SPREADING OUT THE HODGE FILTRATION IN RIGID ANALYTIC GEOMETRY

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Abstract.

Contents

1. Introduction

In this paper, we will provide a rigid analytic construction of the deformation to the normal cone, studied in [4]. Our goal is to use this geometric construction to deduce certain important results concerning both *rigid analytic* and *over-convergent* (Hodge complete) *derived de Rham cohomology* of rigid analytic spaces over a non-archimedean field of characteristic zero. We will then exploit this ideas to come up with analogues concerning *derived rigid cohomology* of finite type schemes over a perfect field in characteristic zero. In particular, our main goal is to extrapolate the main result of [3] to the setting of derived rigid cohomology.

1.1. **Preliminaries.** Let $\mathcal{X}$ be an ∞-topos. The notion of a *local* $\mathcal{T}_{\mathrm{an}}(k)$*-structure on* $\mathcal{X}$ was first introduced in [8, Definition 2.4], see also [1, §2].

Let $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ be a local $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$. Since the pregeometry $\mathcal{T}_{\mathrm{an}}(k)$ is compatible with n-truncations, cf. [8, Theorem 3.23], it follows that $\pi_0(\mathcal{O}) \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, as well.

Denote by $\mathcal{J} \subseteq \pi_0(\mathcal{O})$, the *Jacobson ideal* of $\pi_0(\mathcal{O}^{\mathrm{alg}})$, which can be naturally regarded as an object in the ∞-category

$$\mathrm{Mod}_{\pi_0(\mathcal{O}^{\mathrm{alg}})} \simeq \mathrm{Mod}_{\pi_0(\mathcal{O})},$$

for a justification of the latter equivalence, see for instance [7, Theorem 4.5] . Since the ∞-category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ is a presentable ∞-category we can consider the quotient

$$\pi_0(\mathcal{O})_{\mathrm{red}} \coloneqq \pi_0(\mathcal{O})/\mathcal{J} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}),$$

which we refer to the *reduced* $\mathcal{T}_{\mathrm{an}}(k)$*-structure on* $\mathcal{X}$ *associated to* $\pi_0(\mathcal{O})$. Moreover, the corresponding *underlying algebra* satisfies

$$(\pi_0(\mathcal{O})_{\mathrm{red}})^{\mathrm{alg}} \simeq \pi_0(\mathcal{O})^{\mathrm{alg}}/\mathcal{J} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

One can further prove that $\pi_0(\mathcal{O})_{\mathrm{red}} \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ actually lies in the full subcategory $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$.

Definition 1.1. Let $Z = (\mathcal{Z}, \mathcal{O}_Z) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote a $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos. We define the *reduced* $\mathcal{T}_{\mathrm{an}}(k)$*-structure* ∞*-topos* as

$$Z_{\mathrm{red}} \coloneqq (\mathcal{Z}, \pi_0(\mathcal{O}_Z)_{\mathrm{red}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

We shall denote by $\mathrm{Afd}_k^{\mathrm{red}}$ (resp., $\mathrm{An}_k^{\mathrm{red}}$) the full subcategory of dAfd_k (resp., dAn_k) spanned by reduced k-affinoid (resp., k-analytic spaces).

Notation 1.2. Let $(-)^{\mathrm{red}} \colon \mathrm{dAn}_k \to \mathrm{An}_k^{\mathrm{red}}$ denote the functor obtained by the formula

$$Z = (\mathcal{Z}, \mathcal{O}_Z) \in \mathrm{dAn}_k \mapsto Z_{\mathrm{red}} = (\mathcal{Z}, \pi_0(\mathcal{Z})_{\mathrm{red}}) \in \mathrm{An}_k^{\mathrm{red}}.$$

We shall refer to it as the *underlying reduced k-analytic space.*

Lemma 1.3. *Let $f \colon X \to Y$ be a Zariski open immersion of derived k-analytic spaces. Then $f^{\mathrm{red}} \colon X^{\mathrm{red}} \to Y^{\mathrm{red}}$ is also a Zariski open immersion.*

Proof. By the definitions, it is clear that the truncation

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y),$$

is a Zariski open immersion of ordinary k-analytic spaces. In the case of ordinary k-analytic spaces it is clear from the construction that the reduction of Zariski open immersions is again a Zariski open immersion. □

Definition 1.4. In [7, Definition 5.41] the authors introduced the notion of a square-zero extension between $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi. In particular, given a morphism $f \colon Z \to Z'$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, we shall say that f *has the structure of a square-zero extension* if f exhibits Z' as a square-zero extension of Z.

Recall the definition of the ∞-categories of derived k-affinoid and derived k-analytic spaces given in [8, Definition 7.3 and Definition 2.5.], respectively.

Remark 1.5. Let $X \in \mathrm{An}_k$. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be an ideal satisfying $\mathcal{J}^2 = 0$. Consider the fiber sequence

$$\mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J},$$

in the ∞-category $\mathrm{Coh}^+(X)$. It corresponds to a well defined morphism $d \colon \mathcal{O}_X/\mathcal{J} \to \mathcal{J}[1]$ admitting $\mathcal{O}_X$ as fiber. The morphism d defines a derivation $d \colon \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{J}} \to \mathcal{J}[1]$, by pre-composing with the natural map $\mathcal{O}_X/\mathcal{J} \to \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathcal{J}}$. In particular, we can consider the square-zero extension of $\mathcal{O}_X$ by $\mathcal{J}$ induced by $\mathcal{J}$ defined by d. The latter object must then be equivalent to $\mathcal{O}_X$ itself. We conclude that $\mathcal{O}_X$ is a square-zero extension of $\mathcal{O}_X/\mathcal{J}$.

Lemma 1.6. *Let $Z \coloneqq (\mathcal{Z}, \mathcal{O}_Z) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ denote a $\mathcal{T}_{\mathrm{an}}(k)$-structure ∞-topos. Suppose that the reduction Z_{red} is equivalent to a derived k-affinoid space. Then the truncation $\mathrm{t}_0(Z)$ is isomorphic to an ordinary k-affinoid space. If we assume further that for every $i > 0$, the homotopy sheaves $\pi_i(\mathcal{O}_Z)$ are coherent $\pi_0(\mathcal{O}_Z)$-modules, then Z itself is equivalent to a derived k-affinoid space.*

Proof. We first observe that the second claim of the Lemma follows readily from the first one. We thus are thus reduced to prove that $\mathrm{t}_0(Z)$ is isomorphic to an ordinary k-affinoid space. Let $\mathcal{J} \subseteq \pi_0(\mathcal{O}_Z)$, denote the coherent ideal sheaf associated to the closed immersion $Z_{\mathrm{red}} \hookrightarrow Z$. Notice that the ideal $\mathcal{J}$ agrees with the Jacobson ideal of $\pi_0(\mathcal{O}_Z)$. Since derived k-analytic spaces are Noetherian, it follows that there exists a sufficiently large integer $n \geq 2$ such that

$$\mathcal{J}^n = 0.$$

Arguing by induction we can suppose that $n = 2$, that is to say that

$$\mathcal{J}^2 = 0.$$

In particular, Remark 1.5 implies that the above map has the natural morphism $Z_{\mathrm{red}} \to Z$ has the structure of a square zero extension. The assertion now follows from [7, Proposition 6.1] and its proof. □

Remark 1.7. We observe that the converse of Lemma 1.6 holds true. Indeed, the natural morphism $Z_{\mathrm{red}} \to Z$ is a closed immersion. In particular, if $Z \in \mathrm{dAfd}_k$ we deduce readily from that $Z_{\mathrm{red}} \in \mathrm{dAfd}_k$, as well.

Definition 1.8. Let $f \colon X \to Y$ be a morhpism in the ∞-category dAn_k. We shall say that f is an *affine morphism* if for every morphism $Z \to Y$ in dAn_k such that Z is equivalent to a derived k-affinoid space, the pullback

$$Z' \coloneqq Z \times_Y X \in \mathrm{dAn}_k,$$

is also equivalent to a derived k-affinoid space.

Notation 1.9. Let $f \colon X \to Y$ be a morphism of derived k-analytic spaces. We shall denote by

$$f^{\#} \colon \mathcal{O}_Y \to f_* \mathcal{O}_X,$$

the induced morphism at the level of $\mathcal{T}_{\mathrm{an}}(k)$-structures.

Lemma 1.10. *Let $f \colon X \to Y$ be an affine morphism in dAn_k. Suppose that we are given a Zariski open immersion $g \colon Z \to Y$ such that $Z \in \mathrm{dAfd}_k$ which corresponds to the completement of the zero locus of a section $s \in \pi_0(\mathcal{O}_Y)$. Then the fiber product*

$$Z' \coloneqq Z \times_Y X \in \mathrm{dAn}_k,$$

is equivalent to a derived k-affinoid space and moreover $\Gamma(Z', \mathcal{O}^{\mathrm{alg}}_{Z'}) \simeq B[1/f^{\#}(s)]$, where $B \coloneqq \Gamma(X, \mathcal{O}^{\mathrm{alg}}_X)$.

Proof. The first assertion of the Lemma follows readily from the definition of affine morphisms. We shall now prove the second claim. Let $A \coloneqq \Gamma(Y, \mathcal{O}^{\mathrm{alg}}_Y)$. In this case, we have a natural equivalence of derived k-algebras

$$A[1/f] \simeq \Gamma(Z, \mathcal{O}^{\mathrm{alg}}_Z).$$

Since Zariski open immersions are stable under pullbacks, it follows that the natural morphism $g' \colon Z' \to X$ is itself a Zariski open immersion. In particular, it follows that we can identify

$$\Gamma(Z', \mathcal{O}_{Z'}) \simeq B[1/t],$$

where $t \in \pi_0(B)$. In order to conclude the proof, we observe that the 0-th truncation, $\mathrm{t}_0(g)$, is again a Zariski open immersion. For this reason, one should have forcibly that $t = f^{\#}(s)$, by the universal property of fiber products of ordinary k-analytic spaces. □

2. Non-archimedean differential geometry

2.1. **Analytic formal moduli problems under a base.** In this §, we will study the notion of *analytic formal moduli problems under* a fixed derived k-analytic space. The results presented here will prove to be crucial for the study of the deformation to the normal cone in the k-analytic setting, presented in the next section. We start with the following definition:

Definition 2.1. Let $f \colon X \to Y$ be a morphism in dAn_k. We say that f is a *nil-isomorphism* if $f_{\mathrm{red}} \colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is an isomorphism of k-analytic spaces. We denote by $\mathrm{AnNil}_{/X}$ the full subcategory of $(\mathrm{dAn}_k)^{\mathrm{ft}}_{X/}$ spanned by nil-isomorphisms $X \to Y$ of finite type.

Lemma 2.2. *Let $f \colon X \to Y$ be a nil-isomorphism in dAn_k. Then:*

(1) Given any morphism $Z \to Y$ in dAn_k, the induced morphism

$$Z \times_X Y \to Z,$$

is again an nil-isomoprhism.

(2) f is an affine morphism.

(3) f is a finite morphism.

Proof. To prove (i), it suffices to prove that the functor $(\text{-})^{\mathrm{red}} \colon \mathrm{dAn}_k \to \mathrm{An}_k^{\mathrm{red}}$ commutes with finite limits. The truncation functor

$$\mathrm{t}_0 \colon \mathrm{dAn}_k \to \mathrm{An}_k,$$

commutes with finite limits. So we further reduce ourselves to the prove that the usual underlying reduced functor

$$(-)^{\mathrm{red}} \colon \mathrm{An}_k \to \mathrm{An}_k^{\mathrm{red}},$$

commutes with finite limits. By construction, the latter assertion is equivalent to the claim that the complete tensor product of ordinary k-affinoid algebras commutes with the operation of taking the quotient by the Jacobson radical, which is immediate.

We now prove (ii). Let $Z \to Y$ be a Zariski open immersion such that Z is a derived k-affinoid space. Then we claim that the pullback $Z \times_X Y$ is again a derived k-affinoid space. Thanks to Lemma 1.6 we reduced to prove that $(Z \times_X Y)_{\mathrm{red}}$ is equivalent to an ordinary k-affinoid space. Thanks to (i), we deduce that the induced morphism

$$(Z \times_X Y)_{\mathrm{red}} \to Z_{\mathrm{red}},$$

is an isomorphism of ordinary k-analytic spaces. In particular, $(Z \times_X Y)_{\mathrm{red}}$ is a k-affinoid space. The result now follows from Lemma 1.6.

To prove (iii), we shall show that the induced morphism on the 0-th truncations $\mathrm{t}_0(X) \to \mathrm{t}_0(Y)$ is a finite morphism of ordinary k-affinoid spaces. But this follows immeaditely from the fact that both $\mathrm{t}_0(X)$ and $\mathrm{t}_0(Y)$ can be obtained from the reduced X_{red} by means of a finite sequence of finite coherent X_{red}-modules. □

Definition 2.3. A morphism $X \to Y$ be a morphism in dAn_k is called a *nil-embedding* if the induced map of ordinary k-analytic spaces $\mathrm{t}_0(X) \to \mathrm{t}_0(Y)$ is a closed immersion, such that the ideal of $\mathrm{t}_0(X)$ in $\mathrm{t}_0(Y)$ is nilpotent.

Proposition 2.4. *Let $f \colon X \to Y$ be a nil-embedding of derived k-analytic spaces. Then there exists a sequence of morphisms*

$$X = X_0^0 \hookrightarrow X_0^1 \hookrightarrow \cdots \hookrightarrow X_0^n = X_0 \hookrightarrow X_1 \dots X_n \hookrightarrow \cdots \hookrightarrow Y,$$

such that for each $0 \leq i \leq n$ the morphism $X_0^i \hookrightarrow X_0^{i+1}$ has the structure of a square zero extension. Similarly, for every $i \geq 0$, the morphism $X_i \hookrightarrow X_{i+1}$ has the structure of a square-zero extension. Furthermore, the induced morphisms $\mathrm{t}_{\leq i}(X_i) \to \mathrm{t}_{\leq i}(Y)$ are equivalences of derived k-analytic spaces.

Proof. The proof follows the same scheme of reasoning as of [4, Proposition 5.5.3]. For the sake of completeness we present the complete here. Consider the induced morphism on the underlying truncations

$$\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y).$$

By construction, there exists a sufficiently large integer $n \geq 0$ such that

$$\mathcal{J}^{n+1} = 0,$$

where $\mathcal{J} \subseteq \pi_0(\mathcal{O}_Y)$ denotes the ideal associated to the nil-embedding $\mathrm{t}_0(f)$. Therefore, we can factor the latter as a finite sequence of square-zero extensions of ordinary k-analytic spaces

$$\mathrm{t}_0(X) \hookrightarrow X_0^{\mathrm{ord},0} \hookrightarrow \cdots \hookrightarrow X_0^{\mathrm{ord},n} = \mathrm{t}_0(Y),$$

as in the proof of Lemma 1.6. For each $0 \leq i \leq n$, we set

$$X_0^i \coloneqq X \coprod_{\mathrm{t}_0(X)} X_0^{\mathrm{ord},i}.$$

By construction, we have that the natural morphism $\mathrm{t}_0(X_0^n) \to \mathrm{t}_0(Y)$ is an isomorphism of ordinary k-analytic spaces. We now argue by induction on the Postnikov towers associated to the morphism $f \colon X \to Y$. Suppose that for a certain integer $i \geq 0$, we have constructed a derived k-analytic space X_i together with morphisms $g_i \colon X \to X_i$ and $h_i \colon X_i \to Y$ such that $f \simeq h_i \circ g_i$ and the induced morphism

$$\mathrm{t}_{\leq i}(X_i) \to \mathrm{t}_{\leq i}(Y)$$

is an equivalence of derived k-analytic spaces. We shall proceed as follows: by the assumption that h_i is $(i+1)$-connective, we deduce from [7, Proposition 5.34] the existence of a natural equivalence

$$\tau_{\leq i}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}) \simeq 0,$$

in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. Consider the natural fiber sequence

$$h_i^* \mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_{X_i} \to \mathbb{L}^{\mathrm{an}}_{X_i/Y},$$

in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. The natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X_i/Y} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1],$$

induces a morphism $\mathbb{L}^{\mathrm{an}}_{X_i} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y})[i+1]$, such that the composite

$$h_i^* \mathbb{L}^{\mathrm{an}}_Y \to \mathbb{L}^{\mathrm{an}}_{X_i} \to \pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}), \tag{2.1}$$

is null-homotopic, in $\mathrm{Mod}_{\mathcal{O}_{X_i}}$. The existence of (2.1) produces a square-zero extension

$$X_i \to X_{i+1},$$

together with a morphism $h_{i+1} \colon X_{i+1} \to Y$, factoring $h_i \colon X_i \to Y$. We are reduced to show that the morphism

$$\mathcal{O}_Y \to h_{i+1,*}(\mathcal{O}_{X_{i+1}}),$$

is $(i+2)$-connective. Consider the commutative diagram

$$\begin{tikzcd}
h_{i,*}(\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}))[i] \arrow[r] & h_{i+1,*}(\mathcal{O}_{X_{i+1}}) \arrow[r] & h_{i,*}(\mathcal{O}_{X_i}) \\
\mathcal{I} \arrow[r] \arrow[u, "s_i"] & \mathcal{O}_Y \arrow[r] \arrow[u] & h_*(\mathcal{O}_{X_i}) \arrow[u] \\
\mathcal{J} \arrow[r] \arrow[u] & \mathcal{J} \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}, \tag{2.2}$$

where both the vertical and horizontal composites are fiber sequences. Thanks to [7, Proposition 5.34] we can identify the natural morphism

$$s_i \colon \mathcal{I} \to h_{i,*}(\pi_{i+1}(\mathbb{L}^{\mathrm{an}}_{X_i/Y}))[i]$$

with the natural morphism $\mathcal{I} \to \tau_{\geq i}(I)$. We deduce that the fiber of the morphism s_i must be necessarily $(i+1)$-connective. The latter observation combined with the structure of (2.2) implies that $h_{i+1} \colon X_{i+1} \to Y$ induces an equivalence of derived k-analytic spaces

$$\mathrm{t}_{\leq i+1}(X_{i+1}) \to \mathrm{t}_{\leq i+1}(Y),$$

as desired. □

Corollary 2.5. *Let $X \in \mathrm{dAn}_k$. Then the natural morphism*

$$X_{\mathrm{red}} \to X,$$

in dAn_k*, can be* approximated *by successive square zero extensions.*

Proof. The assertion of the Corollary follows readily from Proposition 2.4 by observing that the canonical morphism $X_{\mathrm{red}} \to X$ has the structure of a nil-embedding. □

Lemma 2.6. *Let $f \colon S \to S'$ be a nil-isomorphism between derived k-analytic spaces. Then the pullback functor*

$$f^* \colon \mathrm{Coh}^+(S') \to \mathrm{Coh}^+(S),$$

admits a well defined right adjoint, f_*.

Proof. Since $f \colon S \to S'$ is a nil-isomorphism, we conclude from Lemma 2.2 that f is an affine morphism between derived k-analytic spaces. By Zariski descent of Coh^+, cf. [2, Theorem 3.7], together with Lemma 1.10 we reduce the statement of the Lemma to the case where both S and S' are equivalent to derived k-affinoid spaces. In this case, by Tate acyclicity theorem we reduce ourselves to show that the usual base change functor

$$f^* \colon \mathrm{Coh}^+(A) \to \mathrm{Coh}^+(B),$$

where $A \coloneqq \Gamma(S, \mathcal{O}_S^{\mathrm{alg}})$ and $B \coloneqq \Gamma(S', \mathcal{O}_{S'}^{\mathrm{alg}})$, admits a right adjoint. The result now follows from the observation that the canonical induced morphism $\pi_0(A) \to \pi_0(B)$ is a finite morphism of ordinary rings. Indeed, the latter morphism can be obtained by means of a finite sequence of (classical) square-zero extensions with respect to the corresponding Jacobson ideals of both $\pi_0(A)$ and $\pi_0(B)$. Such ideals are necessarily finitely generated as $\pi_0(A)$-modules, and the result follows. □

Lemma 2.7. *Let $f \colon S \to S'$ be a square-zero extension and $g \colon S \to T$ a nil-isomorphism in* dAn_k*. Suppose we are given a pushout diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd},$$

in dAn_k*. Then the induced morphism $T \to T'$ is a square-zero extension.*

Proof. Since g is a nil-isomorphism of derived k-analytic spaces, Lemma 2.6 implies that the pullback functor $g^* \colon \mathrm{Coh}^+(T) \to \mathrm{Coh}^+(S)$ admits a well defined right adjoint

$$g_* \colon \mathrm{Coh}^+(S) \to \mathrm{Coh}^+(T).$$

Let $\mathcal{F} \in \mathrm{Coh}^+(S)^{\geq 0}$ and $d \colon \mathbb{L}_S^{\mathrm{an}} \to \mathcal{F}$ be a derivation associated with the morphism $f \colon S \to S'$. Consider now the natural composite

$$d' \colon \mathbb{L}_T^{\mathrm{an}} \to g_*(\mathbb{L}_S^{\mathrm{an}}) \xrightarrow{g_*(d)} g_*(\mathcal{F}),$$

in the ∞-category $\mathrm{Coh}^+(T)$. By the universal property of the analytic cotangent complex, we deduce the existence of a square-zero extension

$$T \to T',$$

in the ∞-category dAn_k. Let $X \in \mathrm{dAn}_k$ together with morphisms $S' \to X$ and $T \to X$ compatible with both f and g. By the universal property of the relative analytic cotangent complex, the morphism $S' \to X$ induces a uniquely defined (up to a contractible indeterminacy space)

$$\mathbb{L}^{\mathrm{an}}_{S/X} \to \mathcal{F},$$

in $\mathrm{Coh}^+(S)$, such that the compositve $\mathbb{L}^{\mathrm{an}}_S \to \mathbb{L}^{\mathrm{an}}_{S/X} \to \mathcal{F}$ agrees with d. By applying the right adjoint g_* above we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{L}^{\mathrm{an}}_T \arrow[r, "\mathrm{can}"] \arrow[d] & \mathbb{L}^{\mathrm{an}}_{T/X} \arrow[d] \arrow[dr, "d''"] & \\
g_*(\mathbb{L}^{\mathrm{an}}_S) \arrow[r] & g_*(\mathbb{L}^{\mathrm{an}}_{S/X}) \arrow[r] & g_*(\mathcal{F}),
\end{tikzcd}$$

in the ∞-category $\mathrm{Coh}^+(T)$. From this, we conclude again by the universal property of the relative analytic cotangent complex the existence of a natural morphism $T' \to X$ extending both $T \to X$ and $S' \to X$ and compatible with the restriction to S. The latter assertion is equivalent to state that the commutative square

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & S' \arrow[d] \\
T \arrow[r] & T',
\end{tikzcd}$$

is a pushout diagram in dAn_k. The proof is thus concluded. $\square$

Proposition 2.8. *Let $f\colon X \to Y$ be a nil-embedding of derived k-analytic spaces. Let $g\colon X \to Z$ be a finite morphism in dAn_k. The the diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \\
Z &
\end{tikzcd}$$

admits a colimit in dAn_k, denoted Z'. Moreover, the natural morphism $Z \to Z'$ is also a nil-embedding.

Proof. The ∞-category of $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ is a presentable ∞-category. Consider the pushout diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \\
Z \arrow[r] & Z',
\end{tikzcd}$$

in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. By construction, the underlying ∞-topos of Z' can be computed as the pushout in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ of the induced diagram on the underlying ∞-topoiof X, Z and Y. Moreover, since g is a nil-isomorphism it induces an equivalence on underlying ∞-topoiof both X and Y. It follows that the induced morphism $Z \to Z'$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ induces an equivalence on the underlying ∞-topoi. Moreover, it follows essentially by construction that we have a natural equivalence

$$\mathcal{O}_{Z'} \simeq g_*(\mathcal{O}_Y) \times_{g_*(\mathcal{O}_Y)} \mathcal{O}_Z \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}\mathrm{loc}(Z).$$

As effective epimorphisms are preserved under fiber products in an ∞-topos, it follows that the natural morphism

$$\mathcal{O}_{Z'} \to \mathcal{O}_Z,$$

is an effective epimorphism (since $g_*(\mathcal{O}_Y) \to g_*(\mathcal{O}_X)$ it is so). Consider now the commutative diagram of fiber sequences

$$\begin{tikzcd} \mathcal{J}' \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[d] \\ \mathcal{J} \arrow[r] & g_*(\mathcal{O}_Y) \arrow[r] & g_*(\mathcal{O}_X), \end{tikzcd}$$

in the stable ∞-category $\mathrm{Mod}_{\mathcal{O}'_Z}$. Since the right commutative square is a pullback square it follows that the morphism

$$\mathcal{J}' \to \mathcal{J},$$

is an equivalence. In particular, $\pi_0(\mathcal{J}')$ is a finitely generated nilpotent ideal of $\pi_0(\mathcal{O}^{\mathrm{alg}}_{\mathcal{J}'})$. Indeed, finitely generation follows from our assumption that g is a finite morphism. Thanks to Lemma 1.6, it follows that $\mathrm{t}_0(Z')$ is an ordinary k-analytic space and the morphism $\mathrm{t}_0(Z') \to \mathrm{t}_0(Z)$ is a nil-embedding. We are thus reduced to show that for every $i > 0$, the homotopy sheaf $\pi_i(\mathcal{O}_{Z'}) \in \mathrm{Coh}^+(\mathrm{t}_0(Z'))$. But this follows immediately from the existence of a fiber sequence

$$\mathcal{O}_{Z'} \to g_*(\mathcal{O}_Y) \oplus \mathcal{O}_Z \to g_*(\mathcal{O}_X),$$

in the ∞-category $\mathrm{Mod}_{\mathcal{O}_{Z'}}$ together with the fact that $g_*(\mathcal{O}_Y)$ and $g_*(\mathcal{O}_Z)$ have coherent homotopy sheaves, by our assumption that g is a finite morphism combined with Lemma 2.2. □

Definition 2.9. An *analytic formal moduli problem under X* corresponds to the datum of a functor

$$F\colon (\mathrm{AnNil}_{X/})^{\mathrm{op}} \to \mathcal{S},$$

satisfying the following two conditions:

(1) $F(X) \simeq *$ in $\mathcal{S}$;
(2) $F \simeq \mathbf{res}^{<\infty} \circ F$, where $\mathbf{res}_!^{<\infty}$ denotes the right Kan extension along the natural inclusion
(3) Given any pushout diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T', \end{tikzcd}$$

in $\mathrm{AnNil}_{X/}$ for which f is has the structure of a square zero extension, the induced morphism

$$F(T') \to F(T) \times_{F(S)} F(S),$$

is an equivalence in $\mathcal{S}$.

We shall denote by $\mathrm{AnFMP}_{X/}$ the full subcategory of $\mathrm{Fun}((\mathrm{AnNil}_{X/})^{\mathrm{op}}, \mathcal{S})$ spanned by analytic formal moduli problems under X.

Construction 2.10. We have a composite diagram

$$h\colon \mathrm{AnNil}_{X/} \to \mathrm{dAn}_k \hookrightarrow \mathrm{AnPreStk}.$$

Therefore, given any analytic pre-stack regarded as a limit-preserving functor $F\colon \mathrm{AnPreStk}^{\mathrm{op}} \to \mathcal{S}$, one can consider its restriction to the ∞-category $\mathrm{AnNil}_{X/}$:

$$F \circ h\colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathcal{S}.$$

We have thus a natural restriction functor

$$h_* \colon \mathrm{AnPreStk} \to \mathrm{Fun}(\mathrm{AnNil}_{X/}^{\mathrm{op}}, \mathcal{S}).$$

Example 2.11. Let $X \in \mathrm{dAn}_k$. As in the algebraic case, we can consider the *de Rham pre-stack associated to* X, $X_{\mathrm{dR}} \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$, determined by the formula

$$X_{\mathrm{dR}}(Z) \coloneqq X(Z_{\mathrm{red}}), \quad Z \in \mathrm{dAfd}_k.$$

We have a natural morphism $X \to X_{\mathrm{dR}}$ induced from the natural morphism $Z_{\mathrm{red}} \to Z$. We claim that $h_*(X_{\mathrm{dR}}) \in \mathrm{Fun}(\mathrm{AnNil}_{X/}^{\mathrm{op}}, \mathcal{S})$ belongs to the full subcategory $\mathrm{AnFMP}_{X/}$. Indeed, in this case it is clear that $h_*(X_{\mathrm{red}})$ is the final object in $\mathrm{AnFMP}_{X/}$ which clearly satisfies conditions i) and ii) in Definition 2.9.

Notation 2.12. We set $\mathrm{AnNil}_{X/}^{\mathrm{cl}} \subseteq \mathrm{AnNil}_{X/}$ to be the full subcategory spanned by those objects corresponding to nil-embeddings of the form

$$X \to S,$$

in dAn_k.

Proposition 2.13. *Let $Y \in \mathrm{AnNil}_{X/}$. The following assertions hold:*

(1) Then the inclusion functor

$$\mathrm{AnNil}_{X//Y}^{\mathrm{cl}} \hookrightarrow \mathrm{AnNil}_{X//Y},$$

is cofinal.

(2) The natural morphism

$$\operatorname*{colim}_{Z \in \mathrm{AnNil}_{X//Y}^{\mathrm{cl}}} Z \to Y,$$

is an equivalence in $\mathrm{Fun}((\mathrm{AnNil}_{X//Y})^{\mathrm{op}}, \mathcal{S})$.

(3) The ∞-category $\mathrm{AnNil}_{X//Y}^{\mathrm{cl}}$ is filtered.

Proof. We start by proving claim (i). Consider the usual restriction along the natural morphism $X_{\mathrm{red}} \to X$ functor

$$\mathbf{res} \colon \mathrm{AnNil}_{X/} \to \mathrm{AnNil}_{X_{\mathrm{red}}/}.$$

Such functor admits a well defined left adjoint

$$\mathbf{push} \colon \mathrm{AnNil}_{X_{\mathrm{red}}/} \to \mathrm{AnNil}_{X/},$$

which is determined by the formula

$$(X_{\mathrm{red}} \to T) \in \mathrm{AnNil}_{X_{\mathrm{red}}/} \mapsto (X \to T') \in \mathrm{AnNil}_{X/},$$

where we set

$$T' \coloneqq X \coprod_{X_{\mathrm{red}}} T \in \mathrm{AnNil}_{X/}. \tag{2.3}$$

We claim that $T' \in \mathrm{AnNil}_{X/}$ belongs to the full subcategory $\mathrm{AnNil}_{X/}^{\mathrm{cl}} \subseteq \mathrm{AnNil}_{X/}$. Indeed, since the structural morphism $X_{\mathrm{red}} \to T$, is necessarily a nil-embedding we deduce that the claim follows readily from Proposition 2.8. We shall denote

$$\mathbf{res}_!(Y) \colon \mathrm{AnNil}_{X_{\mathrm{red}}/}^{\mathrm{op}} \to \mathcal{S},$$

the left Kan extension of Y along the functor $\mathbf{res}$ above. By the colimit formula for left Kan extensions, c.f. [6, Lemma 4.3.2.13], it follows that $\mathbf{res}_!(Y)$ is given by the formula

$$(X_{\mathrm{red}} \to T) \in \mathrm{AnNil}_{X_{\mathrm{red}}/} \mapsto Y(T') \in \mathcal{S},$$

where T' is as in (2.3). Let $g\colon X_{\mathrm{red}} \to T$ in $\mathrm{AnNil}_{X_{\mathrm{red}}/}$ and assume that g factors through the natural morphism $X_{\mathrm{red}} \to X$. Then we have a natural morphism

$$i_{T,*}\colon Y(T) \to \mathbf{res}_!(Y)(T),$$

in $\mathcal{S}$, which exhibits the former as a retract of the latter. Denote by

$$p_{T,*}\colon \mathbf{res}_!(Y)(T) \to Y(T),$$

be a right inverse to $i_{S,*}$. Consider the functor

$$\mathbf{res}_Y\colon \mathrm{AnNil}_{X//Y} \to \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)},$$

given by the formula

$$(X \to S \to Y) \in \mathrm{AnNil}_{X//Y} \mapsto (X_{\mathrm{red}} \to S \xrightarrow{f} \mathbf{res}_!(Y)),$$

where $f\colon S \to \mathbf{res}_!(Y)$ corresponds to the morphism

$$S_X \xrightarrow{p_S} S \to Y,$$

where $S_X \coloneqq X \coprod_{X_{\mathrm{red}}} S$. We claim that the functor $\mathbf{res}_Y$ is a right adjoint to the functor

$$\mathbf{push}_Y\colon \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \to \mathrm{AnNil}_{X//Y},$$

the latter given by the formula

$$(X_{\mathrm{red}} \to T \to \mathbf{res}_!(Y)) \in \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \mapsto (X \to T_X \to Y) \in \mathrm{AnNil}_{X//Y}.$$

Indeed, the datum of a morphism

$$(X_{\mathrm{red}} \to T \to \mathbf{res}_!(Y)) \to \mathbf{res}_Y(X \to S \to Y),$$

in $\mathrm{AnNil}_{X_{\mathrm{red}}/\mathbf{res}_!(Y)}$ corresponds to the datum of a commutative diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & \mathbf{res}_!(Y) \arrow[d, "="] \\
X_{\mathrm{red}} \arrow[r] & S \arrow[r] & \mathbf{res}_!(Y),
\end{tikzcd}$$

where the right bottom morphism corresponds to the composite $S_X \to S \to Y$. For this reason, the given datum is equivalent to a commutative diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r] \arrow[d] & T \arrow[r] & T_X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[dr, "="] & \\
X_{\mathrm{red}} \arrow[r] & S \arrow[r] & S_X \arrow[r] & S \arrow[r] & Y,
\end{tikzcd}$$

which on the other hand is equivalent to the datum of a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "="] & T_X \arrow[r] \arrow[d] & Y \arrow[d, "="] \\
X \arrow[r] & S \arrow[r] & Y
\end{tikzcd}$$

The previous observations combined together then imply that we have a well defined adjunction

$$\mathbf{res}\colon \mathrm{AnNil}_{X//Y} \rightleftarrows \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)} \colon \mathrm{push}.$$

We thus conclude that $\mathrm{AnNil}_{X//Y} \to \mathrm{AnNil}_{X_{\mathrm{red}}//\mathbf{res}_!(Y)}$ is a cofinal functor (as it admits a left adjoint). Claim (i) now follows immediately from the observation that the functor

$$\mathrm{push}\colon \mathrm{AnNil}_{X_{\mathrm{red}}/\mathbf{res}_!(Y)} \to \mathrm{AnNil}_{X//Y},$$

factors through the natural inclusion $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y} \to \mathrm{AnNil}_{X//Y}$. given by the formulas

$$(X \to S \to Y) \in \mathrm{AnNil}_{X//Y} \mapsto (X_{\mathrm{red}} \to X \to S \to Y) \in \mathrm{AnNil}_{X_{\mathrm{red}}//Y}$$

We further observe that it suffices to prove that $(\mathrm{AnNil}^{\mathrm{cl}}_{X//Y})_{S/}$ admits an initial object. Consider the pushout diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r] \arrow[d] & S \arrow[d] \\ X \arrow[r] & S' \end{tikzcd},$$

computed in the ∞-category dAn_k, whose existence is guaranteed by Proposition 2.8. Indeed, the natural morphism

$$X_{\mathrm{red}} \simeq S_{\mathrm{red}} \to S$$

is a nil-isomorphism and $X_{\mathrm{red}} \to X$ is finite, thanks to Lemma 2.2). Thanks to c2.8 we further deduce that the morphism $X \to X'$ is a closed nil-isomorphism. These observations combined together imply that $X' \in \mathrm{AnNil}^{\mathrm{cl}}_{X/}$.

We further observe that X' admits an induced morphism to Y, in the ∞-category $\mathrm{Fun}((\mathrm{AnNil}_{X/})^{\mathrm{op}}, \mathcal{S})$, compatible with the datum of $(X \to S \to Y)$ in $\mathrm{AnNil}_{X//Y}$. The latter assertion follows immediately from the fact that Y, being an analytic formal moduli problem under X, is required to send pushouts along square zero extensions to fiber products of spaces. together with Proposition 2.4.

Claim (ii) follows immediately from (i) combined with Yoneda Lemma. To prove (iii) we simply notice that the ∞-category $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ admits finite colimits, thanks to Proposition 2.8. ☐

Construction 2.14. Let $X \in \mathrm{dAn}_k$. Consider the natural functor

$$F\colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathrm{dAn}^{\mathrm{op}}_k.$$

Left Kan extension along F induces a functor

$$F_!\colon \mathrm{Fun}(\mathrm{AnNil}^{\mathrm{op}}_{X/}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{dAn}^{\mathrm{op}}_k, \mathcal{S}),$$

and thus an induced functor

$$F_!\colon \mathrm{AnFMP}_{X/} \to \mathrm{Fun}(\mathrm{dAn}^{\mathrm{op}}_k, \mathcal{S}),$$

as well. We denote the latter ∞-category by $\mathrm{AnPreStk}_k$, the ∞-category of *k-analytic pre-stacks*. Proposition 2.13 implies that the functor $F_!$ preserves filtered colimits. In particular, if we regard Y as a k-analytic prestack can be presented as an *ind-inf*-object in the ∞-category dAn_k, i.e., it can be written as a colimit along closed nil-isomorphisms being parametrized by a filtered ∞-category, see for instance [4] for a precise meaning of the latter notion in the algebraic setting.

Definition 2.15. Let $Y \in \mathrm{AnFMP}_{X/}$ denote an analytic formal moduli problem under X. The *relative pro-analytic cotangent complex of Y under X* is defined as the pro-object

$$\mathbb{L}^{\mathrm{an}}_{X/Y} \coloneqq \{\mathbb{L}^{\mathrm{an}}_{X/Z}\}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} \in \mathrm{Pro}(\mathrm{Coh}^+(X)),$$

where, for each $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$, $\mathbb{L}^{\mathrm{an}}_{X/Z} \in \mathrm{Coh}^+(X)$ denotes the usual analytic cotangent complex associated to the structural morphism $X \to Z$ in $\mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$.

Remark 2.16. Let $Y \in \mathrm{AnFMP}_{X/}$. Let $Z \in \mathrm{dAn}_k$, there exists a natural morphism

$$\mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X/Z},$$

in $\mathrm{Coh}^+(X)$. Passing to the limit over $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$, we obtain a natural map

$$\mathbb{L}^{\mathrm{an}}_X \to \mathbb{L}^{\mathrm{an}}_{X//Y},$$

in $\mathrm{Pro}(\mathrm{Coh}^+(X))$, as well.

The following result provides justifies our choice of terminology for the object $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$:

Lemma 2.17. *Let $Y \in \mathrm{AnFMP}_{X/}$. Let $X \hookrightarrow S$ be a square zero extension associated to an analytic derivation*

$$d\colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F},$$

where $\mathcal{F} \in \mathrm{Coh}^+(X)^{\geq 0}$. Then there exists a natural morphism

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \to \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}, \mathcal{F})} \{d\}$$

which is furthermore an equivalence in the ∞-category $\mathcal{S}$.

Proof. Thanks to Proposition 2.13 we can identify the space of liftings of the map $X \to Y$ along $X \to S$ with the mapping space

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \simeq \underset{Z \in \mathrm{AnNil}_{X//Y}}{\mathrm{colim}} \mathrm{Map}_{\mathrm{AnNil}_{X/}}(S, Z).$$

Fix $Z \in \mathrm{AnNil}_{X//Y}$. Then we have a natural identification of mapping spaces

$$\mathrm{Map}_{\mathrm{AnNil}_{X/}}(S, Z) \simeq \mathrm{Map}_{(\mathrm{dAn}_k)_{X/}}(S, Z) \qquad \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_{X/Z}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}_X, \mathcal{F})} \{d\}, \tag{2.4}$$

see [7, §5.4] for a justification of the latter assertion. Notice that the right hand side of (2.4) identifies with the space of null-homotopies of the morphism

$$g_Z^* \mathbb{L}^{\mathrm{an}}_Z \to \mathcal{F},$$

in $\mathrm{Coh}^+(X)$, where $g_Z\colon X \to Z$ denotes the structural morphism. Passing to the colimit over $Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$, we conclude that

$$\mathrm{Map}_{\mathrm{AnFMP}_{X/}}(S, Y) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(X))}(\mathbb{L}^{\mathrm{an}}_{X/Y}, \mathcal{F}) \times_{\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathbb{L}^{\mathrm{an}}, \mathcal{F})} \{d\},$$

as desired. ☐

Remark 2.18. Let $f\colon Y \to Z$ denote a morphism in $\mathrm{AnFMP}_{X/}$. Then, for every $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ the induced morphism

$$S \to Z,$$

in $\mathrm{AnFMP}_{X/}$ factors through some $S' \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z}$. For this reason, we obtain a natural morphism

$$\mathbb{L}^{\mathrm{an}}_{X/S'} \to \mathbb{L}^{\mathrm{an}}_{X/S},$$

in the ∞-category $\mathrm{Coh}^+(X)$. Passing to the limit over $S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}$ we obtain a canonically defined morphism

$$f^*\colon \mathbb{L}^{\mathrm{an}}_{X/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Y},$$

in $\mathrm{Pro}(\mathrm{Coh}^+(X))$.

Proposition 2.19. *Let $f\colon Y \to Z$ be a morphism in the ∞-category $\mathrm{AnFMP}_{X/}$. Suppose that f induces an equivalence of relative pro-analytic cotangent complexes via Remark 2.18. Then f is itself an equivalence of analytic formal moduli problems under X.*

Proof. Thanks to Proposition 2.13 we are reduced to show that given any

$$S \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Z},$$

the structural morphism $g_S \colon X \to S$ admits an extension $S \to Y$ which factors the structural morphism $X \to Y$. Thanks to Proposition 2.4 we can reduce ourselves to the case where $X \to S$ has the structure of a square zero extension. In this case, the result follows from Lemma 2.17 combined with our hypothesis. □

Definition 2.20. Let $Y \in \mathrm{AnPreStk}$, we shall say that Y is *infinitesimally cartesian* if it satisfies [7, Definition 7.3].

Proposition 2.21. *Let $Y \in \mathrm{AnPreStk}_{X/}$. Assume further that Y is infitesimally cartesian and it admits a relative pro-cotangent complex, $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Pro}(\mathrm{Coh}^+(X))$. then Y is equivalent to an analytic formal moduli problem under X.*

Proof. We must prove that given a pushout diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"] & S' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

in the ∞-category $\mathrm{AnNil}_{X/}$ the natural morphism

$$Y(T') \to Y(T) \times_{Y(S)} Y(S'),$$

is an equivalence in the ∞-category $\mathcal{S}$. Suppose further that $S \hookrightarrow S'$ is associated to some derivation $d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}$ for some $\mathcal{F} \in \mathrm{Coh}^+(S)^{\geq 0}$. Notice that the induced morphism $T \to T'$ admits a structure of a square-zero extension (Todo: prove this.) Then, by our assumptions of Y being infinitesimally cartesian and admitting a relative pro-cotangent complex, we have a chain of natural equivalences of the form

$$\begin{aligned} Y(T') &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{T/}(T', Y) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(T))_{\mathbb{L}^{\mathrm{an}}_Z/}}(\mathbb{L}^{\mathrm{an}}_{T/Y}, g_*(\mathcal{F})) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))_{g^*\mathbb{L}^{\mathrm{an}}_T/}}(g^*\mathbb{L}^{\mathrm{an}}_{T/Y}, \mathcal{F}) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(S))_{\mathbb{L}^{\mathrm{an}}_S/}}(\mathbb{L}^{\mathrm{an}}_{S,Y}, \mathcal{F}) \\ &\simeq \coprod_{f \colon T \to Y} \mathrm{Map}_{S/}(S', Y) \\ &\simeq Y(T) \times_{Y(S)} Y(S'), \end{aligned}$$

where the third equivalence follows from the existence of a commutative diagram between fiber sequences

$$\begin{tikzcd} g^* f^* \mathbb{L}^{\mathrm{an}}_Y \arrow[r] \arrow[d, "="] & g^* \mathbb{L}^{\mathrm{an}}_T \arrow[r] \arrow[d] & g^* \mathbb{L}^{\mathrm{an}}_{T/Y} \arrow[d] \\ (f \circ g)^* \mathbb{L}^{\mathrm{an}}_Y \arrow[r] & \mathbb{L}^{\mathrm{an}}_S \arrow[r] & \mathbb{L}^{\mathrm{an}}_{S/Y}, \end{tikzcd}$$

in the ∞-category $\mathrm{Pro}(\mathrm{Coh}^+(S))$ combined with the fact that the derivation $d_T \colon \mathbb{L}^{\mathrm{an}}_T \to g_*(\mathcal{F})$ is induced from

$$d \colon \mathbb{L}^{\mathrm{an}}_S \to \mathcal{F}.$$

(Todo: one must prove this last assertion + the fact that the pushforward functor g_* preserves almost perfect complexes.) $\square$

2.2. **Analytic formal moduli problems over a base.** Let $X \in \mathrm{dAn}_k$ denote a derived k-analytic space. In [9, Definition 6.11] the authors introduced the ∞-category of *analytic formal moduli problems over* X, which we shall denote by $\mathrm{AnFMP}_{/X}$.

Definition 2.22. We shall denote by $\mathrm{AnNil}^{\mathrm{cl}}_{/X} \subseteq \mathrm{AnNil}_{/X}$ the faithful subcategory in which we only allows morphisms

$$S \to S'$$

in $\mathrm{AnNil}_{/X}$ which are closed nil-isomorphisms.

We start with the analogue of Proposition 2.13 in the setting of analytic formal moduli problems over X:

Proposition 2.23. *Let $Y \in \mathrm{AnFMP}_{/X}$. The following assertions hold:*

(1) The inclusion functor

$$(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y} \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y},$$

is cofinal.

(2) The natural morphism

$$\operatorname*{colim}_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} Z \to Y,$$

is an equivalence in the ∞-category $\mathrm{AnFMP}_{/X}$.

(3) The ∞-category $\mathrm{AnNil}^{\mathrm{cl}}_{/X}$ is filtered.

Proof. We first prove assertion (i). Let $S \to Z$ be a morphism in $(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}$. Consider the pushout diagram

$$\begin{tikzcd} S_{\mathrm{red}} \arrow[r] \arrow[d] & S \arrow[d] \\ Z \arrow[r] & Z', \end{tikzcd} \tag{2.5}$$

in the ∞-category $\mathrm{AnNil}_{/X}$ whose existence is guaranteed by Proposition 2.8. Since the upper horizontal morphism in (2.5) is a closed nil-isomorphism we deduce, again by Proposition 2.8, that $Z \to Z'$ is a closed nil-isomorphism, as well. Therefore, we can factor the diagram

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

via a closed nil-isomorphism $Z \to Z'$. We conclude that the inclusion functor $(\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y} \to (\mathrm{AnNil}_{/X})_{/Y}$ is cofinal. It is clear that assertion (ii) follows immediately from (i). We now prove (iii). Let

$$\theta \colon K \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y},$$

be a functor where K is a finite ∞-category. We must show that θ can be extended to a functor

$$\theta^{\triangleright} \colon K^{\triangleright} \to (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}.$$

Thanks to Proposition 2.4 we are allowed to reduce ourselves to the case where morphisms indexed by K are square-zero extensions. The result now follows from the fact that Y being an analytic moduli problem sends finite colimits along square-zero extensions to finite limits. $\square$

Just as in the previous section we deduce that every analytic formal moduli problem over X admits the structure of an *ind-inf*-object in $\mathrm{AnPreStk}_k$:

Corollary 2.24. *Let $Y \in (\mathrm{AnPreStk}_k)_{/X}$. Then Y is equivalent to an analytic formal moduli problem over X if and only if there exists a presentation $Y \operatorname{colim}_{i\in I} Z_i$, where I is a filtered ∞-category and for every $i \to j$ in I, the induced morphism*

$$Z_i \to Z_j,$$

is a closed embedding of derived k-affinoid spaces that are nil-isomorphic to X.

Definition 2.25. Let $Y \in \mathrm{AnFMP}_{/X}$. We define the ∞-category of *cohrent modules on Y*, denoted $\mathrm{Coh}^+(Y)$, as the limit

$$\mathrm{Coh}^+(Y) := \lim_{Z \in (\mathrm{AnNil}_{/X})_{/Y}} \mathrm{Coh}^+(Z).$$

Definition 2.26. Let $Y \in \mathrm{AnFMP}_{/X}$, $Z \in \mathrm{dAfd}_k$ and let $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$. Suppose furthermore that we are given a morphism $f \colon Z \to Y$. We define the *tangent space of Y at f twisted by $\mathcal{F}$* as the fiber

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} := \mathrm{fib}_f\big(Y(Z[\mathcal{F}]) \to Y(Z)\big) \in \mathcal{S}.$$

Remark 2.27. Let $Y \in \mathrm{AnFMP}_{/X}$. Since $Y \simeq \operatorname{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} S$ as an ind-object in dAn_k, it follows that given $Z \in \mathrm{dAfd}_k$ one has an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{AnPreStk}}(Z, Y) \simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^c l_{/X})_{/Y}} \mathrm{Map}_{\mathrm{AnPreStk}}(Z, S).$$

For this reason, given any morphism $f \colon Z \to Y$ and $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$, we can identify the tangent space $\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}}$ with the filtered colimit of spaces

$$\begin{aligned}
\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} &\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathrm{fib}\big(S(Z[\mathcal{F}]) \to S(Z)\big) \\
&\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathbb{T}^{\mathrm{an}}_{S,Z,\mathcal{F}} \\
&\simeq \operatorname*{colim}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{Z//Y}} \mathrm{Map}_{\mathrm{Coh}^+(Z)}(f^* \mathbb{L}^{\mathrm{an}}_S, \mathcal{F}),
\end{aligned}$$

where the latter equivalence follows from [7, Lemma 7.7]. Therefore, it follows that the analytic formal moduli problem $Y \in \mathrm{AnFMP}_{/X}$ admits an *absolute pro-cotangent complex* given as

$$\mathbb{L}^{\mathrm{an}}_Y := \{\mathbb{L}^{\mathrm{an}}_S\}_{S \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \in \mathrm{Pro}(\mathrm{Coh}^+(Y)).$$

Corollary 2.28. *Let $Y \in \mathrm{AnFMP}_{/X}$. Then its absolute cotangent complex $\mathbb{L}^{\mathrm{an}}_Y$ classifies analytic deformations on Y. More precisely, given $Z \to Y$ a morphism where $Z \in \mathrm{dAfd}_k$ and $\mathcal{F} \in \mathrm{Coh}^+(Z)^{\geq 0}$ one has a natural equivalence of mapping spaces*

$$\mathbb{T}^{\mathrm{an}}_{Y,Z,\mathcal{F}} \simeq \mathrm{Map}_{\mathrm{Pro}(\mathrm{Coh}^+(Y))}(\mathbb{L}^{\mathrm{an}}_Y, \mathcal{F}).$$

Proof. It follows immediately from the description of mapping spaces in ∞-categories of pro-objects. ☐

2.3. **Non-archimedean nil-descent for almost perfect complexes.** In this §, we prove that the ∞-category $\mathrm{Coh}^+(X)$, for $X \in \mathrm{dAn}_k$ satisfies nil-descent with respect to morphims $Y \to X$, which exhibit the former as an analytic formal moduli problem over X.

Proposition 2.29. *Let $f\colon Y \to X$, where $X \in \mathrm{dAn}_k$ and $Y \in \mathrm{AnFMP}_{/X}$. Consider the associated simplicial object $Y^\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnPreStk}$, induced by f. Then the natural functor*

$$f_\bullet^*\colon \mathrm{Coh}^+(X) \to \mathrm{Tot}(\mathrm{Coh}^+(Y^\bullet)),$$

is an equivalence of ∞-categories.

Proof. Consider the natural equivalence of k-analytic prestacks

$$Y \simeq \operatorname*{colim}_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} Z.$$

Then, by definition one has a natural equivalence

$$\mathrm{Coh}^+(Y) \simeq \lim_{Z \in (\mathrm{AnNil}^{\mathrm{cl}}_{/X})_{/Y}} \mathrm{Coh}^+(Z),$$

of ∞-categories. In particular, since totalizations commute with cofiltered limits in Cat_∞, it follows that we can suppose from the beginning that $Y \simeq Z$ for some $Z \in \mathrm{AnNil}_{/X}$. In this case, the morphism $f\colon Y \to X$ is affine. In particular, by flat descent of $\mathrm{Coh}^+(-)$, it follows that we can suppose that X is equivalent to a derived k-affinoid space and therefore so it is Y. In this case, by Tate acyclicity theorem it follows that letting $A \coloneqq \Gamma(X, \mathcal{O}_X^{\mathrm{alg}})$ and $B \coloneqq \Gamma(Y, \mathcal{O}_Y^{\mathrm{alg}})$ the pullback functor f^* can be identified with the base change functor

$$\mathrm{Coh}^+(A) \to \mathrm{Coh}^+(B).$$

In this case, it follows that B is nil-isomophic to A and the result follows from [5, Theorem 3.3.1]. ☐

2.4. **Non-archimedean formal groupoids.** Let $X \in \mathrm{dAfd}_k$ denote a derived k-affinoid space. We denote by $\mathrm{AnFGrpd}(X)$ the full subcategory of the ∞-category of simplicial objects

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{AnFMP}_{/X}),$$

spanned by those objects $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{AnFMP}_{/X}$ satisfiying the following requirements:

(1) $F([0]) \simeq X$;
(2) For each $n \geq 1$, the morphism

$$F([n]) \to F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]),$$

induced by the morphisms $s^i\colon [1] \to [n]$ given by $(0,1) \mapsto (i, i+1)$, is an equivalence in $\mathrm{AnFMP}_{/X}$.

(Todo: Put the above as a definition + introduce analytic formal moduli problems over.)

Lemma 2.30. *Let $X \in \mathrm{dAn}_k$. Given any $Y \in \mathrm{AnFMP}_{X/}$, then for each $i = 0, 1$ the i-th projection morphism*

$$p_0\colon X \times_Y X \to X,$$

computed in the ∞-category $\mathrm{AnPreStk}_k$ lies in the essential image of $\mathrm{AnFMP}_{/X}$ via Construction 2.14.

Proof. Consider the pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_0"] & X \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

computed in the ∞-category AnPreStk. Thanks to Proposition 2.13 together with the fact that fiber products commute with filtered colimis in the ∞-category $\mathrm{AnPreStk}_k$, we deduce that

$$X \times_Y X \simeq \operatorname*{colim}_{Z \in \mathrm{AnNil}^{\mathrm{cl}}_{X//Y}} X \times_Z X,$$

in $\mathrm{AnPreStk}_k$. It is clear that $(p_i \colon X \times_Z X \to X)$ lies in the essential image of $\mathrm{AnFMP}_{/X}$, for $i = 0, 1$. Thus also the filtered colimit

$$(p_i \colon X \times_Y X) \in \mathrm{AnFMP}_{/X}, \quad \text{for } i = 0, 1,$$

as desired. ☐

Construction 2.31. Thanks to Lemma 2.30, there exists a well defined functor $\Phi \colon \mathrm{AnFMP}_{X/} \to \mathrm{AnFGrpd}(X)$ given by the formula

$$(X \to Y) \in \mathrm{AnFMP}_{X/} \mapsto Y^{\wedge}_X \in \mathrm{AnFGrpd}(X),$$

where $Y^{\wedge}_X \in \mathrm{AnFGrpd}(X)$ denotes the analytic formal groupoid over X whose presentation is given by

$$\dots \ \substack{\longrightarrow\\[-1em] \longrightarrow\\[-1em] \longrightarrow\\[-1em] \longrightarrow} \ X \times_Y X \times_Y X \ \substack{\longrightarrow\\[-1em] \longrightarrow\\[-1em] \longrightarrow} \ X \times_Y X \ \substack{\longrightarrow\\[-1em] \longrightarrow} \ X \ .$$

Moreover, given any $\mathcal{G} \in \mathrm{AnFGrpd}(X)$, we can associate it an analytic formal moduli problem under X, denoted $\mathrm{B}_X(\mathcal{G})$, as follows: let $X \to S$ be an object in $\mathrm{AnNil}_{X/}$, then we let

$$\mathrm{B}_X(\mathcal{G})(S) \coloneqq \big\{ (\widetilde{S} \to S) \in \mathrm{AnFMP}_{/S}, \widetilde{S} \to X, \text{a morphism of groupoid objects } \widetilde{S} \times_S \widetilde{S} \to \mathcal{G} \text{ satisfying } (*) \big\}$$

where condition (*) is determined by requiring that the commutative squares

$$\begin{tikzcd} \widetilde{S} \times_S \widetilde{S} \arrow[r] \arrow[d, "p_i"] & \mathcal{G} \arrow[d, "p_i"] \\ \widetilde{S} \arrow[r] & X \end{tikzcd}$$

for $i = 0, 1$ are cartesian. Such association is functorial in $(X \to S) \in \mathrm{AnNil}_{X/}$ and thus it defines a well defined functor

$$\mathrm{B}_X(\mathcal{G}) \colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathcal{S}.$$

Remark 2.32. Let $X \in \mathrm{dAn}_k$ and $\mathcal{G} \in \mathrm{AnFGrpd}(X)$. There exists a canonical morphism $X \to \mathrm{B}_X(\mathcal{G})$ given by associating every

$$Z \in \mathrm{dAfd}_k$$

Lemma 2.33. *The functor* $\mathrm{B}_X(\mathcal{G}) \colon \mathrm{AnNil}^{\mathrm{op}}_{X/} \to \mathcal{S}$ *is equivalent to an analytic formal moduli problem.*

Proof. Thanks to Proposition 2.21 it suffices to prove that $\mathrm{B}_X(\mathcal{G})$ is infinitesimally cartesian and it admits furthermore a pro-cotagent complex. Infinitesimally cartesian follows from the modular description of $\mathrm{B}_X(\mathcal{G})$ combined with the fact that $\mathcal{G}$ is infinitesimally cartesian, as well. We are thus required to show that $\mathrm{B}_X(\mathcal{G})$ admits a *global* pro-cotangent complex. ☐

References

[1] Jorge António. p-adic derived formal geometry and derived raynaud localization theorem. *arXiv preprint arXiv:1805.03302*, 2018.

[2] Jorge António and Mauro Porta. Derived non-archimedean analytic hilbert space. *arXiv preprint arXiv:1906.07044*, 2019.

[3] Bhargav Bhatt. Completions and derived de rham cohomology. *https://arxiv.org/abs/1207.6193*, 2012.

[4] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[5] Daniel Halpern-Leistner and Anatoly Preygel. Mapping stacks and categorical notions of properness. *arXiv preprint arXiv:1402.3204*, 2014.

[6] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[7] Mauro Porta and Tony Yue Yu. Representability theorem in derived analytic geometry. *arXiv preprint arXiv:1704.01683*, 2017. To appear in *Journal of the European Mathematical Society*.

[8] Mauro Porta and Tony Yue Yu. Derived non-archimedean analytic spaces. *Selecta Math. (N.S.)*, 24(2):609–665, 2018.
[9] Mauro Porta and Tony Yue Yu. Non-archimedean quantum k-invariants. *arXiv preprint arXiv:2001.05515*, 2020.

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