§11.4: The Comparison Tests

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

Dr. Jorge Eduardo Basilio

Department of Mathematics & Computer Science
Pasadena City College

Class #19 Notes

May 2, 2019
Spring 2019

Outline

1. Guiding Questions
2. Introduction
3. Comparison Test
4. Limit Comparison Test

Guiding Questions for §11.4

Guiding Question(s)

1. What is the comparison test?
2. What is the limit comparison test?

Introduction

- Finding the exact value (sum) of a general series is very difficult. We typically can only estimate the series with a partial sum, S_n.
- Still, we must know whether or not the series converges before we try to use the estimate (say, in solving an applied problem).
- Our goal in this section is to develop two more tools that helps us determine whether a series converges or diverges.
- The tools we learn in this section are called the comparison test and the limit comparison test.
- This question is so important that we will learn additional tools in future sections.

Comparison Test

- We focus on series with positive terms only. I.e. $a_n > 0$ (or non-negative $a_n \geq 0$)
- The basic idea of the comparison test is to use basic inequalities to compare with series we already know converge or diverge (so far: geometric series, harmonic series, p-series)
- We use the informal "squint test."
- If the squint test suggest convergence, estimate with a series that's *known to converge that's larger*
- If the squint test suggest divergence, estimate with a series that's *known to diverge that's smaller*

Comparison Test

- $\sum_{j=1}^{\infty} \frac{1}{2^j+2j+1}$ "squint test" should compare with $\sum_{j=1}^{\infty} \frac{1}{2^j}$ (which converges by geometric series with $r = 1/2$)

$$\text{each term: } \frac{1}{2^j + 2j + 1} \le \frac{1}{2^j}, \text{ for each } j = 1, 2, 3, \ldots$$

- $\sum_{m=1}^{\infty} \frac{1}{4m-3}$ "squint test" should compare with $\sum_{m=1}^{\infty} \frac{1}{m}$ (which diverges since harmonic)

$$\text{each term: } \frac{1}{4m - 3} \ge \frac{1}{4m}, \text{ for each } j = 1, 2, 3, \ldots$$

Comparison Test

Theorem 1: Comparison Test

Assume that $a_n > 0$ and $b_n > 0$, i.e. the series $\sum a_n$ and $\sum b_n$ have positive terms.

(a) If $a_n \leq b_n$ for all $n \geq 1$ and $\sum b_n < \infty$ [C] THEN $\sum a_n < \infty$ [C]

(b) If $b_n \leq a_n$ for all $n \geq 1$ and $\sum b_n = \infty$ [D] THEN $\sum a_n = \infty$ [D]

Remark: the comparison test still works even if we ignore a finite number of terms. That is, if we know the comparison work *after ignoring a finite number of terms*, $n \geq N$ (for some large N).

Comparison Test

Comparison Test

Comparison Test

Activity 1:

Use the comparison test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Comparison Test

Comparison Test

Activity 2:

Use the comparison test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or diverges.

(Hint: ignore the first few terms then compare.)

Comparison Test

Limit Comparison Test

- Hard to make comparisons with inequalities with series like:

$$\sum_{n=0}^{\infty} \frac{1}{3^n - 4} \qquad \text{or} \qquad \sum_{k=0}^{\infty} \frac{2k^3 - 5k - 3}{\sqrt{7 + k^8}}$$

- "Squint Test" says the first is like $\sum \frac{1}{3^n}$ so should converge (geometric series with $r = 1/3$) and the second is like $\sum \frac{k^3}{k^{8/2}} = \sum \frac{1}{k}$ so should diverge (harmonic).
- The limit comparison test turns the squint test into a rigorous test!

Limit Comparison Test

Theorem 2: Limit Comparison Test

Assume that $a_n > 0$ and $b_n > 0$, i.e. the series $\sum a_n$ and $\sum b_n$ have positive terms. Define

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If $0 < L < \infty$, THEN either both series converge, or both series diverge.
If $L = 0$ or $L = \infty$ then the test is inconclusive.

Remark: the limit comparison test still works even if we ignore a finite number of terms. That is, if the first few terms are negative, but are positive after then we can still use this test.

Limit Comparison Test

Limit Comparison Test

Activity 3:

Use the limit comparison test to determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n - 4}$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{2k^3 - 5k - 3}{\sqrt{7 + k^8}}$

Limit Comparison Test

Limit Comparison Test