MODULI OF ℓ-ADIC REPRESENTATIONS (CONTINUATION)

JORGE ANTÓNIO

ABSTRACT. In this text we prove that if we take G is a more general profinite group, for example an absolute Galois group, G, the moduli $\mathrm{LocSys}^{\Gamma}_{G,n}$ is representable by a rigid ℓ-analytic space, provided we fix the inertia action at infinity.

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INTRODUCTION

1. PREVIOUS WORKS AND RAPPELS

2. SETTING THE STAGE

2.1. **Recall on the monodromy of (local) inertia.** Let K be a local field, whose residual characteristic we suppose different from ℓ. Let $\bar{K}$ be an algebraic closure of K and define $\mathrm{G}_K := \mathrm{Gal}\left(\overline{K}/K\right)$ to be the absolute Galois group of K. Suppose we are given a finite Galois extension L of K, then we can consider its Galois group $\mathrm{Gal}\,(L/K)$ together with its inertia subgroup $I_{L/K}$ which is the subgroup spanned by those elements of $\mathrm{Gal}\,(L/K)$ which fix $\mathcal{O}_L/\mathfrak{m}_L$. The subgroup $I_{L/K}$ can be equivalently defined as the kernel of the surjective group homomorphism $\mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k)$, thus we have a short exact sequence

$$1 \to I_{L/K} \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k) \to 1, \tag{1}$$

where l and k denote the residue fields of both L and K, respectively. We define the (absolute) inertia group of $K$to be the inverse limit

$$I_K := \lim_{L/K \text{ finite}} I_{L/K}.$$

It is canonically a subgroup of G_K. Another important ingredient for us is wild inertia. Given L/K as above we can consider the subgroup $P_{L/K}$ of $I_{L/K}$ spanned by those elements which act trivially on $\mathcal{O}_L/\mathfrak{m}_L^2$ and we call it the wild inertia group (or simply wild inertia). Define then $P_K := \lim_{L \text{ finite}} P_{L/K}$, the absolute wild inertia group of K. It is a consequence of **??** that we have a short exact sequence

$$1 \to I_K \to \mathrm{G}_K \to G_k \to 1 \tag{2}$$

where $G_k := \mathrm{Gal}(\bar{k}/k)$ and $\bar{k}$ is defined as the residue field of $\overline{K}$. Notice that P_K is a normal subgroup of I_K and we have a short exact sequence of groups:

$$1 \to P_K \to I_K \to I_K/P_K \to 1. \tag{3}$$

The wild inertia group P_K is generally huge but it turns out that the quotient I_K/P_K is much more amenable:

Proposition 2.1.1. [**?**, Corollary 13] *The quotient I_K/P_K is canonically isomorphic to $\hat{\mathbb{Z}}'(1)$, where the latter denotes the profinite group $\prod_{q \neq p} \mathbb{Z}_q(1)$, where $p = \mathrm{char}(k)$ is the residual characteristic.*

Define $P_{K,\ell}$ to be the inverse image of $\prod_{q\neq \ell,p} \mathbb{Z}_q$ in I_K. We have thus an exact sequence of groups

$$1 \to P_K \to P_{K,\ell} \to \prod_{q\neq \ell,p} \mathbb{Z}_q \to 1.$$

Define $G_{K,\ell}$ to denote the quotient $G_K/P_{K,\ell}$ and notice that we have short exact sequences of groups as,

$$1 \to P_{K,\ell} \to \mathrm{G}_K \to G_{K,\ell} \to 1,$$

Date: January 16, 2019.
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$$1 \to \mathbb{Z}_\ell(1) \to G_{K,\ell} \to G_k \to 1.$$

As a consequence, the quotient $G_{K,\ell}$ is topologically of finite type.

Suppose we are now given a continuous representation

$$\rho\colon \mathrm{G}_K \to \mathrm{GL}_n(\mathbb{Q}_\ell),$$

(we can also consider ρ with values in a finite extension E of $\mathbb{Q}_\ell$, without changing the exposition). Up to conjugation we can suppose that ρ actually preserves a lattice inside the vector space underlying ρ, thus we have a commutative diagram

$$\begin{tikzcd} G_K \arrow[r, "\rho"] \arrow[dr] & \mathrm{GL}_n(\mathbb{Z}_\ell) \arrow[d] \\ & \mathrm{GL}_n(\mathbb{Q}_\ell) \end{tikzcd}$$

And $\rho\,(G_K)$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_\ell)$. We have moreover, a short exact sequence

$$1 \to N_1 \to \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell) \to 1,$$

where N_1 is a pro-ℓ-subgroup of $\mathrm{GL}_n(\mathbb{Z}_\ell)$, (it is the subgroup of matrices congruent to Id mod ℓ). As $P_{K,\ell}$ is a profinite group which is, by construction, an inverse limit of finite groups of order prime to ℓ we must have, necessarily, $\rho\,(G_K) \cup N_1 = \{1\}$. Therefore $\rho(G_K)$ injects into $\mathrm{GL}_n(\mathbb{F}_\ell)$ which implies that the former is a finite group, thus the wild inertia of G_K acts on $\mathrm{GL}_n(\mathbb{Q}_\ell)$ via a finite quotient. It is, in this way, natural to consider the moduli of ℓ-adic representations of G_K where we fix the level by which $P_{K,\ell}$ acts on, i.e., the finite quotient of $P_{K,\ell}$ over which $\rho_{|P_{K,\ell}}$ should factor.

2.2. **Definition of the functor.** Let X be a smooth curve not necessarily proper. Its étale fundamental group $\pi_1^{\text{ét}}(X)$ is a profinite group not necessarily of topological finite generation. Let $\overline{X}$ be a smooth proper curve such that we have an open immersion $X \hookrightarrow \overline{X}$ and let $D = \overline{X} \backslash X = \coprod_i x_i$ be the divisor at infinity. We have an exact sequence of groups of the form

$$1 \to \prod_i \mathrm{Gal}(\overline{K}_i/K_i) \to \pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}\left(\overline{X}\right) \to 1, \text{checkthis!}$$

where K_i denotes the residue field of $x_i \hookrightarrow \overline{X}$, and $\overline{K}_i$ a fixed algebraic closure. The field K_i is a local field as in the previous section, so the discussion of it goes through and moreover we have that $\pi_1^{\text{ét}}\left(\bar{(}X)\right)$ is topologically of finite generation.

In [?] it is proven that the moduli spaces $\mathrm{Hom}_{\text{cont}}\left(\pi_1^{\text{ét}}(\overline{X}), \mathrm{GL}_n\,(-)\right)$ and $\mathrm{LocSys}_{\ell,n}\,\left(\overline{X}\right)$ parametrizing continuous group homomorphims $\pi_1^{\text{ét}}(\overline{X}) \to \mathrm{GL}_n(A)$ and pro-étale local systems on $\overline{X}$ are representable by (derived) $\mathbb{Q}_\ell$-analytic stacks. We cannot expect the analogue functors for X are representable by a geometric $\mathbb{Q}_\ell$-analytic space as the group $\pi_1^{\text{ét}}(X)$ is too big, as it contains the inertia at infinity. Nevertheless, we can give a definition of representable functors by imposing reasonable conditions at infinity, such imposition is something natural to do after the discussion in the previous section.

For simplicity of exposition we assume that D consists of only one point $x \in \overline{X}$, the general construction goes through in the same fashion. We call K the residue field of x and we let $G_K := \mathrm{Gal}(\overline{K}_x/K_x)$ as in the previous section. Let Γ be a finite quotient group of $P_{K,\ell}$ and define the functor

$$\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(-)\right) : \mathrm{Afd}^{\mathrm{op}}_{\mathbb{Q}_\ell} \to \mathrm{Set},$$

given by the assignement

$$A \in \mathrm{Afd}^{\mathrm{op}}_{\mathbb{Q}_\ell} \mapsto \mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\right),$$

where we consider $\mathrm{GL}_n(A)$ as a topological group making use of the natural topology on the affinoid algebra A and $\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\right)$ denotes the fiber product of the diagram

$$\begin{tikzcd} \mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\right) \arrow[r] \arrow[d] & \mathrm{Hom}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\right) \arrow[d] \\ \mathrm{Hom}_{\text{cont}}\left(P_{K,\ell}, \mathrm{GL}_n(A)\right)_\Gamma \arrow[r] & \mathrm{Hom}_{\text{cont}}\left(P_{K,\ell}, \mathrm{GL}_n(A)\right), \end{tikzcd}$$

where the right vertical map is the restriction along the inclusion $P_{K,\ell} \hookrightarrow G_K$ and the bottom left term denotes the set of those continuous group homomorphisms $P_{K,\ell} \to \mathrm{GL}_n(A)$ that factor through the finite quotient Γ when restricted to a group homomorphism $P_{K,\ell} \to \mathrm{GL}_n(A)$.

Theorem 2.2.1. *The functor* $\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{ét}(X), \mathrm{GL}_n\right) : \mathrm{Afd}^{\text{op}}_{\mathbb{Q}_\ell} \to \mathrm{Set}$ *is representable by a rigid* $\mathbb{Q}_\ell$*-analytic space.*

Proof. Choose a continuous group homomorphism $\varphi\colon \hat{\mathbb{Z}}^r \to \pi_1^{\text{ét}}(X)$ such that the image of the (chosen) topological generators of the former under φ form a set of generators for Γ seen as a quotient of $P_{K,\ell}$ and for the (topologically finite generated) quotient $G_{K,\ell}$. Restriction under φ induces a closed immersion of functors $\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n\right) \hookrightarrow \mathrm{Hom}_{\text{cont}}\left(\hat{\mathbb{Z}}^r, \mathrm{GL}_n\right)$. Thanks to [**?**] the latter is representable by a rigid $\mathbb{Q}_\ell$-analytic space and therefore also $\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n\right)$. □

The functor $\mathrm{Hom}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n\right)$ admits a natural action of the analytified general linear group, $\mathrm{GL}_n^{\text{an}}$, given by conjugation of morphisms $\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$. Moreover, this action preserves the condition that a given such morphism ρ factors through a finite quotient Γ, when restricted to $P_{K,\ell}$. Therefore, such action descends to the rigid $\mathbb{Q}_\ell$-analytic space $\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n\right)$.

Definition 2.1. Define $\mathrm{LocSys}^{\Gamma}_{\ell,n} \in \mathrm{St}\left(\mathrm{Afd}^{\text{op}}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, P_{\text{sm}}\right)$ to be the quotient stack $[\mathrm{Hom}^{\Gamma}_{\text{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n\right)/\mathrm{GL}_n^{\text{an}}]$, with respect to the rigid $\mathbb{Q}_\ell$-analytic context, where we consider the conjugation action as above.

Thanks to **??** we obtain the following important result:

Corollary 2.2.1. *The stack* $\mathrm{LocSys}^{\Gamma}_{\ell,n}$ *is representable by a rigid* $\mathbb{Q}_\ell$*-analytic stack.*

Remark 2.1. **??** is an important result as it tells us that $\mathrm{LocSys}^{\Gamma}_{\ell,n}$ is an object with sufficient geometric behaviour. The reader should think of it as an analogue of an Artin stack, in the context of algebraic geometry.

2.3. **Higher dimensional case.** Let X now be a smooth scheme over a field k. One can proceed similarly to the case of curves to show the representability of the moduli of ℓ-adic local systems on X. In order to do so, one also needs to bound the ramification at infinity. It turns out that method is very similar to the one used in dimension 1, and can be used to treat homogeneously both cases. Before going through constructions we need to recall the notion of the tame fundamental group of a scheme.

It corresponds to the group of automorphisms of the functor fibre, when restricted to those tamely ramified at infinity (finite) étale coverings of X. More precisely, let $\overline{X}$ be a smooth compactification of X and define the tame fundamental group of X to be the inverse limit over all finite étale coverings

Let $f\colon Y \to X$ be a finite Galois covering, with (finite) group of automorphism Γ. We define,

$$\mathrm{Hom}_{\text{cont},f}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(-)\right) : \mathrm{Afd}^{\text{op}}_{\mathbb{Q}_\ell} \to \mathrm{Set},$$

as the functor which associates to each ℓ-affinoid algebra A the set $\mathrm{Hom}_{\text{cont},f}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(-)\right)$ of those continuous group homomorphisms $\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$ such that when restricted, under f, to $\pi_1^{\text{ét}}(Y)$ factor through the tame fundamental group $\pi_1^{\text{tame}}(Y)$. It is a fact that the latter is topologically of finite generation:

Proposition 2.3.1. *Let* Y *be a smooth variety over a field. Then the tame fundamental group of* Y*,* $\pi_1^{\text{tame}}(Y)$ *is topologically of finite generation.*

Proof. This is a formal consequence of [**?**, Appendix, Theorem 1.2], indeed one can find a smooth, geometrically connected curve C over the base field such that the induced morphism at the level of fundamental groups

$$\pi_1^{\text{ét}}(C) \to \pi_1^{\text{ét}}(X) \twoheadrightarrow \pi_1^{\text{tame}}(X)$$

is surjective and factors through the tame quotient $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{tame}}(X)$, the latter profinite group being topologically of finite generation, Need reference here. One thus concludes that also $\pi_1^{\text{tame}}(X)$ is topological of finite generation. □

Using **??** together with the reasoning done in the case of relative dimension 1, we obtain the following important results:

Theorem 2.3.1. *The functor* $\mathrm{Hom}_{\mathrm{cont},f}\left(\pi_1^{\acute{e}t}(X), \mathrm{GL}_n(-)\right) : \mathrm{Afd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathrm{Set}$ *is representable by a rigid* $\mathbb{Q}_\ell$*-analytic space. Moreover, its quotient stack, under the conjugation action of* $\mathrm{GL}_n^{\mathrm{an}}$ *is representable by a* $\mathbb{Q}_\ell$*-analytic stack.*

Definition 2.2. Define $\mathrm{LocSys}_{\ell,n}^{\Gamma} \in \mathrm{St}\left(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, P_{\mathrm{sm}}\right)$ as the quotient stack of $\mathrm{Hom}_{\mathrm{cont},f}\left(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(-)\right)$ under the conjugation action of $\mathrm{GL}_n^{\mathrm{an}}$ on it.

Corollary 2.3.1. *The stack* $\mathrm{LocSys}_{\ell,n}^{\Gamma}$ *is representable by a* $\mathbb{Q}_\ell$*-analytic stack.*

Remark 2.2. This construction works uniformly in all dimensions and it turns out that it is equivalent to our previous discussion in the case of curves.

3. DERIVED STRUCTURE

Let X be a smooth scheme over a separably closed field of positive characteristic prime to ℓ. To X we can attach its étale homotopy type $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}\left(\mathcal{S}^{\mathrm{fc}}\right)$, which is a pro-object in the ∞-category of finite constructible spaces, $\mathcal{S}^{\mathrm{fc}}$, see [**?**] for more details. We can consider the stack in the context of derived $\mathbb{Q}_\ell$-analytic geometry,

$$d\mathrm{LocSys}_{\ell,n}(X) \colon \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathcal{S},$$

given informally on objects, by the formula

$$A \in \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \mapsto \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{grp}}(\mathcal{C})}\left(\mathrm{Sh}^{\text{ét}}(X), \mathrm{BGL}_n(A)\right),$$

where $\mathcal{C}$ is the ∞-category of ind-pro-objects in the ∞-category $\mathcal{S}$.

The space $\mathrm{Map}_{\mathrm{Mon}(\mathcal{C})}\left(\mathrm{Sh}^{\text{ét}}(X), \mathrm{BGL}_n(A)\right)$ corresponds precisely to the ∞-groupoid of continuous representations of $\mathrm{Sh}^{\text{ét}}(X)$ with values in $\mathrm{BGL}_n(A)$ equipped with its canonical *ind-pro-topology*, see [**?**, section 4.4] for more details about such notions. We have moreover, that

$$t_0\left(d\mathrm{LocSys}_{\ell,n}(X)(A)\right) \in \mathrm{St}\left(d\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, P_{\mathrm{sm}}\right)$$

is naturally equivalent to the stack $\mathrm{LocSys}_{\ell,n}(X) \colon \mathrm{Afd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathcal{S}$ introduced previously. The derived stack $d\mathrm{LocSys}_{\ell,n}$ is not representable as its underlying 0-truncation is not representable by a $\mathbb{Q}_\ell$-analytic stack. However, it admits a tangent complex. Given $\rho \in d\mathrm{LocSys}_{\ell,n}(X)(A)$, for some $A \in d\mathrm{Afd}_{\mathbb{Q}_\ell}$ we can compute its tangent complex at ρ $\mathbb{T}_{d\mathrm{LocSys}_{\ell,n}(X),\rho}$ as:

$$\mathbb{T}_{d\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq C^*_{\text{ét}}\left(X, \mathrm{Ad}\left(\rho\right)\right)[1] \in \mathrm{Mod}_A,$$

where $\mathrm{Ad}\left(\rho\right) \simeq \rho \otimes \rho^\vee$ denotes the adjoint representation and $C^*_{\text{ét}}\left(X, \mathrm{Ad}\left(\rho\right)\right) \in \mathrm{Mod}_A$ the (pro-)étale cohomology of X with $\mathrm{Ad}(\rho)$-coefficients.

Given $f \colon Y \to X$ a finite Galois covering of X we can define $d\mathrm{LocSys}_{\ell,n}^{\Gamma}$ as follows:

Let Γ be the (finite) group of automorphisms of the étale covering $f \colon Y \to X$ and consider the following fiber sequence of profinite spaces:

$$\Gamma \to \mathrm{Sh}^{\text{ét}}(Y) \to \mathrm{Sh}^{\text{ét}}(X). \tag{4}$$

Define now the wild homotopy type of a general scheme Y by means of a fiber sequence of the form:

$$\mathrm{Sh}^{w}(Y) \to \mathrm{Sh}^{\text{ét}}(Y) \to \mathrm{B}\pi_1^{\mathrm{tame}}(Y), \tag{5}$$

where the displayed map $\mathrm{Sh}^{\text{ét}}(Y) \to \mathrm{B}\pi_1^{\mathrm{tame}}(Y)$ in **??** is the map induced at the level of profinite homotopy types of the (continuous) group epimorphism:

$$\pi_1^{\text{ét}}(Y) \to \pi_1^{\mathrm{tame}}(Y),$$

introduced above.

Remark 3.1. Notice that, by construction, we have natural (continuous) group isomorphisms

$$\pi_i\left(\mathrm{Sh}^{w}(Y)\right) \simeq \pi_i\left(\mathrm{Sh}^{\text{ét}}(Y)\right)$$

for $i > 1$ and $\pi_1 \mathrm{Sh}^{w}(Y) \simeq \pi_1^{w}(Y)$, the wild ramification étale fundamental group of Y.

We can thus consider the functor $d\mathrm{LocSys}^w_{\ell,n}(Y)\colon d\mathrm{Afd}^{\mathrm{op}}_{\mathbb{Q}_\ell} \to \mathcal{S}$ given informally by the association

$$A \in d\mathrm{Afd}^{\mathrm{op}}_{\mathbb{Q}_\ell} \mapsto \mathrm{Map}_{\mathrm{cont}}\left(\mathrm{Sh}^w(Y), \mathrm{BGL}_n(A)\right),$$

which we refer to the derived moduli stack of wild (pro)-étale local systems on Y. We have thus obvious restriction functors

$$d\mathrm{LocSys}_{\ell,n}(Y) \to d\mathrm{LocSys}^w_{\ell,n}(Y),$$

Suppose now we have a surjective continuous group homomorphism $q\colon \pi_1^w(Y) \to \Gamma$, where Γ is a finite group. Such morphism induces then a well defined morphism (up to contractible indeterminacy)

$$\varphi\colon \mathrm{Sh}^w(Y) \to \mathrm{B}\Gamma,$$

which induces an obvious restriction functor $\varphi^*\colon \mathrm{LocSys}^w_{\ell,n}(Y) \to \mathrm{LocSys}_{\ell,n}(\Gamma)$, where $\mathrm{LocSys}_{\ell,n}(\Gamma)\colon d\mathrm{Afd}_{\mathbb{Q}_\ell} \to \mathcal{S}$ is the functor informally defined by the association

$$A \in d\mathrm{Afd}_{\mathbb{Q}_\ell} \mapsto \mathrm{Map}_{\mathrm{cont}}\left(\mathrm{B}\Gamma, \mathrm{BGL}_n(A)\right).$$

Remark 3.2. By unwinding the definitions it follows that, for each $A \in d\mathrm{Afd}_{\mathbb{Q}_\ell}$ we have a natural equivalence of mapping spaces

$$\mathrm{Map}_{\mathcal{S}}\left(\mathrm{B}\Gamma, \mathrm{BGL}_n(A^{\mathrm{alg}})\right),$$

where $A \in \mathrm{CAlg}_{\mathbb{Q}_\ell}$ denotes the underlying algebra associated to $A \in d\mathrm{Afd}_{\mathbb{Q}_\ell}$.

Definition 3.1. Let X be a smooth scheme over a field K. We define the (derived) moduli stack of derived (pro)-étale local systems on X whose wild ramification is bounded by Γ at infinity as the fiber product

$$d\mathrm{LocSys}^\Gamma_{\ell,n}(X) := d\mathrm{LocSys}_{\ell,n}(X) \times_{d\mathrm{LocSys}_{\ell,n}(\mathrm{B}\Gamma)} d\mathrm{LocSys}^w_{\ell,n}(X)$$

Proposition 3.0.1. *Let $q\colon \pi_1^w(X) \to \Gamma$ be a surjective continuous group homomorphism such that Γ is finite. Then the 0-truncation of $d\mathrm{LocSys}^\Gamma_{\ell,n}(X)$ is naturally equivalent to $\mathrm{LocSys}^\Gamma_{\ell,n}(X)$ and therefore it is representable by a $\mathbb{Q}_\ell$-analytic moduli stack.*

The novelty about the derived version $d\mathrm{LocSys}^\Gamma_{\ell,n}(X)$ is that we know how to compute its tangent complex and therefore understand the obstruction theory of those continuous group representation

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n\left(\bar{\mathbb{Q}}_\ell\right)$$

Proposition 3.0.2. *Let $A \in d\mathrm{Afd}_{\mathbb{Q}_\ell}$ and $\rho \in d\mathrm{LocSys}^\Gamma_{\ell,n}(X)(A)$, the tangent complex of $d\mathrm{LocSys}^\Gamma_{\ell,n}(X)$ at ρ is naturally equivalent to*

$$\mathbb{T}_{\mathrm{LocSys}^\Gamma_{\ell,n},\rho} \simeq C^*\left(X, \mathrm{Ad}\left(\rho\right)\right)[1] \in \mathrm{Mod}_A.$$

Proof. □

Thanks to **??** we know how to compute the tangent complex of $d\mathrm{LocSys}^\Gamma_{\ell,n}$, however we care most about its *dual*, the cotangent complex. Therefore, we need $\mathbb{T}_{\mathrm{LocSys}^\Gamma_{\ell,n},\rho}$ to be a dualizable object in the corresponding derived ∞-category of modules over the coefficient (derived) ring. We now state the conditions on ρ for this to hold:

Theorem 3.0.1. *The (derived) moduli stack $d\mathrm{LocSys}^\Gamma_{\ell,n}(X)$ is representable by a derived $\mathbb{Q}_\ell$-analytic stack.*

4. Comparison statements

Let G be a profinite group and let $\rho\colon G \to \mathrm{GL}_n(K)$ be a continuous ℓ-adic representation of G, where K is a finite extension of $\mathbb{Q}_\ell$. We can define the functor of derived deformation of ρ,

$$\mathrm{Def}_\rho\colon \mathrm{CAlg}^{\mathrm{art}}_K \to \mathcal{S},$$

where $\mathrm{CAlg}^{\mathrm{art}}_K$ denotes the ∞-category of Artinian simplicial K-algebras. The object Def_ρ is an example of an (algebraic) formal moduli problem. Thanks to our previous considerations on the tangent complex for $\mathrm{LocSys}_{\ell,n}(G)$ we can now easily compute the tangent complex of Def_ρ as

$$\mathbb{T}_{\mathrm{Def}_\rho} \simeq C^*_{\mathrm{cont}}\left(G, \mathrm{Ad}(\rho)\right)[1] \in \mathrm{Mod}_K,$$

where $C^*_{\mathrm{cont}}\left(G, \mathrm{Ad}(\rho)\right)$ denotes the object of continuous group cohomology with $\mathrm{Ad}(\rho)$-coefficients. It follows now that we have

Theorem 4.0.1. *The formal moduli problem* Def_ρ *is pro-representable as a functo* $\mathrm{CAlg}_K^{\mathrm{art}} \to \mathcal{S}$.

Proof. □

Thanks to **??** one can naturally identify

$$\mathrm{Def}_\rho \simeq \mathrm{Map}_{\mathrm{CAlg}_K^{\mathrm{art}}}\left(\lim_n \mathcal{R}_n, -\right) : \mathrm{CAlg}_K^{\mathrm{art}} \to \mathcal{S},$$

where each $\mathcal{R}_n \in \mathrm{CAlg}_K^{\mathrm{art}}$ and $\lim_n \mathcal{R}_n$ denotes the corresponding object of the ∞-category $\mathrm{Pro}\left(\mathrm{CAlg}_K^{\mathrm{art}}\right)$. One can then naturally consider Def_ρ as a functor defined on the ∞-category of derived formal schemes over K, which we denote dfSch_K, and in such case, thanks to the Spf-construction, it is representable by the derived formal scheme $\mathrm{Spf}\,(\lim_n \mathcal{R}_n)$.

Construction 4.0.2. *Let* $\mathcal{R}'_n \in \mathrm{CAlg}_{\mathcal{O}_K}^{\mathrm{art}}$ *denote a simplicial Artinian algebra considered as a formal model over* $\mathcal{O}_K$ *for* $\mathcal{R}_n \in \mathrm{CAlg}_K^{\mathrm{art}}$. *We can form then the pro-simplicial algebra object*

$$\lim_{m,n} \mathcal{R}'_{m,n} := \lim_{m,n} \mathcal{R}'_n/(\pi^m) \in \mathrm{CAlg}_{\mathcal{O}_K}^{\mathrm{art}},$$

where π *is a uniformizer for* K, *fixed from the beginning of our discussion and* $\mathcal{R}'_n/(\pi^m)$ *denotes the following homotopy pushout:*

$$\begin{array}{ccc} \mathcal{R}'_n[t] & \xrightarrow{t\mapsto 0} & \mathcal{R}'_n \\ \big\downarrow{\scriptstyle t\mapsto \pi^m} & & \big\downarrow \\ \mathcal{R}'_n & \longrightarrow & \mathcal{R}'_{n,m}. \end{array}$$

We can thus consider $\mathcal{R}' := \lim_{m,n} \mathcal{R}'_n/(\pi^m)$ *as a p-complete simplicial* $\mathcal{O}_K$*-algebra, i.e.,* $\mathcal{R}' \in \mathrm{CAlg}_{\mathcal{O}_K}^{\mathrm{ad}}$. *Therefore,* $\mathrm{Spf}\mathcal{R}' \in \mathrm{dfSch}_K$ *admits a generic fiber, which we denote*

$$\mathrm{Def}_\rho^{\mathrm{rig}} := (\mathrm{Spf}\mathcal{R}')^{\mathrm{rig}} \in \mathrm{dAn}_K.$$

By construction, we have a natural map $\Psi(\rho)\colon \mathrm{Def}_\rho^{\mathrm{rig}} \to \mathrm{LocSys}_{\ell,n}(G)$ *and passing to the colimit we obtain a natural morphism:*

$$\Psi := \coprod_{\{\rho\colon G \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)\}} : \mathrm{Def}_\rho^{\mathrm{rig}} \to \mathrm{LocSys}_{\ell,n}(G) \tag{6}$$

in the ∞*-category of derived* $\mathbb{Q}_\ell$*-analytic stacks*

Proposition 4.0.1. *The morphism of derived* $\mathbb{Q}_\ell$*-analytic stacks displayed in* **??** *is an open immersion.*

Proof. □

Somewhat surprisingly, the morphism Ψ is not an epimorphism of derived stacks, as the following example illustrates:

Example 4.0.3.

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JORGE ANTÓNIO, IMT TOULOUSE, 118 RUE DE NARBONNE 31400 TOULOUSE

E-mail address: `jtiago1993@gmail.com`