Lecture 9: Bayesian Estimation & Empirical Bayes

Big Data and Machine Learning for Applied Economics
Econ 4676

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Announcement

- **Next Thursday September 10, I'll be teaching the class**
- **Problem Set 1 is due next Tuesday September 15 at 11:00**
- At some point over the weekend I'll send what points everyone should present
- Assignment would be based on the groups created on `Github`
- You should consider class presentations as mini-seminars, just 2-5 minutes using one or two transparencies
- Attempt to make a concise interpretation of the relevant material, making effective use of supporting numerical and graphical evidence.

Agenda

Bayes Theorem

Bayes th

prior $\theta \sim p(\ )$

posterior

$$\overbrace{\pi(\theta|X)}^{\text{posterior}} = \frac{\overbrace{f(X|\theta)}^{\text{likelihood}}\overbrace{p(\theta)}^{\text{prior}}}{m(X)} \tag{1}$$

likelihood

$X \sim f(X|\theta)$

Normalizado q' depende solo de los datos

with $m(X)$ is the marginal distribution of X, i.e.

$$m(X) = \int f(X|\theta)p(\theta)d\theta \tag{2}$$

→ integrate out

It is important to note that Bayes' theorem does not tell us what our beliefs should be, it tells us how they should change after seeing new information.

A Simple Covid Example

- Suppose we are interested in the prevalence of COVID in a small city. The higher the prevalence, the more public health precautions we would recommend be put into place.
- A small random sample of 20 individuals from the city will be checked for the presence of the virus.
- Interest is in θ, the fraction of infected individuals in the city. Roughly speaking, the parameter space includes all numbers between zero and one. The data X records the total number of people in the sample who are infected.
- Before the sample is obtained the number of infected individuals in the sample is unknown.

A Simple Covid Example

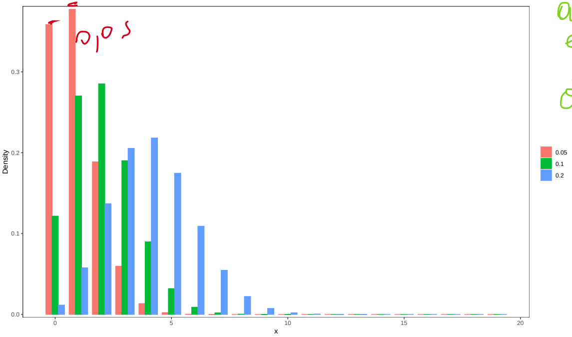
If the value of θ were known, a reasonable sampling model would be

$$X|\theta \sim Binomial(20, \hat{\theta}) \quad (3)$$

→ prob, no observamos θ

average estimado

$\hat{\theta}_{MLE} = \frac{X}{N} = 0$



θ true = 0,05

$$Pr(X = x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} \quad (4)$$

$$Pr(X = 0) = \binom{20}{0} 0.05^0 (1 - 0.05)^{20-0} \approx 0.36 \quad (5)$$

A Simple Covid Example

Prior distribution

- Other studies from various parts of the country indicate that the infection rate in comparable cities ranges from about 0.05 to 0.20, with an average prevalence of 0.10.
- We will therefore use a prior distribution $p(\theta)$

$\theta \in [0,1]$

$$\theta \sim Beta(a, b) \tag{6}$$

where the density of a Beta takes the form of

$\Gamma(a) = (a-1)!$

$$p(\theta) = \frac{\Gamma(a+b)}{\underbrace{\Gamma(a)\,\Gamma(b)}_{B(a,b)}}\theta^{a-1}(1-\theta)^{b-1} \tag{7}$$

$E(\theta) = \int_0^1 \theta\, p(\theta)\, d\theta$

with $a = 2$ and $b = 20$. Note that

$$E(\theta) = \frac{a}{a+b} = 0.09 \approx 0.10 \tag{8}$$

H/W $$= Pr(0.05 < \theta < 0.20) = 0.66 \tag{9}$$

A Simple Covid Example

Posterior distribution

binomial

Beta

$$\pi(\theta|X) = \frac{f(X|\theta)p(\theta)}{m(X)} \tag{10}$$

$$\pi(\theta|X) = \binom{n}{x}\theta^{x}(1-\theta)^{n-x} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}\frac{1}{m(x)} \tag{11}$$

The marginal

$\theta \in [0, 1]$

$$m(x) = \int f(X|\theta)p(\theta)d\theta \tag{12}$$

$$= \int_0^1 \binom{n}{x}\theta^{x}(1-\theta)^{n-x} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}d\theta \tag{13}$$

$$= \binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \theta^{x+a-1}(1-\theta)^{n-x+b-1}d\theta \tag{14}$$

A Simple Covid Example

Posterior distribution

The marginal (cont)

$$m(x) = \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+x)\Gamma(b+n-x)}{\Gamma(a+b+n)} \underbrace{\int_0^1 \frac{\Gamma(a+b+n)}{\Gamma(a+x)\Gamma(b+n-x)} \theta^{x+a-1}(1-\theta)^{n-x+b-1} d\theta}_{=1} \tag{15}$$

B(a+x, b+n-x)

$$= \underbrace{\binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+x)\Gamma(b+n-x)}{\Gamma(a+b+n)}}_{\text{no depende de } \theta} \tag{16}$$

The posterior

$$\pi(\theta|X) = \frac{\Gamma(a+b+n)}{\Gamma(a+x)\Gamma(b+n-x)} \theta^{x+a-1}(1-\theta)^{n-x+b-1} \tag{17}$$

$$\sim Beta(a+x, b+n-x) \tag{18}$$

A Simple Covid Example

With the posterior we can calculate then any moment of the posterior distribution. For example suppose that for our study none of the sample of individuals is infected (x=0). Then the posterior is

$$\pi(\theta|X=0) \sim Beta(2,40) \tag{19}$$

$a = 2, b = 20, n = 20$. Then

$B(a,b) \to E(\ ) = \frac{a}{a+b}$

$B(a+x, n-x+b)$

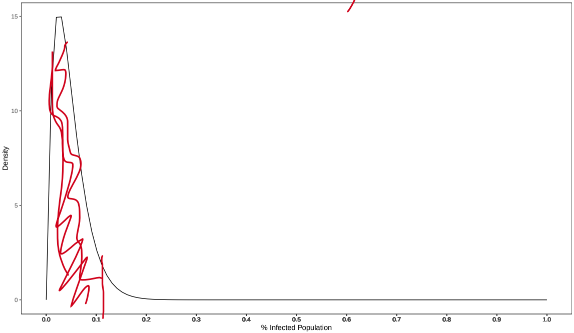
$$
\begin{aligned}
E(\theta|X=0) &= \frac{a+x}{a+b+n} && (20)\\
&= \frac{n}{a+b+n}\frac{x}{n} + \frac{a+b}{a+b+n}\frac{a}{a+b} && (21)\\
&= \frac{n}{a+b+n}\bar{x}_{MLE} + \frac{a+b}{a+b+n}\theta_{prior} && (22)\\
&= \frac{n}{a+b+n}0 + \frac{a+b}{a+b+n}\frac{2}{22} && (23)\\
&= 0.048 && (24)
\end{aligned}
$$

A Simple Covid Example

Since we have the full distribution we could calculate for example:

$$mode(\theta|X \overset{=0}{}) = 0.025 \tag{25}$$

$$\underbrace{Pr(\theta < 0.10|X = 0)}_{Pr(\theta > 0.5 / X=0)} = 0.93 \tag{26}$$



Bayes Theorem

Conjugate Priors. Basic idea:

λ' comb of λ

- $X \sim D(\theta)$ and $\theta \sim P(\lambda) \rightarrow \theta|X \sim P(\lambda')$
- $X \sim Bernoulli(\theta)$ and $\theta \sim Beta(a, b) \rightarrow \theta|X \sim Beta(a', b')$
- $X \sim N(\mu, \sigma)$ and $\theta \sim N(\mu_0, \sigma_0) \rightarrow \theta|X \sim N(\mu', \sigma')$

Batting Averages

We can model

$$\text{Batting Average} \sim Binomial(\overset{N}{\cancel{n}}, \theta) \tag{27}$$

- where n is the times at bat and θ is the proportion of successes
- We use a conjugate prior for simplicity

$$p(\theta) \sim Beta(\alpha_0, \beta_0) \tag{28}$$

The posterior is:

$$\pi(\theta) \sim Beta(\alpha_0 + \overset{x}{hits}, \beta_0 + \overset{N - x}{N - hits}) \tag{29}$$

Batting Averages

Using last class data:

```
## # A tibble: 6 x 4
##   name                H    AB average
##   <chr>           <int> <int>   <dbl>
## 1 Hank Aaron       3771 12364  0.305
## 2 Tommie Aaron      216   944  0.229
## 3 Andy Abad           2    21  0.0952
## 4 John Abadie        11    49  0.224
## 5 Ed Abbaticchio    772  3044  0.254
## 6 Fred Abbott       107   513  0.209
```

→ mucha info

→ poca info

Batting Averages

We are using batting averages to assess who are the best and worst batters

- Best?

```
## # A tibble: 6 x 4
##   name             H    AB average
##   <chr>        <int> <int>   <dbl>
## 1 Roe Skidmore     1     1       1
## 2 Charlie Snow     1     1       1
## 3 Matt Tupman      1     1       1
## 4 Allie Watt       1     1       1
## 5 Al Wright        1     1       1
## 6 George Yantz     1     1       1
```

Batting Averages

We are using batting averages to assess who are the best and worst batters

- Worst?

```
## # A tibble: 6 x 4
##   name                   H    AB average
##   <chr>              <int> <int>   <dbl>
## 1 Frank Abercrombie      0     4       0
## 2 Horace Allen           0     7       0
## 3 Pete Allen             0     4       0
## 4 Walter Alston          0     1       0
## 5 Bill Andrus            0     9       0
## 6 Wyman Andrus           0     4       0
```

Batting Averages

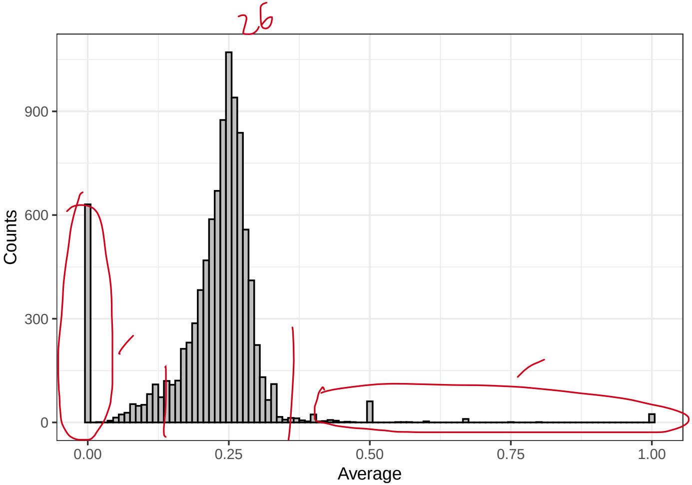
Question: Can we use Bayesian stats to get a better estimate?

$$X \sim Beta(\alpha_0, \beta_0) \tag{30}$$

- We don't know α_0 and β_0. We could use the fact that most batting averages are between .210 and .360. Select α_0 and β_0 accordingly.
- Or we can use Empirical Bayes: estimate these parameters from the data

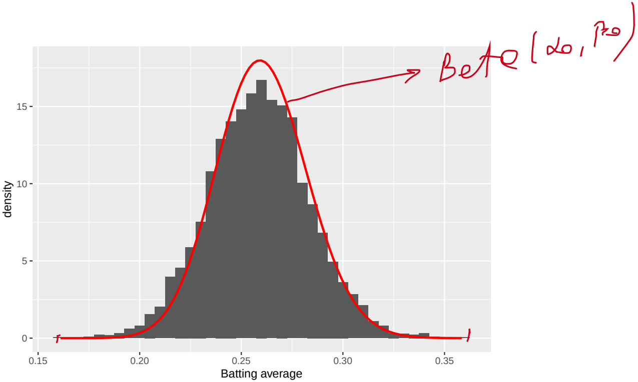
Batting Averages

Histogram of batting averages



Batting Averages

Restrict our sample to those data points that are informative (individuals that have gone at bat at least 500 times)



Batting Averages

How we find the parameters that find the red line $\rightarrow$ MLE! We know that

$$f(x_i|\alpha_0, \beta_0) = \frac{\Gamma(\alpha_0 + \beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)} x_i^{\alpha_0 - 1} (1 - x_i)^{\beta_0 - 1} \tag{31}$$

The log likelihood

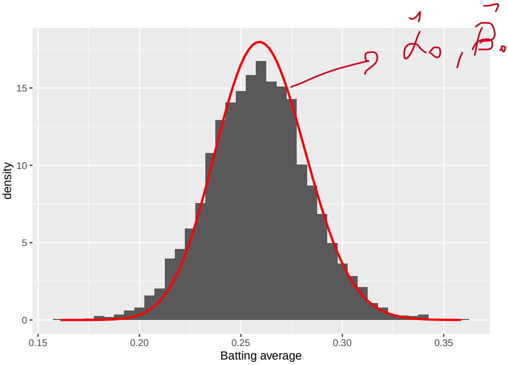
$$l(\alpha_0, \beta_0|X) = n.log(\frac{\Gamma(\alpha_0 + \beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)}) + \sum_{i=1}^{n}((\alpha_0 - 1)log(x_i) + (\beta_0 - 1)log(1 - x_i)) \tag{32}$$

In R

```
# log-likelihood function
ll <- function(alpha, beta) {
-sum(VGAM::dbetabinom.ab(x, total, alpha, beta, log = TRUE))
}
# maximum likelihood estimation
m <- mle(ll, start = list(alpha = 1, beta = 10),
method = "L-BFGS-B", lower = c(0.0001, .1))
ab <- coef(m)
```

Batting Averages

```
alpha0 <- ab[1]
101.7319
beta0 <- ab[2]
289.046
```



Batting Averages

We can use the estimated average based on the posterior mean

$$E(\theta|X) = \frac{\alpha_0 + hits}{\alpha_0 + \beta_0 + N} \tag{33}$$

- And ask again: who are the best batters by this improved estimate?

```
## # A tibble: 5 x 5
##   name                     H    AB average eb_estimate
##   <chr>                <int> <int>   <dbl>       <dbl>
## 1 Rogers Hornsby        2930  8173   0.358       0.354
## 2 Shoeless Joe Jackson  1772  4981   0.356       0.349
## 3 Ed Delahanty          2597  7510   0.346       0.342
## 4 Billy Hamilton        2164  6283   0.344       0.339
## 5 Willie Keeler         2932  8591   0.341       0.338
```

Batting Averages

We can use the estimated average based on the posterior mean

$$E(\theta|X) = \frac{\alpha_0 + hits}{\alpha_0 + \beta_0 + N} \tag{34}$$

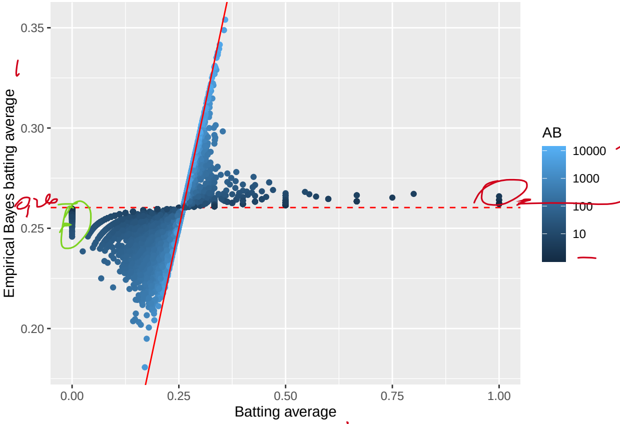
- Who are the *worst* batters?

```
## # A tibble: 5 x 5
##   name                 H    AB average eb_estimate
##   <chr>            <int> <int>   <dbl>       <dbl>
## 1 Bill Bergen        516  3028   0.170       0.181
## 2 Ray Oyler          221  1265   0.175       0.195
## 3 Henry Easterday    203  1129   0.180       0.201
## 4 John Vukovich       90   559   0.161       0.202
## 5 George Baker        74   474   0.156       0.203
```

Batting Averages

x=y 45°

We can see how EB changed all of the batting average estimates:



Predicting Batting Averages

- Now supposed you want to know the end of season final batting average of players, after observing them their 45 first times at bat.

180 part. bal

Player	Observed	Final
1	0.395	0.346
2	0.355	0.279
3	0.313	0.276
4	0.291	0.266
5	0.247	0.271
6	0.224	0.266
7	0.175	0.318

Predicting Batting Averages

- Recall that we can think each time at bat can be thought as a binomial trial, with θ the probability of success equal to the player's true batting average.
- With 45 trials, we can "reasonably" use a Normal Approximation.

$\theta_i \to$ desconocido

$$X_i \sim N(\theta_i, \sigma^2) \tag{35}$$

$\hookrightarrow \sigma^2$ conocido

where

- θ_i is the true batting average for player i
- σ^2 is the known variance that equals $(0.0659)^2$

We are going to use also a normal prior

$$\theta_i \sim N(\mu, \tau^2) \tag{36}$$

Predicting Batting Averages

With this model the posterior mean for θ_i is $E(\theta_i|X_i)$

$$E(\theta_i|X_i) = \frac{\sigma^2}{\sigma^2+\tau^2}\mu + \frac{\tau^2}{\sigma^2+\tau^2}X_i \tag{37}$$

$\rightarrow$ MLE

prior

Note that the marginal of X_i

$\sigma^2+\tau^2$ -?

$$m(X_i) \sim N(\mu, \sigma^2+\tau^2) \quad i = 1,\dots,n \tag{38}$$

with these we can construct estimates of $E(\theta_i|X_i)$, note that

$S^2 = \frac{\sum (x_i - \bar{x})^2}{n} \rightarrow$ sesgado

$E(S^2) = \sigma^2+\tau^2$

$$E(\bar{X}) = \mu \tag{39}$$

$S^2 = \frac{(n-1)}{(n-1)} \frac{\sum (x_i - \bar{x})^2}{n}$

$$E\left[\frac{(n-3)\sigma^2}{\sum(X_i-\bar{X})^2}\right] = \frac{\sigma^2}{\sigma^2+\tau^2} \tag{40}$$

$\rightarrow$ conocido

$n > 3$

$n < 3$ no funciona

Predicting Batting Averages

The empirical Bayes estimator of θ_i is then

$$\delta(X_i) = \left[\frac{(n-3)\sigma^2}{\sum((X_i - \bar{X})^2}\right]\bar{X} + \left[1 - \frac{(n-3)\sigma^2}{\sum((X_i - \bar{X})^2}\right]X_i \qquad (41)$$

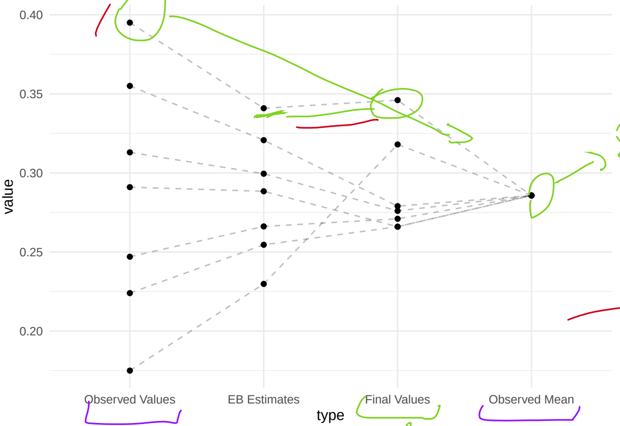
$\frac{\sigma^2}{\tau^2+\sigma^2}$ MLE

Player	Observed	Final	Empirical Bayes
1	0.395	0.346	0.341
2	0.355	0.279	0.321
3	0.313	0.276	0.299
4	0.291	0.266	0.288
5	0.247	0.271	0.266
6	0.224	0.266	0.255
7	0.175	0.318	0.230

- RMSE Observed 6.861903 $\to$ MLE
- RMSE EB 3.918203

Predicting Batting Averages

$\hat{\theta} = X\hat{\beta}$



$\bar{X}$ de todos los jugadores

MLE
bateo
45

Final
Temporada

$\bar{X}$

Review & Next Steps

- Recap Bayesian
- Empirical Bayes Examples
- Next couple of classes we are going to focus on
  - Concepts underlying spatial data: points, lines, polygons, reference systems
  - Plotting and describing spatial data
  - Econometric models for spatial data
- Questions? Questions about software?

Further Readings

- Casella, G., & Berger, R. L. (2002). Statistical inference (Vol. 2, pp. 337-472). Pacific Grove, CA: Duxbury. Chapter 7
- Casella, G. (1985). An introduction to empirical Bayes data analysis. The American Statistician, 39(2), 83-87.
- Robinson, D. (2017). Introduction to Empirical Bayes: Examples from Baseball Statistics. 2017.