SPREADING OUT THE HODGE FILTRATION IN RIGID ANALYTIC GEOMETRY

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Abstract.

Contents

1. Introduction

In this paper, we will provide a rigid analytic construction of the deformation to the normal cone, studied in [4]. Our goal is to use this geometric construction to deduce certain important results concerning both *rigid analytic* and *over-convergent* (Hodge complete) *derived de Rham cohomology* of rigid analytic spaces over a non-archimedean field of characteristic zero. We will then exploit this ideas to come up with analogues concerning *derived rigid cohomology* of finite type schemes over a perfect field in characteristic zero. In particular, our main goal is to extrapolate the main result of [3] to the setting of derived rigid cohomology.

1.1. **Preliminaries.** Let $\mathcal{X}$ be an ∞-topos. The notion of a *local* $\mathcal{T}_{\mathrm{an}}(k)$*-structure on* $\mathcal{X}$ was first introduced in [8, Definition 2.4], see also [1, §2].

Let $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ be a local $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$. Since the pregeometry $\mathcal{T}_{\mathrm{an}}(k)$ is compatible with n-truncations, cf. [8, Theorem 3.23], it follows that $\pi_0(\mathcal{O}) \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, as well.

Denote by $\mathcal{J} \subseteq \pi_0(\mathcal{O})$, the *Jacobson ideal* of $\pi_0(\mathcal{O}^{\mathrm{alg}})$, which can be naturally regarded as an object in the ∞-category

$$\mathrm{Mod}_{\pi_0(\mathcal{O}^{\mathrm{alg}})} \simeq \mathrm{Mod}_{\pi_0(\mathcal{O})},$$

for a justification of the latter equivalence, see for instance [7, Theorem 4.5] . Since the ∞-category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ is a presentable ∞-category we can consider the quotient

$$\pi_0(\mathcal{O})_{\mathrm{red}} \coloneqq \pi_0(\mathcal{O})/\mathcal{J} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}),$$

which we refer to the *reduced* $\mathcal{T}_{\mathrm{an}}(k)$*-structure on* $\mathcal{X}$ *associated to* $\pi_0(\mathcal{O})$. Moreover, the corresponding *underlying algebra* satisfies

$$(\pi_0(\mathcal{O})_{\mathrm{red}})^{\mathrm{alg}} \simeq \pi_0(\mathcal{O})^{\mathrm{alg}}/\mathcal{J} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

One can further prove that $\pi_0(\mathcal{O})_{\mathrm{red}} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ actually lies in the full subcategory $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$.

Definition 1.1. Let $Z = (\mathcal{Z}, \mathcal{O}_Z) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote a $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos. We define the *reduced $\mathcal{T}_{\mathrm{an}}(k)$-structure ∞-topos* as

$$Z_{\mathrm{red}} \coloneqq (\mathcal{Z}, \pi_0(\mathcal{O}_Z)_{\mathrm{red}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

We shall denote by $\mathrm{Afd}_k^{\mathrm{red}}$ (resp., $\mathrm{An}_k^{\mathrm{red}}$) the full subcategory of dAfd_k (resp., dAn_k) spanned by reduced k-affinoid (resp., k-analytic spaces).

Notation 1.2. Let $(-)^{\mathrm{red}} \colon \mathrm{dAn}_k \to \mathrm{An}_k^{\mathrm{red}}$ denote the functor obtained by the formula

$$Z = (\mathcal{Z}, \mathcal{O}_Z) \in \mathrm{dAn}_k \mapsto Z_{\mathrm{red}} = (\mathcal{Z}, \pi_0(\mathcal{Z})_{\mathrm{red}}) \in \mathrm{An}_k^{\mathrm{red}}.$$

We shall refer to it as the *underlying reduced k-analytic space.*

Lemma 1.3. *Let $f \colon X \to Y$ be a Zariski open immersion of derived k-analytic spaces. Then $f^{\mathrm{red}} \colon X^{\mathrm{red}} \to Y^{\mathrm{red}}$ is also a Zariski open immersion.*

Proof. By the definitions, it is clear that the truncation

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y),$$

is a Zariski open immersion of ordinary k-analytic spaces. In the case of ordinary k-analytic spaces it is clear from the construction that the reduction of Zariski open immersions is again a Zariski open immersion. □

Definition 1.4. In [7, Definition 5.41] the authors introduced the notion of a square-zero extension between $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi. In particular, given a morphism $f \colon Z \to Z'$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, we shall say that f *has the structure of a square-zero extension* if f exhibits Z' as a square-zero extension of Z.

Recall the definition of the ∞-categories of derived k-affinoid and derived k-analytic spaces given in [8, Definition 7.3 and Definition 2.5.], respectively.

Remark 1.5. Let $X \in \mathrm{An}_k$. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be an ideal satisfying $\mathcal{J}^2 = 0$. Consider the fiber sequence

$$\mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J},$$

in the ∞-category $\mathrm{Coh}^+(X)$. It corresponds to a well defined morphism $d \colon \mathcal{O}_X/\mathcal{J} \to \mathcal{J}[1]$ admitting $\mathcal{O}_X$ as fiber. The morphism d defines a derivation $d \colon \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{J}} \to \mathcal{J}[1]$, by pre-composing with the natural map $\mathcal{O}_X/\mathcal{J} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{J}}$. In particular, we can consider the square-zero extension of $\mathcal{O}_X$ by $\mathcal{J}$ induced by $\mathcal{J}$ defined by d. The latter object must then be equivalent to $\mathcal{O}_X$ itself. We conclude that $\mathcal{O}_X$ is a square-zero extension of $\mathcal{O}_X/\mathcal{J}$.

Lemma 1.6. *Let $Z \coloneqq (\mathcal{Z}, \mathcal{O}_Z) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote a $\mathcal{T}_{\mathrm{an}}(k)$-structure ∞-topos. Suppose that the reduction Z_{red} is equivalent to a derived k-affinoid space. Then the truncation $\mathrm{t}_0(Z)$ is isomorphic to an ordinary k-affinoid space. If we assume further that for every $i > 0$, the homotopy sheaves $\pi_i(\mathcal{O}_Z)$ are coherent $\pi_0(\mathcal{O}_Z)$-modules, then Z itself is equivalent to a derived k-affinoid space.*

Proof. We first observe that the second claim of the Lemma follows readily from the first one. We thus are thus reduced to prove that $\mathrm{t}_0(Z)$ is isomorphic to an ordinary k-affinoid space. Let $\mathcal{J} \subseteq \pi_0(\mathcal{O}_Z)$, denote the coherent ideal sheaf associated to the closed immersion $Z_{\mathrm{red}} \hookrightarrow Z$. Notice that the ideal $\mathcal{J}$ agrees with the Jacobson ideal of $\pi_0(\mathcal{O}_Z)$. Since derived k-analytic spaces are Noetherian, it follows that there exists a sufficiently large integer $n \geq 2$ such that

$$\mathcal{J}^n = 0.$$

Arguing by induction we can suppose that $n = 2$, that is to say that

$$\mathcal{J}^2 = 0.$$

In particular, Remark 1.5 implies that the above map has the natural morphism $Z_{\mathrm{red}} \to Z$ has the structure of a square zero extension. The assertion now follows from [7, Proposition 6.1] and its proof. $\square$

Remark 1.7. We observe that the converse of Lemma 1.6 holds true. Indeed, the natural morphism $Z_{\mathrm{red}} \to Z$ is a closed immersion. In particular, if $Z \in \mathrm{dAfd}_k$ we deduce readily from that $Z_{\mathrm{red}} \in \mathrm{dAfd}_k$, as well.

Definition 1.8. Let $f \colon X \to Y$ be a morhpism in the ∞-category dAn_k. We shall say that f is an *affine morphism* if for every morphism $Z \to Y$ in dAn_k such that Z is equivalent to a derived k-affinoid space, the pullback

$$Z' \coloneqq Z \times_Y X \in \mathrm{dAn}_k,$$

is also equivalent to a derived k-affinoid space.

Notation 1.9. Let $f \colon X \to Y$ be a morphism of derived k-analytic spaces. We shall denote by

$$f^\# \colon \mathcal{O}_Y \to f_* \mathcal{O}_X,$$

the induced morphism at the level of $\mathcal{T}_{\mathrm{an}}(k)$-structures.

Lemma 1.10. *Let $f \colon X \to Y$ be an affine morphism in dAn_k. Suppose that we are given a Zariski open immersion $g \colon Z \to Y$ such that $Z \in \mathrm{dAfd}_k$ which corresponds to the completement of the zero locus of a section $s \in \pi_0(\mathcal{O}_Y)$. Then the fiber product*

$$Z' \coloneqq Z \times_Y X \in \mathrm{dAn}_k,$$

is equivalent to a derived k-affinoid space and moreover $\Gamma(Z', \mathcal{O}_{Z'}^{\mathrm{alg}}) \simeq B[1/f^\#(s)]$, where $B \coloneqq \Gamma(X, \mathcal{O}_X^{\mathrm{alg}})$.

Proof. The first assertion of the Lemma follows readily from the definition of affine morphisms. We shall now prove the second claim. Let $A \coloneqq \Gamma(Y, \mathcal{O}_Y^{\mathrm{alg}})$. In this case, we have a natural equivalence of derived k-algebras

$$A[1/f] \simeq \Gamma(Z, \mathcal{O}_Z^{\mathrm{alg}}).$$

Since Zariski open immersions are stable under pullbacks, it follows that the natural morphism $g' \colon Z' \to X$ is itself a Zariski open immersion. In particular, it follows that we can identify

$$\Gamma(Z', \mathcal{O}_{Z'}) \simeq B[1/t],$$

where $t \in \pi_0(B)$. In order to conclude the proof, we observe that the 0-th truncation, $\mathrm{t}_0(g)$, is again a Zariski open immersion. For this reason, one should have forcibly that $t = f^\#(s)$, by the universal property of fiber products of ordinary k-analytic spaces. $\square$

2. Non-archimedean differential geometry

2.1. **Analytic formal moduli problems under a base.** In this §, we will study the notion of *analytic formal moduli problems under* a fixed derived k-analytic space. The results presented here will prove to be crucial for the study of the deformation to the normal cone in the k-analytic setting, presented in the next section. We start with the following definition:

Definition 2.1. Let $f \colon X \to Y$ be a morphism in dAn_k. We say that f is a *nil-isomorphism* if $f_{\mathrm{red}} \colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is an isomorphism of k-analytic spaces. We denote by $\mathrm{AnNil}_{/X}$ the full subcategory of $(\mathrm{dAn}_k)^{\mathrm{ft}}_{X/}$ spanned by nil-isomorphisms $X \to Y$ of finite type.

Lemma 2.2. *Let $f \colon X \to Y$ be a nil-isomorphism in dAn_k. Then:*

(1) Given any morphism $Z \to Y$ in dAn_k, the induced morphism

$$Z \times_X Y \to Z,$$

is again an nil-isomoprhism.
(2) f is an affine morphism.
(3) f is a finite morphism.

Proof. To prove (i), it suffices to prove that the functor $(\text{-})^{\mathrm{red}} \colon \mathrm{dAn}_k \to \mathrm{An}_k^{\mathrm{red}}$ commutes with finite limits. The truncation functor

$$\mathrm{t}_0 \colon \mathrm{dAn}_k \to \mathrm{An}_k,$$

commutes with finite limits. So we further reduce ourselves to the prove that the usual underlying reduced functor

$$(-)^{\mathrm{red}} \colon \mathrm{An}_k \to \mathrm{An}_k^{\mathrm{red}},$$

commutes with finite limits. By construction, the latter assertion is equivalent to the claim that the complete tensor product of ordinary k-affinoid algebras commutes with the operation of taking the quotient by the Jacobson radical, which is immediate.

We now prove (ii). Let $Z \to Y$ be a Zariski open immersion such that Z is a derived k-affinoid space. Then we claim that the pullback $Z \times_X Y$ is again a derived k-affinoid space. Thanks to Lemma 1.6 we reduced to prove that $(Z \times_X Y)_{\mathrm{red}}$ is equivalent to an ordinary k-affinoid space. Thanks to (i), we deduce that the induced morphism

$$(Z \times_X Y)_{\mathrm{red}} \to Z_{\mathrm{red}},$$

is an isomorphism of ordinary k-analytic spaces. In particular, $(Z \times_X Y)_{\mathrm{red}}$ is a k-affinoid space. The result now follows from Lemma 1.6.

To prove (iii), we shall show that the induced morphism on the 0-th truncations $\mathrm{t}_0(X) \to \mathrm{t}_0(Y)$ is a finite morphism of ordinary k-affinoid spaces. But this follows immeaditely from the fact that both $\mathrm{t}_0(X)$ and $\mathrm{t}_0(Y)$ can be obtained from the reduced X_{red} by means of a finite sequence of finite coherent X_{red}-modules. $\square$

Definition 2.3. A morphism $X \to Y$ be a morphism in dAn_k is called a *nil-embedding* if the induced map of ordinary k-analytic spaces $\mathrm{t}_0(X) \to \mathrm{t}_0(Y)$ is a closed immersion, such that the ideal of $\mathrm{t}_0(X)$ in $\mathrm{t}_0(Y)$ is nilpotent.

Proposition 2.4. *Let $f \colon X \to Y$ be a nil-embedding of derived k-analytic spaces. Then there exists a sequence of morphisms*

$$X = X_0^0 \hookrightarrow X_0^1 \hookrightarrow \cdots \hookrightarrow X_0^n = X_0 \hookrightarrow X_1 \dots X_n \hookrightarrow \cdots \hookrightarrow Y,$$

such that for each $0 \leq i \leq n$ the morphism $X_0^i \hookrightarrow X_0^{i+1}$ has the structure of a square zero extension. Similarly, for every $i \geq 0$, the morphism $X_i \hookrightarrow X_{i+1}$ has the structure of a square-zero extension. Furthermore, the induced morphisms $\mathrm{t}_{\leq i}(X_i) \to \mathrm{t}_{\leq i}(Y)$ are equivalences of derived k-analytic spaces.

Proof. The proof follows the same scheme of reasoning as of [4, Proposition 5.5.3]. For the sake of completeness we present the complete here. Consider the induced morphism on the underlying truncations

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y).$$

By construction, there exists a sufficiently large integer $n \geq 0$ such that

$$\mathcal{J}^{n+1} = 0,$$

where $\mathcal{J} \subseteq \pi_0(\mathcal{O}_Y)$ denotes the ideal associated to the nil-embedding $\mathrm{t}_0(f)$. Therefore, we can factor the latter as a finite sequence of square-zero extensions of ordinary k-analytic spaces

$$\mathrm{t}_0(X) \hookrightarrow X_0^{\mathrm{ord},0} \hookrightarrow \cdots \hookrightarrow X_0^{\mathrm{ord},n} = \mathrm{t}_0(Y),$$

as in the proof of Lemma 1.6. For each $0 \leq i \leq n$, we set

$$X_0^i \coloneqq X \bigsqcup_{\mathrm{t}_0(X)} X_0^{\mathrm{ord},i}.$$

By construction, we have that the natural morphism $\mathrm{t}_0(X_0^n) \to \mathrm{t}_0(Y)$ is an isomorphism of ordinary k-analytic spaces. We now argue by induction on the Postnikov towers associated to the morphism $f \colon X \to Y$. Suppose that for a certain integer $i \geq 0$, we have constructed a derived k-analytic space X_i together with morphisms $g_i \colon X \to X_i$ and $h_i \colon X_i \to Y$ such that $f \simeq h_i \circ g_i$ and the induced morphism

$$\mathrm{t}_{\leq i}(X_i) \to \mathrm{t}_{\leq i}(Y)$$

is an equivalence of derived k-analytic spaces. We shall proceed as follows: by the assumption that h_i is $(i+1)$-connective, we deduce from [7, Proposition 5.34] the existence of a natural equivalence

$$\tau_{\leq i}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}) \simeq 0,$$

in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. Consider the natural fiber sequence

$$h_i^* \mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_{X_i} \to \mathbb{L}^{\mathrm{an}}_{X_i/Y},$$

in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. The natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X_i/Y} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1],$$

induces a morphism $\mathbb{L}^{\mathrm{an}}_{X_i} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1]$, such that the composite

$$h_i^* \mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_{X_i} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}), \tag{2.1}$$

is null-homotopic, in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. The existence of (2.1) produces a square-zero extension

$$X_i \to X_{i+1},$$

together with a morphism $h_{i+1} \colon X_{i+1} \to Y$, factoring $h_i \colon X_i \to Y$. We are reduced to show that the morphism

$$\mathcal{O}_Y \to h_{i+1,*}(\mathcal{O}_{X_{i+1}}),$$

is $(i+2)$-connective. Consider the commutative diagram

$$\begin{tikzcd}
h_{i,*}(\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}))[i] \arrow[r] & h_{i+1,*}(\mathcal{O}_{X_{i+1}}) \arrow[r] & h_{i,*}(\mathcal{O}_{X_i}) \\
\mathcal{I} \arrow[u, "s_i"] \arrow[r] & \mathcal{O}_Y \arrow[u] \arrow[r] & h_*(\mathcal{O}_{X_i}) \arrow[u] \\
\mathcal{J} \arrow[u] \arrow[r] & \mathcal{J} \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}, \tag{2.2}$$

where both the vertical and horizontal composites are fiber sequences. Thanks to [7, Proposition 5.34] we can identify the natural morphism

$$s_i \colon \mathcal{I} \to h_{i,*}(\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}))[i]$$

with the natural morphism $\mathcal{I} \to \tau_{\geq i}(I)$. We deduce that the fiber of the morphism s_i must be necessarily $(i+1)$-connective. The latter observation combined with the structure of (2.2) implies that $h_{i+1} \colon X_{i+1} \to Y$ induces an equivalence of derived k-analytic spaces

$$\mathrm{t}_{\leq i+1}(X_{i+1}) \to \mathrm{t}_{\leq i+1}(Y),$$

as desired. □

Corollary 2.5. *Let $X \in \mathrm{dAn}_k$. Then the natural morphism*

$$X_{\mathrm{red}} \to X,$$

in dAn_k*, can be* approximated *by successive square zero extensions.*

Proof. The assertion of the Corollary follows readily from Proposition 2.4 by observing that the canonical morphism $X_{\mathrm{red}} \to X$ has the structure of a nil-embedding. □

Lemma 2.6. *Let $f \colon S \to S'$ be a nil-isomorphism between derived k-analytic spaces. Then the pullback functor*

$$f^* \colon \mathrm{Coh}^+(S') \to \mathrm{Coh}^+(S),$$

admits a well defined right adjoint, f_*.

Proof. Since $f \colon S \to S'$ is a nil-isomorphism, we conclude from Lemma 2.2 that f is an affine morphism between derived k-analytic spaces. By Zariski descent of Coh^+, cf. [2, Theorem 3.7], together with Lemma 1.10 we reduce the statement of the Lemma to the case where both S and S' are equivalent to derived k-affinoid spaces. In this case, by Tate acyclicity theorem we reduce ourselves to show that the usual base change functor

$$f^* \colon \mathrm{Coh}^+(A) \to \mathrm{Coh}^+(B),$$

where $A \coloneqq \Gamma(S, \mathcal{O}_S^{\mathrm{alg}})$ and $B \coloneqq \Gamma(S', \mathcal{O}_{S'}^{\mathrm{alg}})$, admits a right adjoint. The result now follows from the observation that the canonical induced morphism $\pi_0(A) \to \pi_0(B)$ is a finite morphism of ordinary rings. Indeed, the latter morphism can be obtained by means of a finite sequence of (classical) square-zero extensions with respect to the corresponding Jacobson ideals of both $\pi_0(A)$ and $\pi_0(B)$. Such ideals are necessarily finitely generated as $\pi_0(A)$-modules, and the result follows. □

Lemma 2.7. *Let $f \colon S \to S'$ be a square-zero extension and $g \colon S \to T$ a nil-isomorphism in* dAn_k*. Suppose we are given a pushout diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd},$$

in dAn_k*. Then the induced morphism $T \to T'$ is a square-zero extension.*

Proof. Since g is a nil-isomorphism of derived k-analytic spaces, Lemma 2.6 implies that the pullback functor $g^* \colon \mathrm{Coh}^+(T) \to \mathrm{Coh}^+(S)$ admits a well defined right adjoint

$$g_* \colon \mathrm{Coh}^+(S) \to \mathrm{Coh}^+(T).$$

Let $\mathcal{F} \in \mathrm{Coh}^+(S)^{\geq 0}$ and $d \colon \mathbb{L}_S^{\mathrm{an}} \to \mathcal{F}$ be a derivation associated with the morphism $f \colon S \to S'$. Consider now the natural composite

$$d' \colon \mathbb{L}_T^{\mathrm{an}} \to g_*(\mathbb{L}_S^{\mathrm{an}}) \xrightarrow{g_*(d)} g_*(\mathcal{F}),$$

in the ∞-category $\mathrm{Coh}^+(T)$. By the universal property of the analytic cotangent complex, we deduce the existence of a square-zero extension

$$T \to T',$$

in the ∞-category dAn_k. Let $X \in \mathrm{dAn}_k$ together with morphisms $S' \to X$ and $T \to X$ compatible with both f and g. By the universal property of the relative analytic cotangent complex, the morphism $S' \to X$ induces a uniquely defined (up to a contractible indeterminacy space)

$$\mathbb{L}^{\mathrm{an}}_{S/X} \to \mathcal{F},$$

in $\mathrm{Coh}^+(S)$, such that the compositve $\mathbb{L}^{\mathrm{an}}_S \to \mathbb{L}^{\mathrm{an}}_{S/X} \to \mathcal{F}$ agrees with d. By applying the right adjoint g_* above we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{L}^{\mathrm{an}}_T \arrow[r, "\mathrm{can}"] \arrow[d] & \mathbb{L}^{\mathrm{an}}_{T/X} \arrow[d] \arrow[dr, "d''"] & \\ g_*(\mathbb{L}^{\mathrm{an}}_S) \arrow[r] & g_*(\mathbb{L}^{\mathrm{an}}_{S/X}) \arrow[r] & g_*(\mathcal{F}), \end{tikzcd}$$

in the ∞-category $\mathrm{Coh}^+(T)$. From this, we conclude again by the universal property of the relative analytic cotangent complex the existence of a natural morphism $T' \to X$ extending both $T \to X$ and $S' \to X$ and compatible with the restriction to S. The latter assertion is equivalent to state that the commutative square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T', \end{tikzcd}$$

is a pushout diagram in dAn_k. The proof is thus concluded. □

Proposition 2.8. *Let $f\colon X \to Y$ be a nil-embedding of derived k-analytic spaces. Let $g\colon X \to Z$ be a finite morphism in dAn_k. The the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z & \end{tikzcd}$$

admits a colimit in dAn_k, denoted Z'. Moreover, the natural morphism $Z \to Z'$ is also a nil-embedding.

Proof. The ∞-category of $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ is a presentable ∞-category. Consider the pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z \arrow[r] & Z', \end{tikzcd}$$

in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. By construction, the underlying ∞-topos of Z' can be computed as the pushout in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ of the induced diagram on the underlying ∞-topoiof X, Z and Y. Moreover, since g is a nil-isomorphism it induces an equivalence on underlying ∞-topoiof both X and Y. It follows that the induced morphism $Z \to Z'$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ induces an equivalence on the underlying ∞-topoi. Moreover, it follows essentially by construction that we have a natural equivalence

$$\mathcal{O}_{Z'} \simeq g_*(\mathcal{O}_Y) \times_{g_*(\mathcal{O}_Y)} \mathcal{O}_Z \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}\mathrm{loc}(Z).$$

As effective epimorphisms are preserved under fiber products in an ∞-topos, it follows that the natural morphism

$$\mathcal{O}_{Z'} \to \mathcal{O}_Z,$$

is an effective epimorphism (since $g_*(\mathcal{O}_Y) \to g_*(\mathcal{O}_X)$ it is so). Consider now the commutative diagram of fiber sequences

$$\begin{tikzcd} \mathcal{J}' \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[d] \\ \mathcal{J} \arrow[r] & g_*(\mathcal{O}_Y) \arrow[r] & g_*(\mathcal{O}_X), \end{tikzcd}$$

in the stable ∞-category $\mathrm{Mod}_{\mathcal{O}'_Z}$. Since the right commutative square is a pullback square it follows that the morphism

$$\mathcal{J}' \to \mathcal{J},$$

is an equivalence. In particular, $\pi_0(\mathcal{J}')$ is a finitely generated nilpotent ideal of $\pi_0(\mathcal{O}^{\mathrm{alg}}_{\mathcal{J}'})$. Indeed, finitely generation follows from our assumption that g is a finite morphism. Thanks to Lemma 1.6, it follows that $\mathrm{t}_0(Z')$ is an ordinary k-analytic space and the morphism $\mathrm{t}_0(Z') \to \mathrm{t}_0(Z)$ is a nil-embedding. We are thus reduced to show that for every $i > 0$, the homotopy sheaf $\pi_i(\mathcal{O}_{Z'}) \in \mathrm{Coh}^+(\mathrm{t}_0(Z'))$. But this follows immediately from the existence of a fiber sequence

$$\mathcal{O}_{Z'} \to g_*(\mathcal{O}_Y) \oplus \mathcal{O}_Z \to g_*(\mathcal{O}_X),$$

in the ∞-category $\mathrm{Mod}_{\mathcal{O}_{Z'}}$ together with the fact that $g_*(\mathcal{O}_Y)$ and $g_*(\mathcal{O}_Z)$ have coherent homotopy sheaves, by our assumption that g is a finite morphism combined with Lemma 2.2. $\square$

Definition 2.9. An *analytic formal moduli problem under X* corresponds to the datum of a functor

$$F \colon (\mathrm{AnNil}_{X/})^{\mathrm{op}} \to \mathcal{S},$$

satisfying the following two conditions:

(1) $F(X) \simeq *$ in $\mathcal{S}$;
(2) $F \simeq \mathbf{res}^{<\infty} \circ F$, where $\mathbf{res}_!^{<\infty}$ denotes the right Kan extension along the natural inclusion
(3) Given any pushout diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T', \end{tikzcd}$$

in $\mathrm{AnNil}_{X/}$ for which f is has the structure of a square zero extension, the induced morphism

$$F(T') \to F(T) \times_{F(S)} F(S),$$

is an equivalence in $\mathcal{S}$.

We shall denote by $\mathrm{AnFMP}_{X/}$ the full subcategory of $\mathrm{Fun}((\mathrm{AnNil}_{X/})^{\mathrm{op}}, \mathcal{S})$ spanned by analytic formal moduli problems under X.

Construction 2.10. We have a composite diagram

$$h \colon \mathrm{AnNil}_{X/} \to \mathrm{dAn}_k \hookrightarrow \mathrm{AnPreStk}.$$

Therefore, given any analytic pre-stack regarded as a limit-preserving functor $F \colon \mathrm{AnPreStk}^{\mathrm{op}} \to \mathcal{S}$, one can consider its restriction to the ∞-category $\mathrm{AnNil}_{X/}$:

$$F \circ h \colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathcal{S}.$$

We have thus a natural restriction functor

$$h_* \colon \mathrm{AnPreStk} \to \mathrm{Fun}(\mathrm{AnNil}_{X/}^{\mathrm{op}}, \mathcal{S}).$$

Example 2.11. Let $X \in \mathrm{dAn}_k$. As in the algebraic case, we can consider the *de Rham pre-stack associated to* X, $X_{\mathrm{dR}} \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$, determined by the formula

$$X_{\mathrm{dR}}(Z) \coloneqq X(Z_{\mathrm{red}}), \quad Z \in \mathrm{dAfd}_k.$$

We have a natural morphism $X \to X_{\mathrm{dR}}$ induced from the natural morphism $Z_{\mathrm{red}} \to Z$. We claim that $h_*(X_{\mathrm{dR}}) \in \mathrm{Fun}(\mathrm{AnNil}_{X/}^{\mathrm{op}}, \mathcal{S})$ belongs to the full subcategory $\mathrm{AnFMP}_{X/}$. Indeed, in this case it is clear that $h_*(X_{\mathrm{red}})$ is the final object in $\mathrm{AnFMP}_{X/}$ which clearly satisfies conditions i) and ii) in Definition 2.9.

Notation 2.12. We set $\mathrm{AnNil}_{X/}^{\mathrm{cl}} \subseteq \mathrm{AnNil}_{X/}$ to be the full subcategory spanned by those objects corresponding to nil-embeddings of the form

$$X \to S,$$

in dAn_k.

Proposition 2.13. *Let $Y \in \mathrm{AnNil}_{X/}$. The following assertions hold:*

1. *Then the inclusion functor*
$$\mathrm{AnNil}_{X//Y}^{\mathrm{cl}} \hookrightarrow \mathrm{AnNil}_{X//Y},$$
is cofinal.
2. *The natural morphism*
$$\operatorname*{colim}_{Z \in \mathrm{AnNil}_{X//Y}^{\mathrm{cl}}} Z \to Y,$$
is an equivalence in $\mathrm{Fun}((\mathrm{AnNil}_{X//Y})^{\mathrm{op}}, \mathcal{S})$.
3. *The ∞-category $\mathrm{AnNil}_{X//Y}^{\mathrm{cl}}$ is filtered.*

Proof. We start by proving claim (i). Consider the usual restriction along the natural morphism $X_{\mathrm{red}} \to X$ functor

$$\mathbf{res} \colon \mathrm{AnNil}_{X/} \to \mathrm{AnNil}_{X_{\mathrm{red}}/}.$$

Such functor admits a well defined left adjoint

$$\mathbf{push} \colon \mathrm{AnNil}_{X_{\mathrm{red}}/} \to \mathrm{AnNil}_{X/},$$

which is determined by the formula

$$(X_{\mathrm{red}} \to T) \in \mathrm{AnNil}_{X_{\mathrm{red}}/} \mapsto (X \to T') \in \mathrm{AnNil}_{X/},$$

where we set

$$T' \coloneqq X \bigsqcup_{X_{\mathrm{red}}} T \in \mathrm{AnNil}_{X/}. \tag{2.3}$$

We claim that $T' \in \mathrm{AnNil}_{X/}$ belongs to the full subcategory $\mathrm{AnNil}_{X/}^{\mathrm{cl}} \subseteq \mathrm{AnNil}_{X/}$. Indeed, since the structural morphism $X_{\mathrm{red}} \to T$, is necessarily a nil-embedding we deduce that the claim follows readily from Proposition 2.8. We shall denote

$$\mathbf{res}_!(Y) \colon \mathrm{AnNil}_{X_{\mathrm{red}}/}^{\mathrm{op}} \to \mathcal{S},$$

the left Kan extension of Y along the functor $\mathbf{res}$ above. By the colimit formula for left Kan extensions, c.f. [6, Lemma 4.3.2.13], it follows that $\mathbf{res}_!(Y)$ is given by the formula

$$(X_{\mathrm{red}} \to T) \in \mathrm{AnNil}_{X_{\mathrm{red}}/} \mapsto Y(T') \in \mathcal{S},$$

where T' is as in (2.3). Let $g\colon X_{\mathrm{red}} \to T$ in $\mathrm{AnNil}_{X_{\mathrm{red}}/}$ and assume that g factors through the natural morphism $X_{\mathrm{red}} \to X$. Then we have a natural morphism

$$i_{T,*}\colon Y(T) \to \mathbf{res}_!(Y)(T),$$

in $\mathcal{S}$, which exhibits the former as a retract of the latter. Denote by

$$p_{T,*}\colon \mathbf{res}_!(Y)(T) \to Y(T),$$

be a right inverse to $i_{S,*}$. Consider the functor

$$\mathbf{res}_Y\colon \mathrm{AnNil}_{X//Y} \to \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)},$$

given by the formula

$$(X \to S \to Y) \in \mathrm{AnNil}_{X//Y} \mapsto (X_{\mathrm{red}} \to S \xrightarrow{f} \mathbf{res}_!(Y)),$$

where $f\colon S \to \mathbf{res}_!(Y)$ corresponds to the morphism

$$S_X \xrightarrow{p_S} S \to Y,$$

where $S_X \coloneqq X \bigsqcup_{X_{\mathrm{red}}} S$. We claim that the functor $\mathbf{res}_Y$ is a right adjoint to the functor

$$\mathbf{push}_Y\colon \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \to \mathrm{AnNil}_{X//Y},$$

the latter given by the formula

$$(X_{\mathrm{red}} \to T \to \mathbf{res}_!(Y)) \in \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \mapsto (X \to T_X \to Y) \in \mathrm{AnNil}_{X//Y}.$$

Indeed, the datum of a morphism

$$(X_{\mathrm{red}} \to T \to \mathbf{res}_!(Y)) \to \mathbf{res}_Y(X \to S \to Y),$$

in $\mathrm{AnNil}_{X_{\mathrm{red}}/\mathbf{res}_!(Y)}$ corresponds to the datum of a commutative diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & \mathbf{res}_!(Y) \arrow[d, "="] \\
X_{\mathrm{red}} \arrow[r] & S \arrow[r] & \mathbf{res}_!(Y),
\end{tikzcd}$$

where the right bottom morphism corresponds to the composite $S_X \to S \to Y$. For this reason, the given datum is equivalent to a commutative diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r] \arrow[d] & T \arrow[r] & T_X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[dr, "="] & \\
X_{\mathrm{red}} \arrow[r] & S \arrow[r] & S_X \arrow[r] & S \arrow[r] & Y,
\end{tikzcd}$$

which on the other hand is equivalent to the datum of a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "="] & T_X \arrow[r] \arrow[d] & Y \arrow[d, "="] \\
X \arrow[r] & S \arrow[r] & Y
\end{tikzcd}$$

The previous observations combined together then imply that we have a well defined adjunction

$$\mathbf{res}\colon \mathrm{AnNil}_{X//Y} \rightleftarrows \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)}\colon \mathrm{push}.$$

We thus conclude that $\mathrm{AnNil}_{X//Y} \to \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)}$ is a cofinal functor (as it admits a left adjoint). Claim (i) now follows immediately from the observation that the functor

$$\mathrm{push}\colon \mathrm{AnNil}_{X_{\mathrm{red}}/\mathbf{res}_!(Y)} \to \mathrm{AnNil}_{X//Y},$$

factors through the natural inclusion $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y} \to \mathrm{AnNil}_{X//Y}$. Claim (ii) follows immediately from (i) combined with Yoneda Lemma. To prove (iii) we shall make use of [6, Lemma 5.3.1.12]. Let

$$F\colon \partial\Delta^n \to \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}.$$

For each $[m] \in \Delta^n$, denote by $S_m := F([m])$ in $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$. We then have that the pushout

$$S_n \bigsqcup_X S_{n-1},$$

exists in $\mathrm{AnNil}^{\mathrm{cl}}_{X/}$. We wish to show that $S_n \bigsqcup_X S_n$ admits a morphism

$$S_n \bigsqcup_X S_{n-1} \to Y,$$

compatible with the diagram F. In order to show this, we can filter the diagram F by diagrams $F_i \to F$ such that $X \to F_0$ is formed by square-zero extensions and so are each $F_i \to F_{i+1}$. Moreover, by the fact that Y satisfies condition (ii) in Definition 2.9 it follows that we can find a well defined morphism

$$S_n \bigsqcup_X S_{n-1} \to Y,$$

which is compatible with F, as desired. □

Construction 2.14. Let $X \in \mathrm{dAn}_k$. Consider the natural functor

$$F\colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathrm{dAn}^{\mathrm{op}}_k.$$

Left Kan extension along F induces a functor

$$F_!\colon \mathrm{Fun}(\mathrm{AnNil}^{\mathrm{op}}_{X/}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{dAn}^{\mathrm{op}}_k, \mathcal{S}),$$

and thus an induced functor

$$F_!\colon \mathrm{AnFMP}_{X/} \to \mathrm{Fun}(\mathrm{dAn}^{\mathrm{op}}_k, \mathcal{S}),$$

as well. We denote the latter ∞-category by $\mathrm{AnPreStk}_k$, the ∞-category of *k-analytic pre-stacks*. Proposition 2.13 implies that the functor $F_!$ preserves filtered colimits. In particular, if we regard Y as a k-analytic prestack can be presented as an *ind-inf*-object in the ∞-category dAn_k, i.e., it can be written as a filtered colimit of nil-embeddings $X \to Z$. We refer the reader to [4] for a precise meaning of the latter notion in the algebraic setting.

Definition 2.15. Let $Y \in \mathrm{AnFMP}_{X/}$ denote an analytic formal moduli problem under X. The *relative pro-analytic cotangent complex of* Y *under* X is defined as the pro-object

$$\mathbb{L}^{\mathrm{an}}_{X/Y} := \{\mathbb{L}^{\mathrm{an}}_{X/Z}\}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \in \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

where, for each $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$

$$\mathbb{L}^{\mathrm{an}}_{X/Z} \in \mathrm{Coh}^+(X),$$

denotes the usual analytic cotangent complex associated to the structural morphism $X \to Z$ in $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$.

Remark 2.16. Let $Y \in \mathrm{AnFMP}_{X/}$. Let $Z \in \mathrm{dAn}_k$, there exists a natural morphism

$$\mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/Z},$$

in $\mathrm{Coh}^+(X)$. Passing to the limit over $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$, we obtain a natural map

$$\mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/Y},$$

in $\mathrm{Pro}(\mathrm{Coh}^+(X))$, as well.

The following result provides justifies our choice of terminology for the object $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$:

Lemma 2.17. *Let $Y \in \mathrm{AnFMP}_{X/}$. Let $X \hookrightarrow S$ be a square zero extension associated to an analytic derivation*

$$d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F},$$

where $\mathcal{F} \in \mathrm{Coh}^+(X)^{\geq 0}$. Then there exists a natural morphism

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \to \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}, \mathcal{F})} \{d\}$$

which is furthermore an equivalence in the ∞-category $\mathcal{S}$.

Proof. Thanks to Proposition 2.13 we can identify the space of liftings of the map $X \to Y$ along $X \to S$ with the mapping space

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \simeq \operatorname*{colim}_{Z \in \mathrm{AnNil}_{X//Y}} \mathrm{Map}_{\mathrm{AnNil}_{X/}}(S, Z).$$

Fix $Z \in \mathrm{AnNil}_{X//Y}$. Then we have a natural identification of mapping spaces

$$\begin{aligned} \mathrm{Map}_{\mathrm{AnNil}_{X/}}(S, Z) &\simeq \mathrm{Map}_{(\mathrm{dAn}_k)_{X/}}(S, Z) && (2.4) \\ &\simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Z}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_X, \mathcal{F})} \{d\}, && (2.5) \end{aligned}$$

see [7, §5.4] for a justification of the latter assertion. Passing to the colimit over $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$, we conclude that

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}, \mathcal{F})} \{d\},$$

as desired. ☐

Construction 2.18. Let $f \colon Y \to Z$ denote a morphism in $\mathrm{AnFMP}_{X/}$. Then, for every $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ the induced morphism

$$S \to Z,$$

in $\mathrm{AnFMP}_{X/}$ factors through some $S' \in \mathrm{AnNil}^{\mathrm{cl}}_{X/Z}$. For this reason, we obtain a natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X/S'} \to \mathbb{L}^{\mathrm{an}}_{X/S},$$

in the ∞-category $\mathrm{Coh}^+(X)$. Passing to the limit over $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ we obtain a canonically defined morphism

$$\theta(f) \colon \mathbb{L}^{\mathrm{an}}_{X/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Y},$$

in $\mathrm{Pro}(\mathrm{Coh}^+(X))$.

Proposition 2.19. *Let $f \colon Y \to Z$ be a morphism in the ∞-category $\mathrm{AnFMP}_{X/}$. Suppose that f induces an equivalence of relative pro-analytic cotangent complexes via Construction 2.18. Then f is itself an equivalence of analytic formal moduli problems under X.*

Proof. Thanks to Proposition 2.13 we are reduced to show that given any

$$S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z},$$

the structural morphism $g_S \colon X \to S$ admits a unique extension $S \to Y$ which factors the structural morphism $X \to Y$. Thanks to Proposition 2.4 we can reduce ourselves to the case where $X \to S$ has the structure of a square zero extension. In this case, the result follows from Lemma 2.17 combined with our hypothesis. □

Definition 2.20. Let $Y \in \mathrm{AnPreStk}$, we shall say that Y is *infinitesimally cartesian* if it satisfies [7, Definition 7.3].

Proposition 2.21. *Let $Y \in \mathrm{AnPreStk}^{<\infty}_{X/}$. Assume further that Y is infinitesimally cartesian and it admits a relative pro-cotangent complex, $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$. Then Y is equivalent to an analytic formal moduli problem under X.*

Proof. We must prove that given a pushout diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"] & S' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

in the ∞-category $\mathrm{AnNil}_{X/}$, where f has the structure of a square-zero extension, then the natural morphism

$$Y(T') \to Y(T) \times_{Y(S)} Y(S'),$$

is an equivalence in the ∞-category $\mathcal{S}$. Suppose further that $S \hookrightarrow S'$ is associated to some derivation $d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}$ for some $\mathcal{F} \in \mathrm{Coh}^+(S)^{\geq 0}$. Thanks to Lemma 2.7 we deduce that the induced morphism $T \to T'$ admits a structure of a square-zero extension. Then, by our assumptions of Y being infinitesimally cartesian and admitting a relative pro-cotangent complex, we have a chain of natural equivalences of the form.

$$\begin{aligned} Y(T') &\simeq \coprod_{f\colon T \to Y} \mathrm{Map}_{T/}(T', Y) \\ &\simeq \coprod_{f\colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(T))_{\mathbb{L}^{\mathrm{an}}_Z/}}(\mathbb{L}^{\mathrm{an}}_{T/Y}, g_*(\mathcal{F})) \\ &\simeq \coprod_{f\colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))_{g^*\mathbb{L}^{\mathrm{an}}_T/}}(g^*\mathbb{L}^{\mathrm{an}}_{T/Y}, \mathcal{F}) \\ &\simeq \coprod_{f\colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))_{\mathbb{L}^{\mathrm{an}}_S/}}(\mathbb{L}^{\mathrm{an}}_{S,Y}, \mathcal{F}) \\ &\simeq \coprod_{f\colon T \to Y} \mathrm{Map}_{S/}(S', Y) \\ &\simeq Y(T) \times_{Y(S)} Y(S'), \end{aligned}$$

where the third equivalence follows from the existence of a commutative diagram between fiber sequences

$$\begin{tikzcd} g^* f^* \mathbb{L}^{\mathrm{an}}_Y \arrow[r] \arrow[d, "="] & g^*\mathbb{L}^{\mathrm{an}}_T \arrow[r] \arrow[d] & g^*\mathbb{L}^{\mathrm{an}}_{T/Y} \arrow[d] \\ (f\circ g)^*\mathbb{L}^{\mathrm{an}}_Y \arrow[r] & \mathbb{L}^{\mathrm{an}}_S \arrow[r] & \mathbb{L}^{\mathrm{an}}_{S/Y}, \end{tikzcd}$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(S))$ combined with the fact that the derivation $d_T \colon \mathbb{L}^{\mathrm{an}}_T \to g_*(\mathcal{F})$ is induced from

$$d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F},$$

as in the proof of Lemma 2.7. The result now follows. □

2.2. **Analytic formal moduli problems over a base.** Let $X \in \mathrm{dAn}_k$ denote a derived k-analytic space. In [9, Definition 6.11] the authors introduced the ∞-category of *analytic formal moduli problems over* X, which we shall denote by $\mathrm{AnFMP}_{/X}$.

Notation 2.22. Let $X \in \mathrm{dAn}_k$. We shall denote by $\mathrm{AnNil}_{/X}$ the full subcategory of $(\mathrm{dAn}_k)_{/X}$ spanned by nil-isomorphisms

$$Z \to X.$$

Definition 2.23. We shall denote by $\mathrm{AnNil}^{\mathrm{cl}}_{/X} \subseteq \mathrm{AnNil}_{/X}$ the faithful subcategory in which we only allow morphisms

$$S \to S'$$

in $\mathrm{AnNil}_{/X}$ which are closed nil-isomorphisms.

We start with the analogue of Proposition 2.13 in the setting of analytic formal moduli problems over X:

Proposition 2.24. *Let $Y \in \mathrm{AnFMP}_{/X}$. The following assertions hold:*

(1) The inclusion functor

$$(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y} \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y},$$

is cofinal.

(2) The natural morphism

$$\operatorname*{colim}_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} Z \to Y,$$

is an equivalence in the ∞-category $\mathrm{AnFMP}_{/X}$.

(3) The ∞-category $\mathrm{AnNil}^{\mathrm{cl}}_{/X}$ is filtered.

Proof. We first prove assertion (i). Let $S \to Z$ be a morphism in $(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}$. Consider the pushout diagram

$$\begin{tikzcd} S_{\mathrm{red}} \arrow[r] \arrow[d] & S \arrow[d] \\ Z \arrow[r] & Z', \end{tikzcd} \tag{2.6}$$

in the ∞-category $\mathrm{AnNil}_{/X}$ whose existence is guaranteed by Proposition 2.8. Since the upper horizontal morphism in (2.6) is a closed nil-isomorphism, we can reduce ourselves to the case where the latter is an actual square-zero extension. Indeed, the latter assertion follows by arguing by induction combined with Proposition 2.4. Since Y is assumed to be an analytic formal moduli problem over X we then deduce that the canonical morphism

$$\begin{aligned} Y(Z') &\to Y(Z) \times_{Y(S_{\mathrm{red}})} Y(S) \\ &\simeq Y(Z) \times Y(S), \end{aligned}$$

is an equivalence (we implicitly used above the fact that $S_{\mathrm{red}} \simeq X_{\mathrm{red}}$). As a consequence the object $(Z' \to X)$ in $\mathrm{AnNil}_{/X}$ admits an induced morphism $Z' \to Y$ making the required diagram commute. Thanks Proposition 2.8 we deduce that both $S \to Z'$ and $Z \to Z'$ are closed nil-isomorphisms. Therefore, we can factor the diagram

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

via a closed nil-isomorphism $Z \to Z'$. We conclude that the inclusion functor $(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y} \to (\mathrm{AnNil}_{/X})_{/Y}$ is cofinal. It is clear that assertion (ii) follows immediately from (i). We now prove (iii). Let

$$\theta \colon K \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y},$$

be a functor where K is a finite ∞-category. We must show that θ can be extended to a functor

$$\theta^{\triangleright} \colon K^{\triangleright} \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}.$$

Thanks to Proposition 2.4 we are allowed to reduce ourselves to the case where morphisms indexed by K are square-zero extensions. The result now follows from the fact that Y being an analytic moduli problem sends finite colimits along square-zero extensions to finite limits. ☐

Lemma 2.25. *Let $X \in \mathrm{dAn}_k$. Given any $Y \in \mathrm{AnFMP}_{X/}$, then for each $i = 0, 1$ the i-th projection morphism*

$$p_0 \colon X \times_Y X \to X,$$

computed in the ∞-category $\mathrm{AnPreStk}_k$ lies in the essential image of $\mathrm{AnFMP}_{/X}$ via Construction 2.14.

(Personal: This lemma might be as well erase, as it is a direct consequence of the previous proposition. We will need nonetheless this result in the construction of $\mathrm{B}_X(Y)$.)

Proof. Consider the pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_0"] & X \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

computed in the ∞-category AnPreStk. Thanks to Proposition 2.13 together with the fact that fiber products commute with filtered colimis in the ∞-category $\mathrm{AnPreStk}_k$, we deduce that

$$X \times_Y X \simeq \operatorname*{colim}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} X \times_Z X,$$

in $\mathrm{AnPreStk}_k$. It is clear that $(p_i \colon X \times_Z X \to X)$ lies in the essential image of $\mathrm{AnFMP}_{/X}$, for $i = 0, 1$. Thus also the filtered colimit

$$(p_i \colon X \times_Y X) \in \mathrm{AnFMP}_{/X}, \quad \text{for } i = 0, 1,$$

as desired. ☐

Just as in the previous section we deduce that every analytic formal moduli problem over X admits the structure of an *ind-inf*-object in $\mathrm{AnPreStk}_k$:

Corollary 2.26. *Let $Y \in (\mathrm{AnPreStk}_k)_{/X}$. Then Y is equivalent to an analytic formal moduli problem over X if and only if there exists a presentation $Y \operatorname{colim}_{i \in I} Z_i$, where I is a filtered ∞-category and for every $i \to j$ in I, the induced morphism*

$$Z_i \to Z_j,$$

is a closed embedding of derived k-affinoid spaces that are nil-isomorphic to X.

Proof. It follows immediately from Proposition 2.24 (ii). ☐

Definition 2.27. Let $Y \in \mathrm{AnFMP}_{/X}$. We define the ∞-category of *coherent modules on Y*, denoted $\mathrm{Coh}^+(Y)$, as the limit

$$\mathrm{Coh}^+(Y) := \lim_{Z \in (\mathrm{dAn}_k)_{/Y}} \mathrm{Coh}^+(Z),$$

computed in the ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$. We define the ∞-category of *pro-coherent modules on* Y, denoted $\mathrm{Pro}(\mathrm{Coh}^+(Y))$, as

$$\mathrm{Pro}(\mathrm{Coh}^+(Y)) \coloneqq \lim_{Z \in (\mathrm{dAn}_k)_{/Y}} \mathrm{Pro}(\mathrm{Coh}^+(Z)),$$

where the limit is computed in the ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$.

Definition 2.28. Let $Y \in \mathrm{AnFMP}_{/X}$, $Z \in \mathrm{dAfd}_k$ and let $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$. Suppose furthermore that we are given a morphism $f \colon Z \to Y$. We define the *tangent space of* Y *at* f *twisted by* $\mathcal{F}$ as the fiber

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F},f} \coloneqq \mathrm{fib}_f\big(Y(Z[\mathcal{F}]) \to Y(Z)\big) \in \mathcal{S}.$$

Whenever the morphism f is clear from the context, we shall drop the subscript f above and denote the tangent space of Y at f simply by $\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}}$.

Remark 2.29. Let $Y \in \mathrm{AnFMP}_{/X}$. The equivalence of ind-objects

$$Y \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} S,$$

in the ∞-category dAn_k, implies that, for any $Z \in \mathrm{dAfd}_k$, one has an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{AnPreStk}}(Z, Y) \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \mathrm{Map}_{\mathrm{AnPreStk}}(Z, S).$$

For this reason, given any morphism $f \colon Z \to Y$ and any $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$, we can identify the tangent space $\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}}$ with the filtered colimit of spaces

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathrm{fib}_f\big(S(Z[\mathcal{F}]) \to S(Z)\big) \tag{2.7}$$

$$\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathbb{T}^{\mathrm{an}}_{S,Z,\mathcal{F}} \tag{2.8}$$

$$\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathrm{Map}_{\mathrm{Coh}^+(Z)}(f^*_{S,Z}(\mathbb{L}^{\mathrm{an}}_S), \mathcal{F}), \tag{2.9}$$

where we have denoted by $f_{S,Z} \colon Z \to S$ any morphism, in $(\mathrm{dAn}_k)_{/X}$, factoring $f \colon Z \to Y$. Moreover, the latter equivalence follows readily from [7, Lemma 7.7]. Therefore, we deduce that the analytic formal moduli problem $Y \in \mathrm{AnFMP}_{/X}$ admits an *absolute pro-cotangent complex* given as

$$\mathbb{L}^{\mathrm{an}}_Y \coloneqq \{f^*_{S,Z}(\mathbb{L}^{\mathrm{an}}_S)\}_{Z \in (\mathrm{dAn}_k)_{/Y}, S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \in \mathrm{Pro}(\mathrm{Coh}^+(Y)).$$

Corollary 2.30. *Let* $Y \in \mathrm{AnFMP}_{/X}$*. Then its absolute cotangent complex* $\mathbb{L}^{\mathrm{an}}_Y$ *classifies analytic deformations on* Y*. More precisely, given* $Z \to Y$ *a morphism where* $Z \in \mathrm{dAfd}_k$ *and* $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$ *one has a natural equivalence of mapping spaces*

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(Y))}(\mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}).$$

Proof. It follows immediately from the natural equivalences displayed in (2.7) combined with the description of mapping spaces in ∞-categories of pro-objects. □

We now introduce the notion of square-zero extensions of analytic formal moduli problems over X:

Construction 2.31. Let $(f \colon Y \to X) \in \mathrm{AnFMP}_{/X}$. Let $d \colon \mathbb{L}^{\mathrm{an}}_Y \to \mathcal{F}[1]$ be an *analytic derivation* in $\mathrm{Pro}(\mathrm{Coh}^+(Y))$, where $\mathcal{F} \in \mathrm{Coh}^+(Y)^{\geq 0}$, such that

$$\mathcal{F} \simeq f^*(\mathcal{F}'),$$

for some suitable object $\mathcal{F}' \in \mathrm{Coh}^+(X)^{\geq 0}$. Thanks to Remark 2.29 one has the following natural equivalences of mapping spaces

$$\begin{aligned} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(Y))}(\mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}[1]) &\simeq \lim_{S \in (\mathrm{AnNil}_{/X})_{/Y}} \operatorname*{colim}_{S' \in (\mathrm{AnNil}_{/X})_{S//Y}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))}(f^*_{S,S'}(\mathbb{L}^{\mathrm{an}}_S), g^*_S(\mathcal{F}')[1]) \\ &\simeq \lim_{S \in (\mathrm{AnNil}_{/X})_{/Y}} \operatorname*{colim}_{S' \in (\mathrm{AnNil}_{/X})_{S//Y}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))}(\mathbb{L}^{\mathrm{an}}_S, (f_{S,S'})_* g^*_S(\mathcal{F}')[1]), \end{aligned}$$

where $g_S \colon S \to X$ denotes the structural morphism in $\mathrm{AnNil}_{/X}$ and $f_{S,S'} \colon S \to S'$ a given transition morphism in the ∞-category $(\mathrm{AnNil}_{/X})_{/Y}$. For this reason, we can form the filtered colimit

$$Y' \coloneqq \operatorname*{colim}_{S \in (\mathrm{AnNil}_{/X})_{/Y}} \operatorname*{colim}_{S' \in (\mathrm{AnNil}_{/X})_{S//Y}} \overline{S}' \in \mathrm{AnPreStk}.$$

By construction, one has a natural morphism $Y \hookrightarrow Y'$ in the ∞-category AnPreStk. Moreover, thanks to Proposition 2.21 it follows that $Y' \in \mathrm{AnFMP}_{/X}$.

Definition 2.32. Let $Y \in \mathrm{AnFMP}_{/X}$. Suppose we are given an analytic derivation

$$d \colon \mathbb{L}^{\mathrm{an}}_Y \to \mathcal{F}[1],$$

in $\mathrm{Pro}(\mathrm{Coh}^+(Y))$ where $\mathcal{F} \in \mathrm{Coh}^+(Y)^{\geq 0}$ is such that $\mathcal{F} \simeq f^*(\mathcal{F}')$, for some $\mathcal{F}' \in \mathrm{Coh}^+(X)^{\geq 0}$. We shall say that the induced morphism

$$h \colon Y \to Y',$$

defined in Construction 2.31, is a *square-zero extension* of Y associated to the analytic derivation d.

Corollary 2.33. *Let* $Y \in \mathrm{AnFMP}_{/X}$*. Given any square-zero extension* $h \colon X \hookrightarrow S$ *in* dAn_k*. Then the space of cartesian squares*

$$\begin{tikzcd} Y \arrow[r, "h'"] \arrow[d, "f"] & Y' \arrow[d, "g"] \\ X \arrow[r, "h"] & S, \end{tikzcd}$$

such that $h' \colon Y \to Y'$ *is a square-zero extension and* $g \colon Y' \to S$ *exhibits the former as an analytic formal moduli problem over* S *is naturally equivalent to the space of factorizations*

$$f^* \mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_Y \to f^*(\mathcal{F}')[1],$$

in $\mathrm{Pro}(\mathrm{Coh}^+(Y))$*, of the analytic derivation* $d \colon \mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}'[1]$ *associated to the morphism* h *above.*

Proof. By the universal property of ind-objects we reduce the statement to the case where $Y \in \mathrm{AnNil}_{/X}$ and thus $Y' \in \mathrm{AnNil}_{/S}$, in which case the statement follows immediately by the universal property of the cotangent complex. (Todo: Add details in this proof.) ☐

Corollary 2.34. *Let* $f \colon Z \to X$ *be a morphism in the* ∞*-category* dAn_k*. Suppose we are given analytic formal moduli problems*

$$f \colon Y \to X \quad \text{and} \quad g \colon \widetilde{Z} \to Z$$

together with a commutative diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "s"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "f"] & X, \end{tikzcd}$$

in the ∞*-category* AnPreStk*. Let* $d \colon \mathbb{L}^{\mathrm{an}}_Z \to \mathcal{F}[1]$ *be an analytic derivation corresponding to a square-zero extension morphism* $Z \to Z'$ *in the* ∞*-category* dAn_k*. Denote by* $\widetilde{d} \colon \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \to \mathcal{F}[1]$ *the induced analytic derivation* *as in Construction 2.31 and let* $h\colon \widetilde{Z} \hookrightarrow \widetilde{Z}'$ *the induced square-zero extension in* AnPreStk *such that we have a cartesian diagram*

$$\begin{tikzcd} \widetilde{Z} \arrow[r] \arrow[d] & \widetilde{Z}' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

in the ∞*-category* AnPreStk*. Then the space of factorizations*

$$s\colon \widetilde{Z} \to \widetilde{Z}' \to Y,$$

is naturally equivalent to the space of factorizations

$$\widetilde{d}\colon \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \to \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}/Y} \to \mathcal{F}[1],$$

in the ∞*-category* $\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}))$.

Proof. The statement holds true in the case where $\widetilde{Z} \in \mathrm{AnNil}_{/Z}$ and $Y \in \mathrm{AnNil}_{/X}$, by the universal property of the relative cotangent complex. The general case is reduced to the previous one by a standard argument with ind-objects in AnPreStk. □

2.3. **Non-archimedean nil-descent for almost perfect complexes.** In this §, we prove that the ∞-category $\mathrm{Coh}^+(X)$, for $X \in \mathrm{dAn}_k$ satisfies nil-descent with respect to morphims $Y \to X$, which exhibit the former as an analytic formal moduli problem over X.

Proposition 2.35. *Let* $f\colon Y \to X$*, where* $X \in \mathrm{dAn}_k$ *and* $Y \in \mathrm{AnFMP}_{/X}$*. Consider the Čech nerve* $Y^\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnPreStk}$ *associated to* f*. Then the natural functor*

$$f^*_\bullet\colon \mathrm{Coh}^+(X) \to \mathrm{Tot}(\mathrm{Coh}^+(Y^\bullet)),$$

is an equivalence of ∞*-categories.*

Proof. Consider the natural equivalence of k-analytic prestacks

$$Y \simeq \operatorname*{colim}_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} Z.$$

Then, by definition one has a natural equivalence

$$\mathrm{Coh}^+(Y) \simeq \lim_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \mathrm{Coh}^+(Z),$$

of ∞-categories. In particular, since totalizations commute with cofiltered limits in Cat_∞, it follows that we can suppose from the beginning that $Y \simeq Z$ for some $Z \in \mathrm{AnNil}_{/X}$. In this case, the morphism $f\colon Y \to X$ is affine. In particular, the fact that $\mathrm{Coh}^+(-)$ satisfies Zariski descent combined with Lemma 1.10 we further reduce ourselves to the case where we might assume both X and Y to be both equivalent to derived k-affinoid spaces. In this case, by Tate acyclicity theorem it follows that letting $A \coloneqq \Gamma(X, \mathcal{O}_X^{\mathrm{alg}})$ and $B \coloneqq \Gamma(Y, \mathcal{O}_Y^{\mathrm{alg}})$ the pullback functor f^* can be identified with the base change functor

$$\mathrm{Coh}^+(A) \to \mathrm{Coh}^+(B).$$

In this case, it follows that B is nil-isomophic to A. Moreover, since the latter are derived noetherian rings the statement of the proposition follows due to [5, Theorem 3.3.1]. □

Corollary 2.36. *Let $X \in \mathrm{dAn}_k$ and $f \colon Y \to X$ a morphism in* AnPreStk *which exhibits Y as an analytic formal moduli problem over X. Then the natural functor*

$$f_\bullet^* \colon \mathrm{Pro}(\mathrm{Coh}^+(X)) \to \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))),$$

is fully faithful, where $Y^\bullet$ denotes the Čech nerve of the morphism f.. Moreover, the essential image of the functor $f_\bullet^*$ identifies canonically with the full subcategory

$$\mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(X))) \subseteq \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))),$$

spanned by those $\{\mathcal{F}_{i,[n]}\}_{i \in I_{[n]}^{\mathrm{op}},[n]} \in \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X)))$, such that for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, the ∞-categories $I_{[n]} = I$, for some fixed filtered ∞-category I.

Proof. By the very definition of the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(Y))$, we reduce ourselves as in Proposition 2.35 to the case where $Y = S$, for some $S \in \mathrm{AnNil}_{/X}$. In this case, it follows readily from Proposition 2.35 that the natural functor

$$f_\bullet^* \colon \mathrm{Pro}(\mathrm{Coh}^+(X)) \to \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))),$$

is fully faithful. We now proceed to prove the second claim of the corollary. Notice that, Lemma 2.6 implies that there exists a well defined right adjoint

$$f_* \colon \mathrm{Coh}^+(S) \to \mathrm{Coh}^+(X),$$

to the usual pullback functor $f^* \colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(S)$. We can extend the right adjoint f_* to a well defined functor

$$f_* \colon \mathrm{Pro}(\mathrm{Coh}^+(S)) \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

which commutes with cofiltered limits. For this reason, we have a well defined functor

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*} \colon \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

which further commutes with cofiltered limits. We claim that $\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*}$ is a right adjoint to $f_\bullet^*$ above. Indeed, given any $\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$ and $\{\mathcal{G}_{j,[n]}\}_{j \in J_{[n]}^{\mathrm{op}}, [n] \in \mathbf{\Delta}^{\mathrm{op}}} \in \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X)))$, we compute

$$\mathrm{Map}_{\mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X)))}(f_\bullet^*(\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}}), \{\mathcal{G}_{j,[n]}\}_{j \in J_{[n]}^{\mathrm{op}}, [n] \in \mathbf{\Delta}^{\mathrm{op}}}) \simeq \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(Y^{[n]}))}(\{f_{[n]}^\bullet(\mathcal{F}_i)\}_{i \in I^{\mathrm{op}}}, \{\mathcal{G}_{i,[n]}\}_{i \in I_{[n]}^{\mathrm{op}},[n]})$$

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \lim_{j \in J_{[n]}^{\mathrm{op}}} \mathrm{colim}_{i \in I} \mathrm{Map}_{\mathrm{Coh}^+(Y^{[n]})}(f_{[n]}^\bullet(\mathcal{F}_i), \mathcal{G}_{i,[n]}) \simeq \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \lim_{j \in J_{[n]}^{\mathrm{op}}} \operatorname*{colim}_{i \in I} \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}_i, f_{[n],*}(\mathcal{G}_{i,[n]}))$$

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}}, \{f_{[n],*}(\mathcal{G}_{i,[n]})\}_{i \in I_{[n]}^{\mathrm{op}}}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\{\mathcal{F}_i\}_{i \in I^{\mathrm{op}}}, \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \{f_{[n],*}(\mathcal{G}_{i,[n]})\}_{i \in I_{[n]}^{\mathrm{op}}}),$$

as desired. It is clear that the functor $f_\bullet^*$ above factors through the full subcategory

$$\mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \subseteq \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))).$$

For this reason, the pair $(f_\bullet^*, \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*})$ restricts to a well defined adjunction

$$f_\bullet^* \colon \mathrm{Pro}(\mathrm{Coh}^+(X)) \rightleftarrows \mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \colon \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*}.$$

In order to conclude, we will show that the functor

$$\lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*} \colon \mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))) \to \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

is conservative. Since both the ∞-categories $\mathrm{Pro}(\mathrm{Coh}^+(X))$ and $\mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X)))$ are stable, we are reduced to prove that given any

$$\{\mathcal{G}_{i,[n]}\}_{i \in I^{\mathrm{op}}} \in \mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X))),$$

such that

$$\lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} f_{\bullet,*}(\{\mathcal{G}_{i,[n]}\}_{i\in I^{\mathrm{op}}})\simeq 0, \tag{2.10}$$

we necessarily have

$$\{\mathcal{G}_{i,[n]}\}_{i\in I^{\mathrm{op}}}\simeq 0,$$

in $\mathrm{Tot}'(\mathrm{Pro}(\mathrm{Coh}^+(Y^\bullet/X)))$. Assume then (2.10). Since the the object $\{\mathcal{G}_{i,[n]}\}_{i\in I^{\mathrm{op}},[n]\in\mathbf{\Delta}^{\mathrm{op}}}$ has fixed cofiltered ∞-category I^{op} at the pro-level, we have an equivalence

$$\begin{aligned}\lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}\{\mathcal{G}_{i[n]}\}_{i\in I^{\mathrm{op}}} &\simeq \{\lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}\mathcal{G}_{i,[n]}\}_{i\in I^{\mathrm{op}}}\\ &\simeq \{\mathcal{G}_i\}_{i\in I^{\mathrm{op}}},\end{aligned}$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(X))$, where the $\mathcal{G}_i\in\mathrm{Coh}^+(X)$, for each $i\in I$, satisfying

$$f^*_{[n]}(\mathcal{G}_i)\simeq\mathcal{G}_{i,[n]}\in\mathrm{Coh}^+(Y^{[n]}),$$

thanks to Proposition 2.35. For this reason, we conclude that

$$f^*_\bullet(\{\mathcal{G}_i\}_{i\in I^{\mathrm{op}}})\simeq\{\mathcal{G}_{i,[n]}\}_{i\in I^{\mathrm{op}},[n]}\simeq 0,$$

as desired. □

We now use the *pseudo-nil-descent* for $\mathrm{Pro}(\mathrm{Coh}^+(X))$ (Todo: define what this means), to compute relative cotangent complexes of analytic formal moduli problems over X:

Corollary 2.37. *Let $f\colon Z\to X$ be a morphism in dAn_k. Suppose we are given a pullback square*

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "h"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X, \end{tikzcd}$$

in the ∞-category $\mathrm{AnPreStk}$, where $(Y\to X)\in\mathrm{AnFMP}_{/X}$ and $(g\colon\widetilde{Z}\to Z)\in\mathrm{AnFMP}_{/Z}$. Then the lax-limit object

$$\{\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/Y^{[n]}}\}\in\mathrm{Tot}^{\mathrm{lax}}(\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}^\bullet/Z))),$$

defines an actual cartesian section

$$\{\mathbb{L}^{\mathrm{an}}_{(\widetilde{Z})^{[n]}/Y^{[n]}}\}\in\mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z))),$$

which furthermore belongs to the essential image of the natural functor

$$g^*_\bullet\colon\mathrm{Pro}(\mathrm{Coh}^+(Z))\to\mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}^\bullet/Z)))$$

Proof. We first show that the $\{\mathbb{L}^{\mathrm{an}}_{(\widetilde{Z})^{[n]}/Y^{[n]}}\}\in\mathrm{Tot}^{\mathrm{lax}}(\mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z)))$ defines a cartesian section in the totalizations

$$\mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z))).$$

In order to show this assertion, it is sufficient to prove for every $[n]\in\mathbf{\Delta}^{\mathrm{op}}$ we have a natural equivalence

$$h^*(\mathbb{L}^{\mathrm{an}}_{(\widetilde{Z})^{[n]}/Y^{[n]}})\simeq\mathbb{L}^{\mathrm{an}}_{(\widetilde{Z})^{[n+1]}/Y^{[n+1]}},$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^{[n+1]}))$. The latter claim is an immediate consequence of the base change property for the analytic cotangent complex in the case where $Y\in\mathrm{AnNil}_{/X}$ (and thus so do $\widetilde{Z}\in\mathrm{AnNil}_{/Z}$)), which follows readily from [7, Proposition 5.12]. In the general case where $Y\in\mathrm{AnFMP}_{/X}$, we reduce to the case where $Y \in \mathrm{AnNil}_{/X}$ by combining Proposition 2.24 with the observation that filtered colimits commute with finite limits in the ∞-category AnPreStk.

We now prove the second assertion. Thanks to characterization of the essential image of natural functor

$$g_\bullet^* \colon \mathrm{Pro}(\mathrm{Coh}^+(Z)) \to \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z))),$$

provided in Corollary 2.36, we are reduced to show that for each $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, we have a natural equivalence of pro-objects

$$\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/Y^{[n]}} \simeq \{\mathbb{L}^{\mathrm{an}}_{\widetilde{S}^{[n]}/S^{[n]}}\}_{\widetilde{S} \in (\mathrm{AnNil}_{/Z})_{/\widetilde{Z}}, S \in (\mathrm{AnNil}_{/X})_{/Y}}.$$

The latter statement follows readily from the first part of the proof by a direct inductive argument. □

2.4. **Non-archimedean formal groupoids.** Let $X \in \mathrm{dAn}_k$. We start with the definition of the notion of *analytic formal groupoids over* X:

Definition 2.38. We denote by $\mathrm{AnFGrpd}(X)$ the full subcategory of the ∞-category of simplicial objects

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{AnFMP}_{/X}),$$

spanned by those objects $F \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnFMP}_{/X}$ satisfiying the following requirements:

(1) $F([0]) \simeq X$;
(2) For each $n \geq 1$, the morphism

$$F([n]) \to F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]),$$

induced by the morphisms $s^i \colon [1] \to [n]$ given by $(0,1) \mapsto (i, i+1)$, is an equivalence in $\mathrm{AnFMP}_{/X}$.

We shall refer to objects in $\mathrm{AnFGrpd}(X)$ as *analytic formal groupoids over* X.

Remark 2.39. Note that Proposition 2.24 implies that the the fiber products exist in $\mathrm{AnFMP}_{/X}$. Therefore, the previous definition is reasonable.

Construction 2.40. Thanks to Lemma 2.25, there exists a well defined functor $\Phi \colon \mathrm{AnFMP}_{X/} \to \mathrm{AnFGrpd}(X)$ given by the formula

$$(X \to Y) \in \mathrm{AnFMP}_{X/} \mapsto Y^\wedge_X \in \mathrm{AnFGrpd}(X),$$

where $Y^\wedge_X \in \mathrm{AnFGrpd}(X)$ denotes the analytic formal groupoid over X admitting

$$\cdots \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} X \times_Y X \times_Y X \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} X \times_Y X \rightrightarrows X \ ,$$

as simplicial presentation.

Construction 2.41. Let $\mathcal{G} \in \mathrm{AnFGrpd}(X)$. Consider the *$k$-analytic classifying pre-stack*, $\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}} \in \mathrm{AnPreStk}$, obtained as the geometric realization of the simplicial object

$$\mathcal{G} \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnPreStk}.$$

Given any $Z \in \mathrm{dAfd}_k$, the *space of Z-points of* $\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}$,

$$\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}(Z),$$

can be identified with the space whose objects correspond to the datum of:

(1) A morphism $\widetilde{Z} \to X$, where $\widetilde{Z} \in \mathrm{AnPreStk}$, such that

$$\widetilde{Z} \simeq Z \times_{\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}} X;$$

(2) A morphism of groupoid-objects in AnPreStk $\widetilde{Z} \times_Z \widetilde{Z} \to \mathcal{G}$.

We now define $\mathrm{B}_X(\mathcal{G})$ as the sub-object spanned by those connected components of $\mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}$ corresponding to $\widetilde{Z} \to Z$ which exhibit $\widetilde{Z} \in \mathrm{AnFMP}_{/Z}$. Denote by

$$\mathrm{can}\colon \mathrm{B}_X(\mathcal{G}) \to \mathrm{B}_X(\mathcal{G})^{\mathrm{pre}},$$

the canonical morphism. It follows from the constructions that the natural morphism

$$X \to \mathrm{B}_X(\mathcal{G})^{\mathrm{pre}},$$

factors as $X \to \mathrm{B}_X(\mathcal{G}) \xrightarrow{\mathrm{can}} \mathrm{B}_X(\mathcal{G})^{\mathrm{pre}}$.

Lemma 2.42. *The natural morphism $X \to \mathrm{B}_X(\mathcal{G})$ exhibits the latter as an object in the ∞-category $\mathrm{AnFMP}_{X/}$ of analytic formal moduli problems under X.*

Proof. Thanks to Proposition 2.21 it suffices to prove that $\mathrm{B}_X(\mathcal{G})$ is infinitesimally cartesian and it admits furthermore a pro-cotagent complex. The fact that $\mathrm{B}_X(\mathcal{G})$ is infinitesimally cartesian follows from the modular description of $\mathrm{B}_X(\mathcal{G})$ combined with the fact that $\mathcal{G}$ is infinitesimally cartesian, as well. We are thus required to show that $\mathrm{B}_X(\mathcal{G})$ admits a *global* pro-cotangent complex. Let $Z \in \mathrm{dAn}_k$ and suppose we are given an arbitrary morphism

$$q\colon Z \to \mathrm{B}_X(\mathcal{G}),$$

in the ∞-category AnPreStk. Thanks to Corollary 2.36 combined with Corollary 2.37 it follows that the object

$$\{\mathbb{L}^{\mathrm{an}}_{\widetilde{Z}^{[n]}/\mathcal{G}^{[n]}}\}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \in \mathrm{Tot}(\mathrm{Pro}(\mathrm{Coh}^+((\widetilde{Z})^\bullet/Z)))),$$

defines a well defined object $\mathbb{L}^{\mathrm{an}'}_{Z/\mathrm{B}_X(\mathcal{G})} \in \mathrm{Pro}(\mathrm{Coh}^+(Z))$. Moreover, it is clear that we there exists a natural morphism

$$\theta\colon \mathbb{L}^{\mathrm{an}}_Z \to \mathbb{L}^{\mathrm{an}'}_{Z/\mathrm{B}_X(\mathcal{G})},$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(Z))$. Let

$$q^*\mathbb{L}^{\mathrm{an}'}_{\mathrm{B}_X(\mathcal{G})} := \mathrm{fib}(\theta),$$

computed in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(Z))$. We claim that $q^*\mathbb{L}^{\mathrm{an}'}_{\mathrm{B}_X(\mathcal{G})}$ identifies with the analytic cotangent complex of $\mathrm{B}_X(\mathcal{G})$ at the point $q\colon Z \to \mathrm{B}_X(\mathcal{G})$. Let

$$Z \hookrightarrow Z',$$

denote a square-zero extension which corresponds to a certain analytic derivation

$$d\colon \mathbb{L}^{\mathrm{an}}_Z \to \mathcal{F}[1],$$

for some $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$. Using Corollary 2.33 we deduce that the space of cartesian squares of the form

$$\begin{tikzcd} \widetilde{Z} \arrow[r] \arrow[d] & \widetilde{Z}' \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

where the morphism $\widetilde{Z} \to \widetilde{Z}'$ is a square-zero extension in the ∞-category AnPreStk is equivalent to the space of factorizations

$$d\colon g^*\mathbb{L}^{\mathrm{an}}_Z \to \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \xrightarrow{d'} g^*(\mathcal{F})[1],$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}))$. Apply the same reasoning to the each object in the Čech nerve

$$\widetilde{Z}^\bullet \to Z.$$

Furthermore, Corollary 2.34 implies that the space of factorizations

$$\widetilde{Z}^\bullet \to (\widetilde{Z}')^\bullet \to \mathcal{G}^\bullet,$$

identifies with the space of factorizations

$$d' \colon \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}} \to \mathbb{L}^{\mathrm{an}}_{\widetilde{Z}/\mathcal{G}} \to g^*(\mathcal{F})[1],$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(\widetilde{Z}))$. By pseudo-pro-nildescent we then deduce that the above factorization space identified with the space of factorizations

$$d \colon \mathbb{L}^{\mathrm{an}}_{Z} \to \mathbb{L}^{\mathrm{an}}_{Z/\mathrm{B}_X(\mathcal{G})} \to \mathcal{F}[1],$$

as desired. □

Theorem 2.43. *The functor* $\Phi\colon \mathrm{AnFMP}_{/X} \to \mathrm{AnFGrpd}(X)$ *of Construction 2.40 is an equivalence of* ∞*-categories.*

Proof. Let $\mathcal{G} \in \mathrm{AnFGrpd}$. Thanks to (1) in Construction 2.41 it follows that one has a canonical equivalence

$$X \times_{\mathrm{B}_X(\mathcal{G})} X \simeq \mathcal{G},$$

in AnPreStk. This shows that the construction

$$\mathrm{B}_X(\mathcal{G}) \colon \mathrm{AnFGrpd}(X) \to \mathrm{AnFMP}_{X/},$$

is a right inverse to Φ. As a consequence the functor Φ is essentially surjective. By the same reasoning we deduce that given $X \to Y$ in $\mathrm{AnFMP}_{X/}$ the natural morphism

$$Y \to \mathrm{B}_X(Y \times_X Y),$$

is also an equivalence in AnPreStk. □

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