§11.8: Power Series

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Outline

Guiding Questions for §11.8

Guiding Question(s)

1. How does a calculator compute (approximations) to transcendental functions like
$$\sin(1), \pi, e, \ln(3), \ldots$$
2. What are power series?
3. When do power series define a function?

Introduction

- Review: what the meaning of $\sin(1)$ is.

- Review: what is the definition of π?

- Review: what is the "official definition" of e?

- Review: what is the definition of $\ln(3)$?

Introduction

- Review: state when a geometric series converges and diverges.

- For what values of x does the series
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$
converge and diverge? Don't forget to consider the endpoints!

- What equivalent function, $f(x)$, equals $\sum_{n=0}^{\infty} x^n$ when it converges?

Power Series

Definition 1: Power Series

- Given a sequence $\{c_n\}_{n=0}^{\infty}$, the associated series
$$\sum_{n=0}^{\infty} c_n x^n$$
is called the power series centered at $x = 0$ associated to the sequence of coefficients $\{c_n\}_{n=0}^{\infty}$.
- The values of x for which the above series converges is called the domain or interval of convergence.
- We can define power series centered at any point we wish: the series
$$\sum_{n=0}^{\infty} c_n (x-a)^n$$
is called the power series centered at $x = a$ associated to the sequence $\{c_n\}_{n=0}^{\infty}$.

Power Series

Remarks

- If the power series is centered at $x = a$, then a is always in the domain for any sequence $\{c_n\}_{n=0}^{\infty}$.
- There are power series for which the domain is just $\{a\}$.
- There are power series for which the domain is $(-\infty, \infty)$.
- Polynomials are power series! Just let $c_n = 0$ for all $n >$ degree of polynomial.
- There are other ways to make (new) functions out of a sequence:
  - Fourier Series: $\sum c_n \sin(nx)$ and $\sum c_n \cos(nx)$
  - Dirichlet Series: $\sum \frac{c_n}{n^x}$
- Important Point Power series are functions on their domain!

Activity 1:

Determine the **interval of convergence** for:

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

Power Series

Activity 2:

Determine the **interval of convergence** for:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$$

Power Series

Power Series

Activity 3:

Determine the **interval of convergence** for:

$$\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n$$

Power Series

Activity 4:

Determine the **interval of convergence** for:

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Power Series

Power Series

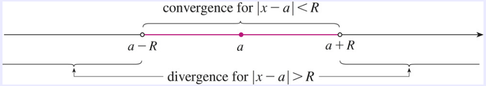
Theorem 1: Interval and Radius of Convergence Theorem

For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities for the domain:

1. There is a positive number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

   R is called the radius of convergence.

   We say $|x-a| < R$ or $(a-R, a+R)$ is the interval of convergence.

   Warning! At the endpoints, anything can happen! They must be checked individually for each series.

2. The series converges only "at a point" $x = a$. We say: $R = 0$.
3. The series converges at every real number x. We say: $R = \infty$.

Power Series

- Act 1: $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$. Centered at $a = 0$.
  - Interval of Convergence: $(-3, 3)$.
  - Radius of Convergence: $R = 3$.
- Act 2: $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$. Centered at $a = -3$.
  - Interval of Convergence: $(-7, 1)$.
  - Radius of Convergence: $R = 4$.
- Act 3: $\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n$. Centered at $a = 2$ (why?).
  - Interval of Convergence: $[\frac{15}{8}, \frac{17}{8})$.
  - Radius of Convergence: $R = \frac{1}{8}$ (why?).
- Act 4: $\sum_{n=0}^{\infty} n!(2x+1)^n$. Centered at $a = -\frac{1}{2}$ (why?).
  - Interval of Convergence: $\{-\frac{1}{2}\}$.
  - Radius of Convergence: $R = 0$ (why?).

Activity 5:

Determine the **interval of convergence** and the **radius of convergence** for:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Activity 6:

Determine the **interval of convergence** and the **radius of convergence** for:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1}$$

Power Series

Where are we going?

- Soon, we will find power series that are equivalent to:
    - $\sin(x)$
    - e^x
    - $\ln(x)$
- Using these power series, we can approximate $\sin(1)$, π, e, and $\ln(3)$ to any accuracy desired by simple arithmetic!
- For π, we can use either Liebniz' or Euler's formulas mentioned at the beginning of the chapter.