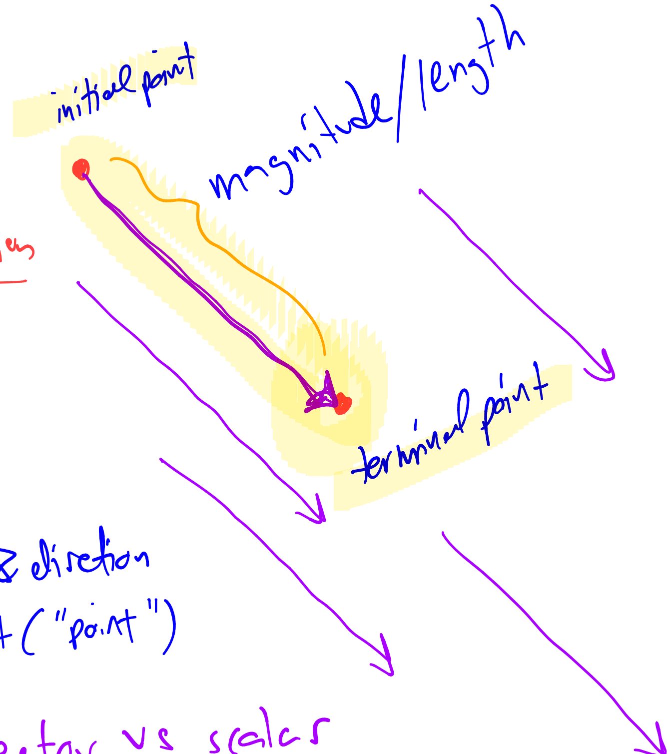
Section 9.1 - Vectors in 2D

Objectives:
- **Geometric Description of Vectors**
- **Vectors in the Coordinate Plane**
- **Using Vectors to model velocity and force**

Geometric Description of Vectors



Definition(s): We start by defining a few terms:

"arrow"

- **Vector:** has a magnitude & direction
- **Initial point:** tail vector
- **Terminal point:** tip vector

Notation $\vec{\ }$ $\vec{v}$

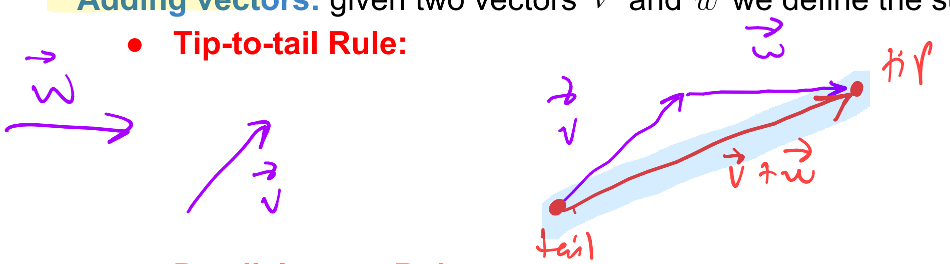
- **Magnitude** or **length** of a vector:
- **Direction** of a vector: two points describe directions
- **Equal vectors:** same if same magnitude & direction
- **Zero vector:** initial point = terminal point ("point")
- **Scalar:** a real #

↳ so clearly talk about vector vs scalar #

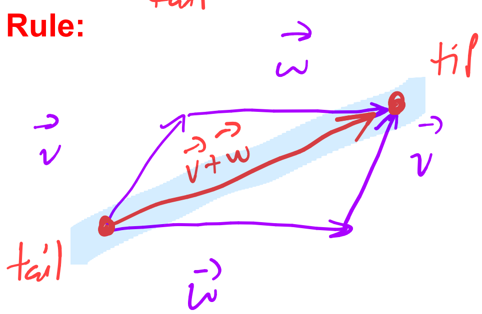
Vector Arithmetic

Adding vectors: given two vectors $\vec{v}$ and $\vec{w}$ we define the sum $\vec{v}+\vec{w}$ as follows:

- **Tip-to-tail Rule:**

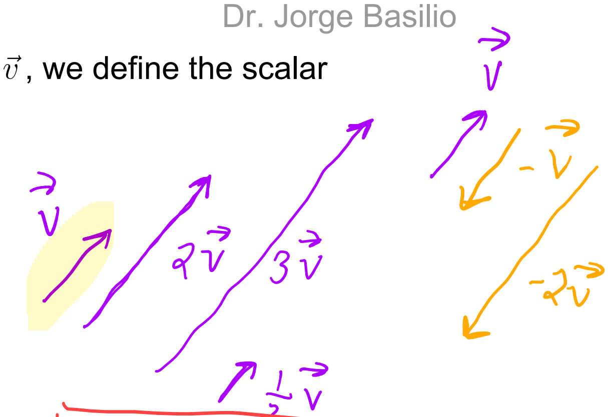


- **Parallelogram Rule:**



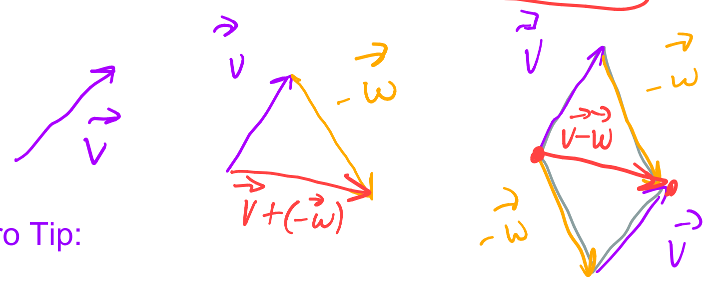
scalar = real #

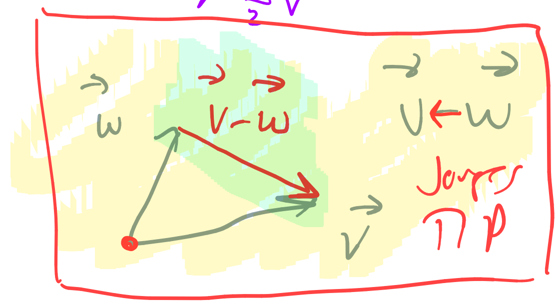
Scalar Multiplication: for any constant c and a vector $\vec{v}$, we define the scalar multiplication $c \cdot \vec{v}$:



- **For positive c:** multiply length of $\vec{v}$ by c
- **For negative c:** multiply length of $\vec{v}$ by $|c|$ & make opposite direction

Subtracting vectors: draw $\vec{v} - \vec{w}$ as follows:

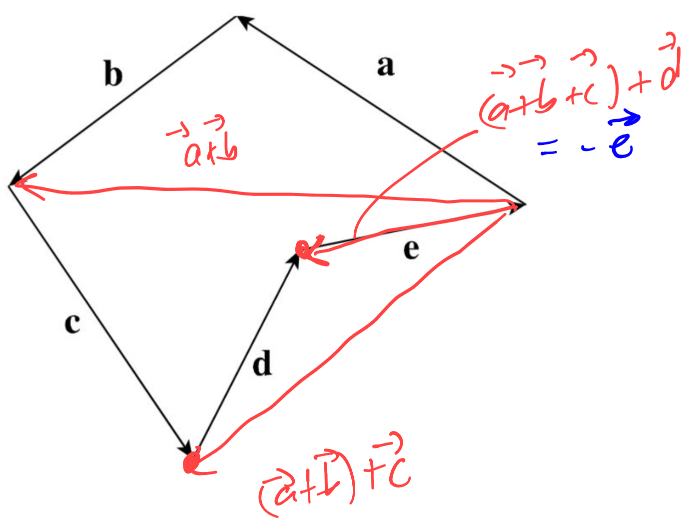




Pro Tip:

Important Note: notice this is a bit awkward. This is a real number times a vector. It is impossible to define multiplication using two vectors!

Ex 1: Compute $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}-\mathbf{e}$

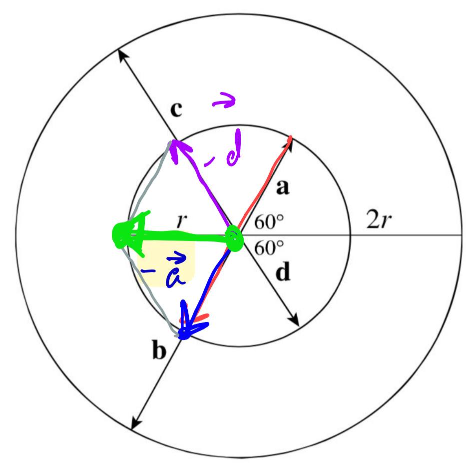


$$(\vec{a}+\vec{b}+\vec{c}+\vec{d}) = -\vec{e}$$

$$\vec{a}+\vec{b}+\vec{c}+\vec{d}-\vec{e} = -\vec{e}-\vec{e}$$

$$\boxed{-2\vec{e}}$$

Ex 2: Compute $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$



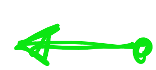
- $\vec{b} = -2\vec{a}$

  $\vec{a}+\vec{b} = \vec{a} - 2\vec{a} = -\vec{a}$

- $\vec{a}+\vec{b}+\vec{c}+\vec{d} = -\vec{a} + \vec{c}+\vec{d}$

  $\vec{c}+\vec{d} = -\vec{d}$

  $= -\vec{a} - \vec{d}$



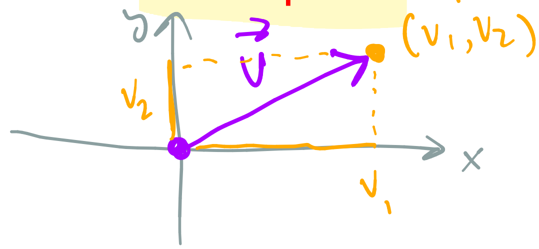
• Vectors in the Coordinate Plane

We want a more precise description of vectors that doesn't rely purely on drawing pictures. To do this, we give components to a vector as follows:

→ starting point = origin

Definition(s): If we place a vector with its tail at the origin, then the tip can be described with the coordinates of a single point. We say that a vector is in **standard position**. point TIP

Notation: $\boxed{\vec{v} = \langle v_1, v_2 \rangle}$.

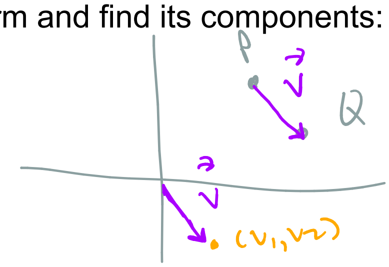


Horizontal component of $\vec{v}$ = v_1
Vertical component of $\vec{v}$ = v_2

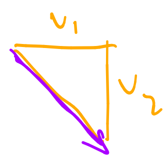
Key point: "pointy bracket" notation expresses all vectors with the same starting point at the origin $(0, 0)$.

Theorem: Equality of vectors in Components
If $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, then $\vec{v} = \vec{w}$ *if and only if* $v_1 = w_1$ & $v_2 = w_2$

Given two points P and Q we can form the vector $\vec{v} = \vec{PQ}$. We can put $\vec{v}$ in standard form and find its components:

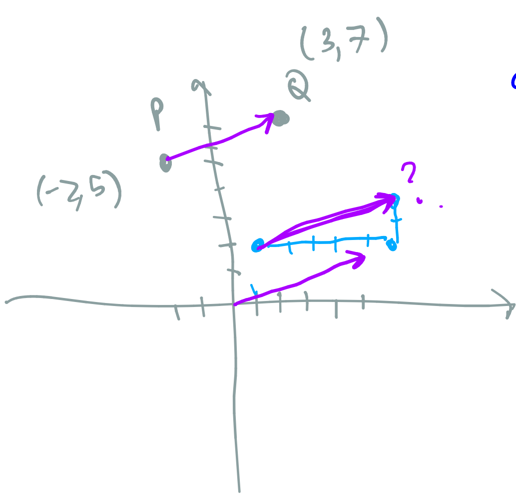


$P = (p_1, p_2)$
$Q = (q_1, q_2)$



$$\boxed{\begin{aligned}\vec{v} &= \vec{PQ} \\ &= \langle q_1 - p_1, q_2 - p_2 \rangle\end{aligned}}$$

Ex 3: Let $P = (-2, 5)$ and $Q = (3, 7)$. If $\vec{v} = \vec{PQ}$,
(a) find the components of $\vec{v}$ in standard form
(b) If $\vec{v}$ is moved to have initial point $(1, 2)$, what is its terminal point?



a) $\boxed{\vec{v} = \langle 5, 2 \rangle}$ standard form

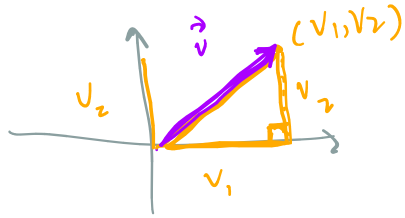
b) terminal point. $\boxed{(6, 4)}$

pointy brackets

Theorem: Vector Length in Components

If $\vec{v} = \langle v_1, v_2 \rangle$ then the magnitude or length of $\vec{v}$ is denoted by $|\vec{v}|$ and given by:

Proof



length $= \sqrt{v_1^2 + v_2^2}$

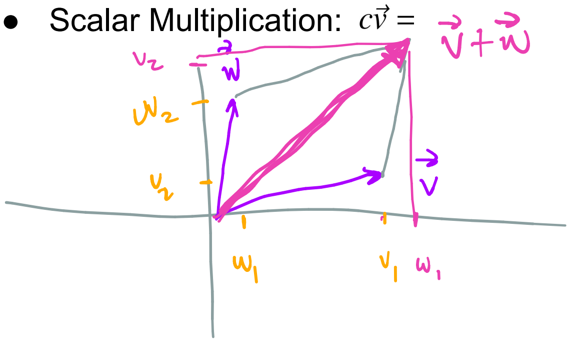
$|\vec{v}| =$ length

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2}}$$

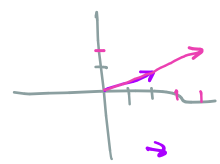
Theorem: Vector Arithmetic in Components

horizontal comp. vert comp.

- Addition: $\vec{v} + \vec{w} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \boxed{\langle v_1 + w_1, v_2 + w_2 \rangle} = \vec{v} + \vec{w}$
- Subtraction: $\vec{v} - \vec{w} = \langle v_1, v_2 \rangle - \langle w_1, w_2 \rangle = \boxed{\langle v_1 - w_1, v_2 - w_2 \rangle} = \vec{v} - \vec{w}$
- Scalar Multiplication: $c\vec{v} =$



$c\vec{v} = c\langle v_1, v_2 \rangle = \boxed{\langle cv_1, cv_2 \rangle} = c\vec{v}$



$\vec{v} = \langle 2, 1 \rangle$ $\quad 2\vec{v} = \langle 4, 2 \rangle$

<u>**Ex 4:**</u> Let $\vec{v} = \langle 2, 1 \rangle$ and $\vec{w} = \langle 1, -1 \rangle$. Find:

(a) $\vec{v} + \vec{w} = \langle 2+1, 1+(-1) \rangle = \boxed{\langle 3, 0 \rangle}$

(b) $\vec{v} - \vec{w} = \langle 2-1, 1-(-1) \rangle = \boxed{\langle 1, 2 \rangle}$

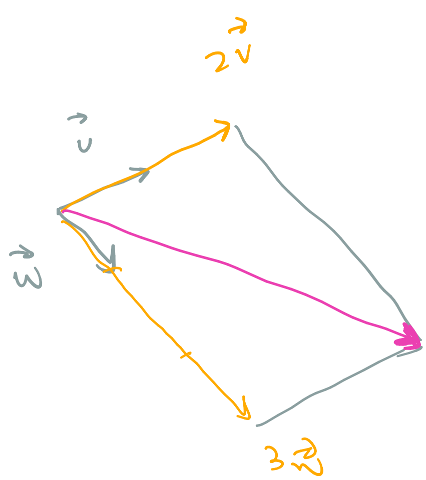
(c) $2\vec{v} = \boxed{\langle 4, 2 \rangle}$

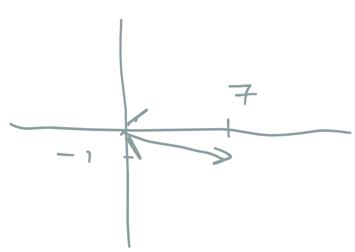
(d) $3\vec{w} = \boxed{\langle 3, -3 \rangle}$

(e) $2\vec{v} + 3\vec{w} = 2\langle 2, 1 \rangle + 3\langle 1, -1 \rangle$

$= \langle 4, 2 \rangle + \langle 3, -3 \rangle$

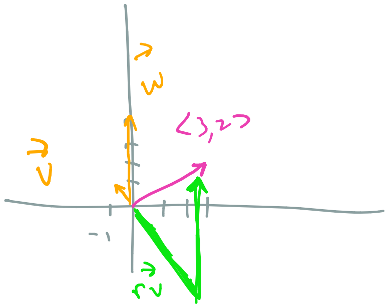
$= \boxed{\langle 7, -1 \rangle}$





Ex 5: Let $\vec{v} = \langle -1, 1\rangle$ and $\vec{w} = \langle 0, 4\rangle$.

(a) Find r and s such that $\langle 3, 2\rangle = r\vec{v} + s\vec{w}$. $= r\langle -1,1\rangle + s\langle 0,4\rangle$



$\langle 3,2\rangle = \langle -r, r\rangle + \langle 0, 4s\rangle$

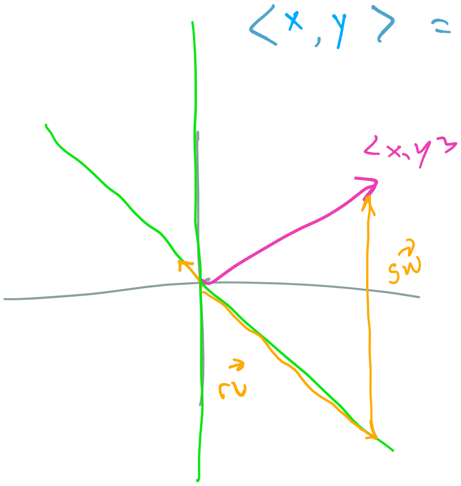
$\langle 3, 2\rangle = \langle -r+0, r+4s\rangle$

$\begin{cases} 3 = -r \longrightarrow r = -3 \\ 2 = r+4s \longrightarrow 2 = (-3)+4s \\ \qquad 5 = 4s \longrightarrow s = 5/4 \end{cases}$

$\boxed{r = -3 \quad s = 5/4}$

(b) Does a vector $\langle x, y\rangle$ exist for which no r or s can be found for $\langle x, y\rangle = r\vec{v} + s\vec{w}$?

$\langle x, y\rangle = \langle -r, r+4s\rangle$



$\begin{cases} x = -r \longrightarrow r = -x \\ y = r+4s \longrightarrow y = -x+4s \end{cases}$

$s = \dfrac{x+y}{4}$

$\boxed{r = -x \quad s = \dfrac{x+y}{4}}$

bold

$\mathbf{v} = \vec{v}$

PROPERTIES OF VECTORS

Vector addition	Multiplication by a scalar
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ ✓	$1\mathbf{u} = \mathbf{u}$
Length of a vector	$0\mathbf{u} = \mathbf{0}$
$\lvert c\mathbf{u}\rvert = \lvert c\rvert\,\lvert\mathbf{u}\rvert$	$c\mathbf{0} = \mathbf{0}$

Zero Vector $\vec{0} = \langle 0, 0\rangle$

$\vec{u} + (-\vec{u}) = \langle u_1, u_2\rangle + (-1)\langle u_1, u_2\rangle$

$= \langle u_1, u_2\rangle + \langle -u_1, -u_2\rangle$

$= \langle u_1 - u_1, u_2 - u_2\rangle$

$= \langle 0, 0\rangle = \vec{0}$

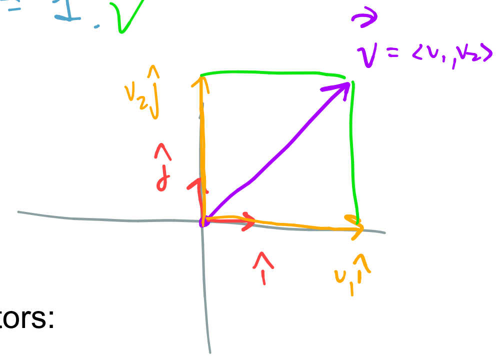
- ## Unit Vectors

$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$

Definition: A **unit vector** is a vector with magnitude or length equal to 1.

Ex6: Check that $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ is a unit vector.

$|\vec{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1.$ ✓



Standard Basis Vectors: special unit vectors

$\boxed{\hat{\imath} = \langle 1, 0 \rangle}$ and $\boxed{\hat{\jmath} = \langle 0, 1 \rangle}$

Theorem: Any vector can be written using the standard basis vectors:

$$\boxed{\vec{v} = \langle v_1, v_2 \rangle = v_1 \hat{\imath} + v_2 \hat{\jmath}}$$

Ex7: (a) Write $\vec{v} = \langle 7, -3 \rangle$ in terms of $\hat{\imath}$ and $\hat{\jmath}$. $\boxed{\vec{v} = 7\hat{\imath} - 3\hat{\jmath}}$

(b) If $\vec{v} = 2\hat{\imath} - 3\hat{\jmath}$ and $\vec{w} = -5\hat{\imath} + 2\hat{\jmath}$, find $2\vec{v} + 3\vec{w}$.

$2\vec{v} + 3\vec{w} = 2(2\hat{\imath} - 3\hat{\jmath}) + 3(-5\hat{\imath} + 2\hat{\jmath})$

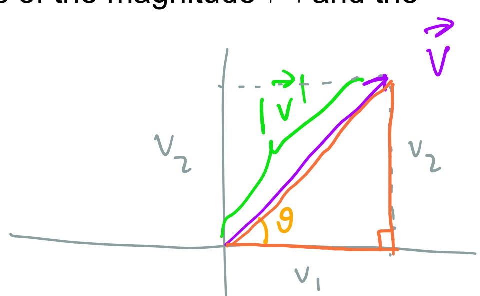
$= 4\hat{\imath} - 6\hat{\jmath} - 15\hat{\imath} + 6\hat{\jmath} = -11\hat{\imath} + 0\hat{\jmath} = \boxed{-11\hat{\imath}}$

- ## Polar Form of Vectors

We studied vectors in standard form essentially using rectangular coordinates. In many ways, however, vectors are more naturally expressed using polar coordinates: length and direction (angle). Given $|\vec{v}|$ & θ

Theorem: A vector $\vec{v} = \langle v_1, v_2 \rangle$ can be expressed in terms of the magnitude $|\vec{v}|$ and the direction θ as follows:

The horizontal component: $v_1 = |\vec{v}| \cos\theta$
The vertical component: $v_2 = |\vec{v}| \sin\theta$



$\cos\theta = \frac{v_1}{|\vec{v}|}$ $\sin\theta = \frac{v_2}{|\vec{v}|}$

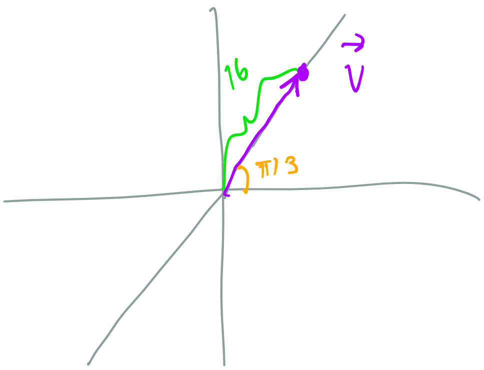
$|\vec{v}|\cos\theta = v_1$ $|\vec{v}|\sin\theta = v_2$

Using i and j:

$\vec{v} = <v_1, v_2> = (|\vec{v}|\cos\theta)\hat{\imath} + (|\vec{v}|\sin\theta)\hat{\jmath}$

$$\boxed{\vec{v} = |\vec{v}|(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})}$$

Ex8: A vector $\vec{v}$ has length 16 and direction $\pi/3$. Find both components of $\vec{v}$ and express $\vec{v}$ in standard form.



$v_1 = |\vec{v}|\cos\theta = 16\cos\frac{\pi}{3} = 16\left(\frac{1}{2}\right) = 8$

$v_2 = |\vec{v}|\sin\theta = 16\sin\frac{\pi}{3} = 16\left(\frac{\sqrt{3}}{2}\right) = 8\sqrt{3}$

$\boxed{\vec{v} = \langle 8, 8\sqrt{3}\rangle}$ standard form.

$\vec{v} = 16\left(\cos\frac{\pi}{3}\,\hat{\imath} + \sin\frac{\pi}{3}\,\hat{\jmath}\right)$

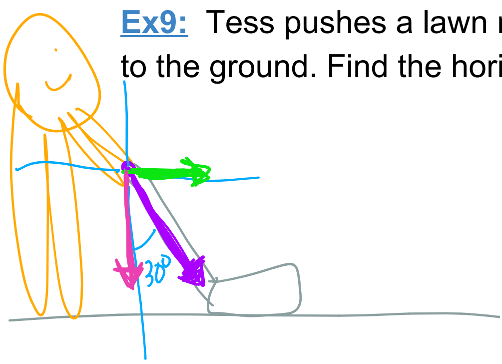
$r = 16$
$\theta = \pi/3$

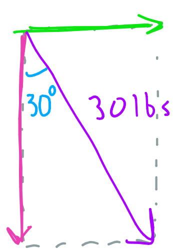
- **Using vectors to model velocity and force**

Vectors are everywhere and help model many situations. A few examples:

- Velocities: (magnitude + direction)
  - Winds in weather
  - ships in the ocean
  - Planes in the sky
- Forces

Ex9: Tess pushes a lawn mower with a force of 30 pounds exerted at an angle of $30°$ to the ground. Find the horizontal and vertical components of the force.





$|\vec{F}| = 30$ lbs

Horizontal: $|\vec{F}|\cos(30°) = 30\cos(30°)$

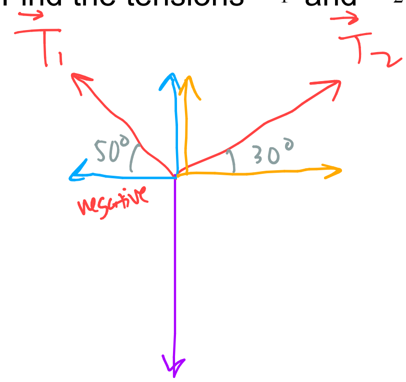
$= \boxed{25.98 \text{ lbs}}$ Horizontal

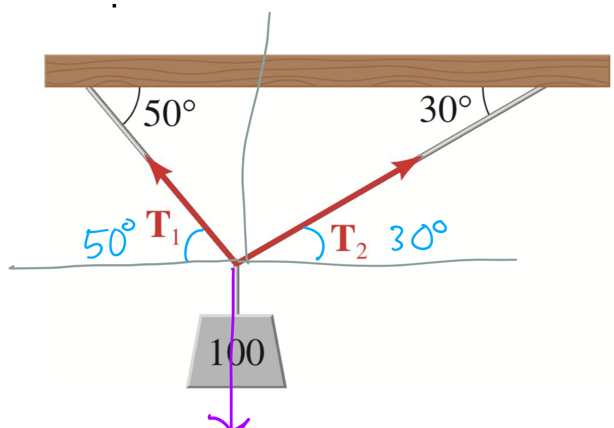
Vertical: $|\vec{F}|\sin(30°) = 30\sin(30°)$

$= \boxed{15 \text{ lbs}}$ Vertical

Bonus: A 100-lb weight hangs from a string as shown in the figure. The weight is a force pointing downward: $\vec{F} = -100\hat{j}$. Due to the tension in the ropes, all three forces are in equilibrium. This means that $\vec{T}_1 + \vec{T}_2 + \vec{F} = \vec{0}$.

Find the tensions $\vec{T}_1$ and $\vec{T}_2$ in the string.





$\vec{F}$ = weight $= -100\hat{j}$

- $\vec{T}_1 = -|\vec{T}_1|\cos 50^\circ\,\hat{i} + |\vec{T}_1|\sin 50^\circ\,\hat{j}$
- $\vec{T}_2 = |T_2|\cos 30^\circ\,\hat{i} + |T_2|\sin 30^\circ\,\hat{j}$
- KEY $\vec{T}_1 + \vec{T}_2 + \vec{F} = \vec{0} \Longrightarrow \vec{T}_1 + \vec{T}_2 = -\vec{F} = -(-100\hat{j}) = 0\hat{i} + 100\hat{j}$

$$\begin{cases} \text{make LHS } \hat{i} = 0 \\ \text{make LHS } \hat{j} = 100 \end{cases}$$

$$\begin{cases} -|T_1|\cos 50^\circ + |T_2|\cos 30^\circ = 0 \\ |T_1|\sin 50^\circ + |T_2|\sin 30^\circ = 100 \end{cases}$$

Now solve for $|T_1|$ & $|T_2|$

Let $x = |T_1|$, $y = |T_2|$, $a = \cos 50^\circ$, $c = \sin 50^\circ$, $b = \cos 30^\circ$, $d = \sin 30^\circ$

$$\begin{cases} ① \; -ax + by = 0 \\ ② \; cx + dy = 100 \end{cases}$$

Solve ① for x: $-ax + by = 0$

$$-ax = -by$$

$$x = \frac{by}{a} \Big\} \text{ sub into } ②$$

sub x into ②: $a\left(c\left(\frac{by}{a}\right) + dy\right) = (100)a$

$$bcy + ady = 100a$$

$$y(ad + bc) = 100a$$

$$\boxed{y = \frac{100a}{ad + bc}}$$

plug-in a, b, c, & d & we have y!

Recall

Let $x = |T_1|$, $y = |T_2|$, $a = \cos 50^\circ$, $c = \sin 50^\circ$
$b = \cos 30^\circ$, $d = \sin 30^\circ$

- $y = \dfrac{100a}{ad+bc} = \dfrac{100\cos 50^\circ}{(\cos 50^\circ)(\sin 30^\circ) + (\cos 30^\circ)(\sin 50^\circ)} = \dfrac{100\cos(50^\circ)}{\sin(80^\circ)} = |T_2|$

use this ID to simplify!

- $x = \dfrac{by}{a} = \dfrac{\cos 30^\circ}{\cancel{\cos 50^\circ}} \cdot \dfrac{100\cancel{\cos(50^\circ)}}{\sin 80^\circ} = \dfrac{100\cos 30^\circ}{\sin 80^\circ} = |T_1|$

Finally:

- $\vec{T}_1 = -|\vec{T}_1| \cos 50^\circ \hat{i} + |\vec{T}_1| \sin 50^\circ \hat{j}$
- $\vec{T}_2 = |T_2| \cos 30^\circ \hat{i} + |T_2| \sin 30^\circ \hat{j}$

- $\vec{T}_1 = -\left(\dfrac{100\cos 30^\circ}{\sin 80^\circ}\right)\cos 50^\circ \hat{i} + \left(\dfrac{100\cos 30^\circ}{\sin 80^\circ}\right)\sin 50^\circ \hat{j}$

$$\boxed{\vec{T}_1 = -56.52579\ldots \hat{i} + 67.36481\ldots \hat{j}}$$

remember to use degree mode on calc

- $\vec{T}_2 = \left(\dfrac{100\cos 50^\circ}{\sin 80^\circ}\right)\cos 30^\circ \hat{i} + \left(\dfrac{100\cos 50^\circ}{\sin 80^\circ}\right)\sin 30^\circ \hat{j}$

$$\boxed{\vec{T}_2 = 56.52579\ldots \hat{i} + 32.63518\ldots \hat{j}}$$