PC $(r, \theta)$ RC $(x, y)$

$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$

$\boxed{\begin{aligned} r &= \pm\sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}}$

# Section 8.2 - Graphs of Polar Equations

**Objectives:**

- **Graphing polar equations: circles,**
- **Symmetry**
- **Graphing polar equations using Desmos**

## Graphing Polar Equations

only r  only θ  θ = input

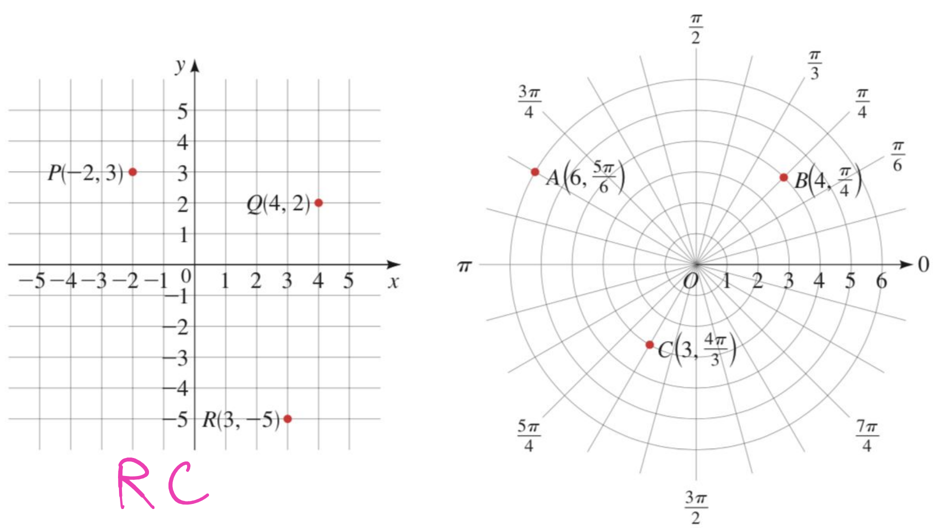
**Definition(s):** The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation $(r, \theta)$ whose coordinates satisfy the equation.

Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations than by rectangular equations.

**Important notes:**

1. Polar functions do not need to pass the vertical line test!
2. But they are still functions using the definition: for each unique input, there's only one output.

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure (a)). To plot points in polar coordinates, it is convenient to use a grid consisting of circles centered at the pole and rays emanating from the pole, as in Figure (b).

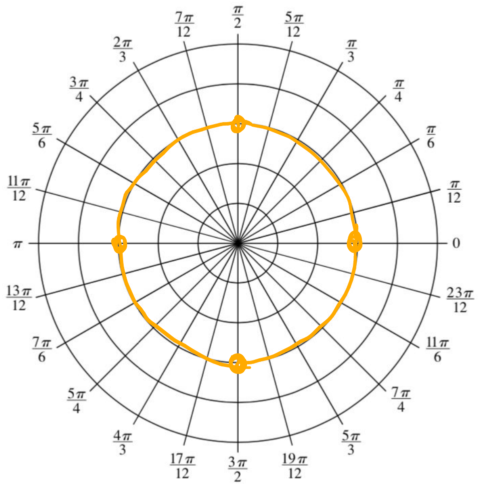



(a) Grid for rectangular coordinates  (b) Grid for polar coordinates

Desmos demonstrations: https://www.desmos.com/calculator/8vvjscjxto

- **Graphing Circles**

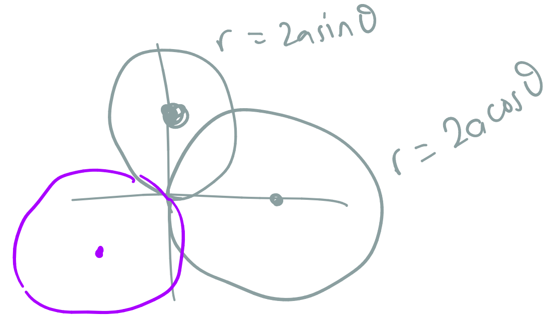
**Ex 1:** Sketch the graph of $r = 3$ and express the equation in RC.



$r = \pm\sqrt{x^2+y}$

$(3)^2 = \left(\pm\sqrt{x^2+y^2}\right)^2$

$\boxed{9 = x^2+y^2}$ RC




M

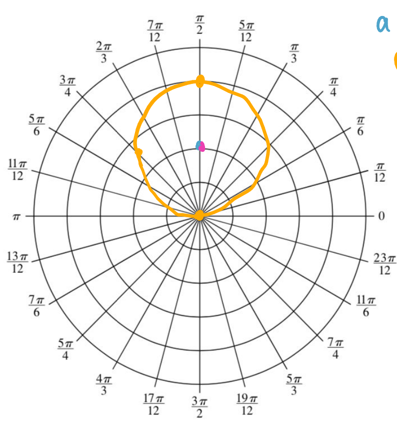
Theorem: Circles

1. Circles: $r = a$. Have radius $|a|$ and centered at origin.
2. Circles: $r = 2a\sin(\theta)$. Have radius $|a|$ and centered at $(a, \pi/2)$. r θ — PC
3. Circles: $r = 2a\cos(\theta)$. Have radius $|a|$ and centered at $(a, 0)$. — PC
4. Circles: $r = 2a\sin(\theta) + 2b\cos(\theta)$. Have radius $\sqrt{a^2+b^2}$ and centered at $(a, b)$.

Note: these will always pass through the origin eventually.

*Proof: 1. Obvious. For 2. and 3., multiply both sides by r and convert to RC. Then put the resulting eq in graphing form using complete the square. 4. Similar to the previous two, but complete the square twice.*

**Ex 2:** Sketch the graph of $r = 4\sin(\theta)$ and express the equation in RC.



a = 2
circle
center
$(a, \frac{\pi}{2})$
$(2, \frac{\pi}{2})$

$r(r) = (4\sin\theta)r$

$\underbrace{r^2}_{x^2+y^2} = 4(\underbrace{r\sin\theta}_{y})$

$x^2+y^2 = 4y$

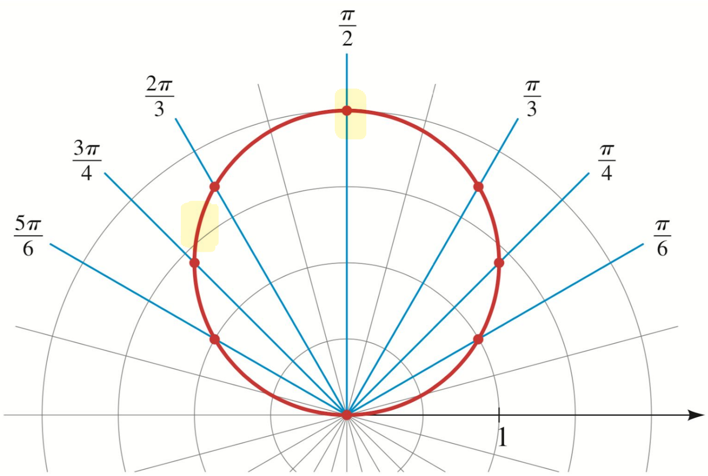
$x^2+y^2-4y = 0$

↓ CTS

$x^2+Bx = \left(x+\frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$

$x^2+(y-2)^2-4 = 0$

$\boxed{x^2+(y-2)^2 = 4}$ EQ circle w/ center $(0,2)$




Tips for sketching circles: when you know you're dealing with a circle, try $\theta = 0, \pi/2, \pi$ and $3\pi/2$ as values when plotting.

- **Lines**

In RC, lines are of the form: $x = a$, $y = a$, or $y = mx + b$. In PC:

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$\tan\theta = \frac{y}{x}$
$y = (\underbrace{\tan\theta}_{m})x$

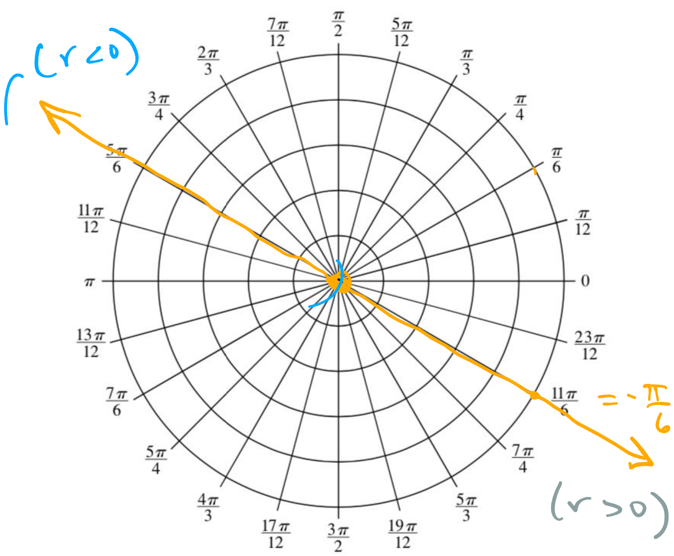
Theorem: Lines

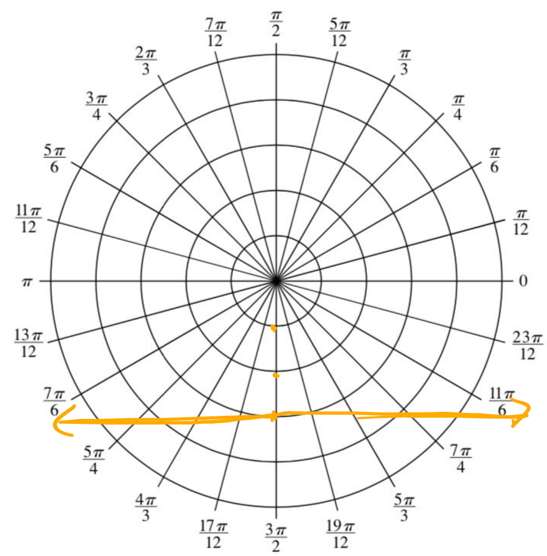
5. Lines with slope: $\theta = a$. Line through origin with slope $\tan(a)$.
6. Vertical Lines: $r\cos(\theta) = a$. Equivalent to $x = a$ in RC. $\to r\cos\theta = x \to x = a$
7. Horizontal Lines: $r\sin(\theta) = a$. Equivalent to $y = a$ in RC. $\to r\sin\theta = y \to y = a$

**Ex 3:** Sketch the graphs of the following and express the equations in RC.

(a) $\theta = -\pi/6$

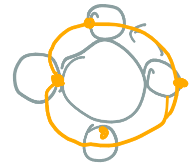
(b) $r\sin(\theta) = -3 \to y = -3$ (HL)



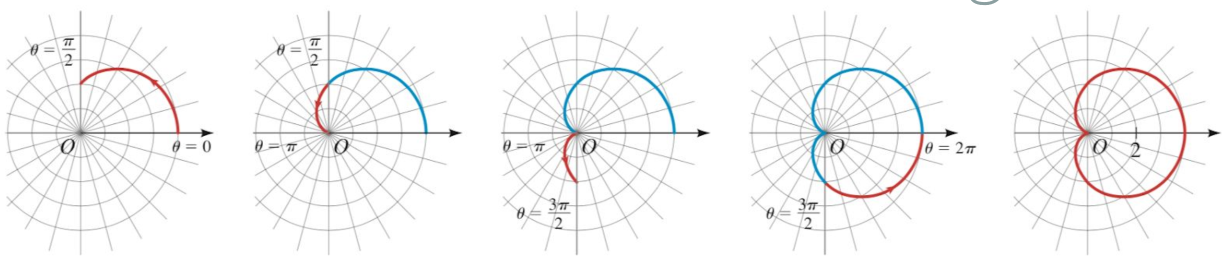



- **Cardioids**

Cardioids are shapes that resemble “hearts”. They are classical shapes from drafting/drawing before computers, of course. Cardioids are derived by “tracing the point of a circle rolling along another circle”.



For example, the graph of $r = 2 + 2\cos(\theta)$ is pictured below.



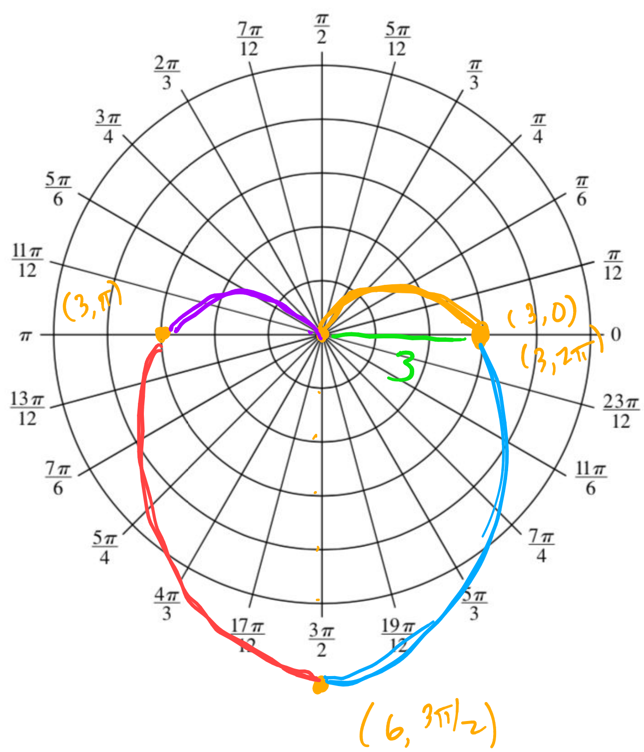

Check out an animation here:
https://upload.wikimedia.org/wikipedia/commons/9/9d/Ani_Epicyloid-cardioid.gif

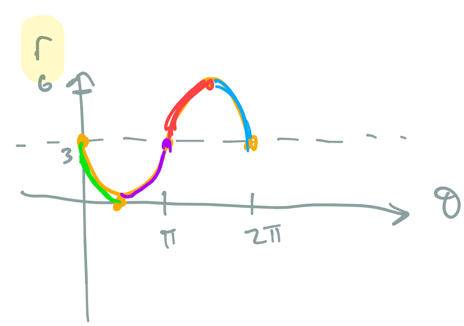
**Ex 4:** Sketch the graph of $\boxed{r = 3 - 3\sin(\theta)}$ the cardioid.

Draw by hand by plotting point at: $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$.

Hint: look at graph of $r$ vs $\theta$ in rectangular coordinates to get idea of what $r$ is doing as $\theta$ increases.

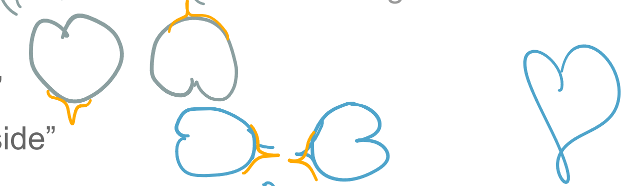
Points
$\theta = 0 \quad r = 3 - 3\sin 0 = 3$
$\theta = \pi/2 \quad r = 3 - 3\sin\frac{\pi}{2} = 0$
$\theta = \pi \quad r = 3 - 3\sin\pi = 3$
$\theta = \frac{3\pi}{2} \quad r = 3 - 3\sin\frac{3\pi}{2} = 6$
$\theta = 2\pi \quad r = 3 - 3\sin 2\pi = 3$




$r = 3 - 3\sin\theta$

$a(1-\sin\theta)$ $a(1+\sin\theta)$



Theorem: Cardioids

8. Cardioids: $r = a(1 \pm \sin(\theta))$ "up/down"
9. Cardioids: $r = a(1 \pm \cos(\theta))$ "on their side"

$1+\cos\theta$ $1-\cos\theta$

Tips:

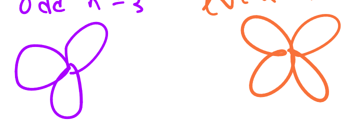
- cardioids with sine and minus sign: heart positioned correctly
- cardioids with sine and plus sign: heart is upside down
- cardioids with cosine and minus sign: heart is on its side with tip pointing left
- cardioids with cosine and plus sign:heart on its side with tip pointing right

- **Roses**

Theorem: Roses

10. Roses: $r = cos(n\theta)$ or $r = \sin(n\theta)$. The number of petals is determined by:
    - $n$ is odd: there are $n$ "petals" in the rose
    - $n$ is even: there are $2n$ "petals" in the rose

odd $n=3$ even $n=2$



**Ex 5:** Sketch the graph of $\boxed{r = 4\sin(3\theta)}$ and express the equation in RC.
Use Desmos. The tip for graphing roses is solving the equations $r = 4$ and $r = -4$ and using the number of petals given by the above theorem. More generally, using $r = \pm \max$

rose → $n=3$ → odd → 3 petals

$r = 4\sin(3\theta) \to r_{max} = 4$

+1

$r=4$

$4\sin(3\theta)=4$

$\sin(3\theta)=1$

$\frac{3\theta}{3} = \frac{\frac{\pi}{2}+2\pi k}{3}$

$\theta = \frac{\pi}{6} + \frac{2\pi}{3}k$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$r=-4$

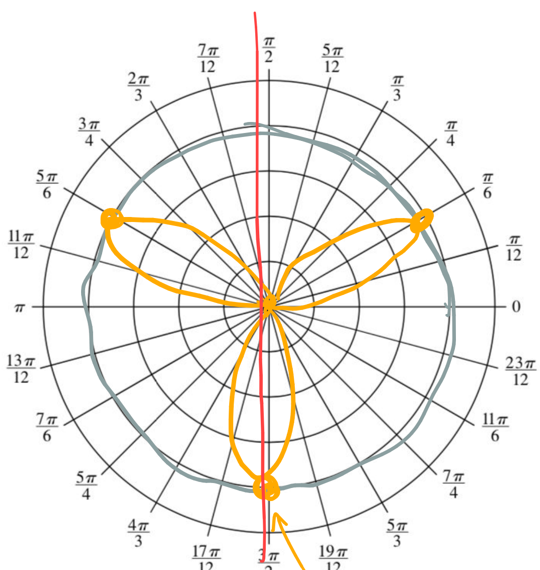
$4\sin(3\theta)=-4$

$\sin(3\theta)=-1$

$\frac{3\theta}{3} = \frac{\frac{3\pi}{2}+2\pi k}{3}$

$\theta = \frac{3\pi}{6} + \frac{2\pi}{3}k$

$\theta = \frac{\pi}{2}$




$(-4, \frac{\pi}{2}) = (4, \frac{3\pi}{2})$

$r=-4,\ \theta=\frac{\pi}{2}$

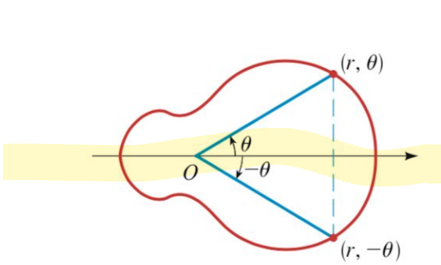
$r=4,\ \theta=\frac{\pi}{2}+\pi$

Symmetry only across $\theta=\pi/2$

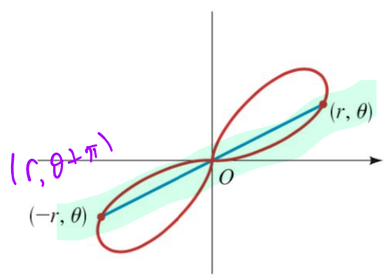
- **Symmetry**

Symmetry Tests

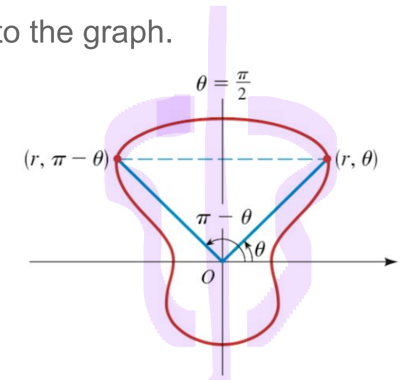
- Symmetry about the polar axis: if graph is unchanged when replacing $\theta$ by $-\theta$. That is, both $(r, \theta)$ and $(r, -\theta)$ belong to the graph.
- Symmetry about the pole: if graph is unchanged when replacing $r$ by $-r$, or by $\theta$ by $\theta \pm \pi$. That is, both $(r, \theta)$ and $(-r, \theta)$ belong to the graph.
- Symmetry about the line $\theta = \pi/2$: if graph is unchanged when replacing $\theta$ by $\pi - \theta$. That is, both $(r, \theta)$ and $(r, \pi - \theta)$ belong to the graph.




(a) Symmetry about the polar axis

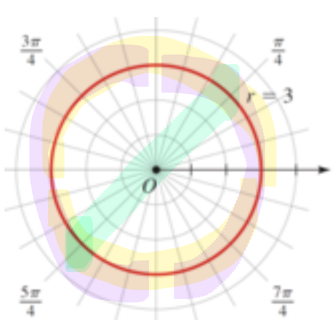



(b) Symmetry about the pole

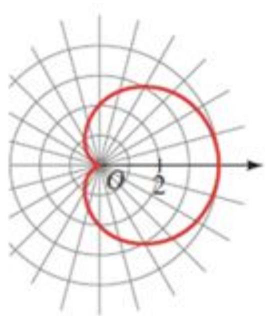



(c) Symmetry about the line $\theta = \frac{\pi}{2}$

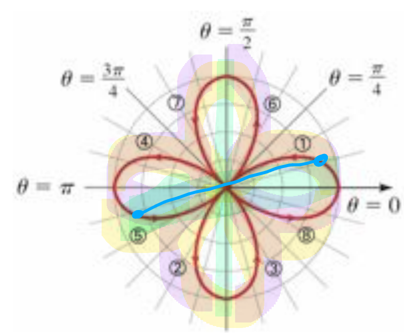
Tip Use symmetry tests when solving problems via matching.




Symmetry:
* about polar axis
* about the pole
* about the line $\theta = \pi/2$




Symmetry:
* about polar axis




Symmetry:
* about polar axis
* about the pole
* about the line $\theta = \pi/2$

- **Limaçons**

Limaçon is French for “snail”. The shapes include cardioids and also shapes that look like cardioids with extra loops. See summary of curves below.

Note when $a = b$ these are Cardioids

Theorem: Limaçons

11. Limaçons: $r = a \pm b\cos(\theta)$ or $r = a \pm b\sin(\theta)$.

- **Graphing polar equations using Desmos**

Use Desmos for problems that ask you to graph equations with technology.

Tip Play around with the domain to see what you need to sketch the entire curve.

## SUMMARY

Know how to sketch and recognize the graphs of the following equations in PC:

1. Circles 2. Lines 3. Cardioids 4. Roses 5. Limaçons 6. Lemniscates

drawin 8.1