A VAN EST THEOREM IN MIXED CHARACTERISTIC

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(Personal: PERSONAL COMMENTS ARE SHOWN!!!)

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1. GEOMETRIC CONTEXT

(Todo: Update the HA bib entry and add Fontaine's reference for the fact that $\mathrm{A}_{\mathrm{inf}}(-)$ provides a pro-universal thickening of perfectoid!)

1.1. Derived $\mathbb{Z}_p$-adic geometric stacks.

In this §we give an overview of the construction of the ∞-category of derived $\mathbb{Z}_p$-adic geometric stacks.

Definition 1.1.1. Let $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}$ denote the ∞-category defined via the pullback square

$$\begin{tikzcd} \mathcal{C}\mathrm{Alg}^{\mathrm{ad}} \arrow[r] \arrow[d] & \mathcal{C}\mathrm{Alg} \arrow[d, "\pi_0(-)"] \\ \mathcal{C}\mathrm{Alg}^{\heartsuit,\mathrm{ad}} \arrow[r] & \mathcal{C}\mathrm{Alg}^{\heartsuit} \end{tikzcd}$$

computed in the ∞-category $\mathcal{C}\mathrm{at}_\infty$. We refer to $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}$ as the ∞-category of *simplicial adic algebras*. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}$ be a simplicial A-adic algebra, we define the ∞-category of simplicial A-adic algebras assertion

$$\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_A := (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}})_{A/}.$$

Of principal importance for us is the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, where we regard $\mathbb{Z}_p \in \mathcal{C}\mathrm{Alg}^{\heartsuit,\mathrm{ad}}$ equipped with its (p)-adic topology.

Let $f\colon A \to B$ be a morphism of simplicial $\mathbb{Z}_p$-adic algebras. The *relative $\mathbb{Z}_p$-adic cotangent complex*, denoted $\mathbb{L}^{\mathrm{ad}}_{B/A} \in \mathrm{Mod}_B$, is defined in [2, §3.4]. It corepresents $\mathbb{Z}_p$-adic derivations of simplicial $\mathbb{Z}_p$-adic algebras. In particular, we have a natural equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}^{\mathrm{ad}}_{B/A}, M) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{A//B}}(B, B \oplus M) \in \mathcal{S},$$

for every $M \in \mathrm{Mod}_B$.

Definition 1.1.2. Let $f\colon A \to B$ be a morphism in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ between p-complete simplicial $\mathbb{Z}_p$-adic algebras. We say that f is *topologically almost of finite presentation* if

$$\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$$

is a topologically of finite presentation morphism between ordinary p-adic complete $\mathbb{Z}_p$-adic algebras and, for every integer $i > 0$,

$$\pi_i(B) \in \mathrm{Mod}_{\pi_0(A)}$$

is a finitely presented $\pi_0(A)$-module.

Definition 1.1.3. Let $f\colon A \to B$ be a morphism in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. The morphism f is said to be *étale* (resp., *smooth*) if it is *almost topologically of finite presentation* and the relative cotangent complex

$$\mathbb{L}^{\mathrm{ad}}_{B/A} \in \mathrm{Mod}_B.$$

vanishes (resp. it is equivalent to a free B-module of finite rank concentrated in degree 0).

Definition 1.1.4. Let P_{sm} denote the class of smooth morphisms in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. The triplet $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ forms a *geometric context*, which we refer to as the *derived $\mathbb{Z}_p$-adic geometric context*. For the definition of the notion of geometric stack we refer the reader to [1, Definition 2.3.1].

Remark 1.1.5. Notice that the conditions (ii) and (iii) in [1, Definition 2.3.1] are automatic. Condition (iv) follows from the local structure for smooth morphisms and condition (i) (and furhter faithfully flat descent) follows from [2, Proposition 3.2.9] together with [8, Theorem 5.15] combined with [11, Proposition 8.1.2.1].

Definition 1.1.6. The ∞-category of *derived $\mathbb{Z}_p$-adic geometric stacks* is defined as the full subcategory of $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ spanned by geometric stacks with respect to the geometric context $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$,

$$\mathrm{dSt}^{\mathrm{ad}} := \mathrm{dSt}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}).$$

Remark 1.1.7. The ∞-category $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ is *closed under $\tau_{\text{ét}}$-descent*. This assertion is a consequence of [8, Proposition 5.16, (3)].

Example 1.1.8. In [2, Definition 3.2.10] it was introduced the notion of a *derived $\mathbb{Z}_p$-adic Deligne-Mumford stack*. The collection of such objects forms naturally an ∞-category, denoted $\mathrm{dfDM}_{\mathbb{Z}_p}$. Let $\mathsf{X} \in \mathrm{dfDM}_{\mathbb{Z}_p}$, then its associated functor of points is naturally an object in the ∞-category $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ and even an object in the ∞-category $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\mathrm{fpqc}})$, thanks to [8, Theorem 5.15] combined with [11, Proposition 8.1.2.1]. Therefore the association

$$\mathsf{X} \in \mathrm{dfDM}_{\mathbb{Z}_p} \mapsto \mathrm{Map}_{\mathrm{dfDM}}(\mathrm{Spf}(-), \mathsf{X}) \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$$

provides us with a fully faithful functor

$$F\colon \mathrm{dfDM}_{\mathbb{Z}_p} \subseteq \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}).$$

In particular, if $\mathsf{X} \in \mathrm{dfSch}_{\mathbb{Z}_p}$ is a derived $\mathbb{Z}_p$-adic scheme, then we can regard it naturally as an étale sheaf on $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$.

We further assert that the fully faithful functor $F\colon \mathrm{dfDM}_{\mathbb{Z}_p} \to \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ factors through the fully faithful natural inclusion $\mathrm{dSt}^{\mathrm{ad}} \subseteq \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$. Indeed, since the ∞-category $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ is closed under $\tau_{\text{ét}}$-descent we are reduced to verify condition (1) in [1, Definition 2.3.3] for derived $\mathbb{Z}_p$-adic Deligne-Mumford stacks. This last assertion is immediate from the fact that derived $\mathbb{Z}_p$-adic Deligne-Mumford stacks admit (affine) étale coverings, by construction. We have thus a natural fully faithful functor

$$G\colon \mathrm{dfDM}_{\mathbb{Z}_p} \to \mathrm{dSt}^{\mathrm{ad}}.$$

Construction 1.1.9. Let $L_p^{\wedge}\colon \mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p} \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ denote the *p-completion functor*, introduced in [11, §8]. Given $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, we define $A_n \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ as the pushout

$$\begin{tikzcd} A[t] \arrow[r, "t \mapsto p^n"] \arrow[d, "t \mapsto 0"'] & A \arrow[d] \\ A \arrow[r] & A_n \end{tikzcd}$$

computed in ∞-category $\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}$. Notice that A_n is naturally an object of the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ via the canonical inclusion $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \subseteq \mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}$. Thanks to [11, Lemma 8.1.2.3], one has a natural equivalence

$$(A)^\wedge_p \simeq \lim_{n\geq 1} A_n,$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, where $(A)^\wedge_p$ denotes the p-completion of A.

Remark 1.1.10. Given a functor $X \colon \mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p} \to \mathcal{S}$, we define its *p-completion* as the functor

$$X^\wedge_p \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \mapsto \mathcal{S}$$

given by the formula

$$A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \mapsto \lim_{n\geq 1} X(A_n) \in \mathcal{S}. \tag{1.1.1}$$

From the above formula (1.1.1), it is clear that if X satisfies étale hyper-descent then so it does $X^\wedge_p$ satisfies descent with respect to $\tau_{\text{ét}}$-hypercoverings in $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$. The same statement holds if we replace étale hyper-descent by fpqc hyper-descent.

Lemma 1.1.11. *Let $X \in \mathrm{dSt}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{ét}, \mathrm{P}_{\mathrm{sm}})$ be a geometric stack with respect to the (derived) algebraic geometric context*

$$(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{ét}, \mathrm{P}_{\mathrm{sm}}).$$

Then the p-completion of X, $X^\wedge_p \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{ét})$, is geometric, i.e., $X^\wedge_p \in \mathrm{dSt}^{\mathrm{ad}}$.

Proof. Suppose X is n-geometric, for a given integer $n \geq 0$. Let

$$\pi \colon P \to X,$$

denote a smooth $(n-1)$-representable covering of X. We shall prove that the *p-adic completion*

$$(\pi)^\wedge_p \colon P^\wedge_p \to X^\wedge_p$$

is itself a $(n-1)$-representable morphism with respect to the geometric context $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$. Consider the pullback square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & P^\wedge_p \arrow[d, "(\pi)^\wedge_p"] \\ \mathrm{Spf}\, A \arrow[r] & X^\wedge_p, \end{tikzcd}$$

computed in the ∞-category $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$. By construction, Y can be realized as a limit diagram of pullback diagrams of the form

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d] & P_n \arrow[d, "\pi_n"] \\ \mathrm{Spec}\, A_n \arrow[r] & X_n, \end{tikzcd}$$

computed in the ∞-category $\mathrm{dSt}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$, where the subscript $(-)_n$ denotes the reduction modulo the ideal $(p^n) \subseteq \mathbb{Z}_p$. As π_m is $(n-1)$-representable, for each $m \geq 1$, it follows that each $Y_m \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ is $(n-1)$-represntable. Therefore, we are reduced to show that the filtered colimit

$$Y \simeq \operatorname*{colim}_{m\geq 1} Y_m$$

is $(n-1)$-representable. As filtered colimits are preserved under finite limites we reduce ourselves, by induction on the geometric level, to the case $n = 0$. When $n = 0$, the result follows from the universal property of the formal spectrum, proved in [11, Proposition 8.1.2.1]. □

Corollary 1.1.12. *Let $\{X_n\}_{n\geq 1}$ denote a compatible* ind-system *in the ∞-category* $\mathrm{dSt}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{ét}, \mathrm{P}_{\mathrm{sm}})$*, where for each $n \geq 1$, X_n denotes a derived $\mathbb{Z}/p^n$-geometric stack. Then the induced functor*

$$\operatorname*{colim}_{n\geq 1} X_n \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \to \mathcal{S}$$

is naturally a derived $\mathbb{Z}_p$-adic geometric stack.

Proof. The proof follows exactly the same lines as the proof of Lemma 1.1.11. □

Example 1.1.13. Both Lemma 1.1.11 and Corollary 1.1.12 allow us to construct many interesting derived $\mathbb{Z}_p$-adic geometric stacks. Indeed, let $X \in \mathrm{dSch}_{\mathbb{Z}_p}$ denote a derived $\mathbb{Z}_p$-scheme, then its p-completion, $X_p^\wedge$ is naturally an object living the ∞-category $\mathrm{dSt}^{\mathrm{ad}}$.

Given a $\mathbb{E}_\infty$-group object in the ∞-category $G \in \mathrm{dSt}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$, the classifying space of the corresponding p-completion $\mathrm{B}G_p^\wedge \in \mathrm{dSt}^{\mathrm{ad}}$. Similarly, one can also consider classifying spaces of formal $\mathbb{Z}_p$-formal groups as derived $\mathbb{Z}_p$-adic geometric stacks.

The p-completion of the derived moduli stack parametrizing perfect complexes, Perf is also naturally an object in the ∞-category $\mathrm{dSt}^{\mathrm{ad}}$. So it is, the (derived) moduli stack classifying formal groups or p-divisible groups in mixed characteristic. Given any formal reductive group G, Bun_G also lives naturally in the ∞-category $\mathrm{dSt}^{\mathrm{ad}}$.

Remark 1.1.14. Let $X \in \mathrm{St}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$. Thanks to Artin-Lurie representability theorem, [9, Theorem 3.1.2], the functor of points associated to X is nilcomplete, infinitesimally cartesian and it admits a global algebraic cotangent complex, which is an almost perfect complex on X. From the definitions, it is clear that $X_p^\wedge$ also possedes these properties, namely its corresponding functor of points is nilcomplete, infinitesimally cartesian and it admits a global almost perfect adic cotangent complex. Moreover, one has a natural equivalence

$$\mathbb{L}^{\mathrm{ad}}_{X_p^\wedge} \simeq (\mathbb{L}^{\mathrm{alg}}_X)_p^\wedge \in \mathrm{Mod}_{X_p^\wedge},$$

where $\mathbb{L}^{\mathrm{alg}}_X$ denotes the (algebraic) cotangent complex of $X \in \mathrm{dSt}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$.

(Personal: One would like to have a Representability theorem in the context of derived $\mathbb{Z}_p$-adic geometry. Otherwise, it will be difficult to state precisely that geometric stacks like Perf satisfy geometricity. Another way around this problem, might be by identifying such moduli with completion along the ideal $(p) \subseteq \mathbb{Z}_p$.) (Personal: Notice that Artin-Lurie representability holds true for Noetherian $\mathbb{E}_\infty$-rings such that $\pi_0(R)$ is a Grothendieck ring. In particular, $\mathbb{Z}_p$ is an example of such so we do have Artin-Lurie Representability theorem for *algebraic* derived geometric $\mathbb{Z}_p$-stacks.)

(Personal: Notice that we would like that our statements are true more generally. In particular, we would like to be able to treat the case of the moduli of p-divisible groups. However, this object is not geometric in our sense, since it does not admit a smooth covering. The only obstruction, it seems at this point, is to prove the existence of the local section, since for this we admit that we have a smooth covering of the form $P \to Y$.)

1.2. **Brief considerations on fpqc-descent.** (Todo: Maybe it is more suitable to put this section in appendix.)

Definition 1.2.1. Let $X \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \to \mathcal{S}$. We say that X satisfies *faithfully flat-descent* if for every faithfully flat morphism

$$f \colon A \to B,$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, one has a canonical equivalence

$$X(\check{\mathrm{C}}(B/A)) \to X(A)$$

in the ∞-category $\mathcal{S}$, where $\check{\mathrm{C}}(B/A)$ denotes the *p-complete Čech nerve* associated to the morphism $f \colon A \to B$. (Todo: Define the p-complete Cech nerve.)

Remark 1.2.2. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ and let $I \subseteq \pi_0(A)$ the defining ideal for A. Since we can identify (naturally) the underlying ∞-topos of $Spec(A/I)$ with the underlying ∞-topos of $\mathrm{Spf}\, A$ we obtain that for each $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, $\mathrm{Spf}\, A \in \mathrm{dfSch}_{\mathbb{Z}_p}$ is quasi-compact.

Definition 1.2.3. Let $X \in \mathrm{dfSch}_{\mathbb{Z}_p}$. An *fpqc-covering* of X is a family of morphisms $\{f_i \colon X_i \to X\}$ of derived $\mathbb{Z}_p$-adic schemes such that each f_i is flat and such that for every affine open $U \subseteq T$ there exists an $n \geq 0$, a map $\alpha \colon \{1, \dots, n\}$ and affine opens $V_i \subseteq X_{\alpha(i)}$, $j = 1, \dots, n$ with $\bigcap_{j=1}^n f_{\alpha(i)}(V_j) = U$.

Definition 1.2.4. Let $Y \in \mathrm{PShv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p})$ denote a pre-sheaf on $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. We say that Y satisfies (hyper)-fpqc-descent if for every $\mathrm{X} \in \mathrm{dfSch}_{\mathbb{Z}_p}$ together with a simplicial object

$$U_\bullet \colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathrm{dfSch}_{\mathbb{Z}_p},$$

such that each transition map is flat and quasi-compact, together with a canonical map $U_\bullet \to X$ wich is surjective, then the natural map

$$Y(X) \to Y(U_\bullet)$$

is an equivalence in the ∞-category $\mathrm{PShv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p})$.

Lemma 1.2.5. *Let $X \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}, \tau_{ét})$ denote an étale sheaf with respect to the site $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{ét})$. Assume further that X satisfies* fpqc*-descent. Then the p-completion*

$$X^\wedge_p \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \to \mathcal{S}$$

also satisfies fpqc*-descent.*

Proof. The result is an immediate consequence of the fact that limits preserve limits. (Todo: expand this proof.) □

Proposition 1.2.6. *The ∞-category of fpqc-sheaves on $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, denoted $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\mathrm{fpqc}})$ is subcanonical.*

Proof. This is an immediate consequence of [8, Theorem 5.15] together with [11, Proposition 8.1.2.1]. □

1.3. **Integral perfectoid algebras.** Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ denote a (discrete) $\mathbb{Z}_p$-algebra. We assume further that R is equipped with an uniformizer $\pi \in R$ such that R is π-adically complete and separated. Denote by

$$\varphi \colon R/pR \to R/pR$$

the *absolute Frobenius* of the reduction R/pR. We define the *tilt* of R, which we shall denote by $R^\flat$, as the (usual) inverse limit

$$R^\flat := \lim_{\varphi} R/pR,$$

in the category of (discrete) $\mathbb{F}_p$-algebras. By construction, the tilt $R^\flat$ is a *perfect $\mathbb{F}_p$-algebra.*

Definition 1.3.1. Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, as above. We define *Fontaine's ring* associated to R as

$$\mathrm{A}_{\mathrm{inf}}(R) := W(R^\flat).$$

(Personal: Maybe one can define $\mathrm{A}_{\mathrm{inf}}(R)$ already as the pro-completion over the kernel of the projection map $\mathrm{A}_{\mathrm{inf}}(R) \to R$.)

Remark 1.3.2. Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ as above. Since $R^\flat$ is a perfect $\mathbb{F}_p$-algebra, it follows that, for each integer $n \geq 1$,

$$\mathrm{A}_{\mathrm{inf}}(R)/p^n \simeq W_n(R^\flat).$$

For this reason, the natural morphism

$$R^\flat \to R/pR,$$

induced by projection on the first factor, induces (by the universal property of p-typical Witt vectors) a canonical morphism

$$\theta \colon \mathrm{A}_{\mathrm{inf}}(R) \to R,$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$.

Definition 1.3.3. Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be a discrete $\mathbb{Z}_p$-adic algebra. We say that R is an *integral perfectoid $\mathbb{Z}_p$-adic algebra* if if satisfies the following conditions:

(i) There exists an element $\pi \in R$ such that R is π-adically complete and separated. We assume further that $\pi^p | \, p$;
(ii) The Frobenius morphism

$$\varphi \colon R/pR \to R/pR$$

is surjective;
(iii) The kernel of the canonical morphism $\theta \colon \mathrm{A}_{\mathrm{inf}}(R) \to R$ is generated by a single element.

Example 1.3.4. The following is a transcription of [4, Example 3.6].

(i) Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be a discrete $\mathbb{Z}_p$-adic algebra. If $pR = 0$ then R is integral perfectoid if and only if R is perfect as a $\mathbb{F}_p$-algebra.
(ii) Let $\mathbb{Z}_p^{\mathrm{cyc}} := (\mathbb{Z}_p[\zeta_{p^\infty}])^\wedge_p$ denote the p-adic completion of the ring of integers of the cyclotomic extension $\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p$. Then $\mathbb{Z}_p^{\mathrm{cyc}}$ is a perfectoid $\mathbb{Z}_p$-algebra.
(iii) The p-adic completion of the ring of integers of any algebraic extension of $\mathbb{Q}_p(\zeta_{p^\infty})$ is an integral perfectoid $\mathbb{Z}_p$-adic algebra.
(iv) The p-adic complete ring $\mathbb{Z}_p^{\mathrm{cyc}}\langle T^{1/p^\infty}\rangle$ is itself a perfectoid $\mathbb{Z}_p$-algebra. So it is the *perfectoid torus*, $\mathbb{Z}_p^{\mathrm{cyc}}\langle T^{\pm 1/p^\infty}\rangle$.

Lemma 1.3.5. *Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be an integral perfect $\mathbb{Z}_p$-adic algebra. Then R is (derived) p-complete.*

Proof. Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be a discrete $\mathbb{Z}_p$-adic algebra. We wish to show that R is p-complete provided that R is p-torsion free (in the usual sense). Thanks to [11, Lemma 8.1.2.3], it suffices to show that the canonical map

$$R \to \lim_{n \geq 1} R_n$$

is an equivalence in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. In our situation, we can identify, for each $n \geq 1$,

$$R_n \simeq R \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}.$$

Therefore, for each integer $i \geq 0$, $\pi_i(R_n) = 0$, except in the case where $i = 0,\ 1$, in which we have $\pi_1(R_n) = R[p^n] = 0$ and $\pi_0(R_n) = R/p^n$. Passing to inverse limits, the Milnor short exact sequence implies that $\pi_i(R^\wedge_p) = 0$ for $i > 1$. For $i = 0$, we have a Milnor short exact sequence of the form

$$0 \to \lim_{n\geq 1}{}^1\ \pi_1(R_n) \to \pi_0(R^\wedge_p) \to \lim_{n \geq 1}\ \pi_0(R_n) = R \to 0, \tag{1.3.1}$$

and for $i = 1$, we have an equivalence $\pi_1(\lim R_n) \simeq \lim_{n\geq 1}{}^1 \pi_1(R_n)$. Therefore, in order to prove the claim it suffices to show that the pro-system $\{\pi_1(R_n)\}_{n\geq 1}$ is trivial. Every integral perfectoid $\mathbb{Z}_p$-algebra has bounded p-torsion, [5, (Todo: Add reference)]. Therefore, for each $m > 0$ there exists a sufficiently large integer $n > m$ such that the transition map $R[p^n] \to R[p^m]$ vanishes. The result then follows, as desired. $\square$

Corollary 1.3.6. *Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ denote an integral perfectoid $\mathbb{Z}_p$-algebra. Then $\mathrm{A}_{\mathrm{inf}}(R) \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is p-adically complete and separated and p-torsion free. In particular, $\mathrm{A}_{\mathrm{inf}}(R)$ is (derived) p-complete.*

Proof. By construction, we have that $\mathrm{A}_{\mathrm{inf}}(R) = W(R^\flat)$. The (discrete) $\mathbb{F}_p$-algebra $R^\flat$ is a perfect $\mathbb{F}_p$-algebra. For this reason, $W(R^\flat)$ is p-adically complete, separated and p-torsion free. $\square$

Remark 1.3.7. We have a natural isomorphism $\mathrm{A}_{\mathrm{inf}}(R)/p \simeq R^\flat$ of (discrete) $\mathbb{Z}_p$-adic algebras.

Lemma 1.3.8. *The morphism $\theta\colon \mathrm{A}_{\mathrm{inf}}(R) \to R$ exhibits the ring $\mathrm{A}_{\mathrm{inf}}(R)$ as the universal pro-thickening of R.*

Proof. This is the content of [15, Proposition 3.14]. (Todo: The reference is not quite right since it is less general, even if the proof applies for any integral perfectoid. Also better to use the original Fontaine reference!) $\square$

The following result is of fundamental importance for our purposes:

Proposition 1.3.9. *Let $R \to S$ be a morphism between integral perfectoid $\mathbb{Z}_p$-adic algebras in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. Then $\mathbb{L}^{\mathrm{ad}}_{S/R} \simeq 0$.*

Proof. This is the content of [4, Proposition 3.14]. $\square$

Remark 1.3.10. Let $f\colon R \to S$ be a morphism between integral perfectoid $\mathbb{Z}_p$-algebras in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. The morphism f induces a morphism, at the level of the tilts,

$$(f)^\flat\colon R^\flat \to S^\flat$$

which is *formally étale*. By applying the p-typical Witt vectors construction one obtains an induced morphism at the level of Fontaine's rings

$$\widetilde{f}\colon \mathrm{A}_{\mathrm{inf}}(R) \to \mathrm{A}_{\mathrm{inf}}(S).$$

Thanks to Lemma 1.3.8 above, [10, Proposition 8.4.2.5, see also Remark 8.4.2.3] together with formal étaleness of f combined with the fact that

$$S \simeq \mathrm{A}_{\mathrm{inf}}(S) \otimes_{\mathrm{A}_{\mathrm{inf}}(R)} R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p},$$

that $\widetilde{f}$ is the unique (up to contractible indeterminacy) such deformation of f, in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$.

The following two results will establish the precise relation between $\mathrm{A}_{\mathrm{inf}}(R)$ and R, via obstruction theory, for perfectoid R:

The following proposition characterizes a (derived) universal property for Spf $A_{\mathrm{dR}}(R)$, at least, when restricted to those simplicial $\mathbb{Z}_p$-adic algebra whose Jacobson ideal of $\pi_0(A)$ is nilpotent.

Proposition 1.3.11. *Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be an integral perfectoid $\mathbb{Z}_p$-algebra. Then, for every p-complete $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, the natural morphism*

$$\theta\colon \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}(\mathrm{A}_{\mathrm{inf}}(R), A) \to \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}(R, \pi_0(A)_{\mathrm{red}}),$$

is an equivalence of mapping spaces.

Proof. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be p-complete and $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ an integral perfectoid $\mathbb{Z}_p$-algebra. For notational purposes, denote by φ_A the Frobenius endomorphism of $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. We have a natural chain of equivalences of the form

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}\big(\mathrm{A}_{\mathrm{inf}}(R), A\big) &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, A \otimes_{\mathbb{Z}_p} \mathbb{F}_p\big)\\ &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big((R^\flat)^{\mathrm{perf}}, A \otimes_{\mathbb{Z}_p} \mathbb{F}_p\big)\\ &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, (A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^\flat\big)\end{aligned}$$

where the first equivalence follos from the universal property of the (derived) p-typical Witt vectors construction combined with the fact that $\mathrm{A}_{\mathrm{inf}}(R) \simeq W(R^\flat)$. The second equivalence follows from the fact that $R^\flat$ is a perfect $\mathbb{F}_p$-simplicial algebra. As a consequence, the natural morphism

$$R^\flat \to (R^\flat)^{\mathrm{perf}}$$

is an isomorphism. The third equivalence, follows from the adjunction $(-)^{\mathrm{perf}} \dashv (-)^\flat$.

Notice that $(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^\flat \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ is a perfect simplicial $\mathbb{F}_p$-algebra. Therefore, thanks to [7, Lemma 11.6], it follows that $A^\flat \simeq \pi_0(A^\flat)$. Moreover, we have a Milnor short exact sequence of the form

$$0 \to \lim_{\varphi_A}{}^1 \pi_1(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \to \pi_0\big((A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^\flat\big) \to \lim_{\varphi_A} \pi_0(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \to 0.$$

We wish to prove that $\lim_{\varphi_A}{}^1 \pi_1(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \simeq 0$ and thus conclude that

$$\pi_0\big((A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^\flat\big) \simeq \lim_{\varphi_A} \pi_0(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p).$$

We have a canonical bijection

$$\pi_1(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \simeq \pi_0\big(\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}(\mathrm{Sym}(\mathbb{F}_p[1]), A)\big).$$

For this reason, we have a canonical equivalence of pro-objects

$$\{\pi_1(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)\}_{\varphi_A} \simeq \{\pi_0\big(\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}(\mathrm{Sym}(\mathbb{F}_p[1]), A)\big)\}_{\varphi_{\mathrm{Sym}(\mathbb{F}_p[1])}}. \tag{1.3.2}$$

But, thanks to the proof of [7, Lemma 11.6], the morphism $\varphi_{\mathrm{Sym}(\mathbb{F}_p[1])}$ sends the generator in degree 1 to the zero element in A. We thus conclude that the pro-system displayed in (1.3.2) is trivial. For this reason, we have

$$\lim_{\varphi_A}{}^1 \pi_1(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \simeq 0.$$

We thus conclude that we have an equivalence

$$A^\flat \simeq \pi_0(A)^\flat.$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. Using this fact, we obtain a natural chain of equivalences of the form

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, (A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^\flat\big) &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, \pi_0(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^\flat\big)\\ &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, \pi_0(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)\big)\\ &\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, \pi_0(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p)\big)\\ &\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}}\big(R^\flat, \mathrm{Tor}^0_{\mathbb{Z}_p}(\pi_0(A), \mathbb{F}_p)\big)\\ &\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}\big(\mathrm{A}_{\mathrm{inf}}(R), \pi_0(A)\big).\end{aligned}$$

The fourth equivalence follows from the Tor-spectral sequence for simplicial rings and the last equivalence follows from the usual universal property of p-typical Witt vectors. We thus conclude that we have an equivalence of mapping spaces,

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}\big(\mathrm{A}_{\mathrm{inf}}(R), A\big) \simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}\big(\mathrm{A}_{\mathrm{inf}}(R), \pi_0(A)\big).$$

By taking the component corresponding to continuous morphisms for the (ε, p)-topology on $\mathrm{A}_{\mathrm{inf}}(R)$, we obtain an equivalence:

$$\operatorname*{colim}_{n \geq 1}\left(\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}\big(\mathrm{A}_{\mathrm{inf}}(R)/(\varepsilon^n), A\big)\right) \simeq \operatorname*{colim}_{n \geq 1}\left(\mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}}\big(\mathrm{A}_{\mathrm{inf}}(R)/(\varepsilon^n), \pi_0(A)\big)\right).$$

Our assumption on $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, which has nilpotent Jacobson ideal, it follows that there exists a sufficiently large integer $k \geq 1$ such that

$$\mathcal{J}^k_{\mathrm{Jac}} = 0.$$

Thanks to [15, Proposition 3.14], it follows then that

$$\begin{aligned}\operatorname*{colim}_{n \geq 1} \Big(\mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}} \big(\mathrm{A}_{\mathrm{inf}}(R)/(\varepsilon^n), \pi_0(A) \big) \Big) &\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}} \big(\mathrm{A}_{\mathrm{inf}}(R)/(\varepsilon^k), \pi_0(A) \big) \\ &\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}} \big(R, \pi_0(A)_{\mathrm{red}} \big).\end{aligned}$$

Putting all the pieces together, we obtain the desired canonical equivalence of mapping spaces

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}} \big(\mathrm{A}_{\mathrm{inf}}(R), A \big) \simeq \mathrm{Hom}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}} \big(R, \pi_0(A)_{\mathrm{red}} \big),$$

as desired. ☐

Corollary 1.3.12. *Let $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be an integral perfectoid $\mathbb{Z}_p$-algebra. Then the canonical morphism*

$$\mathrm{Spf}\, R \to \mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(R)),$$

where we consider $\mathrm{A}_{\mathrm{inf}}(R) \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ equipped with its (p, ε)-topology, exhibits the later as the de Rham *stack of* $\mathrm{Spf}\, R$*, in the ∞-category* $\mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\acute{e}t})$.

Proof. This is a direct consequence of Proposition 1.3.11 and the construction of the de Rham stack associated to a formal spectrum. ☐

Proposition 1.3.13. *Consider a pullback diagram of the form*

$$\begin{tikzcd} \mathrm{Spf}(S) \arrow[r] \arrow[d, "f"] & \widetilde{P} \arrow[d, "\widetilde{f}"] \\ \mathrm{Spf}(R) \arrow[r] & \mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(R)) \end{tikzcd}$$

in the ∞-category $\mathrm{dfSch}_{\mathbb{Z}_p}$, where S and R are p-torsion free integral perfectoid and we consider $\mathrm{A}_{\mathrm{inf}}(R)$ with its natural (p, ε)-adic topology. Then $\widetilde{P} \simeq \mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(S))$.

Proof. Notice first that $\widetilde{P}$ is a thickening of $\mathrm{Spf}(S)$ and the morphism $\widetilde{P} \to \mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(R))$ is a deformation of the morphism $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$. Thanks to [14, Tag 06AD] it follows that $\widetilde{P}$ is affine. Therefore we can write

$$\widetilde{P} \simeq \mathrm{Spf}\, A,$$

where A is a simplicial $\mathbb{Z}_p$-adic algebra. Proposition 1.3.9 implies that $\mathbb{L}^{\mathrm{ad}}_{S/R} \simeq 0$. Moreover, for every $n \geq 1$, [10, Theorem 8.4.2.7] implies that any deformation of the morphism f, over $\mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(R)/\varepsilon^n)$, is controlled by the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_S} \big(\mathbb{L}^{\mathrm{ad}}_{S/R}, S \otimes_R (\varepsilon)/(\varepsilon^n) \big) \simeq 0.$$

Therefore, such deformation must be unique (up to contractible indeterminacy). Since the morphism

$$\mathrm{A}_{\mathrm{inf}}(S)/(\varepsilon_S^n) \to \mathrm{A}_{\mathrm{inf}}(R)/(\varepsilon^n)$$

is such a deformation, we conclude by passage to the limit that we have a canonical equivalence

$$\widetilde{P} \simeq \mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(S))$$

and the morphism $\widetilde{f} \colon \widetilde{P} \to \mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}(R))$ coincides with the morphism induced by the corresponding map

$$\mathrm{A}_{\mathrm{inf}}(R) \to \mathrm{A}_{\mathrm{inf}}(S),$$

(uniquely) induced from $R \to S$. ☐

1.4. de Rham stack.

Definition 1.4.1. Let $X \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}})$ denote an étale sheaf on $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. Then, we defined its associated *de Rham sheaf*

$$X_{\mathrm{dR}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \to \mathcal{S}$$

given on objects by the formula

$$A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \mapsto X(\pi_0(A)_{\mathrm{red}}) \in \mathcal{S}.$$

Lemma 1.4.2. *Let $X \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\acute{e}t})$ and consider its associated de Rham sheaf $X_{\mathrm{dR}} \in \mathrm{Shv}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\acute{e}t})$. If X satisfies fpqc-descent then so does X_{dR}.*

Proof. One is reduced to show that the association $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p} \mapsto A_{\mathrm{red}} \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ satisfies faithfully flat descent. This statement should reduce itself to prove that Jacobson ideal satisfies faithfully flat descent. This last assertion is a special case of fpqc-descent for quasi-coherent sheaves, proved in [14, Tag 023R]. $\square$

1.5. Faithfully flat coverings by perfectoids.

The following result shows that every object $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ which is regular and Noetherian admits a faithfully flat map to a perfectoid ring:

Theorem 1.5.1. *Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be a discrete $\mathbb{Z}_p$-adic algebra. Then A is Noetherian and regular if and only if exists a faithfully flat morphism*

$$A \to A_\infty,$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$, where $A_\infty \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is an integral perfectoid $\mathbb{Z}_p$-algebra.

Proof. This is precisely the content of the main result [3, Theorem 4.7]. $\square$

Remark 1.5.2. Let $\mathbb{T}^n := \mathrm{Spf}(\mathbb{Z}_p\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$ denote the formal torus of dimension n. We define

$$\mathbb{T}^n(p^\infty) := \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cyc}}\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}).$$

We have a canonical morphism

$$\pi_n \colon \mathbb{T}^n(p^\infty) \to \mathbb{T}^n$$

which is an fpqc-covering and exhibits $\mathbb{T}^n(p^\infty)$ as a $\Gamma := \mathbb{Z}_p(1)^n$-torsor over $\mathbb{T}^n$. (Todo: justify this with a refenrece. Does Γ really have the correct dimension?) In this case, we construct an explicit faithfully flat covering of $\mathbb{T}^n$. Moreover, suppose that we have an étale morphism (Todo: Does étale really suffices or do we really finite étale? Check this.)

$$\mathrm{Spf}\, R \to \mathbb{T}^n,$$

where $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. Then we have a perfectoid covering

$$\mathrm{Spf}(R_\infty) \to \mathrm{Spf}(R)$$

where $R_\infty := R \otimes_{\mathbb{Z}_p\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle} \mathbb{Z}_p^{\mathrm{cyc}}\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle$. Then R_∞ is an integral perfectoid $\mathbb{Z}_p$-algebra,(Todo: Check this!) which provides us with a faithfully flat morphism

$$R \to R_\infty$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$. (Todo: Check that $\mathrm{Spf}\, R_\infty$ is quasi-compact.)

1.6. Smooth morphisms between derived $\mathbb{Z}_p$-adic geometric stacks.

In this §we wish to prove the following:

Theorem 1.6.1. *Let $f \colon X \to Y$ denote a smooth morphism in the ∞-category $\mathrm{dSt}^{\mathrm{ad}}$. Then, locally on both X and Y, f can be factored as a composite*

$$X \xrightarrow{g} Y \times \mathfrak{A}^n_{\mathbb{Z}_p} \xrightarrow{\mathrm{pr}_1} Y$$

where g is an étale map of derived $\mathbb{Z}_p$-adic geometric stacks and pr_1 denotes the canonical projection.

Proof. Since the result is both local on X and Y, we can assume that $f \colon X \to Y$ is a smooth morphism between derived $\mathbb{Z}_p$-adic schemes, (Todo: Check this!). In this case, the proof of [13, Proposition 5.50] applies. (Todo: Check this and write it better!) $\square$

1.7. **Assumptions.** First of all we will define the context in which we are going to work through. Let us denote $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ the ∞-category of adic $\mathbb{Z}_p$-simplicial algebras, introduced in [2]. We consider the ∞-category $\mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ of *derived $\mathbb{Z}_p$-adic geometric stacks*. Suppose we are given a diagram of the form

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d, "q"] & X \\ Y & \end{tikzcd}$$

where $M \in \mathrm{dfSch}$ is a smooth $\mathbb{Z}_p$-adic scheme and $X,\ Y \in \mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$. Throughout the text, p denotes a smooth surjective morphism and q a smooth morphism of derived $\mathbb{Z}_p$-adic geometric stacks. We assume further that Y satisfies *fpqc descent*. (Personal: The need to assume that Y satisfies fpqc descent boils down to the fact that the étale topology is not sufficient for many of our purposes. Namely, in order to compute the correct mapping spaces of derived $\mathbb{Z}_p$-adic geometric stacks one needs to pass to perfectoid coverings, in a similar vein as in BMS1. More precisely, given a Noetherian regular algebra $\mathbb{Z}_p$-algebra A, by a theorem of Bhatt, Iyegar and Ma one knows that there exists a faithfully flat morphism $A \to A_\infty$, where A_∞ is integral perfectoid. Furthermore, by a celebrated result of Abhyankar one can suppose further that A_∞ admits all p-th power roots of unity, thus living over the perfectoid covering $\mathbb{Z}_p^{\mathrm{cyc}}$ of $\mathbb{Z}_p$. However, it seems that, in general, the morphism $A \to A_\infty$ is not étale or even weak étale in the sense of [6]. Therefore, in order to be able to reduce our local computations to computations involving the perfectoid nature of suitable algebras, one needs to assume Y to satisfy fpqc descent. This is indeed the case when $Y = \mathrm{Perf},\ \mathrm{QCoh}^{\heartsuit},\ \mathrm{QCoh}, \mathrm{Bun}_G$, where G denotes a reductive group, X a formal scheme, $\mathrm{B}G$, for G a formal group scheme or a p-divisible group, $\mathcal{M}_{\mathrm{FM}}$ the moduli space of p-divisible groups, etc.)

Our goal is to prove a mixed characteristic analogue of a theorem of Van Est, in the context of (derived) differential geometry, proved in great generality by J. Nuiten using the Koszul duality for Lie algebroids, see [12]. (Todo: Recall the statement) (Todo: Check that the geometricity of the stack Y is really need or we can relax a bit the assumptions on Y, in order for the result to be compatible with $\mathcal{M}_{\mathrm{FM}}$, for example.)

2. The existence of a local section

We want to prove a connectivity statement for the canonical morphism of mapping spaces

$$\mathrm{Map}_{M/}(X, Y) \to \mathrm{Map}_{\mathrm{dSt}^{\mathrm{ad}}}(X^\wedge_M, Y^\wedge_M).$$

Suppose then that we are given a morphism of sheaves

$$X^\wedge_M \to Y.$$

Assume further that we have a smooth covering $\pi \colon P \to Y$, in the ∞-category $\mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ and form the pullback square

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ X^\wedge_M \arrow[r] & Y. \end{tikzcd}$$

In particular, the natural morphism $P' \to X^\wedge_M$ is a smooth covering. Moreover, we have a natural inclusion morphism

$$M \to X^\wedge_M.$$

Our goal is to lift the morphism $X^\wedge_M \to Y$ to a morphism $M \to P$ which induces a well defined morphism at the level of Čech nerves

$$\check{\mathrm{C}}(M_\bullet/X) \to \check{\mathrm{C}}(P_\bullet/Y).$$

In order to do so, we will construct a section of the smooth morphism

$$P' \to X^\wedge_M,$$

which induces, via composition, a morphism $M \to P$. The construction of the section is a local construction and it is precisely in this situation that we will use the setting of integral perfectoid $\mathbb{Z}_p$-algebras.

Lemma 2.0.1. *If such a section to $X^\wedge_M \to P'$ exists then it induces such a morphism at the level of Čech nerves*

$$\check{C}(M_\bullet/X) \to \check{C}(P_\bullet/Y)$$

2.1. **First reduction step.** Let $r\colon U \to X$ denote a morphism in $\mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_t, \mathrm{P}_{\mathrm{sm}})$ such that

$$U := \mathrm{Spf}\, A$$

is a derived affined $\mathbb{Z}_p$-adic scheme.

(Personal: Do we have to impose assumptions on the morphism $U \to X$, such as being Zariski open, étale?)

Proposition 2.1.1. *Up to shrienking U we can suppose that the pullback of the smooth covering map $p\colon M \to X$ along r can be realized as a composition of the form*

$$U \times \mathrm{Spf}\, R \xrightarrow{g} U \times \mathbb{T}^n \xrightarrow{\mathrm{pr}_1} U,$$

where g is an étale morphism. Moreover, g itself factors as a composite

$$U \times \mathrm{Spf}\, R \xrightarrow{g'} U \times W \xrightarrow{s} U \times \mathbb{T}^n,$$

where g' corresponds to a rational domain inclusion and s is a finite étale map.

Remark 2.1.2. The second part of the above statement might not be necessary for our purposes. However I am not sure yet.

Notation 2.1.3. We will denote by $\mathbb{Z}_p^{\mathrm{cyc}}$ the integral perfectoid $\mathbb{Z}_p$-algebra obtained from $\mathbb{Z}_p$ by adding all p-th roots of unity to $\mathbb{Z}_p$ and performing a p-adic completion.

Assuming Proposition 2.1.1 holds, one can further proceed to show that:

Lemma 2.1.4. *There exists a commutative diagram, (Personal: possibly cartesian and fpqc coverings?), of the form*

$$\begin{tikzcd}
U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \arrow[r] \arrow[d] & U \times \mathrm{Spf}\, R \arrow[d] \\
U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathbb{T}^n(p^\infty) \arrow[r] \arrow[dr] & U \times \mathbb{T}^n \arrow[d] \\
& U
\end{tikzcd} \tag{2.1.1}$$

in the ∞-category $\mathrm{Shv}_{\mathrm{fpqc}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p})$. Here

$$U^{\mathrm{cyc}} := U \times \mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}$$

and

$$\mathbb{T}(p^\infty) \to \mathbb{T}$$

denotes the perfectoid cover of the torus $\mathbb{T}$, given by adding p-th roots of unity to $\mathbb{Z}_p$ and p-th roots of the (free invertible) variables on $\mathbb{T}$.

Lemma 2.1.5. *The* fpqc*-covering $\mathbb{T}(p^\infty) \to \mathbb{T}$ is a $\Gamma := \prod_n \mathbb{Z}_p(1)$-torsor, for the* fpqc*-topology, which is compatible with the* fpqc*-$\mathbb{Z}_p(1)$-torsor*

$$\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}} \to \mathrm{Spf}\, \mathbb{Z}_p.$$

Proposition 2.1.6. *The pullback square of derived $\mathbb{Z}_p$-adic geometric stacks*

$$\begin{tikzcd}
U \times \mathrm{Spf}\, R \arrow[r] \arrow[d] & M \arrow[d, "p"] \\
U \arrow[r] & X
\end{tikzcd}$$

induces a canonical morphism $U \times (\mathrm{Spf}\, R)_{\mathrm{dR}} \to X^\wedge_M$. Moreover, if the morphism $U \to X$ is surjective or a covering (namely, an effective epimorphism of sheaves) then so it is the canonical morphism

$$U \times (\mathrm{Spf}\, R)_{\mathrm{dR}} \to X^\wedge_M.$$

2.2. **Further reductions.** Our initial situation can be thus translated as follows: consider the chain of pullback diagrams

$$\begin{tikzcd} \widetilde{P}'' \arrow[r] \arrow[d] & \widetilde{P}' \arrow[r] \arrow[d] & \widetilde{P} \arrow[r] \arrow[d] & P \arrow[d] \\ U \times \mathrm{Spf}\, R \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & X_M^\wedge \arrow[r] & Y \\ U \arrow[r] & X \end{tikzcd}.$$

The induced morphism $U \times \mathrm{Spf}\, R \to X_M^\wedge$ factors as

$$U \times \mathrm{Spf}\, R \to U \times (\mathrm{Spf}\, R)_{\mathrm{dR}} \to X_M^\wedge. \tag{2.2.1}$$

Consider the following *pullback* diagram in the ∞-category, $\mathrm{Shv}_{\mathrm{fpqc}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p})$, (Personal: is it really a pullback diagram?),

$$\begin{tikzcd} \mathrm{Spf}\, R_\infty \arrow[r] \arrow[d, "\theta"] & \mathrm{Spf}\, R \arrow[d] \\ \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty) \arrow[r] & (\mathrm{Spf}\, R)_{\mathrm{dR}}, \end{tikzcd}$$

where $\mathrm{A}_{\mathrm{dR}}(R_\infty)$ denotes the completion with respect to the kernel (structural) ring homomorphism

$$\theta \colon \mathrm{A}_{\mathrm{inf}}(R_\infty) \to R_\infty,$$

which, by the perfectoid nature of R_∞, is generated by a single elemet $\varepsilon \in \mathrm{A}_{\mathrm{inf}}(R_\infty$.

Lemma 2.2.1. *Let $R_\infty \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ denote a perfectoid $\mathbb{Z}_p$-algebra and consider the (structural) morphism*

$$\theta \colon \mathrm{A}_{\mathrm{inf}}(R_\infty) \to R_\infty.$$

Then θ exhibits $\mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty)$ as the de Rham stack *associated to $\mathrm{Spf}\, R_\infty$, $(\mathrm{Spf}\, R_\infty)_{\mathrm{dR}}$. Furthermore, if we are given a faithfully flat morphism $R \to R_\infty$, where R is a regular Noetherian $\mathbb{Z}_p$-adic algebra, then $\mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty) \to (\mathrm{Spf}\, R)_{\mathrm{dR}}$ is a fpqc morphism.*

Restricting further along the faithfully flat morphisms $\mathrm{Spf}\, R_\infty \to \mathrm{Spf}\, R$ and $\mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty) \to (\mathrm{Spf}\, R)_{\mathrm{dR}}$ one obtains thus a pullback square of the form

$$\begin{tikzcd} P_\infty \arrow[r] \arrow[d] & \widetilde{P}_\infty \arrow[d] \\ U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \arrow[r] & U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty) \end{tikzcd} \tag{2.2.2}$$

in the ∞-category $\mathrm{Shv}_{\mathrm{fpqc}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p})$.

Remark 2.2.2. Notice that, up to shrienking U mapping into X, one can assume that the smooth map $\widetilde{P}'' \to U \times \mathrm{Spf}\, R$, displayed in diagram (2.1.1), is actually of the form

$$U \times \mathrm{Spf}\, R \times \mathrm{Spf}\, S \to U \times \mathrm{Spf}\, R,$$

where $\mathrm{Spf}\, S$ admits a suitable étale morphism to a torus $\mathrm{Spf}\, S \to \mathbb{T}^m$.

For this reason, building upon what we have discussed so far, one can actually suppose that the diagram (2.2.2) is of the form

$$\begin{tikzcd} U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, S_\infty \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \arrow[r] \arrow[d] & \widetilde{P} \arrow[d] \\ U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \arrow[r] & U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty), \end{tikzcd}$$

where S_∞ is a perfectoid $\mathbb{Z}_p$-algebra which admits a faithfully flat morphism of the form $S \to S_\infty$. Moreover, as before, we can assume that S_∞ admits all p-th roots of unity and p-th roots of the (invertible free) variable overs $\mathbb{T}^m$ ((Todo: make this assertion more precise)). In particular, we deduce that S_∞ is naturally a $\mathbb{Z}_p^{\mathrm{cyc}}$-algebra.

2.3. **Existence and descent properties of local sections.** In order to construct the desired section one would be reduced to show the following:

Notation 2.3.1. We denote by $\widehat{\otimes}$ the p-complete tensor product in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$.

Proposition 2.3.2. *The* adic cotangent complex

$$\mathbb{L}^{\mathrm{ad}}_{S_\infty \widehat{\otimes}_p R_\infty / R_\infty} \simeq 0.$$

In particular, there are no obstructions to find a deformation of the morphism

$$U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, S_\infty \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \to U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty.$$

Corollary 2.3.3. *The deformation morphism* $\widetilde{q} \colon \widetilde{P} \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty)$ *of the projection*

$$U^{\mathrm{cyc}} \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, S_\infty \times_{\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty)$$

corresponds to the trivial deformation. In particular, it admits a section.

Theorem 2.3.4. *There exists a choice of a section* $s \colon \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty) \to \widetilde{P}$ *of* $\widetilde{q}$ *which is* Γ*-equivariant. In particular, it descends to a section of the morphism*

$$\widetilde{q} \colon \widetilde{P} \to U \times (\mathrm{Spf}\, R)_{\mathrm{dR}}$$

(Personal: In order for the section s constructed above to descend, does one need to show that

$$\widetilde{q} \colon \widetilde{P} \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty)$$

$\widetilde{q} \colon \widetilde{P} \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty)$ is also a Γ-torsor?)

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