§9.1: Modeling with Differential Equations

Ch 9: Differential Equations
Math 5B: Calculus II

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Class #6 Notes

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Outline

1. Guiding Questions
2. Introduction to Differential Equations
3. Models of Population Growth
4. Qualitative Behavior of a DE

Guiding Questions for §6.7

Guiding Question(s)

1. How can we apply use mathematical models to study problems and make predictions from the physics, chemistry, biology, economics, and other sciences?
2. What are differential equations?
3. What are some models for population growth?
4. How does a differential equation determine the shape of its solutions?

What is mathematical modeling?

- Mathematical modeling is a process using various mathematical structure (equations, functions, graphs, etc) to represent, describe, or predict real world situations. The aim is to reduce a (usually very complex) problem to a few essential characteristics.
- We'll explore how the derivative is a good tool for modeling many phenomena in the sciences.

Differential Equations

An important class of mathematical models is given by differential equations.

Definition 1: Differential Equations

A Differential Equation (DE) is an equation which involves a function $y = f(t)$ and its derivatives, $y'(t)$, $y''(t)$, etc, and the goal is to find function(s) that satisfy the equation. In other words, the goal in "solving the differential equation" is to "produce a function, or functions, that satisfy the equation."

- All possible solutions to a DE are called general solutions
- A specific solutions to a DE determined by additional information (called initial values) is called a particular solution

Differential Equations

Activity 1:

Verify that $y(t) = \frac{2}{3}e^t + e^{-2t}$ is a solution to the DE $y' + 2y = 2e^t$.

Activity 2:

Which of the following functions are solutions to $y'' + y = \sin(x)$?

(A) $y(x) = \sin(x)$

(B) $y(x) = \cos(x)$

(C) $y(x) = \frac{1}{2}x\sin(x)$

(D) $y(x) = -\frac{1}{2}x\cos(x)$

Example 1:

(a) Exponential Growth/Decay:

The DE is: $\boxed{\dfrac{dy}{dt} = ky}$

The solutions are: $\boxed{y(t) = Ce^{kt} = y_0 e^{kt}}$

(b) Newton's Law of Cooling:

The DE is: $\boxed{\dfrac{dT}{dt} = k(T - T_S)}$

The solutions are: $\boxed{T(t) = T_S + Ce^{kt}}$

Differential Equations

Example 2:

- Given any integrable function, $f(t)$, we can solve a great variety of simple differential equations of the form:

$$\frac{dy}{dt} = f(t)$$

  Where $f(t)$ is known and $y(t)$ is the unknown.

- The (infinitely many) solutions are given by:

$$y(t) = \int f(t)\,dt$$

- That is, the anti-derivative can be viewed as solutions to the corresponding DE. This is was one of the main results of Calc 1.

Example 3:

- In fact, this last example is from Newton. He invented the derivative so that he could define anti-derivatives! That is, he discovered the following differential equations for the equations of motion:

$$\boxed{a(t) = \frac{d^2 s}{dt^2} = -32\ ft/sec^2} \qquad \text{(Galileo's Discovery)}$$

- And solved them:

$$v(t) = \frac{ds}{dt} = \int \frac{d^2 s}{dt^2} dt = \int a(t) dt = \int -32 dt = -32t + v_0$$
$$s(t) = \int \frac{ds}{dt} dt = \int v(t) dt = \int (-32t + v_0) dt = -16t^2 + v_0 t + s_0$$

Activity 3:

Find the (i) general solutions and (ii) particular solutions to the following differential equations:

(a) $\frac{dy}{dt} = t$; $y(0) = 1$

(b) $\frac{dy}{dt} = \cos(t)$; $y(0) = -2$

(c) $\frac{dy}{dt} = \frac{1}{1+t^2}$; $y(1) = \frac{\pi}{4} + 1$

Models of Population Growth

- In §6.5, we studied the law of natural growth/decay: for $y = P(t)$

$$\frac{dP}{dt} = kP$$

- Interpretation is key: the rate of change of the amount $P(t)$ is proportional to the amount present.
    - Imagine: if $P(t)$ starts small and positive, $P(t) \approx 0$ then initially the rate of growth, $P'(t)$ is small. So, the population $P(t)$ grows slowly at first
    - As $P(t)$ grows, then $P'(t)$ grows so the rate of change increases.
    - This is the reason for the basic shape of exponential functions $y = e^{kx}$–i.e. the "boomerang"
- Since $P'(t) = \frac{dP}{dt} > 0$ for all t, $P(t)$ is always increasing!

Models of Population Growth

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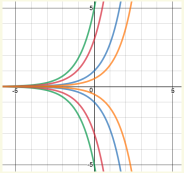


- General Solutions: $P(t) = Ce^{kt} = P_0 e^{kt}$

Models of Population Growth

- In §6.5, we studied the law of natural growth/decay: for $y = P(t)$

$$\frac{dP}{dt} = kP$$

- Very useful for the early growth of a population, but not very realistic in the "long run"
- After a while, the population will grow too big for its environments and the rate of growth will begin to slow down.
- So $P(t)$ is still increasing, but at a decreasing rate, later in time
- We need to modify the DE and come up with a new one with:
  - Early has $P'(t) > 0$
  - Later has $P'(t) < 0$

Models of Population Growth

Definition 2: Logistic Equation

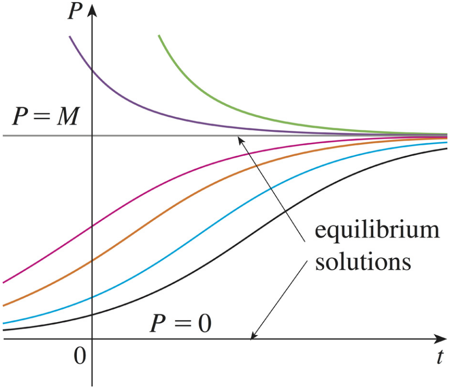
- Assume that there's a carrying capacity, M, where when $P(t) > M$ the rate of growth changes from positive to negative.
- Our new DE needs to have:
  - For small t: $P(t)$ is small compared to M and $\frac{dP}{dt} \approx kP$
  - For $t > M$: $P(t) > M$ and $\frac{dP}{dt} < 0$
- Logistic Differential Equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

- There's two trivial solutions called equilibrium solutions. They are the constant solutions:

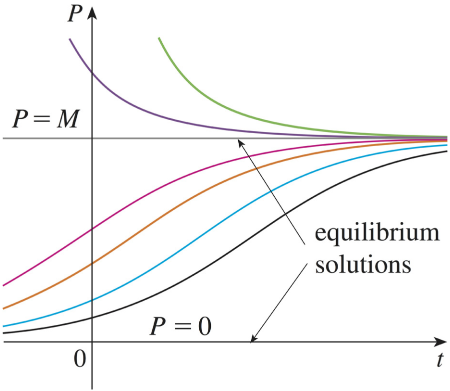
$$P(t) = 0 \qquad \text{and} \qquad P(t) = M$$

Models of Population Growth



- Guess solutions with this shape?
- Built from arctan(t)?
- Solve this later in §9.4
- For now we want to study how the DE determines the shape of its solutions–I call this the qualitative behavior of a DE

Models of Population Growth



$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

- If to start, $P(0)$ lies between 0 and M then $P'(t) > 0$ and $P(t)$ increases
- If $P(t)$ exceeds carrying capacity, $P(t) > M$, then $P'(t) < 0$
- Both cases: $P(t) \to M$ as $t \to \infty$ so $P'(t) \to 0$ as $t \to \infty$

Qualitative Behavior of a DE

We practice using the Qualitative Behavior of a DE to predict the shape of the solutions

Activity 4:

Match the differential equations with the solution graphs labeled I-IV. Give reasons for your choices.

(a) $y' = 1 + x^2 + y^2$

(b) $y' = xe^{-x^2-y^2}$

(c) $y' = \frac{1}{1+e^{x^2+y^2}}$

(d) $y' = \sin(xy)\cos(xy)$