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§7.7: Approximate Integration

Ch 7: Techniques of Integration
Math 5B: Calculus II

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Class #13 Notes

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Outline

1. Guiding Questions
2. Review of definitions of Integration
3. Endpoint Approximations
4. Midpoint Approximations
5. Trapezoid Approximations
6. Simpson's Rule Approximations
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Guiding Questions for §7.7

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Guiding Question(s)

1. What are some common techniques to approximate integrals numerically?
2. What are error bounds for the midpoint, trapezoid, and Simpson's rules?
3. How can we use them to anticipate the number of sums needed to achieve a certain accuracy?

Review Integration

- Definition of anti-derivative. Given a function $f(x)$, we say $F(x)$ is an anti-derivative of $f(x)$ if

$$F'(x) = f(x). \qquad \text{Notation: } F(x) + C = \int f(x)\,dx.$$

  so we could find anti-derivatives $F(x)$ using our knowledge of derivatives and "guessing" $F(x)$.

- Definition of definite integral.

  This is way more complicated. Notation: $\displaystyle\int_a^b f(x)\,dx.$

- Important point: though the notation is similar, they are very different.

$$\int f(x)\,dx = \text{infinitely many functions} \quad \text{and} \quad \int_a^b f(x)\,dx = \text{a single number}$$

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Review Integration

- Definition of definite integral. This is way more complicated.

  First, we cut up the interval $[a, b]$ into n pieces and denote the endpoints of the subintervals $[x_{i-1}, x_i]$ with a partition $a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < x_{i+1} < \cdots < x_n = b$.

  Pick a random sample of $c_i \in [x_{i-1}, x_i]$ and compute the area of the ith sub-rectangle: Area (one rectangle) $= f(c_i)\Delta x_i$ where $\Delta x_i = x_{i-1} - x_i$

  Approximate area under the graph with n rectangles: $S_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i$

  To get the exact area we take a limit:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

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Review Integration

- This is hard and alot of work! But don't ever forget that definite integrals are (infinite) sums which we can always approximate
$$\boxed{\int_a^b f(x)\,dx \approx S_n}.$$
- The Fundamental Theorem of Calculus is a huge short-cut!
$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$$
provided we can find an anti-derivative $F(x)$.
- At this point, you now have many tools in your integration toolbox to compute a lot of integrals. But...
- There's still many, many integrals we can't find anti-derivatives of!
- So we will go back to the definition of the definite integral and settle for approximations.

Approximate Integration Techniques

Essentially, we have many choices for how to pick $c_i \in [x_{i-1}, x_i]$
(Note: the book uses x_i^* instead of c_i).

- **Left-endpoint approximation (LEA)**: $c_i = x_{i-1}$.

  Ex: when f is increasing, this underestimates the exact area.
- **Right-endpoint approximation (REA)**: $c_i = x_i$.

  Ex: when f is increasing, this overestimates the exact area.
- **Midpoint approximation (MPA)**: $c_i = \frac{x_{i-1}+x_i}{2}$.

  this is much better than the LEA and REA, in general, since it averages the them!

In my opinion, it's a bit silly to write out he full formulas for these approximations. If you know the idea, then you can just find these values by hand with patience.

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Endpoint Approximations

Set up for all approximations: n a positive integer. $a, b \in \mathbb{R}$. for $i = 0, 1, 2, \ldots, n$:

$\boxed{\Delta x = \dfrac{b-a}{b}}$ and $\boxed{x_i = a + i \cdot \Delta x}$ and subintervals $[x_{i-1}, x_i]$

Definition 1: Endpoint Approximations

- Left Endpoint approximation (LEA): $S_n(LEA) = L_n$:

$$\int_a^b f(x)\,dx \approx \Delta x\left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right] = S_n(LEA)$$

- Right Endpoint approximation (REA): $S_n(REA) = R_n$:

$$\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right] = S_n(REA)$$

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Midpoint Approximations

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Set up for all approximations: n a positive integer. $a, b \in \mathbb{R}$. for $i = 0, 1, 2, \ldots, n$:

$\boxed{\Delta x = \frac{b-a}{b}}$ and $\boxed{x_i = a + i \cdot \Delta x}$ and subintervals $[x_{i-1}, x_i]$

Definition 2: Midpoint Approximations

- Midpoint approximation (MPA): if we average the endpoints of the subintervals, we get the midpoints, given by $\overline{x_i} = \frac{x_{i-1}+x_i}{2} = a + i \cdot (\Delta x/2)$. Then $S_n(MPA) = M_n$:

$$\int_a^b f(x)\,dx \approx \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + \cdots + f(\overline{x_n})\right] = S_n(MPA)$$

Trapezoid Approximations

Set up for all approximations: n a positive integer. $a, b \in \mathbb{R}$. for $i = 0, 1, 2, \ldots, n$:

$\boxed{\Delta x = \frac{b-a}{b}}$ and $\boxed{x_i = a + i \cdot \Delta x}$ and subintervals $[x_{i-1}, x_i]$

Definition 3: Trapezoid Approximations

- Trapezoid approximation (TrapA): if we average the height of the function at the endpoints of the subintervals, then we get trapezoids. $S_n(TrapA) = T_n$:

$$\int_a^b f(x)\,dx \approx \Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] = S_n(TrapA)$$

Note: Areas(Trap) $= \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x$

Simpson's Rule Approximations

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Definition 4: Simpson's Rule Approximations

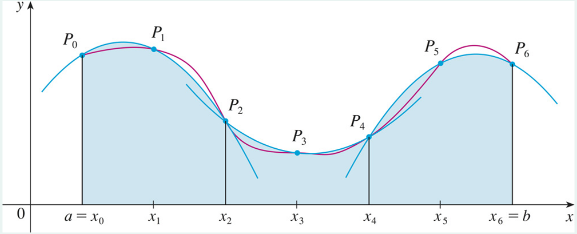
- Assume an even number n of subintervals
- Fact: given three distinct points, there's only one parabola passing through them
- Using this fact, on consecutive subintervals we can use the three points: $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_{i+1}), f(x_{i+1}))$ to find the unique parabola passing through these points. Do this for all pairs of subintervals (thus, why we need an even number).
- Simpson's Rule Approximation (SimpA): $S_n(SimpA) = S_n$:

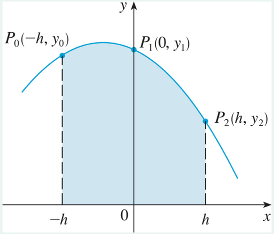
$$\int_a^b f(x)\,dx \approx$$

$$\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Simpson's Rule Approximations

- Notice the pattern: 1 4 2 4 2 ⋯ 2 4 2 4 1





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Definition 5: Error

- We define the error in the following way:

$$\int_a^b f(x)\,dx = \text{Approximation} + \text{Error}$$

  or

$$\text{Error} = \text{Exact} - \text{Approximation}$$

- With this definition the error can be positive or negative (or zero, of course).
- Notation: We will write Err for the error. If we need to be more specific, we can write $\text{Err}(S_n(MPA))$, $\text{Err}(S_n(TrapA))$, etc

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Activity 1:

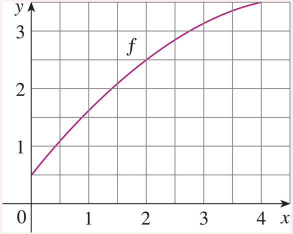
Approximate $\displaystyle\int_0^4 f(x)\,dx$ "by hand" using

(a) $L_2 = S_2(LEA)$

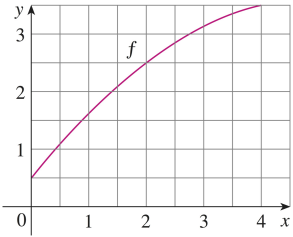
(b) $R_2 = S_2(REA)$

(c) $M_2 = S_2(MPA)$

(d) $T_2 = S_2(TrapA)$



Approximations



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Activity 2:

Using the same graph found in Activity 1, determine whether L_2, R_2, M_2, or T_2 are underestimates, overestimates, or not sure for the exact integral $\int_0^4 f(x)\,dx$.

Activity 3:

Use Sage to approximate $\displaystyle\int_{-2}^{2} \left(1 + x\sin(x^4)\right)\, dx$

(a) $L_{10} = S_{10}(LEA)$

(b) $R_{10} = S_{10}(REA)$

(c) $M_{10} = S_{10}(MPA)$

(d) $T_{10} = S_{10}(TrapA)$

(e) State the error for each of the above

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- Many times, we do not know the exact answer so computing the error of the approximation using our definition is not possible. The following gives us estimates which are extremely useful in many applied settings.
- We will not prove these formulas but you do need to know them.
- General strategy is use the derivatives of f to approximate the error.
- For MPA and TrapA we estimate the **second** derivative using $|f''(x)| \leq K$ for $x \in (a, b)$.
- For SimpA we estimate the **fourth** derivative using $|f^{(4)}(x)| \leq K$ for $x \in (a, b)$.

Error Bounds

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Theorem 1: Error Bounds

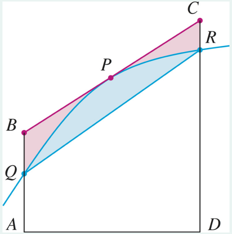
Estimates for the error of approximations are as follows.

- Assume that $|f''(x)| \leq K$ for $x \in (a,b)$. Then
  (a) MPA: $|\text{Err}(S_n(MPA))| \leq \dfrac{K(b-a)^3}{24n^2}$
  (b) TrapA: $|\text{Err}(S_n(TrapA))| \leq \dfrac{K(b-a)^3}{12n^2}$
- Assume that $|f^{(4)}(x)| \leq K$ for $x \in (a,b)$. Then
  (c) SimpA: $|\text{Err}(S_n(SimpA))| \leq \dfrac{K(b-a)^5}{180n^4}$

Error Bounds

General Observations

- Why are these useful? Because you can use these to determine the number of approximations n needed to guarantee a certain amount of accuracy.
- Perhaps surprisingly, the above estimates say we can expect the MPA to be more accurate than the TrapA (by a factor of 2). The following figure illustrates why:



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Activity 4:

How large should n be to guarantee that the approximation of $\int_{-1}^{1} e^{-x^2}$ is accurate to within 0.001 using

(a) MPA?

(b) TrapA?

(c) SimpA?

Error Bounds

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