proof of Corollary 2.9. The result now follows. $\square$

Proposition 2.24. *Let $Z \in (\mathrm{dAnSt}_k)_{X/}$. Suppose further that Z admits a deformation theory. Then the natural morphism*

$$\operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}} S \to Z,$$

in the ∞-category $(\mathrm{dAnSt}_k)_{/X}$, is an equivalence.

Proof. We shall prove that for every derived k-analytic space $T \in \mathrm{dAn}_k$, any diagram of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Z \\ & T \arrow[ur, "h"'] & \end{tikzcd},$$

factors through an object

$$X \to S \to Z,$$

in $\mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$. Consider the commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r] \arrow[d] & T_{\mathrm{red}} \arrow[d] \arrow[dr] & \\ X \arrow[r] & T \arrow[r] & Z, \end{tikzcd} \tag{2.7}$$

in the ∞-category dAnSt_k. Since $Z_{\mathrm{red}} \simeq X_{\mathrm{red}}$ we obtain that $T_{\mathrm{red}} \to Z$ factors necessarily through the colimit

$$\operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}} S \in \mathrm{AnPreStk}.$$

Assume first that T is bounded, i.e. $T \in \mathrm{dAn}_k^{<\infty}$. Then we can construct T out of T_{red} via a finite sequence of square-zero extensions, as in Proposition 2.4. Therefore, in order to construct a factorization

$$T \to S \to Z,$$

in $\mathrm{AnPreStk}_{X/}$, we reduce ourselves to the case where the morphism $T_{\mathrm{red}} \to T$ is itself a square-zero extension. In this case, let

$$d \colon \mathbb{L}^{\mathrm{an}}_{T_{\mathrm{red}}} \to \mathcal{F}[1],$$

where $\mathcal{F} \in \mathrm{Coh}^+(T_{\mathrm{red}})^{\geq 0}$ be the associated derivation. The existence of the diagram (2.7) implies that we have a commutative diagram of the form

$$\begin{tikzcd} g^*\mathbb{L}^{\mathrm{an}}_{T_{\mathrm{red}}} \arrow[r] & \mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}} \arrow[r] & \mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}/T_{\mathrm{red}}} \\ f^*\mathbb{L}^{\mathrm{an}}_{Z} \arrow[r] \arrow[u] & \mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}} \arrow[r] \arrow[u] & \mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}/Z}, \arrow[u] \end{tikzcd}$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(X))$. For this reason, the limit-colimit formula for mapping spaces in pro-∞-categories implies that the natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}/Z} \to \mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}/T_{\mathrm{red}}},$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(X))$ factors necessarily via a morphism of the form

$$\mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}/S} \to \mathbb{L}^{\mathrm{an}}_{X_{\mathrm{red}}/T_{\mathrm{red}}},$$

for some suitable $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$. Thus the existence problem

$$\begin{tikzcd} T_{\mathrm{red}} \arrow[r] \arrow[d] & T \arrow[d, dashed] \arrow[dr] & \\ X_{\mathrm{red}} = S_{\mathrm{red}} \arrow[r] & S \arrow[r] & Z, \end{tikzcd}$$

admits a solution $T \to S \to Z$, as desired. If $T \in \mathrm{dAn}_k$ is a general derived k-analytic space, we reduce ourselves to the bounded case using the fact that Y is nilcomplete. $\square$

Corollary 2.25. *The functor*

$$F \colon \mathrm{AnFMP}_{X/} \to (\mathrm{dAnSt}_k)_{X/},$$

is fully faithful. Moreover, its essential image coincides with those $Z \in (\mathrm{dAnSt}_k)_{X/}$ which admit deformation theory.

Proof. Let $(\mathrm{dAnSt}_k)^{\mathrm{def}}_{X/} \subseteq (\mathrm{dAnSt}_k)_{X/}$ denote the full subcategory spanned by derived k-analytic stacks under X admitting a deformation theory. It is clear that the natural functor

$$F \colon \mathrm{AnFMP}_{X/} \to (\mathrm{dAnSt}_k)_{X/},$$

factors through the full subcategory $(\mathrm{dAnSt}_k)^{\mathrm{def}}_{X/}$. Moreover, the restriction functor

$$\mathrm{res} \colon (\mathrm{dAnSt}_k)^{\mathrm{def}}_{X/} \to \mathrm{Fun}(\mathrm{AnNil}^{\mathrm{op}}_{X/}, \mathcal{S}),$$

factors through $\mathrm{AnFMP}_{X/} \subseteq \mathrm{Fun}(\mathrm{AnNil}^{\mathrm{op}}_{X/}, \mathcal{S})$. Moreover, Proposition 2.24 implies that the restriction functor that F and res are mutually inverse functors, proving the claim. $\square$

2.2. **Analytic formal moduli problems over a base.** Let $X \in \mathrm{dAn}_k$ denote a derived k-analytic space. In [15, Definition 6.11] the authors introduced the ∞-category of *analytic formal moduli problems over X*, which we shall denote by $\mathrm{AnFMP}_{/X}$.

Notation 2.26. Let $X \in \mathrm{dAn}_k$. We shall denote by $\mathrm{AnNil}_{/X}$ the full subcategory of $(\mathrm{dAn}_k)_{/X}$ spanned by nil-isomorphisms

$$Z \to X,$$

in the ∞-category dAn_k.

Definition 2.27. We shall denote by $\mathrm{AnNil}^{\mathrm{cl}}_{/X} \subseteq \mathrm{AnNil}_{/X}$ the faithful subcategory in wich we allow morphisms

$$i \colon S \to S',$$

where i is a nil-embedding in dAn_k.

We start with the analogue of Proposition 2.15 in the setting of analytic formal moduli problems over X:

Proposition 2.28. *Let $Y \in \mathrm{AnFMP}_{/X}$. The following assertions hold:*

(1) The inclusion functor

$$(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y} \to (\mathrm{AnNil}_{/X})_{/Y},$$

is cofinal.

(2) The natural morphism

$$\operatorname*{colim}_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} Z \to Y,$$

is an equivalence in the ∞-category $\mathrm{AnFMP}_{/X}$.

(3) The ∞-category $\mathrm{AnNil}^{\mathrm{cl}}_{/X}$ is filtered.

Proof. We first prove assertion (i). Let $S \to Z$ be a morphism in $(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}$. Consider the pushout diagram

$$\begin{tikzcd} S_{\mathrm{red}} \arrow[r] \arrow[d] & S \arrow[d] \\ Z \arrow[r] & Z', \end{tikzcd} \tag{2.8}$$

in the ∞-category $\mathrm{AnNil}_{/X}$ whose existence is guaranteed by Proposition 2.10. Since the upper horizontal morphism in (2.8) is a nil-embedding, we can reduce ourselves via Proposition 2.4 to the case where the latter is an actual square-zero extension. Since Y is assumed to be an analytic formal moduli problem over X we then deduce that the canonical morphism

$$\begin{aligned} Y(Z') &\to Y(Z) \times_{Y(S_{\mathrm{red}})} Y(S) \\ &\simeq Y(Z) \times Y(S), \end{aligned}$$

is an equivalence (we implicitly used above the fact that $S_{\mathrm{red}} \simeq X_{\mathrm{red}}$). As a consequence the object $(Z' \to X)$ in $\mathrm{AnNil}_{/X}$ admits an induced morphism $Z' \to Y$ making the required diagram commute. Thanks Proposition 2.10 we deduce that both $S \to Z'$ and $Z \to Z'$ are nil-embedding. Therefore, we can factor the diagram

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

via a closed nil-isomorphism $Z \to Z'$. As a consequence, we deduce readily that the inclusion functor

$$(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y} \to (\mathrm{AnNil}_{/X})_{/Y},$$

is cofinal. It is clear that assertion (ii) follows immediately from (i). We now prove (iii). Let

$$\theta \colon K \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y},$$

be a functor where K is a finite ∞-category. We must show that θ can be extended to a functor

$$\theta^{\triangleright} \colon K^{\triangleright} \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}.$$

Thanks to Proposition 2.4 we are allowed to reduce ourselves to the case where morphisms indexed by K are square-zero extensions. The result now follows from the fact that Y being an analytic moduli problem sends finite colimits along square-zero extensions to finite limits. ☐

Lemma 2.29. *Let $X \in \mathrm{dAn}_k$. Given any $Y \in \mathrm{AnFMP}_{X/}$, then for each $i = 0, 1$ the i-th projection morphism*

$$p_0 \colon X \times_Y X \to X,$$

computed in the ∞-category dAnSt_k lies in the essential image of $\mathrm{AnFMP}_{/X}$ via Construction 2.21.

Proof. Consider the pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_0"] & X \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

computed in the ∞-category dAnSt_k. Thanks to Proposition 2.15 together with the fact that fiber products commute with filtered colimis in the ∞-category dAnSt_k, we deduce that

$$X \times_Y X \simeq \operatorname*{colim}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} X \times_Z X,$$

in dAnSt_k. It is clear that $(p_i \colon X \times_Z X \to X)$ lies in the essential image of $\mathrm{AnFMP}_{/X}$, for $i = 0, 1$. Thus also the filtered colimit

$$(p_i \colon X \times_Y X \to X) \in \mathrm{AnFMP}_{/X}, \quad \text{for } i = 0, 1,$$

as desired. $\square$

Just as in the previous section we deduce that every analytic formal moduli problem over X admits the structure of an *ind-inf*-object in $\mathrm{AnPreStk}_k$:

Corollary 2.30. *Let $Y \in (\mathrm{dAnSt}_k)_{/X}$. Then Y is equivalent to an analytic formal moduli problem over X if and only if there exists a presentation*

$$Y \simeq \operatorname*{colim}_{i \in I} Z_i,$$

where I is a filtered ∞-category and for every $i \to j$ in I, the induced morphism

$$Z_i \to Z_j,$$

is a closed embedding of derived k-affinoid spaces that are nil-isomorphic to X.

Proof. It follows immediately from Proposition 2.28 (ii). $\square$

Definition 2.31. Let $Y \in \mathrm{AnFMP}_{/X}$. We define the ∞-category of *coherent modules on* Y, denoted $\mathrm{Coh}^+(Y)$, as the limit

$$\mathrm{Coh}^+(Y) \coloneqq \lim_{Z \in (\mathrm{dAn}_k)_{/Y}} \mathrm{Coh}^+(Z),$$

computed in the ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$. We define the ∞-category of *pseudo-pro-coherent modules on* Y, denoted $\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y))$, as

$$\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y)) \coloneqq \lim_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \mathrm{Pro}(\mathrm{Coh}^+(Z)),$$

where the limit is computed in the ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$.

Definition 2.32. Let $Y \in \mathrm{AnFMP}_{/X}$, $Z \in \mathrm{dAfd}_k$ and let $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$. Suppose furthermore that we are given a morphism $f \colon Z \to Y$. We define the *tangent space of Y at f twisted by* $\mathcal{F}$ as the fiber

$$\mathbb{T}^{\mathrm{an}}_{Y, Z, \mathcal{F}, f} \coloneqq \mathrm{fib}_f\big(Y(Z[\mathcal{F}]) \to Y(Z)\big) \in \mathcal{S}.$$

Whenever the morphism f is clear from the context, we shall drop the subscript f above and denote the tangent space of Y at f simply by $\mathbb{T}^{\mathrm{an}}_{Y, Z, \mathcal{F}}$.

Remark 2.33. Let $Y \in \mathrm{AnFMP}_{/X}$. The equivalence of ind-objects

$$Y \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} S,$$

in the ∞-category dAnSt_k, implies that, for any $Z \in \mathrm{dAfd}_k$, one has an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{dAnSt}_k}(Z, Y) \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \mathrm{Map}_{\mathrm{AnPreStk}}(Z, S).$$

For this reason, given any morphism $f\colon Z \to Y$ and any $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$, we can identify the tangent space $\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}}$ with the filtered colimit of spaces

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathrm{fib}_f\big(S(Z[\mathcal{F}]) \to S(Z)\big) \tag{2.9}$$

$$\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathbb{T}^{\mathrm{an}}_{S,Z,\mathcal{F}} \tag{2.10}$$

$$\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathrm{Map}_{\mathrm{Coh}^+(Z)}(f^*_{S,Z}(\mathbb{L}^{\mathrm{an}}_S), \mathcal{F}), \tag{2.11}$$

where we have denoted by $f_{S,Z}\colon Z \to S$ any morphism, in $(\mathrm{dAn}_k)_{/X}$, factoring $f\colon Z \to Y$ such that

$$(S \to X) \in \mathrm{AnNil}^{\mathrm{cl}}_{/X}.$$

The final equivalence in (2.9), follows from [12, Lemma 7.7]. Therefore, we deduce that the analytic formal moduli problem $Y \in \mathrm{AnFMP}_{/X}$ admits an *absolute pro-cotangent complex* given as

$$\mathbb{L}^{\mathrm{an}}_Y := \{f^*_{S,Z}(\mathbb{L}^{\mathrm{an}}_S)\}_{Z,S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \in \mathrm{Pro}(\mathrm{Coh}^+(Y)).$$

Corollary 2.34. *Let $Y \in \mathrm{AnFMP}_{/X}$. Then its absolute cotangent complex $\mathbb{L}^{\mathrm{an}}_Y$ classifies analytic deformations on Y. More precisely, given any morphism $Z \to Y$ where $Z \in \mathrm{dAfd}_k$ and $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$ one has a natural equivalence of mapping spaces*

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(Y))}(\mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}).$$

Proof. It follows immediately from the natural equivalences displayed in (2.9) combined with the description of mapping spaces in ∞-categories of pro-objects. □

We now introduce the notion of square-zero extensions of analytic formal moduli problems over X:

Construction 2.35. Let $(f\colon Y \to X) \in \mathrm{AnFMP}_{/X}$. Let $d\colon \mathbb{L}^{\mathrm{an}}_Y \to \mathcal{F}[1]$ be an *analytic derivation* in $\mathrm{Pro}(\mathrm{Coh}^+(Y))$, where $\mathcal{F} \in \mathrm{Coh}^+(Y)^{\geq 0}$, such that

$$\mathcal{F} \simeq f^*(\mathcal{F}'),$$

for some suitable object $\mathcal{F}' \in \mathrm{Coh}^+(X)^{\geq 0}$. Thanks to Remark 2.33 one has the following natural equivalences of mapping spaces

$$\mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(Y))}(\mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}[1]) \simeq \lim_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \operatorname*{colim}_{S' \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{S//Y}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))}(f^*_{S,S'}(\mathbb{L}^{\mathrm{an}}_{S'}), g^*_S(\mathcal{F}')[1])$$

$$\simeq \lim_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \operatorname*{colim}_{S' \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{S//Y}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))}(\mathbb{L}^{\mathrm{an}}_{S'}, (f_{S,S'})_* g^*_S(\mathcal{F}')[1]),$$

where $g_S\colon S \to X$ denotes the structural morphism in $\mathrm{AnNil}^{\mathrm{cl}}_{/X}$ and $f_{S,S'}\colon S \to S'$ a given transition morphism in the ∞-category $(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}$. For this reason, we can form the filtered colimit

$$Y' := \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \operatorname*{colim}_{S' \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{S//Y}} \overline{S}' \in \mathrm{dAnSt}_k.$$

By construction, one has a natural morphism $Y \hookrightarrow Y'$ in the ∞-category AnPreStk. Moreover, thanks to Proposition 2.23 it follows that $Y' \in \mathrm{AnFMP}_{/X}$.

Definition 2.36. Let $Y \in \mathrm{AnFMP}_{/X}$. Suppose we are given an analytic derivation

$$d\colon \mathbb{L}^{\mathrm{an}}_Y \to \mathcal{F}[1],$$

in $\mathrm{Pro}(\mathrm{Coh}^+(Y))$ where $\mathcal{F} \in \mathrm{Coh}^+(Y)^{\geq 0}$ is such that $\mathcal{F} \simeq f^*(\mathcal{F}')$, for some $\mathcal{F}' \in \mathrm{Coh}^+(X)^{\geq 0}$. We shall say that the induced morphism

$$h \colon Y \to Y',$$

defined in Construction 2.35, is a *square-zero extension* of Y associated to the analytic derivation d.

Corollary 2.37. *Let $Y \in \mathrm{AnFMP}_{/X}$. Given any square-zero extension $h \colon X \hookrightarrow S$ in dAn_k. Then the space of cartesian squares*

$$\begin{tikzcd} Y \arrow[r, "h'"] \arrow[d, "f"] & Y' \arrow[d, "g"] \\ X \arrow[r, "h"] & S, \end{tikzcd}$$

such that $h' \colon Y \to Y'$ is a square-zero extension and $g \colon Y' \to S$ exhibits the former as an analytic formal moduli problem over S is naturally equivalent to the space of factorizations

$$f^* \mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_Y \to f^*(\mathcal{F}')[1],$$

in $\mathrm{Pro}(\mathrm{Coh}^+(Y))$, of the analytic derivation $d \colon \mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}'[1]$ associated to the morphism h above.

Proof. By the universal property of filtered colimits together with the fact that these preserve fiber products we reduce the statement to the case where $Y \in \mathrm{AnNil}_{/X}$ and thus $Y' \in \mathrm{AnNil}_{/S}$, in which case the statement follows immediately by the universal property of the relative analytic cotangent complex. ☐

Corollary 2.38. *Let $f \colon Z \to X$ be a morphism in the ∞-category dAn_k. Suppose we are given analytic formal moduli problems*

$$f \colon Y \to X \quad \text{and} \quad g \colon \widetilde{Z} \to Z$$

together with a commutative diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "s"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "f"] & X, \end{tikzcd}$$

in the ∞-category dAnSt_k. Let $d \colon \mathbb{L}^{\mathrm{an}}_Z \to \mathcal{F}[1]$, where $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$, be an analytic derivation corresponding to a square-zero extension morphism $Z \to Z'$ in the ∞-category dAn_k. Denote by $\widetilde{d} \colon \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \to \mathcal{F}[1]$ the induced analytic derivation as in Construction 2.35 and let $h \colon \widetilde{Z} \hookrightarrow \widetilde{Z}'$ be the induced square-zero extension in dAnSt_k such that we have a cartesian diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r] \arrow[d] & \widetilde{Z}' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

in the ∞-category dAnSt_k. Then the space of factorizations

$$s \colon \widetilde{Z} \to \widetilde{Z}' \to Y,$$

is naturally equivalent to the space of factorizations

$$\widetilde{d} \colon \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \to \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}/Y} \to \mathcal{F}[1],$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}))$.

Proof. The statement holds true in the case where $\widetilde{Z} \in \mathrm{AnNil}_{/Z}$ and $Y \in \mathrm{AnNil}_{/X}$, by the universal property of the relative cotangent complex. The general case is reduced to the previous one by a standard argument with ind-objects in dAnSt_k. ☐

2.3. **Non-archimedean nil-descent for almost perfect complexes.** In this §, we prove that the ∞-category $\mathrm{Coh}^+(X)$, for $X \in \mathrm{dAn}_k$ satisfies nil-descent with respect to morphims $Y \to X$, which exhibit the former as an analytic formal moduli problem over X.

Proposition 2.39. *Let $f\colon Y \to X$, where $X \in \mathrm{dAn}_k$ and $Y \in \mathrm{AnFMP}_{/X}$. Consider the Čech nerve $Y^\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{dAnSt}_k$ associated to f. Then the natural functor*

$$f_\bullet^*\colon \mathrm{Coh}^+(X) \to \lim_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Coh}^+(Y^\bullet)),$$

is an equivalence of ∞-categories.

Proof. Consider the natural equivalence of derived k-analytic stacks

$$Y \simeq \operatorname*{colim}_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} Z.$$

Then, by definition one has a natural equivalence

$$\mathrm{Coh}^+(Y) \simeq \lim_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \mathrm{Coh}^+(Z),$$

of ∞-categories. In particular, since totalizations commute with cofiltered limits in Cat_∞, it follows that we can suppose from the beginning that $Y \simeq Z$ for some $Z \in \mathrm{AnNil}_{/X}$. In this case, the morphism $f\colon Y \to X$ is affine. In particular, the fact that $\mathrm{Coh}^+(-)$ satisfies descent along admissible open immersions, combined with Lemma 1.10 we further reduce ourselves to the case where both X and Y are derived k-affinoid spaces. In this case, by Tate acyclicity theorem it follows that letting $A := \Gamma(X, \mathcal{O}_X^{\mathrm{alg}})$ and $B := \Gamma(Y, \mathcal{O}_Y^{\mathrm{alg}})$, the pullback functor f^* can be identified with the usual base change functor

$$\mathrm{Coh}^+(A) \to \mathrm{Coh}^+(B).$$

In this case, it follows that B is nil-isomophic to A. Moreover, since the latter are derived noetherian rings the statement of the proposition follows due to [7, Theorem 3.3.1]. $\square$

We now deduce *pseudo-pro-nil-descent* for moprhisms of the form $Y \to X$, which exhibit Y as an analytic formal moduli problem over X:

Corollary 2.40. *Let $X \in \mathrm{dAn}_k$ and $f\colon Y \to X$ a morphism in dAnSt_k which exhibits Y as an analytic formal moduli problem over X. Then the natural functor*

$$f_\bullet^*\colon \mathrm{Pro}(\mathrm{Coh}^+(X)) \to \lim_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Pro}^{\mathrm{ps}}\mathrm{Coh}^+(Y^\bullet/X)),$$

is fully faithful, where $Y^\bullet$ denotes the Čech nerve associated to the morphism f. Moreover, the essential image of the functor $f_\bullet^*$ identifies canonically with the full subcategory

$$\lim_{\mathbf{\Delta}^{\mathrm{op}}}{}' \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(X)) \subseteq \lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)),$$

spanned by those $\{\mathcal{F}_{i,[n]}\}_{i \in I^{\mathrm{op}},[n]} \in \lim_{\mathbf{\Delta}}(\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)))$, for some filtered ∞-category I, which belong to the essential image of the natural functor

$$\lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Coh}^+(Y^\bullet/X)) \to \lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)).$$

Proof. By the very definition of the ∞-category $\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y))$, we reduce ourselves as in Proposition 2.39 to the case where $Y = S$, for some $S \in \mathrm{AnNil}_{/X}$. In this case, it follows readily from Proposition 2.39 that the natural functor

$$f_\bullet^*\colon \mathrm{Pro}(\mathrm{Coh}^+(X)) \to \lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)),$$

is fully faithful. We now proceed to prove the second claim of the corollary. Notice that, Proposition 2.8 implies that there exists a well defined right adjoint

$$f_* \colon \mathrm{Coh}^+(S) \to \mathrm{Coh}^+(X),$$

to the usual pullback functor $f^* \colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(S)$. We can extend the right adjoint f_* to a well defined functor

$$f_* \colon \mathrm{Pro}(\mathrm{Coh}^+(S)) \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

which commutes with cofiltered limits. For this reason, we have a well defined functor

$$f_{\bullet,*} \colon \lim_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

which further commutes with cofiltered limits. We claim that $f_{\bullet,*}$ is a right adjoint to $f_\bullet^*$ above. Indeed, given any $\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$ and $\{\mathcal{G}_{j,[n]}\}_{j \in J^{\mathrm{op}}_{[n]}, [n] \in \mathbf{\Delta}^{\mathrm{op}}} \in \lim_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X)))$, we compute

$$\mathrm{Map}_{\lim_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)))}(f_\bullet^*(\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}}), \{\mathcal{G}_{j,[n]}\}_{j \in J^{\mathrm{op}}_{[n]}, [n] \in \mathbf{\Delta}^{\mathrm{op}}}) \simeq \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^{[n]}))}(\{f_{[n]}^\bullet(\mathcal{F}_i)\}_{i \in I^{\mathrm{op}}}, \{\mathcal{G}_{i,[n]}\}_{i \in I^{\mathrm{op}}_{[n]}}$$

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \lim_{j \in J^{\mathrm{op}}_{[n]}} \mathrm{colim}_{i \in I} \mathrm{Map}_{\mathrm{Coh}^+(Y^{[n]})}(f_{[n]}^*(\mathcal{F}_i), \mathcal{G}_{i,[n]}) \simeq \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \lim_{j \in J^{\mathrm{op}}_{[n]}} \operatorname*{colim}_{i \in I} \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}_i, f_{[n],*}(\mathcal{G}_{i,[n]}))$$

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}}, \{f_{[n],*}(\mathcal{G}_{i,[n]})\}_{i \in I^{\mathrm{op}}_{[n]}}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}}, \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \{f_{[n],*}(\mathcal{G}_{i,[n]})\}_{i \in I^{\mathrm{op}}_{[n]}}),$$

as desired. It is clear that the functor $f_\bullet^*$ above factors through the full subcategory

$$\lim_{\mathbf{\Delta}^{\mathrm{op}}}{}' (\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X))) \subseteq \lim_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))).$$

For this reason, the pair $(f_\bullet^*, f_{\bullet,*})$ restricts to a well defined adjunction

$$f_\bullet^* \colon \mathrm{Pro}(\mathrm{Coh}^+(X)) \rightleftarrows \mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \colon f_{\bullet,*}.$$

In order to conclude, we will show that the functor

$$f_{\bullet,*} \colon \mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

is conservative. Since both the ∞-categories $\mathrm{Pro}(\mathrm{Coh}^+(X))$ and $\lim'_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)))$ are stable, we are reduced to prove that given any

$$\{\mathcal{G}_{i,[n]}\}_{i \in I^{\mathrm{op}}} \in \lim_{\mathbf{\Delta}^{\mathrm{op}}}{}' (\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X))),$$

such that

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*}(\{\mathcal{G}_{i,[n]}\}_{i \in I^{\mathrm{op}}}) \simeq 0, \tag{2.12}$$

we necessarily have

$$\{\mathcal{G}_{i,[n]}\}_{i \in I^{\mathrm{op}}} \simeq 0,$$

in $\lim'_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X)))$. Assume then (2.12). Under our hypothesis, for each index $i \in I$, the object $\{\mathcal{G}_{i,[n]}\}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}}$ satisfies descent datum and thanks to Proposition 2.39 it produces a uniquely well defined object

$$\mathcal{G}_i \in \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

such that for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, one has a natural equivalence of the form

$$\begin{aligned} f_{[n]}^*(\mathcal{G}_i) &\simeq \mathcal{G}_{i,[n]} \\ &\in \mathrm{Coh}^+(Y^{[n]}). \end{aligned}$$

We deduce then that

$$\begin{aligned} f_\bullet^*(\{\mathcal{G}_i\}_{i\in I^{\mathrm{op}}}) &\simeq \{\mathcal{G}_{i,[n]}\}_{i\in I^{\mathrm{op}},[n]} \\ &\simeq 0, \end{aligned}$$

in $\lim'_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(Y^\bullet/X))$, as desired. ☐

We now use the pseudo-pro-nil-descent for $\mathrm{Pro}(\mathrm{Coh}^+(X))$ to compute relative analytic cotangent complexes of analytic formal moduli problems over X:

Corollary 2.41. *Let $f\colon Z \to X$ be a morphism in dAn_k. Suppose we are given a pullback square*

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "h"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X, \end{tikzcd}$$

in the ∞-category dAnSt_k, where $(Y \to X) \in \mathrm{AnFMP}_{/X}$ and $(g\colon \widetilde{Z} \to Z) \in \mathrm{AnFMP}_{/Z}$. Then the lax-limit object

$$\{\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/Y^{[n]}}\} \in \overset{\mathrm{lax}}{\lim}(\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(\widetilde{Z}^\bullet/Z))),$$

defines an actual cartesian section

$$\{\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/Y^{[n]}}\} \in \lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z)),$$

which belongs to the essential image of the natural functor

$$g_\bullet^*\colon \mathrm{Pro}(\mathrm{Coh}^+(Z)) \to \lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}^\bullet/Z)).$$

Proof. We first show that the object

$$\{\mathbb{L}^{\mathrm{an}}_{(\widetilde{Z})^{[n]}/Y^{[n]}}\} \in \overset{\mathrm{lax}}{\lim_{\mathbf{\Delta}^{\mathrm{op}}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z)),$$

defines a cartesian section in

$$\lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z)).$$

In order to show this assertion, it is sufficient to prove for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, that we have a natural equivalence

$$h^*(\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/Y^{[n]}}) \simeq \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n+1]}/Y^{[n+1]}},$$

in the ∞-category $\mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+((\widetilde{Z})^{[n+1]}))$. The latter claim is an immediate consequence of the base change property for the analytic cotangent complex in the case where $Y \in \mathrm{AnNil}_{/X}$ (and thus so do $\widetilde{Z} \in \mathrm{AnNil}_{/Z}$)), which follows readily from [12, Proposition 5.12]. In the general case where $Y \in \mathrm{AnFMP}_{/X}$, we reduce to the previous case by combining Proposition 2.28 with the observation that filtered colimits commute with finite limits in the ∞-category dAnSt_k.

We now prove the second assertion of the Corollary. Thanks to the characterization of the essential image of natural functor

$$g_\bullet^*\colon \mathrm{Pro}(\mathrm{Coh}^+(Z)) \to \lim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z)),$$

provided in Corollary 2.40, we are reduced to show that for each $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, we have a natural equivalence of pro-objects

$$\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/Y^{[n]}} \simeq \{\mathbb{L}^{\mathrm{an}}_{\widetilde{S}^{[n]}/S^{[n]}}\}_{\widetilde{S}\in(\mathrm{AnNil}_{/Z})_{/\widetilde{Z}},\ S\in(\mathrm{AnNil}_{/X})_{/Y}}.$$

The latter statement follows readily from the first part of the proof by a direct inductive argument. ☐

2.4. **Non-archimedean formal groupoids.** Let $X \in \mathrm{dAn}_k$. We start with the definition of the notion of *analytic formal groupoids over* X:

Definition 2.42. We denote by $\mathrm{AnFGrpd}(X)$ the full subcategory of the ∞-category of simplicial objects

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{AnFMP}_{/X}),$$

spanned by those objects $F \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnFMP}_{/X}$ satisfiying the following requirements:

(1) $F([0]) \simeq X$;
(2) For each $n \geq 1$, the morphism

$$F([n]) \to F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]),$$

induced by the morphisms $s^i \colon [1] \to [n]$ given by $(0,1) \mapsto (i, i+1)$, is an equivalence in $\mathrm{AnFMP}_{/X}$.

We shall refer to objects in $\mathrm{AnFGrpd}(X)$ as *analytic formal groupoids over* X.

Remark 2.43. Note that Proposition 2.28 implies that fiber products exist in $\mathrm{AnFMP}_{/X}$. Therefore, the previous definition is reasonable.

Construction 2.44. Thanks to Lemma 2.29, there exists a well defined functor $\Phi \colon \mathrm{AnFMP}_{X/} \to \mathrm{AnFGrpd}(X)$ given by the formula

$$(X \to Y) \in \mathrm{AnFMP}_{X/} \mapsto Y^{\wedge}_X \in \mathrm{AnFGrpd}(X),$$

where $Y^{\wedge}_X \in \mathrm{AnFGrpd}(X)$ denotes the analytic formal groupoid over X admitting

$$\cdots \; \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \; X \times_Y X \times_Y X \; \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \; X \times_Y X \rightrightarrows X \ ,$$

as simplicial presentation.

Construction 2.45. Let $\mathcal{G} \in \mathrm{AnFGrpd}(X)$. Consider the *$k$-analytic classifying pre-stack*, $\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}} \in \mathrm{dAnSt}_k$, obtained as the geometric realization of the simplicial object $\mathcal{G}$, regarded naturally as a functor

$$\mathcal{G} \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{dAnSt}_k.$$

Given any $Z \in \mathrm{dAfd}_k$, the *space of Z-points of* $\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}$,

$$\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}(Z),$$

can be identified with the space whose objects correspond to the datum of:

(1) A morphism $\widetilde{Z} \to X$, where $\widetilde{Z} \in \mathrm{dAnSt}_k$, such that

$$\widetilde{Z} \simeq Z \times_{\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}} X;$$

(2) A morphism of groupoid-objects

$$\widetilde{Z} \times_Z \widetilde{Z} \to \mathcal{G},$$

in the ∞-category dAnSt_k.

We now define $\mathrm{B}_X(\mathcal{G}) \to \mathrm{B}_X^{\mathrm{pre}}(\mathcal{G})$ as the sub-object spanned by those connected components of $\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}$ corresponding to morphisms $\widetilde{Z} \to Z$ in Construction 2.45 (i) which exhibit $\widetilde{Z} \in \mathrm{AnFMP}_{/Z}$. Denote by

$$\mathrm{can} \colon \mathrm{B}_X(\mathcal{G}) \to \mathrm{B}_X(\mathcal{G})^{\mathrm{pre}},$$

the canonical morphism. It follows from the constructions that the natural morphism

$$X \to \mathrm{B}_X(\mathcal{G})^{\mathrm{pre}},$$

factors as $X \to \mathrm{B}_X(\mathcal{G}) \xrightarrow{\mathrm{can}} \mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}$.

Lemma 2.46. *The natural morphism $X \to \mathrm{B}_X(\mathcal{G})$ exhibits the latter as an object in the ∞-category* $\mathrm{AnFMP}_{X/}$ *of analytic formal moduli problems under X.*

Proof. Thanks to Proposition 2.23 it suffices to prove that $\mathrm{B}_X(\mathcal{G})$ is infinitesimally cartesian and it admits furthermore a pro-cotagent complex. The fact that $\mathrm{B}_X(\mathcal{G})$ is infinitesimally cartesian follows from the modular description of $\mathrm{B}_X(\mathcal{G})$ combined with the fact that $\mathcal{G}$ is infinitesimally cartesian, as well. Similarly, $\mathrm{B}_X(\mathcal{G})$ being nilcomplete follows again from its modular description combined with the fact that analytic formal moduli problems are nilcomplete.

We are thus required to show that $\mathrm{B}_X(\mathcal{G})$ admits a *global* pro-cotangent complex. Let $Z \in \mathrm{dAn}_k$ and suppose we are given an arbitrary morphism

$$q\colon Z \to \mathrm{B}_X(\mathcal{G}),$$

in the ∞-category dAnSt_k. Thanks to Corollary 2.40 combined with Corollary 2.41 it follows that the object

$$\{\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/\mathcal{G}^{[n]}}\}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \in \lim_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Pro}^{\mathrm{ps}}(\mathrm{Coh}^+(\widetilde{Z}^{\bullet}/Z)),$$

defines a well defined object $\mathbb{L}^{\mathrm{an}'}_{Z/\mathrm{B}_X(\mathcal{G})} \in \mathrm{Pro}(\mathrm{Coh}^+(Z))$. Moreover, it is clear that there exists a natural morphism

$$\theta\colon \mathbb{L}^{\mathrm{an}}_Z \to \mathbb{L}^{\mathrm{an}'}_{Z/\mathrm{B}_X(\mathcal{G})},$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(Z))$. Let

$$q^*\mathbb{L}^{\mathrm{an}'}_{\mathrm{B}_X(\mathcal{G})} := \mathrm{fib}(\theta),$$

computed in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(Z))$. We claim that $q^*\mathbb{L}^{\mathrm{an}'}_{\mathrm{B}_X(\mathcal{G})}$ identifies with the analytic cotangent complex of $\mathrm{B}_X(\mathcal{G})$ at the point $q\colon Z \to \mathrm{B}_X(\mathcal{G})$. Let

$$Z \hookrightarrow Z',$$

denote a square-zero extension which corresponds to a certain analytic derivation

$$d\colon \mathbb{L}^{\mathrm{an}}_Z \to \mathcal{F}[1],$$

for some $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$. Using Corollary 2.37 we deduce that the space of cartesian squares of the form

$$\begin{tikzcd} \widetilde{Z} \arrow[r] \arrow[d] & \widetilde{Z}' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

where the morphism $\widetilde{Z} \to \widetilde{Z}'$ is a square-zero extension in the ∞-category dAnSt_k is equivalent to the space of factorizations

$$d\colon g^*\mathbb{L}^{\mathrm{an}}_Z \to \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \xrightarrow{d'} g^*(\mathcal{F})[1],$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}))$. Apply the same reasoning to the each object in the Čech nerve

$$\widetilde{Z}^{\bullet} \to Z.$$

Furthermore, Corollary 2.38 implies that the space of factorizations

$$\widetilde{Z}^{\bullet} \to (\widetilde{Z}')^{\bullet} \to \mathcal{G}^{\bullet},$$

identifies with the space of factorizations

$$d'\colon \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \to \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}/\mathcal{G}} \to g^*(\mathcal{F})[1],$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}))$. By pseudo-pro-nil-descent we then deduce that the above factorization space identified with the space of factorizations

$$d\colon \mathbb{L}^{\mathrm{an}}_Z \to \mathbb{L}^{\mathrm{an},'}_{Z/\mathrm{B}_X(\mathcal{G})} \to \mathcal{F}[1].$$

This implies that $\mathbb{L}^{\mathrm{an},'}_{Z/\mathrm{B}_X(\mathcal{G})}$ satisfies the universal property of the relative analytic pro-cotangent complex, as desired. ☐

Theorem 2.47. *The functor* $\Phi\colon \mathrm{AnFMP}_{/X} \to \mathrm{AnFGrpd}(X)$ *of Construction 2.44 is an equivalence of* ∞*-categories.*

Proof. Let $\mathcal{G} \in \mathrm{AnFGrpd}(X)$. Thanks to (1) in Construction 2.45 it follows that one has a canonical equivalence

$$X \times_{\mathrm{B}_X(\mathcal{G})} X \simeq \mathcal{G},$$

in dAnSt_k. This shows that the construction

$$\mathrm{B}_X(\mathcal{G})\colon \mathrm{AnFGrpd}(X) \to \mathrm{AnFMP}_{X/},$$

is a right inverse to Φ. As a consequence the functor Φ is essentially surjective. By the same reasoning we deduce that given $X \to Y$ in $\mathrm{AnFMP}_{X/}$ the natural morphism

$$Y \to \mathrm{B}_X(Y \times_X Y),$$

is also an equivalence in $\mathrm{dAnSt}_k DERIVEDNON-ARCHIMEDEANANALYTICHILBERTSPAC$. ☐

2.5. **The affinoid case.** Let $X \in \mathrm{dAfd}_k$ denote a derived k-affinoid space. Thanks to derived Tate aciclycity theorem, cf. [13, Theorem 3.1] the *global sections functor*

$$\Gamma\colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(A),$$

where $A \coloneqq \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X) \in \mathrm{CAlg}_k$, is an equivalence of ∞-categories. Since ordinary k-affinoid algebras are Noetherian, we deduce that $A \in \mathrm{CAlg}_k$ is a Noetherian derived k-algebra.

Notation 2.48. Let $X \in \mathrm{dAfd}_k$ and $A \coloneqq \Gamma(X, \mathcal{O}_X)$. We denote by

$$\mathrm{FMP}_{/\operatorname{Spec} A} \in \mathrm{Cat}_\infty,$$

the ∞-category of *algebraic formal moduli problems over* $\operatorname{Spec} A$, (c.f. [15, Definition 6.11]).

Theorem 2.49. ([15, Theorem 6.12]) *Let* $X \in \mathrm{dAfd}_k$ *and* $A \coloneqq \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X) \in \mathrm{CAlg}_k$. *Then the induced functor*

$$(-)^{\mathrm{an}}\colon \mathrm{FMP}_{/\operatorname{Spec} A} \to \mathrm{AnFMP}_{/X},$$

is an equivalence of ∞*-categories.*

As an immediate consequence, we obtain the following result:

Corollary 2.50. *Let* $X \in \mathrm{dAfd}_k$ *and* $A \coloneqq \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X)$. *Then one has an equivalence of* ∞*-categories*

$$\mathrm{FMP}_{\operatorname{Spec} A// \operatorname{Spec} A} \to \mathrm{AnFMP}_{X//X},$$

of pointed algebraic formal moduli problems over $\operatorname{Spec} A$ *and pointed analytic formal moduli problems over* X*, respectively.*

Proof. It is an immediate consequence of Theorem 2.49. Indeed, equivalences of ∞-categories with final objects induce natural equivalences on the associated ∞-categories of pointed objects. ☐

Corollary 2.51. *Let* $X \in \mathrm{dAfd}_k$. *Then both the* ∞*-categories* $\mathrm{AnFGrpd}(X)$ *and* $\mathrm{AnFMP}_{X/}$ *admit sifted colimits.*

Proof. Thanks to Theorem 2.47 we are reduced to prove solely that $\mathrm{AnFGrpd}(X)$ admits sifted colimits. Let $F\colon I \to \mathrm{AnFGrpd}(X)$ denote a functor, where I is a sifted ∞-category. Then, for each $i \in I$ and $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ we have that

$$F(i)_{[n]} \in \mathrm{Ptd}(\mathrm{AnFMP}_{/X}).$$

Since the ∞-category $\mathrm{Ptd}(\mathrm{FMP}_{/\operatorname{Spec} A})$ admits sifted colimits, see for instance [6, §5, Corollary1.6.6], we deduce thanks to Corollary 2.50 that so does the ∞-category $\mathrm{Ptd}(\mathrm{AnFMP}_{/X})$. For this reason, we deduce that for each $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, the object

$$Y_{[n]} \coloneqq \operatorname*{colim}_{i \in I} F(i)_{[n]} \in \mathrm{Ptd}(\mathrm{AnFMP}_{/X}),$$

is well defined. Moreover, the simplicial object

$$\{Y_{[n]}\}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}},$$

in the ∞-category $\mathrm{Ptd}(\mathrm{AnFMP}_{/X})$ forms an analytic formal groupoid over X, since sifted colimits commute with finite products. Moreover, it follows from the definitions that the object $\{Y_{[n]}\}_{[n]}$ is a sifted colimit of the diagram $F\colon I \to \mathrm{AnFGrpd}(X)$, as desired. □

Let $Y \in \mathrm{AnFMP}_{X/}$, we can thus identify the associated relative pro-cotangent complex

$$\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(A)).$$

Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, "="'] & X \times_Y X \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_0"] \\ & X \end{tikzcd}$$

in the ∞-category $\mathrm{AnPreStk}$, where $\Delta\colon X \to X \times_Y X$ denotes the usual *diagonal embedding*. We then obtain a natural fiber sequence associated to the above diagram of the form

$$\Delta^* \mathbb{L}^{\mathrm{an}}_{X \times_Y X/X} \to \mathbb{L}^{\mathrm{an}}_{X/X} \to \mathbb{L}^{\mathrm{an}}_{X/X \times_Y X}. \tag{2.13}$$

Notice further that by [12, Proposition 5.12] one has an equivalence

$$\mathbb{L}^{\mathrm{an}}_{X \times_Y X/X} \simeq p_i^* \mathbb{L}^{\mathrm{an}}_{X/Y},$$

for $i = 0, 1$. We further deduce that

$$\mathbb{L}^{\mathrm{an}}_{X \times_Y X/X} \simeq \mathbb{L}^{\mathrm{an}}_{X/Y},$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(A))$. Moreover, since $\mathbb{L}^{\mathrm{an}}_{X/X} \simeq 0$, we obtain from the fiber sequence (2.13) a natural equivalence

$$\mathbb{L}^{\mathrm{an}}_{X/X \times_Y X} \simeq \mathbb{L}^{\mathrm{an}}_{X/Y}[1],$$

in $\mathrm{Pro}(\mathrm{Coh}^+(A))$. Moreover, we can identify the $\mathbb{L}^{\mathrm{an}}_{X \times_Y X/X}$ with the pro-object

$$\mathbb{L}^{\mathrm{an}}_{X/X \times_Y X} \simeq \{\mathbb{L}^{\mathrm{an}}_{X/X \times_S X}\}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}},$$

where $\mathbb{L}^{\mathrm{an}}_{X/X \times_S X} \in \mathrm{Coh}^+(A)$ denotes the relative analytic cotangent complex associated to the closed embedding

$$X \to X \times_S X,$$

for $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X/Y}$. Thanks to [12, Corollary 5.33] we deduce that

$$\mathbb{L}^{\mathrm{an}}_{X/X \times_S X} \simeq \mathbb{L}_{A/A \widehat{\otimes}_{B_S} A},$$

in $\mathrm{Coh}^+(A)$, where $B_S = \Gamma(S, \mathcal{O}_S^{\mathrm{alg}})$. (Personal: Notice that we can remove the hat in the previous tensor product since under our assumptions $A \in \mathrm{Coh}^+(B_S)$, for every such considered S.)

Notation 2.52. Let $X \in \mathrm{dAfd}_k$. We shall denote by

$$\mathrm{QCoh}(X) \coloneqq \mathrm{Mod}_A,$$

where $A \coloneqq \Gamma(X, \mathcal{O}_X^{\mathrm{alg}})$. The latter can be naturally identified with the stable ∞-category $\mathrm{Ind}(\mathrm{Perf}(A))$.

Notation 2.53. Let $X \in \mathrm{dAfd}_k$. We shall denote the *plain duality functor* as

$$(-)^\vee \colon \mathrm{Coh}^+(X)^{\mathrm{op}} \to \mathrm{QCoh}(X),$$

which is given on objects by the formula

$$\mathcal{F} \in \mathrm{Coh}(X)^{\mathrm{op}} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{F}, \mathcal{O}_X^{\mathrm{alg}}) \in \mathrm{Perf}(X).$$

We can extend the latter functor via filtered colimits to a functor on ind-completions:

$$\begin{aligned}(-)^\vee \colon \mathrm{Pro}(\mathrm{Coh}^+(X))^{\mathrm{op}} &\simeq \mathrm{Ind}(\mathrm{Coh}^+(X)^{\mathrm{op}}) \\ &\simeq \mathrm{QCoh}(X).\end{aligned}$$

The latter associates to each ind-colimit

$$\begin{aligned}\mathcal{F} &\simeq \operatorname*{colim}_{i \in I} \mathcal{F}_i \\ &\in \mathrm{Ind}(\mathrm{Coh}^+(X)^{\mathrm{op}}),\end{aligned}$$

where I is a filtered ∞-category and for each $i \in I$, $\mathcal{F}_i \in \mathrm{Coh}^+(X)$, the filtered colimit object

$$\operatorname*{colim}_{i \in I} \mathcal{F}_i^\vee$$

computed in the presentable ∞-category $\mathrm{QCoh}(X)$.

Definition 2.54 (Tangent complex)**.** Let $Y \in \mathrm{AnFMP}_{X/}$. We define the *relative analytic tangent complex* of $X \to Y$ as the object

$$\begin{aligned}\mathbb{T}^{\mathrm{an}}_{X/Y} &\coloneqq (\mathbb{L}^{\mathrm{an}}_{X/Y})^\vee \\ &\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} (\mathbb{L}^{\mathrm{an}}_{X/S})^\vee,\end{aligned}$$

in $\mathrm{QCoh}(X)$.

The following result will play a major role in the study of the deformation to the normal bundle:

Proposition 2.55. *The functor* $\mathbb{T}^{\mathrm{an}}_{X/\bullet} \colon \mathrm{AnFMP}_{X/} \to \mathrm{QCoh}(X)$ *given on objects by the formula*

$$(X \to Y) \in \mathrm{AnFMP}_{X/} \mapsto \mathbb{T}^{\mathrm{an}}_{X/Y} \in \mathrm{QCoh}(X),$$

is conservative and commutes with sifted colimits.

Proof. We start by first proving that $\mathbb{T}^{\mathrm{an}}_{X/\bullet}$ is conservative. Let $Y \in \mathrm{AnFMP}_{X/}$ and let $F \in \mathrm{Perf}(X)$ be a perfect complex on X (as an immediate consequence of derived Tate aciclycity one has that $\mathrm{Perf}(X) \simeq \mathrm{Perf}(A)$).

In this case, we have a chain of natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(X)}(F, \mathbb{T}^{\mathrm{an}}_{X/Y}) &\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \mathrm{Map}_{\mathrm{QCoh}(X)}(F, \mathbb{T}^{\mathrm{an}}_{X/S}) \\
&\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \mathrm{Map}_{\mathrm{QCoh}(X)}(F \otimes_{\mathcal{O}_X} \mathbb{L}^{\mathrm{an}}_{X/S}, \mathcal{O}_X) \\
&\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathbb{L}^{\mathrm{an}}_{X/S}, F^\vee) \\
&\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/S}, F^\vee) \\
&\simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X//Y}, F^\vee),
\end{aligned}$$

in the ∞-category $\mathcal{S}$. Let $f\colon Y \to Z$ be a morphism in the ∞-category $\mathrm{AnFMP}_{X/}$ such that f induces an equivalence at the level of the relative analytic tangent complexes

$$f_*\colon \mathbb{T}^{\mathrm{an}}_{X/Y} \to \mathbb{T}^{\mathrm{an}}_{X/Z}.$$

Then, from the previous considerations we deduce that for every $F \in \mathrm{Perf}(X)$ the induced morphism of mapping spaces

$$\mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, F) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Z}, F)$$

is an equivalence. Moreover, thanks to [9, Corollary 5.5.6.22] we deduce that for every $m \in \mathbb{Z}$ and every $F \in \mathrm{Perf}(X)$ we have a chain of natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \tau_{\leq m}(F)) &\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/S}, \tau_{\leq m}(F)) \\
&\simeq \operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \tau_{\leq m} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/S}, F) \\
&\simeq \tau_{\leq m}\big(\operatorname*{colim}_{S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/S}, F)\big) \\
&\simeq \tau_{\leq m}\big(\mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, F)\big) \\
&\simeq \tau_{\leq m}\big(\mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Z}, F)\big) \\
&\simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Z}, F),
\end{aligned}$$

of mapping spaces. Since every bounded almost perfect $\mathcal{O}_X$-module $\mathcal{F} \in \mathrm{Coh}^+(X)$ is a retract of a truncation of a perfect $\mathcal{O}_X$-module we deduce that one has an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Z}, \mathcal{F}),$$

for every $\mathcal{F} \in \mathrm{Coh}^{\mathrm{b}}(X)$. Conservativity of the functor $\mathbb{T}^{\mathrm{an}}_{X/\bullet}$ now follows immediately from Corollary 2.38 combined with [12, Corollary 5.40] and the fact that both $Y, Z \in \mathrm{AnFMP}_{X/}$ are nilcomplete. We shall now prove that $\mathbb{T}^{\mathrm{an}}_{X/\bullet}$ commutes with sifted colimits. Thanks to Theorem 2.47 we are reduced to show that

$$\mathbb{T}^{\mathrm{an}}_{X/X \times_\bullet X}\colon \mathrm{AnFMP}_{X/} \to \mathrm{Ind}(\mathrm{Coh}^+(A)),$$

preserves sifted colimits. Moreover, given any $Y \in \mathrm{AnFMP}_{X/}$, each projection morphism

$$p_i\colon X \times_Y X \to X,$$

exhibits the latter as an object in $\mathrm{AnFMP}_{/X}$. Since $X \in \mathrm{dAfd}_k$, [15, Theorem 6.12] implies that one has an equivalence of ∞-categories

$$(-)^{\mathrm{an}}\colon \mathrm{FMP}_{/\operatorname{Spec} A} \simeq \mathrm{AnFMP}_{/X}.$$

By (2.13) we are reduced to show that

$$\Delta^! \mathbb{T}_{X \times_\bullet X/X},$$

commutes with sifted colimits. functor $\mathbb{T}^{\mathrm{an}}_{X/\bullet}$ can be identified with the functor

$$\mathbb{T}^{\mathrm{an}}_{X/\bullet}$$

But $\mathbb{T}^{\mathrm{an}}{}_{X/X\times_Y X}$ is defined as the Serre dual of

$$\mathbb{L}^{\mathrm{an}}_{X/X\times_\bullet X} \simeq \mathbb{L}_{A/A\otimes_\bullet A},$$

in $\mathrm{Ind}(\mathrm{Coh}^+(A))$. It follows from [6, §5, Corollary 2.2.4] that the latter preserves sifted colimits. The assertion now follows from the fact that the analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{Ind}(\mathrm{Coh}^+(A)) \to \mathrm{Ind}(\mathrm{Coh}^+(A)),$$

is a left adjoint, and thus commutes with sifted colimits. ☐

3. Non-archimedean Deformation to the normal bundle

In this § we introduce the construction of the deformation to the normal cone in the setting of derived k-analytic geometry. This has already been studied in [15] in the case where we consider the natural morphism $f \colon X_{\mathrm{red}} \to X$, for some $X \in \mathrm{dAn}_k$. The situation in the algebraic case was extensively studied in [6].

3.1. **General construction in the algebraic case.** Consider the object

$$\mathrm{B}^\bullet_{\mathrm{scaled}} \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{dSt}_k),$$

introduced in [6, §9.2.2]. Recall that $\mathrm{B}^n_{\mathrm{scaled}}$ is obtained by gluing $n+1$ copies of $\mathbb{A}^1_k$ together along $0 \in \mathbb{A}^1_k$.

Construction 3.1. Let $f \colon X \to Y$ denote a morphism in the ∞-category $\mathrm{dSt}^{\mathrm{laft}}_k$. In [6, §9.3], the authors introduced the *deformation to the normal bundle associated to the morphism* $f \colon X \to Y$, as the pullback

$$\begin{tikzcd}
\mathcal{D}^\bullet_{X/Y} \arrow[r] \arrow[d] & Y \times \mathbb{A}^1_k \arrow[d] \\
\mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, X \times \mathbb{A}^1_k) \arrow[r] & \mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, Y \times \mathbb{A}^1_k),
\end{tikzcd}$$

where both $X \times \mathbb{A}^1_k$ and $Y \times \mathbb{A}^1_k$ are considered as constant simplicial objects in the ∞-category $\mathrm{dSt}^{\mathrm{laft}}_k$. Suppose now that the morphism $f \colon X \to Y$ is a closed immersion in $\mathrm{dSt}^{\mathrm{laft}}_k$ such that both X and Y admit a *deformation theory* in the sense of [6, §1].

The authors proved in [6, Theorem 9.2.3.4] that each component of the simplicial object $\mathcal{D}^\bullet_{X/Y}$ admits a deformation theory itself. In particular, the object $\mathcal{D}^\bullet_{X/Y}$ defines a formal groupoid over $X \times \mathbb{A}^1_k$. Moreover, Theorem 5.2.3.4 in loc. cit. allows us to associate to $\mathcal{D}^\bullet_{X/Y}$ a formal moduli problem under $X \times \mathbb{A}^1_k$

$$\mathcal{D}_{X/Y} \in \mathrm{FMP}_{X \times \mathbb{A}^1_k/},$$

obtained via the construction provided in §5.2.4 in loc. cit. Moreover, by construction the object $\mathcal{D}_{X/Y} \in (\mathrm{dSt}^{\mathrm{laft}}_k)_{/\mathbb{A}^1_k}$.

Notation 3.2. Notice that the object $\mathcal{D}_{X/Y} \in \mathrm{FMP}_{X\times\mathbb{A}^1_k/}$ admits a natural morphism to $Y \times \mathbb{A}^1_k$. We shall denote from now on the ∞-category of formal moduli problems under $X \times \mathbb{A}^1_k$ together with a morphism to $Y \times \mathbb{A}^1_k$, in $\mathrm{dSt}^{\mathrm{laft}}_k$, by $\mathrm{FMP}_{X\times\mathbb{A}^1_k//Y\times\mathbb{A}^1_k}$.

We shall now describe certain formal properties of the object $\mathcal{D}_{X/Y} \in \mathrm{FMP}_{X\times\mathbb{A}^1_k//Y\times\mathbb{A}^1_k}$:

Proposition 3.3. *Let $f \colon X \to Y$ denote a closed immersion in the ∞-category $\mathrm{dSt}_k^{\mathrm{laft}}$. The following assertions hold:*

(1) The fiber $(\mathcal{D}_{X/Y})_0$ at 0 of the morphism structural morphism $\mathcal{D}_{X/Y} \to \mathbb{A}^1_k$ identifies with the formal moduli problem $\mathrm{T}_{X/Y}[-1]^\wedge$ obtained from the shifted tangent bundle $\mathrm{T}_{X/Y}[-1] \to X$ by completing along the zero section

$$s_0 \colon X \to \mathrm{T}_{X/Y}[-1].$$

(2) The fiber $(\mathcal{D}_{X/Y})_\lambda$ for $\lambda \neq 0$ identifies with the formal completion

$$Y^\wedge_X \in \mathrm{dSt}_k,$$

along the morphism f.

(3) There exists a natural sequence of formal moduli problems

$$X \times \mathbb{A}^1_k = \mathcal{D}^{(0)}_{X/Y} \to \mathcal{D}^{(1)}_{X/Y} \to \cdots \to \mathcal{D}^{(n)}_{X/Y} \to \cdots \to \cdots \to Y,$$

in $\mathrm{FMP}_{X \times \mathbb{A}^1_k // Y \times \mathbb{A}^1_k}$ such that, for each $n \geq 0$, the morphism

$$\mathcal{D}^{(n)}_{X/Y} \to \mathcal{D}^{(n+1)}_{X/Y},$$

has the structure of a square-zero extension by an element in $\mathrm{Coh}^+(\mathcal{D}^{(n)}_{X/Y})^{\geq n}$. In particular, if $X \in \mathrm{dSt}_k^{\mathrm{laft}}$ is a geometric stack, then so it is each $\mathcal{D}^{(n)}_{X/Y}$.

(4) The natural morphism

$$\operatorname*{colim}_n \mathcal{D}^{(n)}_{X/Y} \to \mathcal{D}_{X/Y},$$

in the ∞-category $\mathrm{FMP}_{X \times \mathbb{A}^1_k // Y \times \mathbb{A}^1_k}$ is an equivalence (and thus so it is when computed in the ∞-category $(\mathrm{dSt}_k^{\mathrm{laft}})_{X \times \mathbb{A}^1_k // Y \times \mathbb{A}^k}$). In particular, if $X \in \mathrm{dSt}_k^{\mathrm{laft}}$ is a geometric stack then it follows by (iii) that the colimit

$$\mathcal{D}_{X/Y} \simeq \operatorname*{colim}_n \mathcal{D}^{(n)}_{X/Y},$$

is also geometric.

Remark 3.4. The existence of the object $\{\mathcal{D}^{(n)}_{X/Y}\}_{n \geq 0}$ define a filtration on the global sections of the formal completion $Y^\wedge_X$, which we shall refer to as the *Hodge filtration associated to the morphism f*.

Remark 3.5. (Todo: Probably it is better to have this as a Corollary.) As an immediate of the above Proposition we deduce that the relative algebraic de rham cohomology associated to the morphism $f \colon X \to Y$ can be computed as the derived adic completion global section of the formal completion $Y^\wedge_X$. In particular, the latter does not depend on the underlying derived structure on both X and Y. Nonetheless, the *Hodge filtration* considered depends heavily on the derived structure of hte latter objects in $\mathrm{dSt}_k^{\mathrm{laft}}$.

3.2. **The construction of the deformation in the affinoid case.** The goal in this section is to study the deformation to the normal cone in the non-archimedean setting. Let us assume that we are given a closed immersion

$$f \colon X \to Y,$$

in the ∞-category dAfd_k. Consider the object

$$\mathbf{B}^{\mathrm{an}, \bullet}_{\mathrm{scaled}} \colon \mathbf{\Delta}^{\mathrm{op}} \to (\mathrm{dAnSt}_k)_{/\mathbf{A}^1_k(0)},$$

obtained as the analytification of the simplicial object $\mathrm{B}^{\bullet}_{\mathrm{scaled}} \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{dSt}_k^{\mathrm{laft}})$ described in the previous section. We define the deformation to the normal bundle via the pullback diagram

$$\begin{tikzcd} \mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \arrow[r] \arrow[d] & Y \times \mathbf{A}^1_k \arrow[d] \\ \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\mathrm{an},\bullet}, X \times \mathbf{A}^1_k) \arrow[r] & \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\mathrm{an},\bullet}, Y \times \mathbf{A}^1_k) \end{tikzcd},$$

computed in the ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{dAnSt}_k)$. Consider the natural projection map

$$p \colon \mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \to \mathbf{A}^1_k.$$

We now proceed to identifiy its fiber at $\lambda = 0$.

Notation 3.6. Let $A := \Gamma(Y, \mathcal{O}^{\mathrm{alg}}_Y)$ and $B := \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X)$ and define $Y^{\mathrm{alg}} := \mathrm{Spec}\, A$ and $X^{\mathrm{alg}} := \mathrm{Spec}\, B$.

We have an induced closed immersion

$$f^{\mathrm{alg}} \colon X^{\mathrm{alg}} \to Y^{\mathrm{alg}},$$

of Noetherian affine schemes. Consider the object $\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{dAff}_k)$ as in Construction 3.1. Thanks to [11, Proposition 4.6.1.3] we can find a presentation

$$\lim_{\alpha \in A^{\mathrm{op}}} f_\alpha \colon \lim_{\alpha \in A^{\mathrm{op}}} X^{\mathrm{alg}}_\alpha \to \lim_{\alpha \in A^{\mathrm{op}}} Y^{\mathrm{alg}}_\alpha,$$

such that A is a filtered ∞-category and for each $\alpha \in A$, the morphism

$$f_\alpha \colon X^{\mathrm{alg}}_\alpha \to Y^{\mathrm{alg}}_\alpha,$$

is a closed immersion of almost of finite presentation affine schemes

$$X^{\mathrm{alg}}_\alpha, Y^{\mathrm{alg}}_\alpha \in \mathrm{dAff}^{\mathrm{laft}}.$$

It is clear from the descritption of $\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}$ given in Construction 3.1 that one has a natural equivalence of derived stacks

$$\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \simeq \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}^{\bullet}_{X^{\mathrm{alg}}_\alpha/Y^{\mathrm{alg}}_\alpha},$$

in the ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{dSt}_k)$. Recall further the notion of *relative analytification* introduced in [15, §6.1]. We shall first need a preliminary lemma:

Lemma 3.7. *Let $f \colon X \to Y$ be a closed immersion in dAfd_k. Then one has a natural equivalence*

$$(f^{\mathrm{alg}})^{\mathrm{an}}_Y \simeq f,$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{dAfd}_k)$.

Proof. Let $f^{\mathrm{alg}} \colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ be the induced morphism, where $B := \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X)$ and $A := \Gamma(Y, \mathcal{O}^{\mathrm{alg}}_Y)$. It follows readily from the definitions that the morphism f^{alg} of derived affine schemes is a closed immersion. Moreover, we have a natural identification

$$(\mathrm{Spec}(A)^{\mathrm{alg}})^{\mathrm{an}}_Y \simeq Y.$$

For this reason, the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "\simeq"] \\ (X^{\mathrm{alg}})^{\mathrm{an}}_Y \arrow[r, "(f^{\mathrm{alg}})^{\mathrm{an}}_Y"] & (Y^{\mathrm{alg}})^{\mathrm{an}}_Y \end{tikzcd},$$

induces an equivalence at the level of global sections

$$\Gamma(X, \mathcal{O}_X^{\mathrm{alg}}) \simeq \Gamma((X^{\mathrm{alg}})_Y^{\mathrm{an}}, \mathcal{O}^{\mathrm{alg}}_{(X^{\mathrm{alg}})_Y^{\mathrm{an}}}).$$

The result now follows immediately from the fact that both X and $(X^{\mathrm{alg}})_Y^{\mathrm{an}}$ are derived k-affinoid spaces together with the derived Tate acyclity's theorem, c.f. [13, Theorem 3.1]. □

Lemma 3.8. *Let notations be as above. Then one has a natural equivalence*

$$\mathcal{D}_{X/Y}^{\mathrm{an},\bullet} \simeq (\mathcal{D}^\bullet_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})_Y^{\mathrm{an}},$$

where $(-)_Y^{\mathrm{an}}$ *denotes the relative analytification with respect to* Y*.*

Proof. As before, write the closed immersion of affine schemes

$$f\colon X^{\mathrm{alg}} \to Y^{\mathrm{alg}},$$

as a cofiltered limit

$$\lim_\alpha f_\alpha \colon \lim_\alpha X_\alpha^{\mathrm{alg}} \to \lim_\alpha Y_\alpha^{\mathrm{alg}},$$

where for each index $\alpha \in A$, both X_α and Y_α are affine schemes of almost of finite type. Since the natural projection morphism

$$\mathrm{B}^\bullet_{\mathrm{scaled}} \to \mathbb{A}^1_k,$$

where we regard $\mathbb{A}^1_k$ as a constant simplicial object in the ∞-category dSt_k, is a proper morphism, we deduce from [8, Theorem 6.13] that for every α, the natural morphisms

$$\mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, X_\alpha^{\mathrm{alg}} \times \mathbb{A}^1_k)^{\mathrm{an}} \to \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\bullet,\mathrm{an}}_{\mathrm{scaled}}, (X_\alpha^{\mathrm{alg}})^{\mathrm{an}} \times \mathbf{A}^1_k)$$
$$\mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, Y_\alpha^{\mathrm{alg}} \times \mathbb{A}^1_k)^{\mathrm{an}} \to \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\bullet,\mathrm{an}}_{\mathrm{scaled}}, (Y_\alpha^{\mathrm{alg}})^{\mathrm{an}} \times \mathbf{A}^1_k)$$

are equivalences in the ∞-category dAnSt_k. Therefore, the natural morphism

$$(\mathcal{D}^\bullet_{X_\alpha^{\mathrm{alg}}/Y_\alpha^{\mathrm{alg}}})^{\mathrm{an}} \to \mathcal{D}^{\bullet,\mathrm{an}}_{(X_\alpha^{\mathrm{alg}})^{\mathrm{an}}/(Y_\alpha^{\mathrm{alg}})^{\mathrm{an}}},$$

is an equivalence in the ∞-category dAnSt_k. Observe further that for every $\alpha \in A$ the mapping stacks

$$\mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, X_\alpha^{\mathrm{alg}} \times \mathbb{A}^1_k) \quad \text{and} \quad \mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, Y_\alpha^{\mathrm{alg}} \times \mathbb{A}^1_k)$$

are affine schemes and therefore it follows by the construction of the analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{dAff}_k \to \mathrm{dAnSt}_k,$$

as a right Kan extension of the usual analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{dAff}_k^{\mathrm{laft}} \to \mathrm{dAn}_k,$$

that we have natural equivalences

$$(\lim_\alpha \mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, X_\alpha^{\mathrm{alg}} \times \mathbb{A}^1_k))^{\mathrm{an}} \simeq \mathbf{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^{\bullet,\mathrm{an}}_{\mathrm{scaled}}, (X^{\mathrm{alg}})_Y^{\mathrm{an}} \times \mathbf{A}^1_k)$$
$$(\lim_\alpha \mathrm{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, Y_\alpha^{\mathrm{alg}} \times \mathbb{A}^1_k))^{\mathrm{an}} \simeq \mathbf{Map}_{/\mathbb{A}^1_k}(\mathrm{B}^{\bullet,\mathrm{an}}_{\mathrm{scaled}}, (Y^{\mathrm{alg}})_Y^{\mathrm{an}} \times \mathbf{A}^1_k).$$

in the ∞-category $(\mathrm{dAnSt}_k)_{X \times \mathbb{A}^1 // Y \times \mathbb{A}^1_k}$. The result now follows from the existence of a commutative cube

$$\begin{tikzcd}
(\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})^{\mathrm{an}}_Y \arrow[rr] \arrow[dr] \arrow[dd] & & Y \times \mathbf{A}^1_k \arrow[dr] \arrow[dd] & \\
& (\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})^{\mathrm{an}} \arrow[rr] \arrow[dd] & & (Y^{\mathrm{alg}} \times \mathbb{A}^1_k)^{\mathrm{an}} \arrow[dd] \\
\mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\bullet}, X \times \mathbf{A}^1_k) \arrow[rr] \arrow[dr] & & \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\bullet}, Y \times \mathbf{A}^1_k) \arrow[dr] & \\
& \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\bullet}, (X^{\mathrm{alg}})^{\mathrm{an}} \times \mathbf{A}^1_k) \arrow[rr] & & \mathbf{Map}_{/\mathbf{A}^1_k}(\mathbf{B}^{\bullet}, (Y^{\mathrm{alg}})^{\mathrm{an}} \times \mathbf{A}^1_k)
\end{tikzcd}$$

whose top and front squares are fiber products in dAnSt_k, and thus so it is the back square, as desired. □

Corollary 3.9. *Let $f \colon X \to Y$ be a closed morphism in dAfd_k. Then one has a natural identification*

$$(\mathcal{D}^{\bullet,\mathrm{an}}_{X/Y})_0 \simeq (\mathbf{T}^{\mathrm{an},\bullet}_{X/Y})^{\wedge}[-1],$$

where the latter denotes the commutative group object associated to the formal completion of the tangent bundle of f along the zero section $s_0 \colon X \to \mathbf{T}^{\mathrm{an},\bullet}_{X/Y}[-1]$.

Proof. The result follows from [6, Proposition 9.2.3.6] combined with the fact that relative analytification is defined via objects of almost of finite type plus the fact that analytification commutes with tangent bundles and formal completions, [8, Corollary 5.20]. (Personal: I don't think the latter reference is completely fit.) □

The following result is an immediate consequence of Lemma 3.8:

Lemma 3.10. *For each $[n] \in \mathbf{\Delta}$, the object $\mathcal{D}^{\mathrm{an},[n]}_{X/Y} \in (\mathrm{dAnSt}_k)_{X \times \mathbf{A}^1_k // Y \times \mathbf{A}^1_k}$ admits a deformation theory and it is furthermore affinoid.*

Proof. We shall prove that $\mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{AnFMP}_{X \times \mathbf{A}^1_k // Y \times \mathbf{A}^1_k})$. By [6, Lemma 2.3.2], it follows that $\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}//Y^{\mathrm{alg}}}$ is an object in $\mathrm{FMP}_{X^{\mathrm{alg}}//Y^{\mathrm{alg}}}$. The result now follows by applying the relative analytification functor $(-)^{\mathrm{an}}_Y$ together with [15, Proposition 6.10]. The fact that $\mathcal{D}^{\mathrm{an},[n]}_{X/Y}$ is affinoid, for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, follows readily from the observation that for every $\lambda \in \mathbb{A}^1_k$ the fiber

$$(\mathcal{D}^{\bullet,\mathrm{an}}_{X/Y})_\lambda \in \mathrm{dAfd}_k.$$

□

Construction 3.11. As we proved in §2.5, the ∞-category $\mathrm{AnFMP}_{X \times \mathbf{A}^1_k // Y \times \mathbf{A}^1_k}$ admits sifted colimits. Thanks to Lemma 3.10, it follows that we compute the sifted colimit

$$\mathcal{D}^{\mathrm{an}}_{X/Y} \coloneqq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \in \mathrm{AnFMP}_{X \times \mathbf{A}^1_k // Y \times \mathbf{A}^1_k}.$$

Similarly, we consider the sifted colimit

$$\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \coloneqq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \in \mathrm{FMP}_{X^{\mathrm{alg}} \times \mathbb{A}^1_k // Y^{\mathrm{alg}} \times \mathbb{A}^1_k},$$

computed in the ∞-category $\mathrm{FMP}_{X^{\mathrm{alg}} \times \mathbb{A}^1_k // Y^{\mathrm{alg}} \times \mathbb{A}^1_k}$. We can consider the latter as an object in

$$\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \in \mathrm{dSt}_{X^{\mathrm{alg}} \times \mathbb{A}^1_k // Y^{\mathrm{alg}} \times \mathbb{A}^1_k},$$

and therefore consider its relative analytification $(\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})^{\mathrm{an}}_Y \in \mathrm{dAnSt}_{X^{\mathrm{alg}} \times \mathbf{A}^1_k // Y^{\mathrm{alg}} \times \mathbf{A}^1_k}$. Thanks to Lemma 3.8 it follows that we have a natural morphism

$$\mathcal{D}^{\mathrm{an}}_{X/Y} \to (\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})^{\mathrm{an}}_Y,$$

in the ∞-category $\mathrm{dAnSt}_{X\times\mathbf{A}^1_k//Y\times\mathbf{A}^1_k}$.

Lemma 3.12. *Consider the base change functor*

$$(-)_X\colon \mathrm{AnFMP}_{X//Y}\to\mathrm{AnFMP}_{X//X},$$

given on objects by the formula

$$F\in\mathrm{AnFMP}_{X//Y}\mapsto F\times_Y X\in\mathrm{AnFMP}_{X//X}.$$

Then the functor $(-)_X$ is an equivalence of ∞-categories.

Proof. The functor is clearly a left adjoint to the usual forgetful functor along f

$$\mathrm{AnFMP}_{X//X}\to\mathrm{AnFMP}_{X//Y}.$$

It suffices to prove that for every $F\in\mathrm{AnFMP}_{X//Y}$ and $U\in\mathrm{AnNil}_{X/}$ we have a natural equivalence of mapping spaces

$$\mathrm{Map}_{X//Y}(U,F)\to\mathrm{Map}_{X//X}(U\times_Y X,F\times_Y X),$$

which is an immediate consequence of the universal property of fiber products. □

Proposition 3.13. *The natural morphism*

$$\theta_{X/Y}\colon \mathcal{D}^{\mathrm{an}}_{X/Y}\to(\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})^{\mathrm{an}}_Y,$$

is an equivalence of derived k-analytic stacks.

Proof. Consider the object $(\mathcal{D}_{X^{\mathrm{alg}}//Y^{\mathrm{alg}}})^{\mathrm{an}}_Y\in\mathrm{AnFMP}_{X//Y}$. Under the equivalence of ∞-categories provided in Lemma 3.12, we observe that

$$(\mathcal{D}_{X^{\mathrm{alg}}//Y^{\mathrm{alg}}})^{\mathrm{an}}_Y\times_Y X\simeq(\mathcal{D}_{X^{\mathrm{alg}}//Y^{\mathrm{alg}}})^{\mathrm{an}}_X,$$

where the latter denotes the relative analytification of $\mathcal{D}_{X^{\mathrm{alg}}//Y^{\mathrm{alg}}}$ along the composite morphism

$$\eta\colon X\times\mathbf{A}^1_k\to Y\times\mathbf{A}^1_k\to(Y^{\mathrm{alg}})^{\mathrm{an}}\times\mathbf{A}^1_k.$$

Thanks to Lemma 3.12 we are reduced to show that the natural morphism $\theta_{X/Y}$ induces an equivalence

$$\theta_{X/Y}\times_Y X\colon(\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}})^{\mathrm{an}}_X\to\mathcal{D}^{\mathrm{an}}_{X/Y}\times_Y X,$$

in the ∞-category $\mathrm{AnFMP}_{X\times\mathbf{A}^1_k//X\times\mathbf{A}^1_k}\simeq\mathrm{Ptd}(\mathrm{AnFMP}_{/X\times\mathbf{A}^1_k})$. Moreover, the equivalence of ∞-categories provided in [15, Theorem 6.12] implies that

$$(-)^{\mathrm{an}}_X\colon\mathrm{FMP}_{/X^{\mathrm{alg}}}\to\mathrm{AnFMP}_{/X},$$

is an equivalence of ∞-categories. In particular, we have an induced equivalence of ∞-categories

$$(-)^{\mathrm{an}}_X\colon\mathrm{Ptd}(\mathrm{FMP}_{/X^{\mathrm{alg}}\times\mathbf{A}^1_k})\to\mathrm{Ptd}(\mathrm{AnFMP}_{/X\times\mathbf{A}^1_k}).$$

Since $\mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}$ is computed as the sifted colimit of the object $\mathcal{D}^{\bullet}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}$ in $\mathrm{AnFMP}_{X^{\mathrm{alg}}\times\mathbf{A}^1_k/Y^{\mathrm{alg}}\times\mathbf{A}^1_k}$ and similarly $\mathcal{D}^{\mathrm{an}}_{X/Y}$ is computed as the sifted colimit of $\mathcal{D}^{\mathrm{an},\bullet}_{X/Y}$ in the ∞-category $\mathrm{AnFMP}_{X\times\mathbf{A}^1_k//Y\times\mathbf{A}^1_k}$.

We deduce from Lemma 3.8 that the natural morphism $\theta_{X/Y}\times_Y X$ is an equivalence in $\mathrm{AnFMP}_{X\times\mathbf{A}^1_k//X\times\mathbf{A}^1_k}$, and therefore $\theta_{X/Y}$ is an equivalence in the ∞-category $\mathrm{AnFMP}_{X\times\mathbf{A}^1_k//Y\times\mathbf{A}^1_k}$, as desired. □

Remark 3.14. Consider the natural projection morphism
$$q\colon \mathcal{D}^{\mathrm{an}}_{X/Y} \to \mathbf{A}^1_k.$$
Its fiber at $\lambda \neq 0$ coincides with the formal completion
$$Y^\wedge_X,$$
and its fiber at 0 with the completion along the zero section $s_0\colon X \to \mathbf{T}^{\mathrm{an}}_{X/Y}[-1]$ of the shifted relative tangent bundle,
$$\mathbf{T}^{\mathrm{an}}_{X/Y}[-1]^\wedge.$$

Construction 3.15. Let $g\colon U \to Y$ be a morphism in dAfd_k. Consider the pullback diagram
$$\begin{tikzcd} X_U \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$
computed in the ∞-category dAfd_k. It follows from the definitions that we have a natural pullback square of simplicial objects
$$\begin{tikzcd} \mathcal{D}^{\mathrm{an},\bullet}_{X_U/U} \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \arrow[d] \\ U \arrow[r] & Y, \end{tikzcd}$$
in the ∞-category dAnSt_k. For this reason, we obtain a natural commutative diagram
$$\begin{tikzcd} \mathcal{D}^{\mathrm{an}}_{X_U/U} \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{an}}_{X/Y} \arrow[d] \\ U \arrow[r] & Y, \end{tikzcd}$$
in the ∞-category dAnSt_k. Similarly, consider the pullback diagram
$$\begin{tikzcd} X_U^{\mathrm{alg}} \arrow[r] \arrow[d] & U^{\mathrm{alg}} \arrow[d] \\ X^{\mathrm{alg}} \arrow[r, "f^{\mathrm{alg}}"] & Y^{\mathrm{alg}} \end{tikzcd},$$
in the ∞-category dAff_k. Reasoning as above, we have a natural commutative square
$$\begin{tikzcd} \mathcal{D}_{X_U^{\mathrm{alg}}/U^{\mathrm{alg}}} \arrow[r] \arrow[d] & \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \arrow[d] \\ U^{\mathrm{alg}} \arrow[r] & Y^{\mathrm{alg}}, \end{tikzcd}$$
in the ∞-category dSt_k.

Proposition 3.16. *The commutative square*
$$\begin{tikzcd} \mathcal{D}^{\mathrm{an}}_{X_U/U} \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{an}}_{X/Y} \arrow[d] \\ U \arrow[r] & Y, \end{tikzcd}$$
of Construction 3.15, is a pullback square in the ∞-category dAnSt_k.

Proof. In order to show the assertion of the proposition, we are reduced to show that the natural morphism

$$\mathcal{D}^{\mathrm{an}}_{X_U/U} \to \mathcal{D}^{\mathrm{an}}_{X/Y} \times_Y U,$$

is an equivalence in the ∞-category $\mathrm{AnFMP}_{X_U//U}$. Thanks to Proposition 2.55, we are reduced to show that the induced morphism

$$\mathbf{T}^{\mathrm{an}}_{X_U/\mathcal{D}^{\mathrm{an}}_{X_U/U}} \to \mathbf{T}^{\mathrm{an}}_{X_U/\mathcal{D}_{X/Y}\times_Y U},$$

is an equivalence in the ∞-category $\mathrm{QCoh}(X_U)$. By the fact that the relative analytic tangent complex commutes with sifted colimits we are reduced to show that the natural morphism of simplicial objects

$$\{\mathbf{T}^{\mathrm{an}}_{X_U/\mathcal{D}^{\mathrm{an},\bullet}_{X_U/U}}\} \to \{\mathbf{T}^{\mathrm{an}}_{X_U/\mathcal{D}^{\mathrm{an},\bullet}_{X/Y}\times_Y U}\},$$

is an equivalence in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{QCoh}(X))$. We have a pullback square

$$\begin{tikzcd} \mathcal{D}^{\mathrm{an},\bullet}_{X_U/U} \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \arrow[d] \\ U \arrow[r] & Y \end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{dAfd}_k)$ and therefore, [12, Proposition 5.12] implies that we have necessarily that

$$\{\mathbb{L}^{\mathrm{an}}_{X_U/\mathcal{D}^{\mathrm{an},\bullet}_{X_U/U}}\} \simeq \{\mathbb{L}^{\mathrm{an}}_{X_U/\mathcal{D}^{\mathrm{an},\bullet}_{X/Y}\times_Y U}\},$$

in the ∞-category $\mathrm{QCoh}(X_U)$, and the result follows. $\square$

3.3. **Gluing the Deformation.** In this §, we globalize the results proved so far in §3.2. Let $f\colon X \to Y$ be a closed immersion in the ∞-category dAnSt_k, where we assume Y to be a geometric derived k-analytic stack. Then we can consider as before the *deformation to the normal bundle of the morphism* f constructed via the pullback diagram

$$\begin{tikzcd} \mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \arrow[r] \arrow[d] & Y \times \mathbf{A}^1_k \arrow[d] \\ \mathbf{Map}_{/\mathbf{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, X \times \mathbf{A}^1_k) \arrow[r] & \mathbf{Map}_{/\mathbf{A}^1_k}(\mathrm{B}^\bullet_{\mathrm{scaled}}, Y \times \mathbf{A}^1_k), \end{tikzcd}$$

in the ∞-category dAnSt_k. As in the previous §, one can show that the simplicial object

$$\mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \in \mathrm{AnFMP}_{X\times\mathbf{A}^1_k//Y\times\mathbf{A}^1_k},$$

and in particular we have:

Proposition 3.17. *The simplicial object*

$$\mathcal{D}^{\mathrm{an},\bullet}_{X/Y}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnFMP}_{X//Y},$$

admits a sifted colimit $\mathcal{D}^{\mathrm{an}}_{X/Y} \in \mathrm{AnFMP}_{X/}$.

Proof. Let $U_\bullet \to Y$ be a derived k-affinoid admissible open covering of Y, in dAnSt_k. We have a natural equivalence of ∞-categories

$$\Psi\colon (\mathrm{dAnSt}_k)_{/Y} \to \lim_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{dAnSt}_k)_{/U_\bullet}.$$

It is clear by the construction, that the simplicial object $\mathcal{D}^{\mathrm{an}}_{X/Y}$ satisfies

$$\begin{aligned} \Psi(\mathcal{D}^{\mathrm{an},\bullet}_{X/Y}) &\simeq \{\mathcal{D}^{\mathrm{an},\bullet}_{X_{U_\bullet}/U_\bullet}\} \\ &\in \lim_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{dAnSt}_k)_{/U_\bullet}. \end{aligned}$$

Proposition 3.16 implies that we have a canonically defined object

$$\mathcal{D}^{\mathrm{an}}_{X/Y} \in \mathrm{dAnSt}_k,$$

obtained by gluing the object

$$\{\mathcal{D}^{\mathrm{an}}_{X_{U_\bullet}/U_\bullet}\} \in \lim_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{dAnSt}_k)_{/U_\bullet}.$$

It is clear from the definition that $\mathcal{D}^{\mathrm{an}}_{X/Y} \in \mathrm{AnFMP}_{X//Y}$. We claim that the latter is a colimit of the diagram

$$\mathcal{D}^{\mathrm{an},\bullet}_{X/Y}.$$

We need to show that for every $Z \in \mathrm{AnFMP}_{X//Y}$ together with a morphism

$$\mathcal{D}^{\mathrm{an},\bullet}_{X/Y} \to Z,$$

then there exists a uniquely defined (up to a contractible space of choices) morphism

$$\mathcal{D}^{\mathrm{an}}_{X/Y} \to Z,$$

in the ∞-category $\mathrm{AnFMP}_{X//Y}$. Moreover, we are reduced to check this property locally on Y, in which case the assertion follows immediately from the construction. ☐

3.4. **The Hodge filtration.** In this §, we will describe the construction of the Hodge filtration on the object $Y^\wedge_X$.

Construction 3.18. Let $f\colon X \to Y$ be a closed immersion in the ∞-category dAfd_k. Consider the induced morphism

$$f^{\mathrm{alg}}\colon X^{\mathrm{alg}} \to Y^{\mathrm{alg}},$$

where one sets as usual $X^{\mathrm{alg}} = \operatorname{Spec} A$ and $Y^{\mathrm{alg}} = \operatorname{Spec} B$, where $A \coloneqq \Gamma(X, \mathcal{O}_X^{\mathrm{alg}})$ and $B \coloneqq \Gamma(Y, \mathcal{O}_Y^{\mathrm{alg}})$. Moreover, the morphism f^{alg} is a closed immersion. Consider the induced map

$$g \coloneqq f^{\mathrm{alg}} \times \mathrm{id}_{\mathbb{A}^1_k} \colon X \times \mathbb{A}^1_k \to Y \times \mathbb{A}^1_k,$$

in dAff_k. By Noetherian approximation, we can write g as an inverse limit of the form

$$\lim_{\alpha \in A^{\mathrm{op}}} g_\alpha \colon \lim_{\alpha \in A^{\mathrm{op}}} X_\alpha \times \mathbb{A}^1_k \to \lim_{\alpha \in A^{\mathrm{op}}} Y_\alpha \times \mathbb{A}^1_k,$$

where A is a filtered ∞-category and for each index $\alpha \in A$, we have that

$$g_\alpha \colon X_\alpha \times \mathbb{A}^1_k \to Y_\alpha \times \mathbb{A}^1_k,$$

is a closed immersion in the ∞-category $\mathrm{dAff}_k^{\mathrm{laft}}$. Fix some $\alpha \in A$. Thanks to [6, Theorem 9.5.1.3], there exists a sequence of square-zero extensions of the form

$$X_\alpha \times \mathbb{A}^1_k = X^{(0)} \hookrightarrow X^{(1)}_\alpha \hookrightarrow \cdots \hookrightarrow X^{(n)}_\alpha \hookrightarrow \cdots \to Y_\alpha \times \mathbb{A}^1_k,$$

defined inductively as follows: assume that for $n \geq 0$ we already have defined $X^{(0)}_\alpha \to X^{(n)}_\alpha$ together with a natural morphism to $Y_\alpha \times \mathbb{A}^1_k$. Then [6, Theorem 9.5.1.3] implies that we a naturally defined derivation

$$d^{(n)} \colon \mathbb{L}_{X^{(n)}_\alpha} \to \mathbb{L}_{X^{(n)}_\alpha / Y_\alpha} \to \mathcal{F}^{(n)},$$

where $\mathcal{F}^{(n)} \in \mathrm{Coh}^{\geq n}(X^{(n)}_\alpha)$. We then define $X^{(n)}_\alpha \to X^{(n+1)}_\alpha$ as the square-zero extension associated to $d^{(n)}$. Moreover, as a consequence of the discussion followin [6, §9.5.1], it follows that $X^{(n+1)}$ admits a natural morphism to $Y \times \mathbb{A}^1_k$. We further observe that for each $\alpha \in A$ and $n \geq 0$, the object $X^{(n)}_\alpha$ is a derived affine scheme almost of finite presentation.

Remark 3.19. With notations as in Construction 3.18, we are able to explicitly identify $\mathcal{F}^{(n)} \in \mathrm{Coh}^{+} \geq n(X^{(n)})$ as follows: in [6, §9.5.1] the authors construct the required derivation as the *Serre dual* of a natural morphism of the form

$$(i_{n-1}^*)^{\mathrm{IndCoh}} \operatorname{Sym}^n(\mathbb{T}_{X/Y}[-1])[1]) \to \mathbb{T}_{X^{(n)}/Y} \to \mathbb{T}_{X^{(n)}},$$

in the ∞-category of *ind-coherent sheaves on* $X^{(n)}$. Applying the Serre dual functor we obtain a natural morphism

$$\mathbb{L}_{X^{(n)}} \to \mathbb{L}_{X^{(n)}/Y} \to \mathbf{D}^{\mathrm{Serre}}_{X^{(n)}}((i_{n-1}^*)^{\mathrm{IndCoh}} \operatorname{Sym}^n(\mathbb{T}_{X/Y}[-1])[1])).$$

Since the morphism $i_{n-1}\colon X \times \mathbb{A}^1_k \to X^{(n)}$ is a proper morphism, we are able to identify

$$\mathbf{D}^{\mathrm{Serre}}_{X^{(n)}}((i_{n-1}^*)^{\mathrm{IndCoh}} \operatorname{Sym}^n(\mathbb{T}_{X/Y}[-1])[1])) \simeq i_{n-1,*} \operatorname{Sym}^n(\mathbb{L}_{X/Y}[-1])[1]),$$

in $\mathrm{QCoh}(X^{(n)})$, via [5, Corollary 9.5.9 (b)] combined with and [5, §3.6.6] and [6, Corollary 1.4.4.2].

Lemma 3.20. *Let $n \geq 0$, and let $\alpha \to \beta$ be a morphism in A^{op}. Then the transition morphism*

$$X_\alpha \times \mathbb{A}^1_k \to X_\beta \times \mathbb{A}^1_k,$$

lifts to a well defined induced morphism

$$X_\alpha^{(n)} \to X_\beta^{(n)},$$

in $\mathrm{dAff}_k^{\mathrm{laft}}$.

Proof. The result follows immediately from the naturality of the construction in [6, §9.5.1]. □

Proposition 3.21. ([6, Corollary 9.5.2]) *Fix $\alpha \in A$. Then there exists a natural morphism*

$$\operatorname*{colim}_{n \geq 0} X_\alpha^{(n)} \to \mathcal{D}_{X_\alpha / Y_\alpha},$$

which is furthermore an equivalence in the ∞-category $\mathrm{FMP}_{X_\alpha \times \mathbb{A}^1_k // Y_\alpha \times \mathbb{A}^1_k}$ (and thus in the ∞-category dSt_k).

Corollary 3.22. *The derived stack $\lim_\alpha \operatorname{colim}_{n \geq 0} X_\alpha^{(n)}$ is an affine scheme. Moreover, the natural morphism*

$$\gamma\colon \operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}} \to \lim_{\alpha \in A^{\mathrm{op}}} \operatorname*{colim}_{n \geq 0} X_\alpha^{(n)},$$

is an equivalence of derived affine schemes.

Proof. The first assertion follows immediately from Lemma 3.20 together with the fact that for each $n \geq 0$, $X_\alpha^{(n+1)}$ is a square-zero extension on $X_\alpha^{(n)}$ via an object

$$\mathcal{F}^{(n+1)} \in \mathrm{Coh}^{\geq n}(X_\alpha^{(n)}). \tag{3.1}$$

We further deduce that

$$\operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}} \in \mathrm{dAff}_k.$$

Indeed, the result follows immediately from the fact that for each $X^{(n+1),\mathrm{alg}}$ is a square-zero extension of $X^{(n),\mathrm{alg}}$ by an object

$$\mathcal{G}^{(n+1)} \in \mathrm{Coh}^{\geq n}(X^{(n),\mathrm{alg}}). \tag{3.2}$$

Let $A, B \in \mathrm{CAlg}_k$ denote the derived rings of derived global sections of both

$$\operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}} \quad \text{and} \quad \lim_{\alpha \in A^{\mathrm{op}}} \operatorname*{colim}_{n \geq 0} X_\alpha^{(n)}.$$

Thanks to (3.1) and (3.2) we deduce that

$$\pi_i(A) \simeq \pi_i(A^{(n)}) \quad \text{and} \quad \pi_i(B) \simeq \pi_i(B^{(n)}),$$

for $i \geq 0$. Moreover, we have by construction that

$$\pi_i(A^{(n)}) \simeq \operatorname*{colim}_{\alpha \in A^{\mathrm{op}}} \pi_i(A^{(n)}_\alpha) \quad \text{and} \quad \pi_i(B) \simeq \pi_i(A^{(n)}_\alpha),$$

and the result follows immediately. □

Definition 3.23. Consider the morphism $g \colon X^{\mathrm{alg}} \times \mathbb{A}^1_k \to Y^{\mathrm{alg}} \times \mathbb{A}^1_k$ above. Lemma 3.20 implies that for each $n \geq 0$, the object

$$X^{(n),\mathrm{alg}} := \lim_{\alpha \in A^{\mathrm{op}}} X^{(n)}_\alpha \in \mathrm{dAff},$$

is well defined and it fits into a sequence of square-zero extensions

$$X \times \mathbb{A}^1_k = X^{(0),\mathrm{alg}} \hookrightarrow X^{(1),\mathrm{alg}} \hookrightarrow \dots \hookrightarrow X^{(n),\mathrm{alg}} \hookrightarrow \dots \to Y^{\mathrm{alg}},$$

in the ∞-category dAff_k. We shall refer to this sequence as the *Hodge filtration* associated to the morphism f. Moreover, thanks to Proposition 3.21 it follows that we have a natural morphism

$$\operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}} \to \lim_{\alpha \in ^{\mathrm{op}}} \mathcal{D}_{X_\alpha / Y_\alpha}, \tag{3.3}$$

in the ∞-category $\mathrm{FMP}_{X \times \mathbb{A}^1_k // Y \times \mathbb{A}^1_k}$.

We will need the following auxiliary result:

Lemma 3.24. *The object* $\lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha / Y_\alpha} \in \mathrm{FMP}_{X^{\mathrm{alg}} \times \mathbb{A}^1_k // Y^{\mathrm{alg}} \times \mathbb{A}^1_k}$ *and moreover, the natural morphism*

$$\mathcal{D}_{X^{\mathrm{alg}} / Y^{\mathrm{alg}}} \to \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha / Y_\alpha},$$

is an equivalence in $\mathrm{FMP}_{X^{\mathrm{alg}} \times \mathbb{A}^1_k // Y^{\mathrm{alg}} \times \mathbb{A}^1_k}$.

Proof. For each $\alpha \in A$ let us denote by

$$h_\alpha \colon X \to X_\alpha,$$

the structural morphism. Thanks to [11, Corollary 4.4.1.3] combined with [11, Corollary 4.5.1.3] we conclude that the natural morphisms

$$\theta_\alpha \colon h^*_\alpha \mathbb{L}_{X_\alpha} \to \mathbb{L}_X, \quad \text{foreach} \alpha \in A,$$

assemble to provide an equivalence

$$\theta \colon \operatorname*{colim}_{\alpha \in A} h^*_\alpha \mathbb{L}_{X_\alpha} \to \mathbb{L}_X,$$

in the ∞-category $\mathrm{QCoh}(X^{\mathrm{alg}})$. As a consequence, given any pushout diagram

$$\begin{tikzcd} S \arrow[r, "g_S"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T', \end{tikzcd}$$

in the ∞-category $\mathrm{Nil}_{X/}$, where g_S is a square-zero extension, we can write it as an inverse limit of pushout diagrams of the form

$$\begin{tikzcd} S_\alpha \arrow[r, "g_{S_\alpha}"] \arrow[d] & S'_\alpha \arrow[d] \\ T_\alpha \arrow[r] & T'_\alpha, \end{tikzcd}$$

in $\mathrm{Nil}_{X_\alpha/}$, for each $\alpha \in A$. It then follows that we have a chain of natural equivalences of the form

$$\begin{aligned}\lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(T') &\simeq \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(\lim_{\alpha \in A^{\mathrm{op}}} T'_\alpha)\\ &\simeq \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(T'_\alpha)\\ &\simeq \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(S'_\alpha \times_{S_\alpha} T_\alpha)\\ &\simeq \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(S'_\alpha) \times_{\lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(S_\alpha)} \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}(T_\alpha),\end{aligned}$$

thus $\lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha} \in \mathrm{FMP}_{X//Y}$, as desired. Consider now the canonical morphism

$$\theta \colon \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}} \to \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha},$$

in the ∞-category $\mathrm{FMP}_{X/Y}$. Thanks to the analogue of Proposition 2.20 in the algebraic setting, it suffices to show that the morphism θ induces an equivalence at the level of cotangent complexes

$$\mathbb{L}_{X \times \mathbb{A}^1_k / \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}} \to \mathbb{L}_{X \times \mathbb{A}^1_k / \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}}, \tag{3.4}$$

in the ∞-category $\mathrm{QCoh}(X)$. Since both $\mathcal{D}_{X/Y}$ and $\mathcal{D}_{X_\alpha/Y_\alpha}$ for each $\alpha \in A$, are affine, it follows that we can identify both the left and right hand sides of (3.4) canonically with

$$\begin{aligned}\mathbb{L}_{X \times \mathbb{A}^1_k / \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{D}_{X_\alpha/Y_\alpha}} &\simeq \operatorname*{colim}_{\alpha \in A} \mathbb{L}_{X_\alpha \times \mathbb{A}^1_k / \mathcal{D}_{X_\alpha/Y_\alpha}}\\ &\simeq \operatorname*{colim}_{\alpha \in A} \mathbb{L}_{X_\alpha \times \mathbb{A}^1_k / Y_\alpha \times \mathbb{A}^1_k},\end{aligned}$$

and similarly we deduce that

$$\begin{aligned}\mathbb{L}_{X \times \mathbb{A}^1_k / \mathcal{D}_{X/Y}} &\simeq \mathbb{L}_{X \times \mathbb{A}^1_k / Y \times \mathbb{A}^1_k}\\ &\simeq \operatorname*{colim}_{\alpha \in A} \mathbb{L}_{X_\alpha \times \mathbb{A}^1_k / Y_\alpha \times \mathbb{A}^1_k},\end{aligned}$$

and the result follows. □

Thanks to the above Lemma, we can consider the natural morphism in (3.3) as

$$\beta \colon \operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}} \to \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}},$$

in the ∞-category $\mathrm{FMP}_{X \times \mathbb{A}^1_k // Y \times \mathbb{A}^1_k}$. We have:

Proposition 3.25. *The morphism*

$$\beta \colon \operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}} \to \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}},$$

above is an equivalence in the ∞*-category* $\mathrm{FMP}_{X \times \mathbb{A}^1_k // Y \times \mathbb{A}^1_k}$.

Proof. Thanks to Proposition 2.20, it suffices to show that the natural morphism

$$\mathbb{L}_{X / \operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}}} \to \mathbb{L}_{X / \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}},$$

is an equivalence in $\mathrm{QCoh}(X)$. Moreover, Corollary 3.22 implies that we have a natural equivalence

$$\begin{aligned}\mathbb{L}_{X / \operatorname*{colim}_{n \geq 0} X^{(n),\mathrm{alg}}} &\simeq \operatorname*{colim}_{\alpha \in A} \mathbb{L}_{X_\alpha / \operatorname*{colim}_{n \geq 0} X^{(n)}_\alpha}\\ &\simeq \operatorname*{colim}_{\alpha \in A} \mathbb{L}_{X_\alpha / \mathcal{D}_{X_\alpha/Y_\alpha}}\\ &\simeq \mathbb{L}_{X / \mathcal{D}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}},\end{aligned}$$

where the second equivalence follows from Proposition 3.21. The result is now an immediate consequence of Proposition 2.20. □

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Definition 3.26. Let $f\colon X \to Y$ be a closed immersion in the ∞-category dAfd_k. For each $n \geq 0$, we define the square-zero extension

$$X \times \mathbf{A}^1_k \hookrightarrow X^{(n)},$$

as the relative analytification, $(-)^{\mathrm{an}}_Y$, of the natural square-zero extension

$$X^{\mathrm{alg}} \times \mathbb{A}^1_k \hookrightarrow X^{(n),\mathrm{alg}}.$$

By construction, for each $n \geq 0$, we have natural morphisms

$$X^{(n)} \to Y \times \mathbf{A}^1_k.$$

Putting together the above results we can easily deduce:

Corollary 3.27. *There exists a natural morphism*

$$\operatorname*{colim}_{n \geq 0} X^{(n)} \to \mathcal{D}^{\mathrm{an}}_{X/Y},$$

which is furthermore an equivalence in the ∞-category $\mathrm{AnFMP}_{X \times \mathbf{A}^1_k // Y \times \mathbf{A}^1_k}$.

Proof. It is an immediate consequence of Proposition 3.25 combined with Proposition 3.13. ☐

We now globalize the Hodge filtration on the deformation, $\mathcal{D}^{\mathrm{an}}_{X/Y}$:

Construction 3.28. Let $f\colon X \to Y$ be a closed immersion of derived k-analytic stacks, where Y is assumed to be a geometric derived k-analytic stack. Suppose we are given a morphism

$$U \to Y,$$

in dAnSt_k, where $U \in \mathrm{dAfd}_k$ and form the pullback diagram

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in dAnSt_k. Suppose we are now given a morphism of derived k-affinoid spaces $V \to U$, then we have a natural pullback diagram

$$\begin{tikzcd} X_V \arrow[r] \arrow[d] & X_U \arrow[d] \\ V \arrow[r] & U, \end{tikzcd}$$

in the ∞-category dAfd_k. This provides us for each $n \geq 0$, with natural commutative diagrams

$$\begin{tikzcd} X_V^{(n)} \arrow[r] \arrow[d] & X_U^{(n)} \arrow[d] \\ V \arrow[r] & U, \end{tikzcd} \tag{3.5}$$

of the n-th pieces of the Hodge filtrations on $\mathcal{D}_{X_U/U}$ and $\mathcal{D}_{X_V/V}$, respectively.

Lemma 3.29. *The commutative square in* (3.5) *is a pullback square in the ∞-category* dAfd_k.

Proof. Thanks to Lemma 3.7 it suffices to prove that the induced diagram

$$\begin{tikzcd} X_V^{\mathrm{alg}} \arrow[r] \arrow[d] & X_U^{\mathrm{alg}} \arrow[d] \\ V^{\mathrm{alg}} \arrow[r] & U^{\mathrm{alg}} \end{tikzcd},$$

is a pullback square in dAff_k. By a standard argument of Noetherian approximation, we might assume that $U' := U^{\mathrm{alg}}$ and $V' := V^{\mathrm{alg}}$ are both almost of finite presentation derived affined k-schemes. The result is now a direct consequence of Remark 3.19 combined with [12, Proposition 5.12]. ☐

Construction 3.30. Consider the relative analytification functor

$$(-)^{\mathrm{an}}_Y \colon \lim_{U \in Y^{\mathrm{afd}}_{\text{ét}}} (\mathrm{dSt}_k)_{/U^{\mathrm{alg}} \times \mathbb{A}^1_k} \to \lim_{U \in Y^{\mathrm{afd}}_{\text{ét}}} (\mathrm{dAnSt}_k)_{/U \times \mathbf{A}^1_k}$$
$$\simeq (\mathrm{dAnSt}_k)_{/Y \times \mathbf{A}^1_k}.$$

Thanks to Lemma 3.29, for each $n \geq 0$ the object

$$\{X_U^{(n)}\}_{U \in Y^{\mathrm{afd}}_{\text{ét}}} \in \lim_{n \geq 0} (\mathrm{dSt}_k)_{/U^{\mathrm{alg}}}.$$

Therefore, taking its relative analytification produces well defined objects $X^{(n)} \in (\mathrm{dAnSt}_k)_{Y \times \mathbf{A}^1_k}$ which restricts to the usual Hodge filtration for every morphism

$$g \colon U \to Y,$$

in $Y^{\mathrm{afd}}_{\text{ét}}$. Moreover, by construction we have a natural morphism

$$\operatorname*{colim}_{n \geq 0} X^{(n)} \to \mathcal{D}^{\mathrm{an}}_{X/Y},$$

which is an equivalence in the ∞-category dAnSt_k.

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