Math 243 Discrete Mathematics **Homework 6**

1. In each question part below, I will list a set S and a rule that defines a relation R on S as follows: $(m, n) \in R$ if m and n satisfy the given rule. For each set and rule, do the following five things:

   i List the order pairs in the relation.

   ii Draw the associated arrow diagram.

   iii Give the matrix representation of the relation. Be sure to label the rows and columns with the elements of S.

   iv Identify which properties the relation has: reflexive, antireflexive, or neither; symmetric, antisymmetric, or neither; transitive or not transitive. Be sure to explain your answers.

   v Decide whether the relation is an equivalence relation. If so, identify the equivalence classes.

   (a) $S = \{4,9,17\}$; $m \geq n$.

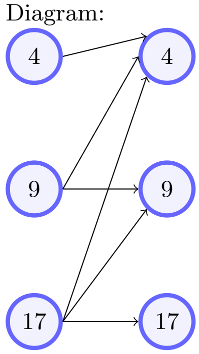
   (b) $S = \{0,2,5\}$; $mn = 0$.

   (c) $S = \{1,2,6,7,11\}$; $m \equiv n \pmod 3$

   <u>Answers for (a):</u>

   - R = $\{(m, n) : m \geq n$ and $m, n \in S\}$
     $\Rightarrow \{(17,4),(9,4),(17,9),(4,4),(9,9),(17,17)\}$

   - Diagram:



   - Matrix:

$$\begin{array}{c} \\ 4 \\ 9 \\ 17 \end{array}\begin{array}{c} \begin{array}{ccc} 4 & 9 & 17 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \text{ Matrix} = \begin{cases} 1 : (m,n) \in R \\ 0 : otherwise \end{cases}$$

   -
     - R is reflexive as $(m, n) \in R \; \forall \; m \in S$
     - R is anti symmetric. As if(a,b) $\in R$ and (b,a) $\in R$ then a = b for:

       a = b = 4

       a = b = 9

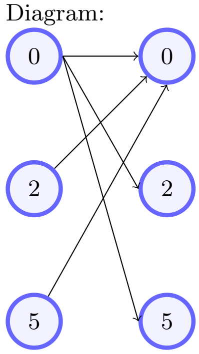
       a = b = 17
     - R is transitive. As if (a,b)$\in R$ and (b,c)$\in R$ then (a,c)$\in R$ holds. $\therefore a, b, c \in S$

   - R is not an equivalence relation based on the reasons mentioned above.

Answers for (b):

- R = $\{(m, n) : mn = 0 \text{ and } m, n \in S\}$
  $\Rightarrow$ {(0,0),(0,2),(0,5),(2,0),(5,0)}

- Diagram:



- Matrix:

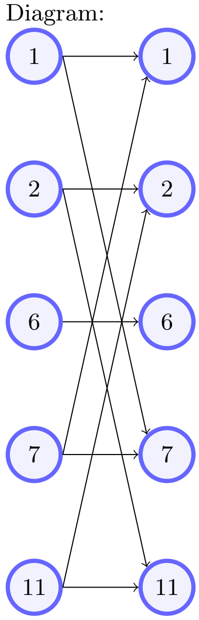
$$\begin{array}{c c} & \begin{array}{ccc} 0 & 2 & 5 \end{array} \\ \begin{array}{c} 0 \\ 2 \\ 5 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \text{Matrix} = \begin{cases} 1 : mn = 0 \\ 0 : otherwise \end{cases}$$

-
  - R is not reflexive as $(2, 2) \notin R$
  - R is symmetric as (0,2) $\in R$ and (2,0) $\in R$
  - R is not transitive. Because (2,0) and (0,5) $\in R$ but (2,5) $\notin R$
- R is not an equivalence relation based on the reasons mentioned above.

Answers for (c):

- R = {(m,n): $m \equiv n \pmod 3$ and $m, n \in S$}
  $\Rightarrow$ {(1,1),(2,2),(6,6),(7,7),(11,11),(1,7),(7,1),(2,11),(11,2)}

- Diagram:



- Matrix:

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 6 & 7 & 11 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 6 \\ 7 \\ 11 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \text{Matrix} = \begin{cases} 1 : m \equiv n \pmod 3 \\ 0 : otherwise \end{cases}$$

-
  - R is reflexive as the diagnol of the matrix consist on only 1's.
  - R is symmetric as since $\forall$ m and n, (m,n) and (n,m) are 1's.
  - R is transitive as $\forall$ m and $n \in R$, if mRn and nRm then mRm and nRn.
- R is an equivalence relation since it is reflexive,symmetric, and transitive for the reasons mentioned above.

2. Suppose X is the set $\{a, b, c, d, e\}$ and $P(X)$ is the power set of X. The domain of each of the relations below is $P(X)$. For each relation, describe the relation in words. Then determine whether it is reflexive, antireflexive, or neither; symmetric, anti-symmetric, or neither; and transitive or not transitive.

(a) A is related to B if $|A-B| = 1$. The vertical bars mean the cardinality of $A - B$.

i. Let $A \in P(X)$:
$\Rightarrow$ Now, $A - A = \emptyset$
$\Rightarrow |A - A| = 0 \neq 1$
$\Rightarrow A \nsim A$.
Since the choice of set A is completely arbritiary, we can say that set A is anit-reflexive.

ii. Let A = {a}, $B = \emptyset$ and $A, B \in P(X)$:
$\Rightarrow A - B$ = {a} and $B - A = \emptyset$
$\Rightarrow |A - B| = 1$ and $|B - A| = 0$
$\Rightarrow$ From this we can say that $A \sim B$ but $B \nsim A$
This results in the conclusion that the relation is not symmetric.

iii. Let A = {a,b}, B = b,a and $A, B \in P(X)$:
$\Rightarrow A - B$ = {a} and $B - A$ = {c}
$\Rightarrow |A - B| = 1$ and $|B - A| = 1$
$\Rightarrow$ From this we can say that $A \sim B$ and $B \sim A$ but $A \neq B$
We can say that the relation is not antisymmetric. $therefore$ the relation is neiter symmetric nor antisymmetric.

iv. Let A = {a,b}, B = b,c and $A, B \in P(X)$:
$\Rightarrow A - B$ = {a} and $B - A$ = {c}
$\Rightarrow |A - B| = 1$ and $|B - A| = 1$
$\Rightarrow$ From this we can say that $A \sim B$ and $B \sim A$
But from the (i) we said that $A \nsim A$,
$\therefore$ the relation is not transitive.

In short, the relation is :

- **antireflexive**
- **neither symmetric and antisymmetric**
- **not transitive**

(b) A is related to B if $A \subset B$. Note that this notation means proper subset.

i. Since every element of a set A is contained in itself, $A \sim A\ \forall A$ and it makes the relation **reflexive**

ii. Let $A \sim B$ and $B \sim A$. Here as per definition, every element of $A \in B$ and $B \in A$ also. Hence A = B.
$\therefore$ The relation is **antisymmetric**.

iii. Let $A \sim B$ and $B \sim C$, then it means that $A \in B$ and $B \in C$. That automatically puts every element of A inside C. So $A \sim C$ and hence the relation is **transitive**.

In short, the relation is :

- **reflexive**
- **antisymmetric**
- **transitive**

3. Determine whether the following relations are equivalence relations. If so, show that the relation fulfills the requirements for an equivalence relation, and describe the equivalence classes. If not, show how the relation fails to fulfill the requirements.

   (a) The relation $\mathcal{C}$ is defined by $(x, y) \in \mathcal{C}$ iff $\cos(x) = \cos(y)$, where $x, y \in \mathbb{R}$.

   - Reflexive:
     Let $y = x$. From this, $(x, x) \in \mathcal{C}$.
     $\Rightarrow \therefore \cos(x) = \cos(x)$ holds.
     Similarly, let $x = y$. From this (y, y)
     $\Rightarrow \therefore \cos(y) = \cos(y)$ also holds.
     Reflexivity holds.

   - Symmetric:
     If $x \sim y$, then it implies that $y \sim x$.
     Assume that $(x, y) \in \mathcal{C}$ then $\cos(x) = \cos(y)$ can be rewritten as $\cos(y) = \cos(x)$.
     This implies that $(y, x) \in \mathcal{C}$.
     Symmetry holds.

   - Transitivity:
     If $(x, y) \in \mathcal{C}$ and $(y, z) \in \mathcal{C}$, then $(x, z) \in \mathcal{C}$.
     Assume that $(x, y) \in \mathcal{C}$, $(y, z) \in \mathcal{C} \therefore$
     $\cos(x) = \cos(y)$
     $\cos(y) = \cos(z)$
     This implies that $\cos(x) = \cos(z)$ and $(x, z) \in \mathcal{C}$.
     Transitivity holds.

   Therefore, it is an equivalence relation. The equivalence classes are:
   $\rightarrow$ If $a \in \mathbb{R}$, the equivalence classes will be $[a] = \{x \in \mathbb{R}\colon \cos(a) = \cos(x)\}$

   (b) The relation $\propto$ on the domain $\mathbb{Z}^+ \times \mathbb{Z}^+$ is defined by $(m, n) \propto (p, q)$ iff $mq = np$.

   - Reflexive:
     Let $n = m$ and $q = p$. Then $(m, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ iff $mp = mp$
     $\therefore$ **Reflexivity holds.**

   - Symmetric:
     If (m, n) and (p, q) are related then it implies (p, q) and (m, n) are related.
     Assume that $(m, n) \propto (p, q)$, this says that $mq = np$ and can be rewritten as $qm = pn$.
     This says, $(q, p) \sim (m, n)$
     $\therefore$ **Symmetry does holds.**

   Because symmetry does not hold, the relation is **not an equivalence relation.** The reason for this conclusion is soley because as long as a relation does not hold at least one of the requirements, the relation fails to be an equivalence relation. In this case, we dont not need to prove or disprove transitivity since symmetry already failed.