Math 243 Discrete Mathematics **Homework 6**

For this homework everyone contruibuted ideas about how to go about the relations. Jorge was responible for the diagrams and maxtix. Emma worked on qustion 1, Matthew worked on qustion 2 and Lucas on question 3. We all meet up a day before submitting to combine our questions and made sure that everyone understood the answer.

1. In each question part below, I will list a set S and a rule that defines a relation R on S as follows: $(m,n) \in R$ if m and n satisfy the given rule. For each set and rule, do the following five things:

   i List the order pairs in the relation.

   ii Draw the associated arrow diagram.

   iii Give the matrix representation of the relation. Be sure to label the rows and columns with the elements of S.

   iv Identify which properties the relation has: reflexive, antireflexive, or neither; symmetric, antisymmetric, or neither; transitive or not transitive. Be sure to explain your answers.

   v Decide whether the relation is an equivalence relation. If so, identify the equivalence classes.

   (a) S = {4,9,17}; $m \geq n$.

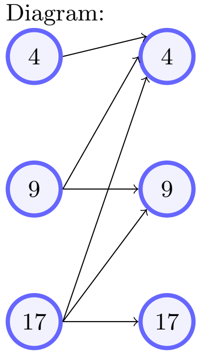
   (b) S = {0,2,5}; $mn = 0$.

   (c) S = {1,2,6,7,11}; $m \equiv n \pmod 3$

Answers for (a):

- R = $\{(m,n) : m \geq n \text{ and } m,n \in S\}$
  $\Rightarrow$ {(17,4),(9,4),(17,9),(4,4),(9,9),(17,17)}

- Diagram:



- Matrix:

$$\begin{array}{c} \\ 4 \\ 9 \\ 17 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 9 & 17 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \text{Matrix} = \begin{cases} 1 : (m,n) \in R \\ 0 : otherwise \end{cases}$$

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  - R is reflexive as $(m,n) \in R \ \forall \ m \in S$
  - R is anti symmetric. As if(a,b) $\in R$ and (b,a) $\in R$ then a = b for:

    a = b = 4

    a = b = 9

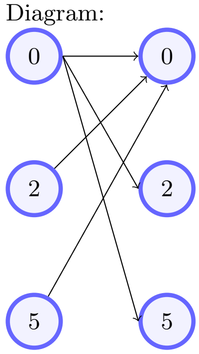
    a = b = 17
  - R is transitive. As if (a,b)$\in R$ and (b,c)$\in R$ then (a,c)$\in R$ holds. $\therefore a, b, c \in S$

- R is not an equivalence relation because R is not symmetric.

Answers for (b):

- R = $\{(m,n) : mn = 0 \text{ and } m,n \in S\}$
  $\Rightarrow$ {(0,0),(0,2),(0,5),(2,0),(5,0)}

- Diagram:



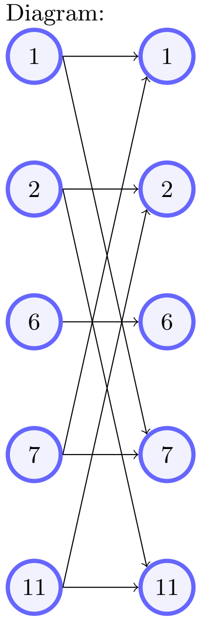
$$\begin{array}{c|c} & \begin{matrix} 0 & 2 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 2 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \text{ Matrix} = \begin{cases} 1 : mn = 0 \\ 0 : otherwise \end{cases}$$

-
  - R is not reflexive as $(2,2) \notin R$
  - R is not antireflexive since 0 points to itself
  - R is symmetric as (0,2) $\in R$ and (2,0) $\in R$
  - R is not transitive. Because (2,0) and (0,5) $\in R$ but (2,5) $\notin R$
- R is not an equivalence relation based on the reasons mentioned above.

Answers for (c):

- R = {(m,n): $m \equiv n \pmod 3$ and $m,n \in S$}
  $\Rightarrow$ {(1,1),(2,2),(6,6),(7,7),(11,11),(1,7),(7,1),(2,11),(11,2)}

- Diagram:



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 6 & 7 & 11 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 6 \\ 7 \\ 11 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \text{ Matrix} = \begin{cases} 1 : m \equiv n \pmod 3 \\ 0 : otherwise \end{cases}$$

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  - R is reflexive as the diagnol consist of only 1's. R is symmetric R is symmetric as since $\forall$ m and n, (m, n) and (n, m) are 1s. For example, in this set, there exists elements (1, 7) and (2,11) in order for the set to be symmetrical, there has to exist elements (7, 1) and (11, 2) and because those elements exist in the set R, we are to claim that set R is symmetrical as this thought process can be applied to all elements of set R..R is transitive as $\forall$ m and $n \in R$, if mRn and nRm then mRm and nRn.
- R is an equivalence relation since it is reflexive, symmetric, and transitive. The equivalence class for R are:
  - $[1] = 1,7$ $[2] = 2,11$ $[6] = 6$

2. Suppose X is the set $\{a, b, c, d, e\}$ and $P(X)$ is the power set of X. The domain of each of the relations below is $P(X)$. For each relation, describe the relation in words. Then determine whether it is reflexive, antireflexive, or neither; symmetric, anti-symmetric, or neither; and transitive or not transitive.

(a) A is related to B if $|A-B| = 1$. The vertical bars mean the cardinality of $A - B$.

i. Let $A \in P(X)$:
$\Rightarrow$ Now, $A - A = \emptyset$
$\Rightarrow |A - A| = 0 \neq 1$
$\Rightarrow A \nsim A$.
Since the choice of relation A is completely arbritiary, we can say that relation A is **anit-reflexive.**

ii. Let $A =$ {a}, $B = \emptyset$ and $A, B \in P(X)$:
$\Rightarrow A - B =$ {a} and $B - A = \emptyset$
$\Rightarrow |A - B| = 1$ and $|B - A| = 0$
$\Rightarrow$ From this we can say that $A \sim B$ but $B \nsim A$
This results in the conclusion that the relation is **not symmetric**.

iii. Let $A =$ {a,b}, $B =$ {a,b} and $A, B \in P(X)$:
$\Rightarrow A - B =$ {a} and $B - A =$ {c}
$\Rightarrow |A - B| = 1$ and $|B - A| = 1$
$\Rightarrow$ From this we can say that $A \sim B$ and $B \sim A$ but $A \neq B$
We can say that the relation is not antisymmetric. $\therefore$ The relation is **neiter symmetric nor antisymmetric**.

iv. Let $A =$ {a,b}, $B =$ {b,c} and $A, B \in P(X)$:
$\Rightarrow A - B =$ {a} and $B - A =$ {c}
$\Rightarrow |A - B| = 1$ and $|B - A| = 1$
$\Rightarrow$ From this we can say that $A \sim B$ and $B \sim A$
But from the (i) we said that $A \nsim A$,
$\therefore$ The relation is **not transitive**.

In short, the relation is :

- **antireflexive**
- **not symmetric**
- **not anti-symmetric**
- **not transitive**

(b) A is related to B if $A \subset B$. Note that this notation means proper subset.

i. Since every element of a set A is contained in itself, $A \sim A\ \forall A$ and it makes the relation **reflexive**

ii. Let $A \sim B$ and $B \sim A$. Here as per definition, every element of $A \subseteq B$ and $B \subseteq A$ also however $A \neq B$.
$\therefore$ The relation is **antisymmetric**.

iii. Let $A \sim B$ and $B \sim C$, then it means that $A \subseteq B$ and $B \subseteq C$. That automatically puts every element of A inside C. So $A \sim C$ and hence the relation is **transitive**.

In short, the relation is :

- **reflexive**
- **antisymmetric**
- **transitive**

3. Determine whether the following relations are equivalence relations. If so, show that the relation fulfills the requirements for an equivalence relation, and describe the equivalence classes. If not, show how the relation fails to fulfill the requirements.

   (a) The relation $\mathcal{C}$ is defined by $(x, y) \in \mathcal{C}$ iff $\cos(x) = \cos(y)$, where $x, y \in \mathbb{R}$.

   - Reflexive:
     Let $y = x$. From this, $(x, x) \in \mathcal{C}$.
     $\Rightarrow \therefore \cos(x) = \cos(x)$ holds.
     Similarly, let $x = y$. From this (y, y)
     $\Rightarrow \therefore \cos(y) = \cos(y)$ also holds.
     Reflexivity holds.

   - Symmetric:
     If $x \sim y$, then it implies that $y \sim x$.
     Assume that $(x, y) \in \mathcal{C}$ then $\cos(x) = \cos(y)$ can be rewritten as $\cos(y) = \cos(x)$.
     This implies that $(y, x) \in \mathcal{C}$.
     Symmetry holds.

   - Transitivity:
     If $(x, y) \in \mathcal{C}$ and $(y, z) \in \mathcal{C}$, then $(x, z) \in \mathcal{C}$.
     Assume that $(x, y) \in \mathcal{C}$, $(y, z) \in \mathcal{C}.\therefore$
     $\cos(x) = \cos(y)$ and $\cos(y) = \cos(z)$
     This implies that $\cos(x) = \cos(z)$ and $(x, z) \in \mathcal{C}$.
     Transitivity holds.

   Therefore, it is an equivalence relation. The equivalence classes are:
   $\rightarrow$ If $a \in \mathbb{R}$, the equivalence classes will be $[a] = \{x \in \mathbb{R}\colon \cos(a) = \cos(x)\}$

   (b) The relation $\propto$ on the domain $\mathbb{Z}^+ \times \mathbb{Z}^+$ is defined by $(m, n) \propto (p, q)$ iff $mq = np$.

   - Reflexive:
     Let $n = m$ and $q = p$. Then $(m, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ iff $mp = mp$
     $\therefore$ **Reflexivity holds.**

   - Symmetric:
     Let (m, n) and (p, q) are related then it implies (p, q) and (m, n) are related.
     Assume that $(m, n) \propto (p, q)$, this says that $mq = np$ and can be rewritten as $qm = pn$. Based on commutative properties, $qm = pn$ can be rewritten as well as $pn = mq$ which shows that **symmetry holds**.

   - Transitive:
     Let $(a, b), (c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(c, d), (e, f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. This says that:
     $(a, b), (c, d)$ is (ad) = (bc) and $(c, d), (e, f)$ is (cf) = (de)
     Multipling the equations, we get: (ad)(cf) = (bc)(de)
     Rearranging the values via commutative properties: (af)(cd) = (be)(cd)
     Divide by (cd), we obtain (af) = (be) and since $(a, b), (e, f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, this proves that the relation is **transitive**.

   Therefore, it is an equivalence relation. The equivalence classes are:
   $\rightarrow [m, n] = \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (mq) = (np)\}$