§7.2: Trigonometric Integrals

Ch 7: Techniques of Integration

Math 5B: Calculus II

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Class #10 Notes

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Guiding Questions for §7.2

Guiding Question(s)

1. What are the methods for evaluating integrals with products of powers of trigonometric functions?

Introduction

We study techniques for evaluating integrals of the form:

$$\int \sin^n(x)\cos^m(x)\,dx, \int \tan^n(x)\sec^m(x)\,dx, \int \sin(mx)\cos(nx)\,dx$$

- Why are these types of integrals interesting or useful?
- Many simple geometric problems lead to integrals of products of trigonometric functions.
- Another example involves digital music. It is made possible because we can solve integrals of the above form. In Chapter 11 (which we will study later) we'll lay the foundations for this application (so called, Fourier series).

Summary of Techniques

Ingredients Needed

Techniques for evaluating integrals of the form:

$$\int \sin^n(x)\cos^m(x)\,dx$$

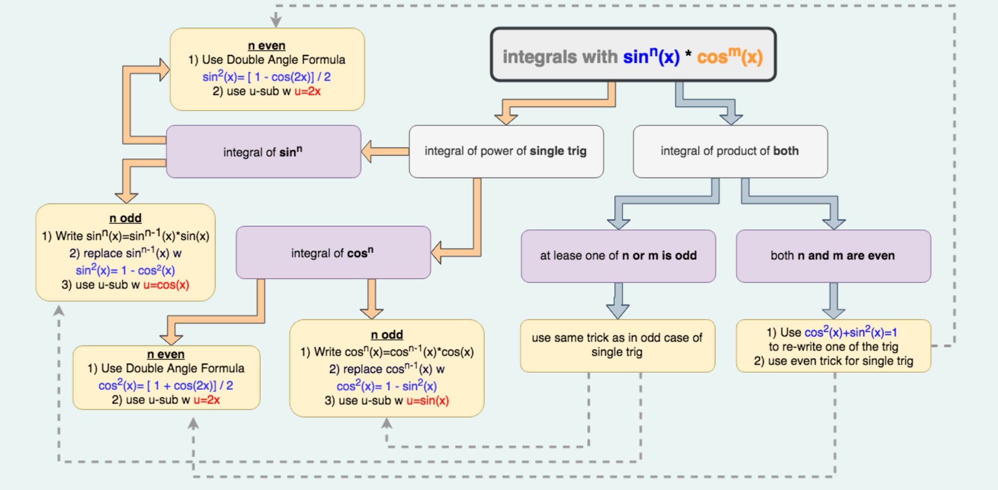
What ingredients are needed?

- **U-Substitution**
- **Trig Formulas**
  - **Pythagorean** $\boxed{\cos^2(x)+\sin^2(x)=1}$
  - **Double Angle** $\boxed{\sin^2(x)=\dfrac{1-\cos(2x)}{2}}$ and $\boxed{\cos^2(x)=\dfrac{1+\cos(2x)}{2}}$
  - **Product-to-Sum** $\boxed{\sin(A)\cos(B)=\dfrac{1}{2}\left(\sin(A-B)+\sin(A+B)\right)}$

    & 2 more formulas

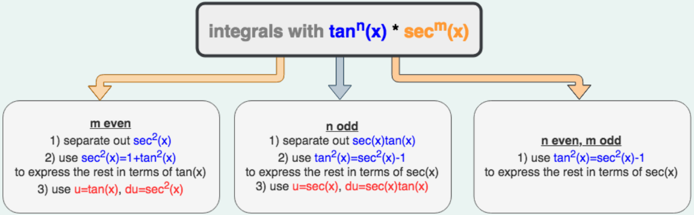
Summary of Techniques

Flow Chart for $\int \sin^n(x)\cos^m(x)\,dx$



Summary of Techniques

Flow Chart for $\int \tan^n(x)\tan^m(x)\,dx$



Keep in mind two basic integrals:

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \quad \text{and} \quad \int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

Powers of trigonometric functions: Odd Case

Activity 1: Case: at least one ODD

Evaluate:

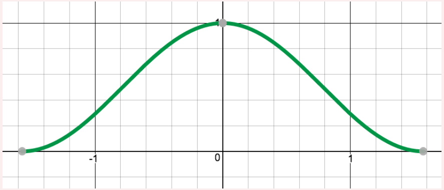
(a) $\displaystyle\int \sin^3(x)\,dx$

(b) $\displaystyle\int \sin^4(x)\cos^5(x)\,dx$

Powers of trigonometric functions: Odd Case

Activity 2: Case: EVEN

(a) Find the area under the curve $y = \cos^2(x)$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.



(b) Find: $\int \cos^4(x)\, dx$

Powers of trigonometric functions: Even Case

Powers of trigonometric functions: $\tan(x)$ and $\sec(x)$

Ingredients Nedded

We'll study techniques for evaluating integrals of the form:

$$\int \tan^n(x)\sec^m(x)\,dx$$

What are the ingredients needed?

- **U-Substitution**
- **Trig Formulas**
  - **Pythagorean** $\boxed{1+\tan^2(x)=\sec^2(x)}$
- See flow chart.

Activity 3:

Evaluate:

(a) Find: $\displaystyle\int \tan^3(x)\sec^5(x)\,dx$

(b) Find: $\displaystyle\int \tan^2(x)\sec^4(x)\,dx$

Powers of trigonometric functions: $\tan(x)$ and $\sec(x)$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

Example 1:

We work this out:

$$\begin{aligned}
\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx \\
&= \int \frac{1}{u} \cdot -du \quad (u = \cos(x), du = -\sin(x)dx) \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C \\
&= \ln|cos^{-1}(x)| + C \\
&= \ln|\sec(x)| + C
\end{aligned}$$

$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

Example 2:

This derivation requires a clever trick that comes out of nowhere:

$$\begin{aligned}\int \sec(x)\,dx &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx\\ &= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx\\ &= \int \frac{1}{u}\cdot du\\ &\qquad (u=\sec(x)+\tan(x), du = (\sec(x)\tan(x)+\sec^2(x))dx)\\ &= \ln|u| + C\\ &= \ln|\sec(x)+\tan(x)| + C\end{aligned}$$

Activity 4:

Use IBP to evaluate: $\displaystyle\int \sec^3(x)\,dx$

Powers of trigonometric functions: tan and sec

We finish with an important class of integrals. These show up in lots of applications.

Ingredients Needed

We'll study techniques for evaluating integrals of the form:

$$\int \sin(nx)\cos(mx)\,dx$$

What are the ingredients needed?

- **Trig Formulas**
  - **Product-to-Sum**

$$\boxed{\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)}$$

$$\boxed{\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)}$$

$$\boxed{\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)}$$

Activity 5:

Find: $\displaystyle\int \sin(4x)\cos(3x)\,dx$

Products of trig functions: Case: $\sin(nx)$ & $\cos(mx)$