TTK4190 Guidance and Control of Vehicles

Assignment 1 Part 1

Written Fall 2017 By Sveinung Haugane and Sondre Midtskogen

Problem 1 - Attitude Control of Satellite

In this part of the assignment we will implement a linearized control law for attitude control of a satelite, using a quaternion parametrization.

Problem 1.1

To find the equilibrium point $\boldsymbol{x_0}$ we first need to find an expression for $\dot{\boldsymbol{x}}$. From equation(2.65)-(2.67) in [1] we have that

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} \frac{1}{2}(\eta p - \epsilon_3 q + \epsilon_2 r) \\ \frac{1}{2}(\epsilon_3 p + \eta q - \epsilon_1 r) \\ \frac{1}{2}(-\epsilon_2 p + \epsilon_1 q + \eta r) \end{bmatrix} \tag{1}$$

Further, from equation (1) in [2], we have that

$$\dot{\boldsymbol{\omega}} = \boldsymbol{I}_{CG}^{-1}(\boldsymbol{\tau} + \boldsymbol{S}(\boldsymbol{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega})$$

which, when written out, yelds

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} \frac{1}{mr^2}K \\ \frac{1}{mr^2}M \\ \frac{1}{mr^2}N \end{bmatrix} \tag{2}$$

By setting $\dot{\boldsymbol{x}}$ equal to 0, and letting $\boldsymbol{q} = [\eta, \epsilon_1, \epsilon_2, \epsilon_3]^T = [1, 0, 0, 0]^T$ and $\boldsymbol{\tau} = 0$ we get that $\dot{\boldsymbol{x}} = \frac{1}{2}[\eta p, \eta q, \eta r, 0, 0, 0]^T = 0$. This implies that $\boldsymbol{\omega} = 0$ at the equilibrium point.

Linearizing around $\dot{\boldsymbol{x}} = 0$ we get

$$\dot{\boldsymbol{x_{lin}}} = \frac{1}{2}\begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}\boldsymbol{x} + \frac{1}{mr^2}\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}\boldsymbol{\tau} \tag{3}$$

The $\mathbf{A}$ and $\mathbf{B}$ matrices are

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$\boldsymbol{B} = \frac{1}{mr^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1.2

Using the control law suggested in the assignment text, we get the system matrix

$$\boldsymbol{A} - \boldsymbol{BK} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{mr^2}k_p & 0 & 0 & -\frac{1}{mr^2}k_d & 0 & 0 \\ 0 & -\frac{1}{mr^2}k_p & 0 & 0 & -\frac{1}{mr^2}k_d & 0 \\ 0 & 0 & -\frac{1}{mr^2}k_p & 0 & 0 & -\frac{1}{mr^2}k_d \end{bmatrix}$$

By inserting numerical values, $m = 100\,\mathrm{kg}, r = 2\,\mathrm{m}, k_p = 1, k_d = 20$, we can calculate the matrix' eigenvalues using the matlab *eig* function. The eigenvalues are

$$\boldsymbol{\lambda} = \begin{bmatrix} -\frac{1}{40} + j\frac{1}{40} \\ -\frac{1}{40} - j\frac{1}{40} \\ -\frac{1}{40} + j\frac{1}{40} \\ -\frac{1}{40} - j\frac{1}{40} \\ -\frac{1}{40} + j\frac{1}{40} \\ -\frac{1}{40} - j\frac{1}{40} \end{bmatrix}, \tag{4}$$

which has real parts in the left half plane, indicating that the closed-loop system is stable.

To make a good controller we need to determine how we want our satellite to respond. Complex eigenvalues will cause the system to oscillate, approaching the desired value faster, but it will also stabilize slower than a critically damped system. Real eigenvalues on the other hand causes no oscillations, but may cause a slow response time if we introduce too much dampening. For this satellite we would like the system to have real eigenvalues, since the satellite will have a directional antenna for communication, nothing will be gained from the oscillations caused by imaginary poles. Oscillations would only disturb the link.

Problem 1.3

The simulation attitude.m was modified to comply with the parameters and control law used in the assignment text. The simulation was run for a period of 150 seconds. The results can be seen in figure 1. As expected, there are some oscillations around the attitude equilibrium, due to having complex eigenvalues.

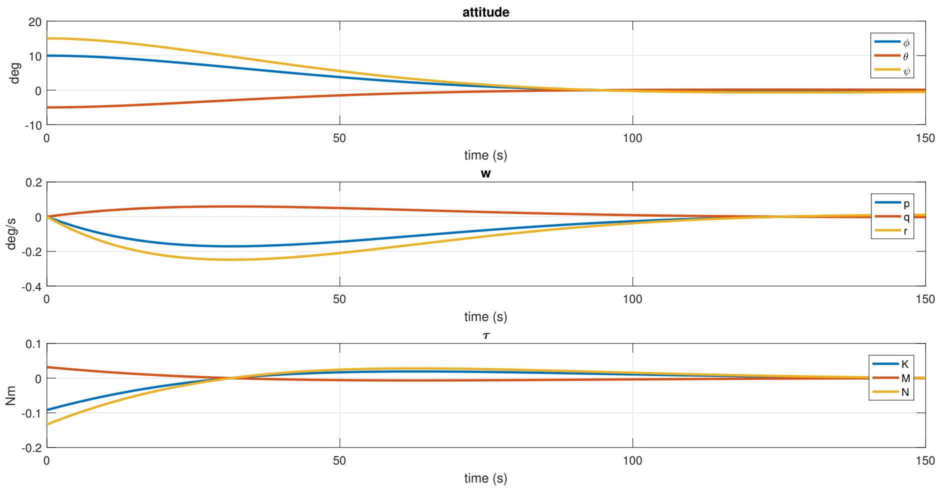

Figure 1

Problem 1.4

With the conjugate of the quaternion being $\bar{\boldsymbol{q}} = \left[\eta, -\boldsymbol{\epsilon}^T\right]^T$. The quaternion error can be written as

$$\tilde{\boldsymbol{q}} := \begin{bmatrix} \tilde{\eta} \\ \tilde{\epsilon} \end{bmatrix} = \bar{\boldsymbol{q}}_d \otimes \boldsymbol{q} = \begin{bmatrix} \eta_d\eta - (-\boldsymbol{\epsilon}_d)^T\boldsymbol{\epsilon} \\ \eta(-\boldsymbol{\epsilon}_d) + \eta_d\boldsymbol{\epsilon} + \boldsymbol{S}(-\epsilon_d)^T\epsilon \end{bmatrix} = \begin{bmatrix} \eta_d\eta + \epsilon_{1d}\epsilon_1 + \epsilon_{2d}\epsilon_2 + \epsilon_{3d}\epsilon_3 \\ \eta_d\epsilon_1 - \eta\epsilon_{1d} + \epsilon_{3d}\epsilon_2 - \epsilon_{2d}\epsilon_3 \\ \eta_d\epsilon_2 - \eta\epsilon_{2d} + \epsilon_{1d}\epsilon_3 - \epsilon_{3d}\epsilon_1 \\ \eta_d\epsilon_3 - \eta\epsilon_{3d} + \epsilon_{2d}\epsilon_1 - \epsilon_{1d}\epsilon_2 \end{bmatrix}. \quad (5)$$

For $\boldsymbol{q} = \boldsymbol{q_d}$ we then have:

$$\tilde{\boldsymbol{q}} = \begin{bmatrix} \eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 \\ \eta\epsilon_1 - \eta\epsilon_1 + \epsilon_2\epsilon_3 - \epsilon_2\epsilon_3 \\ \eta\epsilon_2 - \eta\epsilon_2 + \epsilon_1\epsilon_3 - \epsilon_1\epsilon_3 \\ \eta\epsilon_3 - \eta\epsilon_3 + \epsilon_1\epsilon_2 - \epsilon_1\epsilon_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (6)$$

for $\eta^2 = 1 - \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2$

Problem 1.5

As shown in figure 2, we see that the error in ϕ and ψ oscillates. This is because the reference is changing, and the damping term, $-\boldsymbol{K}_d\boldsymbol{\omega}$, dampens all angular velocity, rather than just the unwanted components. It would therefore be better to base the control design on the angular velocity error $\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_d$.

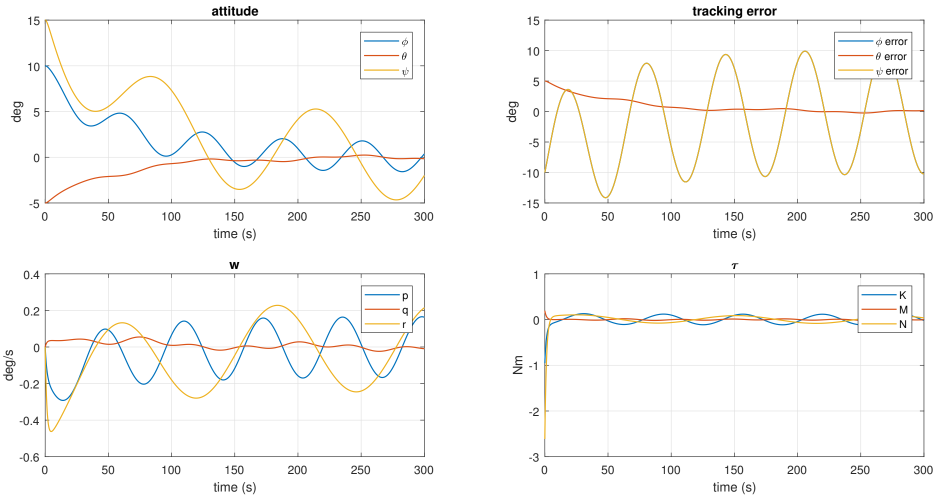

Figure 2: Response

Problem 1.6

We modify the control law to be

$$\boldsymbol{\tau} = -\boldsymbol{K}_d\tilde{\boldsymbol{\omega}} - k_p\tilde{\boldsymbol{\epsilon}}, \quad for \quad \tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d, \tag{7}$$

and set the desired angular velocity to be

$$\boldsymbol{\omega}_d = \boldsymbol{T}_{\Theta d}^{-1}(\boldsymbol{\Theta}_d)\dot{\boldsymbol{\Theta}}_d, \tag{8}$$

where $\boldsymbol{\Theta}_d$ is the desired angular velocity on Euler engles, and $\boldsymbol{T}_{\Theta d}^{-1}(\boldsymbol{\Theta}_d)$ is the coordinate transformation matrix. As suggested in problem 1.5, we now base our control around the error in angular velocity. By doing this we improve our control over systems where the reference point is changing. Comparing the results in figure 3, to the results in figure 2, we see that the error is significantly reduced.

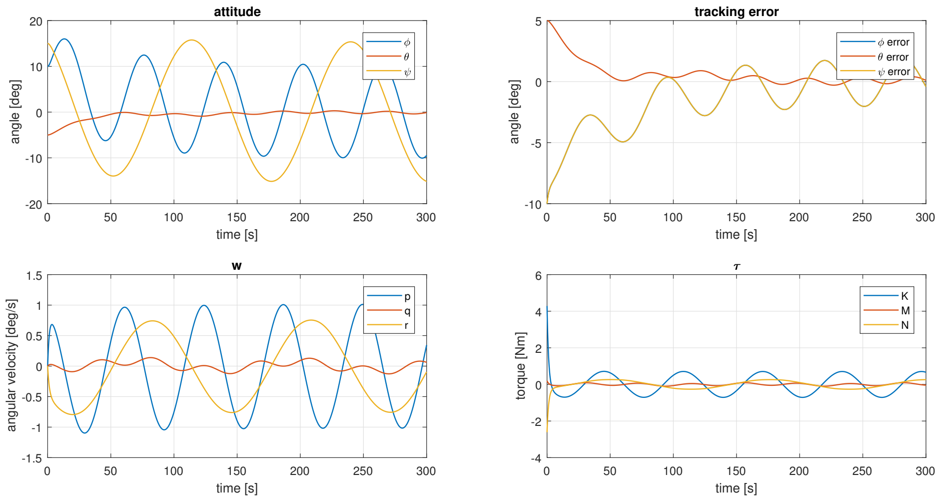

Figure 3: Response

Problem 1.7

The Lyapunov function can be written as

$$V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \boldsymbol{I}_{CG}\tilde{\boldsymbol{\omega}} + 2k_p(1-\tilde{\eta}), \tag{9}$$

The Lyapunov function includes a quadratic and a linear term. From the assignment text, we have that $\boldsymbol{I}_{CG}$ is positive definite, which means that the quadratic term is positive for all values of $\tilde{\boldsymbol{\omega}} \neq \boldsymbol{0}$. Further, $k_p > 0$ and, assuming unit quaternions, $\tilde{\eta} \in [0,1]$. Thus, the linear term is positive for all values of $\tilde{\eta} \neq 1$. At the equilibrium, $\tilde{\boldsymbol{\omega}} = \boldsymbol{0}$ and $\tilde{\eta} = 1$, which means that V is positive definite.

From equation (9) it is clear that $|\tilde{\boldsymbol{\omega}}| \to \infty \implies V \to \infty$, which implies that V is radially unbounded.

$\dot{V}$ is given by

$$\dot{V} = \tilde{\boldsymbol{\omega}}^\top \boldsymbol{I}_{CG}\dot{\tilde{\boldsymbol{\omega}}} - 2k_p\dot{\tilde{\eta}} \tag{10}$$

By inserting for equations (1) and (7) in [2], we get

$$\dot{V} = \tilde{\boldsymbol{\omega}}^\top(\boldsymbol{\tau} + \boldsymbol{S}(\boldsymbol{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega}) + k_p\tilde{\epsilon}^T\tilde{\boldsymbol{\omega}} \tag{11}$$

Using equation (6) in [2], and that the transposed of a scalar is equal to the scalar, we can rewrite equation (11) as

$$\begin{aligned}\dot{V} &= \tilde{\boldsymbol{\omega}}^\top(-\boldsymbol{K_d}\tilde{\boldsymbol{\omega}} - k_p\tilde{\boldsymbol{\epsilon}}) + k_p\tilde{\boldsymbol{\omega}}^\top\tilde{\epsilon} \\ &= -\tilde{\boldsymbol{\omega}}^\top\boldsymbol{K_d}\tilde{\boldsymbol{\omega}}\end{aligned} \tag{12}$$

Since $\boldsymbol{\omega_d} = \mathbf{0} \implies \tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}$, we can write this as

$$\dot{V} = -\boldsymbol{\omega}^\top\boldsymbol{K_d}\boldsymbol{\omega} \tag{13}$$

as it is given i appendix 2.1 in [1] Barbalat's lemma states that if

1. $V \geq 0$
2. $\dot{V} \leq 0$
3. $\dot{V}$ is uniformly continous

then $\dot{V} \to 0$ as $t \to \infty$.

As already explained, 1. is true. 2. and 3. are true because $\boldsymbol{K_d}$ is positive definite. Thus, $\dot{V} \to 0$ as $t \to \infty \implies \boldsymbol{\omega} \to 0$ as $t \to \infty$.

V satisfies the conditions of a strict Lyapunov function, and is radially unbounded. By theorem 4.2 in [3], the equilibrium point is globally asymptotically stable, which also implies that the closed-loop system is globally convergent.

Problem 1.8

Parameterizing attitude can be done in different ways. The most easily understood is probably through Euler angles (roll,pitch,yaw). The resulting model will however include certain singularities, when transforming the model to other coordinate systems. Depending on the order you rotate your system, you will get a singularity in one of the axes at 90 degrees.

When parameterizing using quartanions, one additional varable will be introduced. This does however not change the size of your state vector, because of the definition $|q| = 1$. The main advantage using quaternions is the removal of singular points. As opposed to Euler angles, you will not get these singularities when transforming your coordinate system.

The drawback of using quaternions is shown in the form om normalization errors. when quartanions are integrated over time, in either descrete intervals or continous, the size of $|q|$ might change in value. To ensure that the definition $|q| = 1$ is uphold, one must normalize the quartanion. This may cause numerical errors, which may also reduce stability of the system. Another problem with quaternions is that the rotation $R(q) = R(-q)$, meaning that these two rotations in quaternions represent the same physical rotation. This may cause the controller to choose a longer path than necessary.

References

[1] T. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control.* John Wiley & Sons, 2011.

[2] T. I. Fossen, "Ttk4190 guidance and control of vehicles, assignment 1," 2017.

[3] H. K. Khalil, "Noninear systems," *Prentice-Hall, New Jersey*, vol. 2, no. 5, pp. 5–1, 1996.