A VAN EST THEOREM IN MIXED CHARACTERISTIC

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(Personal: PERSONAL COMMENTS ARE SHOWN!!!)

CONTENTS

1. GEOMETRIC CONTEXT

(Personal: Does one really needs to work locally for the fpqc topology or can we refine the topology further by dropping the quasi-compactness assumption?)

1.1. **Assumptions.** First of all we will define the context in which we are going to work through. Let us denote $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ the ∞-category of adic $\mathbb{Z}_p$-simplicial algebras, introduced in [1]. We consider the ∞-category $\mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ of *derived $\mathbb{Z}_p$-adic geometric stacks*. Suppose we are given a diagram of the form

$$\begin{array}{ccc} M & \xrightarrow{p} & X \\ \big\downarrow{\scriptstyle q} & & \\ Y & & \end{array}$$

where $M \in \mathrm{dfSch}$ is a smooth $\mathbb{Z}_p$-adic scheme and $X,\ Y \in \mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$. Throughout the text, p denotes a smooth surjective morphism and q a smooth morphism of derived $\mathbb{Z}_p$-adic geometric stacks. We assume further that Y satisfies *fpqc descent*. (Personal: The need to assume that Y satisfies fpqc descent boils down to the fact that the étale topology is not sufficient for many of our purposes. Namely, in order to compute the correct mapping spaces of derived $\mathbb{Z}_p$-adic geometric stacks one needs to pass to perfectoid coverings, in a similar vein as in BMS1. More precisely, given a Noetherian regular algebra $\mathbb{Z}_p$-algebra A, by a theorem of Bhatt, Iyegar and Ma one knows that there exists a faithfully flat morphism $A \to A_\infty$, where A_∞ is integral perfectoid. Furthermore, by a celebrated result of Abhyankar one can suppose further that A_∞ admits all p-th power roots of unity, thus living over the perfectoid covering $\mathbb{Z}_p^{\mathrm{cyc}}$ of $\mathbb{Z}_p$. However, it seems that, in general, the morphism $A \to A_\infty$ is not étale or even weak étale in the sense of [2]. Therefore, in order to be able to reduce our local computations to computations involving the perfectoid nature of suitable algebras, one needs to assume Y to satisfy fpqc descent. This is indeed the case when $Y = \mathrm{Perf},\ \mathrm{QCoh}^{\heartsuit},\ \mathrm{QCoh}, \mathrm{Bun}_G$, where G denotes a reductive group, X a formal scheme, $\mathrm{B}G$, for G a formal group scheme or a p-divisible group, $\mathcal{M}_{\mathrm{FM}}$ the moduli space of p-divisible groups, etc.)

Our goal is to prove a mixed characteristic analogue of a theorem of Van Est, in the context of (derived) differential geometry, proved in great generality by J. Nuiten using the Koszul duality for Lie algebroids, see [3]. (Todo: Recall the statement) (Todo: Check that the geometricity of the stack Y is really need or we can relax a bit the assumptions on Y, in order for the result to be compatible with $\mathcal{M}_{\mathrm{FM}}$, for example.)

2. The existence of a local section

We want to prove a connectivity statement for the canonical morphism of mapping spaces

$$\mathrm{Map}_{M/}(X, Y) \to \mathrm{Map}_{\mathrm{dSt}^{\mathrm{ad}}}(X^\wedge_M, Y^\wedge_M).$$

Suppose then that we are given a morphism of sheaves

$$X^\wedge_M \to Y.$$

Assume further that we have a smooth covering $\pi \colon P \to Y$, in the ∞-category $\mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ and form the pullback square

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ X^\wedge_M \arrow[r] & Y. \end{tikzcd}$$

In particular, the natural morphism $P' \to X^\wedge_M$ is a smooth covering. Moreover, we have a natural inclusion morphism

$$M \to X^\wedge_M.$$

Our goal is to lift the morphism $X^\wedge_M \to Y$ to a morphism $M \to P$ which induces a well defined morphism at the level of Čech nerves

$$\check{\mathrm{C}}(M_\bullet/X) \to \check{\mathrm{C}}(P_\bullet/Y).$$

In order to do so, we will construct a section of the smooth morphism

$$P' \to X^\wedge_M,$$

which induces, via composition, a morphism $M \to P$. The construction of the section is a local construction and it is precisely in this situation that we will use the setting of integral perfectoid $\mathbb{Z}_p$-algebras.

Lemma 2.0.1. *If such a section to $X^\wedge_M \to P'$ exists then it induces such a morphism at the level of Čech nerves*

$$\check{C}(M_\bullet/X) \to \check{C}(P_\bullet/Y)$$

2.1. First reduction step.

Let $r \colon U \to X$ denote a morphism in $\mathrm{dSt}^{\mathrm{ad}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}, \tau_t, \mathrm{P}_{\mathrm{sm}})$ such that

$$U := \mathrm{Spf}\, A$$

is a derived affined $\mathbb{Z}_p$-adic scheme.

(Personal: Do we have to impose assumptions on the morphism $U \to X$, such as being Zariski open, étale?)

Proposition 2.1.1. *Up to shrienking U we can suppose that the pullback of the smooth covering map $p \colon M \to X$ along r can be realized as a composition of the form*

$$U \times \mathrm{Spf}\, R \xrightarrow{g} U \times \mathbb{T}^n \xrightarrow{\mathrm{pr}_1} U,$$

where g is an étale morphism. Moreover, g itself factors as a composite

$$U \times \mathrm{Spf}\, R \xrightarrow{g'} U \times W \xrightarrow{s} U \times \mathbb{T}^n,$$

where g' corresponds to a rational domain inclusion and s is a finite étale map.

Remark 2.1.2. The second part of the above statement might not be necessary for our purposes. However I am not sure yet.

Notation 2.1.3. We will denote by $\mathbb{Z}_p^{\mathrm{cyc}}$ the integral perfectoid $\mathbb{Z}_p$-algebra obtained from $\mathbb{Z}_p$ by adding all p-th roots of unity to $\mathbb{Z}_p$ and performing a p-adic completion.

Assuming Proposition 2.1.1 holds, one can further proceed to show that:

Lemma 2.1.4. *There exists a commutative diagram, (Personal: possibly cartesian and fpqc coverings?), of the form*

$$\begin{tikzcd} U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\, R_\infty \arrow[r] \arrow[d] & U \times \mathrm{Spf}\, R \arrow[d] \\ U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathbb{T}^n(p^\infty) \arrow[r] \arrow[dr] & U \times \mathbb{T}^n \arrow[d] \\ & U \end{tikzcd} \tag{2.1.1}$$

in the ∞-category $\mathrm{Shv}_{\mathrm{fpqc}}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}^{\mathrm{ad}})$. Here

$$U^{\mathrm{cyc}} \coloneqq U \times \mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}}$$

and

$$\mathbb{T}(p^\infty) \to \mathbb{T}$$

denotes the perfectoid cover of the torus $\mathbb{T}$, given by adding p-th roots of unity to $\mathbb{Z}_p$ and p-th roots of the (free invertible) variables on $\mathbb{T}$.

Lemma 2.1.5. *The* fpqc*-covering $\mathbb{T}(p^\infty) \to \mathbb{T}$ is a $\Gamma \coloneqq \prod_n \mathbb{Z}_p(1)$-torsor, for the* fpqc*-topology, which is compatible with the* fpqc*-$\mathbb{Z}_p(1)$-torsor*

$$\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cyc}} \to \mathrm{Spf}\, \mathbb{Z}_p.$$

Proposition 2.1.6. *The pullback square of derived $\mathbb{Z}_p$-adic geometric stacks*

$$\begin{tikzcd} U \times \mathrm{Spf}\, R \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

induces a canonical morphism $U \times (\mathrm{Spf}\, R)_{\mathrm{dR}} \to X_M^\wedge$. Moreover, if the morphism $U \to X$ is surjective or a covering (namely, an effective epimorphism of sheaves) then so it is the canonical morphism

$$U \times (\mathrm{Spf}\, R)_{\mathrm{dR}} \to X_M^\wedge.$$

2.2. **Further reductions.** Our initial situation can be thus translated as follows: consider the chain of pullback diagrams

$$\begin{tikzcd} \widetilde{P}'' \arrow[r] \arrow[d] & \widetilde{P}' \arrow[r] \arrow[d] & \widetilde{P} \arrow[r] \arrow[d] & P \arrow[d] \\ U \times \mathrm{Spf}\, R \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & X_M^\wedge \arrow[r] & Y \\ U \arrow[r] & X \end{tikzcd} .$$

The induced morphism $U \times \mathrm{Spf}\, R \to X_M^\wedge$ factors as

$$U \times \mathrm{Spf}\, R \to U \times (\mathrm{Spf}\, R)_{\mathrm{dR}} \to X_M^\wedge. \tag{2.2.1}$$

Consider the following *pullback* diagram in the ∞-category, $\mathrm{Shv}_{\mathrm{fpqc}}(\mathcal{C}\mathrm{Alg}_{\mathbb{Z}_p}^{\mathrm{ad}})$, (Personal: is it really a pullback diagram?),

$$\begin{tikzcd} \mathrm{Spf}\, R_\infty \arrow[r] \arrow[d, "\theta"] & \mathrm{Spf}\, R \arrow[d] \\ \mathrm{Spf}\, \mathrm{A}_{\mathrm{dR}}(R_\infty) \arrow[r] & (\mathrm{Spf}\, R)_{\mathrm{dR}}, \end{tikzcd}$$

where $\mathrm{A}_{\mathrm{dR}}(R_\infty)$ denotes the completion with respect to the kernel (structural) ring homomorphism

$$\theta\colon \mathrm{A}_{\mathrm{inf}}(R_\infty) \to R_\infty,$$

which, by the perfectoid nature of R_∞, is generated by a single elemet $\varepsilon \in \mathrm{A}_{\mathrm{inf}}(R_\infty$.

Lemma 2.2.1. *Let $R_\infty \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$ denote a perfectoid $\mathbb{Z}_p$-algebra and consider the (structural) morphism*

$$\theta \colon \mathrm{A}_{\mathrm{inf}}(R_\infty) \to R_\infty.$$

Then θ exhibits $\mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty)$ *as the* de Rham stack *associated to* $\mathrm{Spf}\,R_\infty$, $(\mathrm{Spf}\,R_\infty)_{\mathrm{dR}}$. *Furthermore, if we are given a faithfully flat morphism $R \to R_\infty$, where R is a regular Noetherian $\mathbb{Z}_p$-adic algebra, then* $\mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty) \to (\mathrm{Spf}\,R)_{\mathrm{dR}}$ *is a fpqc morphism.*

Restricting further along the faithfully flat morphisms $\mathrm{Spf}\,R_\infty \to \mathrm{Spf}\,R$ and $\mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty) \to (\mathrm{Spf}\,R)_{\mathrm{dR}}$ one obtains thus a pullback square of the form

(2.2.2)
$$\begin{tikzcd}
P_\infty \arrow[r] \arrow[d] & \widetilde{P}_\infty \arrow[d] \\
U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,R_\infty \arrow[r] & U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty)
\end{tikzcd}$$

in the ∞-category $\mathrm{Shv}_{\mathrm{fpqc}}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p})$.

Remark 2.2.2. Notice that, up to shrienking U mapping into X, one can assume that the smooth map $\widetilde{P}'' \to U \times \mathrm{Spf}\,R$, displayed in diagram (2.1.1), is actually of the form

$$U \times \mathrm{Spf}\,R \times \mathrm{Spf}\,S \to U \times \mathrm{Spf}\,R,$$

where $\mathrm{Spf}\,S$ admits a suitable étale morphism to a torus $\mathrm{Spf}\,S \to \mathbb{T}^m$.

For this reason, building upon what we have discussed so far, one can actually suppose that the diagram (2.2.2) is of the form

$$\begin{tikzcd}
U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,S_\infty \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,R_\infty \arrow[r] \arrow[d] & \widetilde{P} \arrow[d] \\
U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,R_\infty \arrow[r] & U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty),
\end{tikzcd}$$

where S_∞ is a perfectoid $\mathbb{Z}_p$-algebra which admits a faithfully flat morphism of the form $S \to S_\infty$. Moreover, as before, we can assume that S_∞ admits all p-th roots of unity and p-th roots of the (invertible free) variable overs $\mathbb{T}^m$ ((Todo: make this assertion more precise)). In particular, we deduce that S_∞ is naturally a $\mathbb{Z}_p^{\mathrm{cyc}}$-algebra.

2.3. **Existence and descent properties of local sections.** In order to construct the desired section one would be reduced to show the following:

Notation 2.3.1. We denote by $\widehat{\otimes}$ the p-complete tensor product in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{\mathbb{Z}_p}$.

Proposition 2.3.2. *The* adic cotangent complex

$$\mathbb{L}^{\mathrm{ad}}_{S_\infty \widehat{\otimes}_p R_\infty / R_\infty} \simeq 0.$$

In particular, there are no obstructions to find a deformation of the morphism

$$U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,S_\infty \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,R_\infty \to U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,R_\infty.$$

Corollary 2.3.3. *The deformation morphism $\widetilde{q} \colon \widetilde{P} \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty)$ of the projection*

$$U^{\mathrm{cyc}} \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,S_\infty \times_{\mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,R_\infty \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty)$$

corresponds to the trivial deformation. In particular, it admits a section.

Theorem 2.3.4. *There exists a choice of a section $s \colon \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty) \to \widetilde{P}$ of $\widetilde{q}$ which is Γ-equivariant. In particular, it descends to a section of the morphism*

$$\widetilde{q} \colon \widetilde{P} \to U \times (\mathrm{Spf}\,R)_{\mathrm{dR}}$$

(Personal: In order for the section s constructed above to descend, does one need to show that

$$\widetilde{q} \colon \widetilde{P} \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty)$$

$\widetilde{q} \colon \widetilde{P} \to U^{\mathrm{cyc}} \times_{\mathbb{Z}_p^{\mathrm{cyc}}} \mathrm{Spf}\,\mathrm{A}_{\mathrm{dR}}(R_\infty)$ is also a Γ-torsor?)

References

[1] Jorge António. p-adic derived formal geometry and derived raynaud localization theorem. *arXiv preprint arXiv:1805.03302*, 2018.
[2] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *arXiv preprint arXiv:1309.1198*, 2013.
[3] Joost Nuiten. Koszul duality for lie algebroids. *arXiv preprint arXiv:1712.03442*, 2017.

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