CS 577 . Ass. Ø

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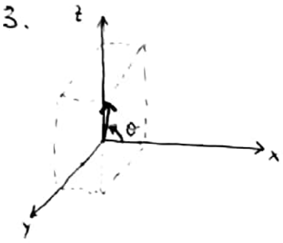
A.

1. $2a - b = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2-4 \\ 4-5 \\ 6-6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

2. $\hat{a} = \frac{a}{\|a\|_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{1^2+2^2+3^2}} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$

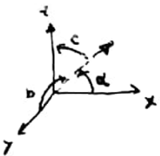
3. 

$\|a\|_2 = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a \cdot i = \|a\| \cdot \|i\| \cdot \cos\theta \rightarrow \theta = \arccos\left(\frac{a \cdot i}{\|a\| \cdot \cancel{\|i\|}}\right) = \arccos\left(\frac{1}{\sqrt{14}}\right) =$

$= 74.5°$

4. $a \cdot b = \|a\| \cdot \|b\| \cdot \cos\theta \rightarrow b \in \{i, j, k\}$ $\quad i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



$\cos\alpha = \frac{a \cdot i}{\|a\| \cdot \|i\|} = \frac{1}{\sqrt{14}}$, $\cos b = \frac{a \cdot j}{\|a\| \cdot \|j\|} = \frac{2}{\sqrt{14}}$, $\cos c = \frac{a \cdot k}{\|a\| \|k\|} = \frac{3}{\sqrt{14}}$

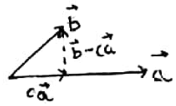
verification: $\cos^2\alpha + \cos^2 b + \cos^2 c = 1 : \frac{1}{14} + \frac{4}{14} + \frac{9}{14}$ ✓

5. $a \cdot b = \|a\| \cdot \|b\| \cdot \cos\theta \rightarrow \cos\theta = \frac{a \cdot b}{\|a\| \cdot \|b\|} = \frac{1 \cdot 4 + 5 \cdot 2 + 6 \cdot 3}{\sqrt{14} \cdot \sqrt{77}} = 0.97463 \Rightarrow \boxed{\theta = 12.93°}$

6. $a \cdot b = b \cdot a = 1 \cdot 4 + 5 \cdot 2 + 6 \cdot 3 = 32$

7. $a \cdot b = \|a\| \cdot \|b\| \cdot \cos\theta = \sqrt{1+4+9} \cdot \sqrt{16+25+36} \cdot \cos(12.93) = 32.0004$

8. Scalar projection $p = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} = \frac{32}{\sqrt{14}} = 8.552$



$\vec{a} \cdot (\vec{b} - c\vec{a}) = 0$
$\vec{a}\vec{b} - c \cdot (\vec{a}\vec{a}) = 0$ $\Big\} \; c = \frac{\vec{a}\vec{b}}{\|\vec{a}\|}$

9. if $t \perp a \Rightarrow t \cdot a = 0 \rightarrow t_1 \cdot 1 + t_2 \cdot 2 + t_3 \cdot 3 = 0$

One perpendicular vector would be: $t = [3 \quad 0 \quad -1]^T$

10. $a \times b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $b \times a = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

can't be multiplied because number of columns in first matrix is different than the number of rows in the second matrix. (vector = 1-column matrix).

11. $\left.\begin{array}{l} t \cdot a = 0 \\ t \cdot b = 0 \end{array}\right\}$ $\begin{array}{l} t_1 + 2t_2 + 3t_3 = 0 \Rightarrow t_1 = -2t_2 - 3t_3 \\ 4t_1 + 5t_2 + 6t_3 = 0 \end{array}$

$-8t_2 - 12t_3 + 5t_2 + 6t_3 = 0 \Leftrightarrow -3t_2 - 6t_3 = 0 \underset{t_3 = 1}{\Leftrightarrow} t_2 = \frac{6}{-3} = -2$

$t_1 = -2 \cdot (-2) - 3 \cdot 1 = 1$

$t = [1 \quad -2 \quad 1]^T \Rightarrow$ verification: $\begin{cases} t \cdot a = 1 - 4 + 3 = 0 \\ t \cdot b = 4 - 10 + 6 = 0 \end{cases}$ ✓

12. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 5 \cdot 3 + 4 \cdot 3 - 12 - (-15 + 6 + 24) = 0 \Rightarrow$ linearly dependents

$a \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} a + 4b - c = 0 \\ 2a + 5b + c = 0 \\ 3a + 6b + 3c = 0 \end{cases} \rightarrow 3a + 9b = 0 \Leftrightarrow a + 3b = 0$

$\left.\begin{array}{l} c = a + 4b = b \\ a = -3b \end{array}\right\}$ $3a + 6b + 3c = -9b + 6b + 3b = 0 \cdot b = 0$ → due to the fact that $b = a = c = 0$ is not the only valid solution, the vectors have a linear dependency.

Linear dependency: $\begin{array}{l} a = -3b \\ b = c \end{array}$

13. $a^T \cdot b = [1 \quad 2 \quad 3] \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$

$a \cdot b^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot [4 \quad 5 \quad 6] = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$

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B)

1. $2A - B = 2 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$

2. $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$

$B \cdot A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$

3. $(A \cdot B)^T = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$

$B^T \cdot A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$ (with matrix properties: $(AB)^T = B^T A^T$

4. $|A| = 1 \cdot (-2) \cdot (-1) + 3 \cdot 2 \cdot 0 + 5 \cdot 4 \cdot 3 - (3 \cdot (-2) \cdot 0 + 5 \cdot 3 \cdot 1 + 2 \cdot 4 \cdot (-1)) = 55$

$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = 15 - 12 + 12 - (-15 + 6 + 24) = 15 - 15 = 0$ → linearly dependent $\boxed{\text{SINGULAR MATRIX}}$

5. $A \cdot A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 9 & 7 \\ 9 & 29 & -13 \\ 7 & -13 & 26 \end{bmatrix}$ → rows are not orthogonal

$B \cdot B^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 14 \end{bmatrix}$ ⇒ rows form an orthogonal set (B)

6. $A^{-1} = \frac{1}{|A|} \cdot Adj(A) = \frac{1}{55} \cdot \begin{bmatrix} -13 & 4 & 20 \\ 17 & -1 & -5 \\ 12 & 9 & -10 \end{bmatrix}^T = \begin{bmatrix} -13/55 & 17/55 & 12/55 \\ 4/55 & -1/55 & 9/55 \\ 20/55 & -5/55 & -10/55 \end{bmatrix} = \begin{bmatrix} 0.236 & 0.309 & 0.218 \\ 0.073 & -0.018 & 0.164 \\ 0.364 & -0.091 & -0.182 \end{bmatrix}$

normalized!!

$B^{-1} = B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \circ \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/21 & 0 \\ 0 & 0 & 1/14 \end{bmatrix} = \begin{bmatrix} 0.167 & 0.095 & 0.214 \\ 0.333 & 0.048 & -0.143 \\ 0.167 & -0.190 & 0.071 \end{bmatrix}$

orthogonal

7. $\nexists C^{-1}$ ⇒ C is a singular matrix and $|C| = 0$.

8. $Ad = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$

9. Projection b onto a: $p = \frac{a \cdot b}{\|a\|}$ ⇒ $P = \frac{A \cdot d}{\|d\|} = \frac{\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\sqrt{14}} = \begin{bmatrix} \sqrt{14} \\ 2.405 \\ 1.871 \end{bmatrix} = \begin{bmatrix} 3.742 \\ 2.405 \\ 1.871 \end{bmatrix}$

projection rows A onto d.

10. $A \cdot d = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} + 3 \cdot \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$

11. $Bx = d \Leftrightarrow B^{-1} \cdot B \cdot x = B^{-1} \cdot d \Leftrightarrow x = B^{-1} \cdot d \overset{B \text{ normalized}}{=} B^T \cdot d = \begin{bmatrix} 0.167 & 0.095 & 0.214 \\ 0.333 & 0.048 & -0.143 \\ 0.167 & -0.19 & 0.071 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

12. $Cx = d \Rightarrow \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ -x_1 + x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow$

$\left. \begin{array}{l} x_2 = -3x_3 + 3 + x_1 \\ x_1 = 1 - 2x_2 - 3x_3 \end{array} \right\} \begin{array}{l} x_2 = 3 - 3x_3 + 1 - 2x_2 - 3x_3 \Leftrightarrow x_2 = 1 - 2x_3 \\ x_1 = 1 - 2 + 4x_3 - 3x_3 \Leftrightarrow x_1 = -1 + x_3 \end{array}$

$\boxed{\text{undetermination}}$ $x_3 \simeq \infty$

$4(-1 + x_3) + 5 \cdot (1 - 2x_3) + 6x_3 = 2 \Leftrightarrow -4 + 4x_3 + 5 - 10x_3 + 6x_3 = 2 \Leftrightarrow 0 \cdot x_3 \neq 1$

This result was expected as C was a singular matrix which doesn't have an inverse.

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C)

1. $Dv = \lambda v = \lambda \cdot I \cdot v \iff Dv - \lambda I v = 0$

$$|D - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 \underset{\text{if } v \neq 0}{=} 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0 \iff \lambda^2 - 3\lambda - 4 = 0 \rightarrow \lambda = \frac{+3 \pm \sqrt{9 - 4 \cdot (-4)}}{2} = \begin{cases} 4 \\ -1 \end{cases}$$

Eigenvalues of D: $\lambda = 4$ and $\lambda = -1$

$$D \cdot v = \lambda \cdot v \rightarrow \begin{cases} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 4 \cdot \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{matrix} x + 2y = 4x \iff 2y = 3x \\ 3x + 2y = 4y \iff 3x = 2y \end{matrix} \rightarrow v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ (and any non-zero multiple)} \\ \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{matrix} x + 2y = -x \iff 2y = -2x \\ 3x + 2y = -y \iff 3x = -3y \end{matrix} \rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ (and any non-zero multiple)} \end{cases}$$

2. $v_1 \cdot v_2 = 2 \cdot 1 + 3 \cdot (-1) = -1$

3. $|E - \lambda I| = \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4 = 0 \rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0 \iff \lambda^2 - 7\lambda + 6 = 0$

$$\lambda = \frac{7 \pm \sqrt{49 - 4 \cdot 6}}{2} = \begin{cases} 6 \Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{matrix} 2x - 2y = 6x \\ -2x + 5y = 6y \end{matrix} \rightarrow \begin{matrix} -y = 2x \\ -2x = y \end{matrix} \Big\} v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ 1 \Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{matrix} 2x - 2y = x \\ -2x + 5y = y \end{matrix} \rightarrow \begin{matrix} x = 2y \\ x = 2y \end{matrix} \Big\} v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$$

$v_1 \cdot v_2 = 1 \cdot 2 - 2 \cdot 1 = 0$

4. The eigenvectors of E are orthogonal between each other. ($v_1 \cdot v_2 = 0$)

5. $|F| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 2 \cdot 2 = 0$ $\Rightarrow$ It doesn't have a trivial solution (unique solution when the determinant $\neq \emptyset$)

6. $Fx = 0 \rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\begin{matrix} x_1 = -2 \\ x_2 = 1 \end{matrix}}$ $\quad x = [-2 \quad 1]^T$

Another solution could be: $x = [-6 \quad 3]^T$

7. $|D| = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 3 \cdot 2 = -4$ $\Rightarrow$ as the determinant is non-zero; the matrix, which could be interpreted as two lines, has different slopes and, therefore just one solution (interception between both lines).

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -2x_1 = 0 \rightarrow x_1 = 0 \\ 2x_2 = 0 \rightarrow x_2 = 0 \end{cases} \rightarrow x = [0 \quad 0]^T$$

D) $f(x) = x^2 + 3$, $g(x) = x^2$, $q(x,y) = x^2 + y^2$

1) $f'(x) = 2x$ and $f''(x) = 2$

2) $\frac{\partial q}{\partial x} = 2x$ and $\frac{\partial q}{\partial y} = 2y$

3) $$\nabla q(x,y) = \begin{bmatrix} \frac{\partial q}{\partial x} \\ \frac{\partial q}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

4) $$\frac{df(g(x))}{dx} = \frac{d}{dx}(x^4 + 3) = 4x^3$$

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = 2g(x) \cdot 2x = 4x^2 \cdot x = 4x^3$$