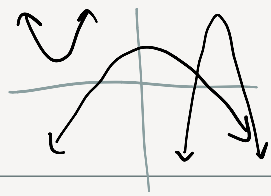
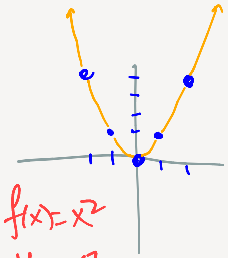





# Section 12.1 Parabolas

**Objectives**
- Geometric Definition of a Parabola
- Equations and Graphs of Parabolas
- Applications

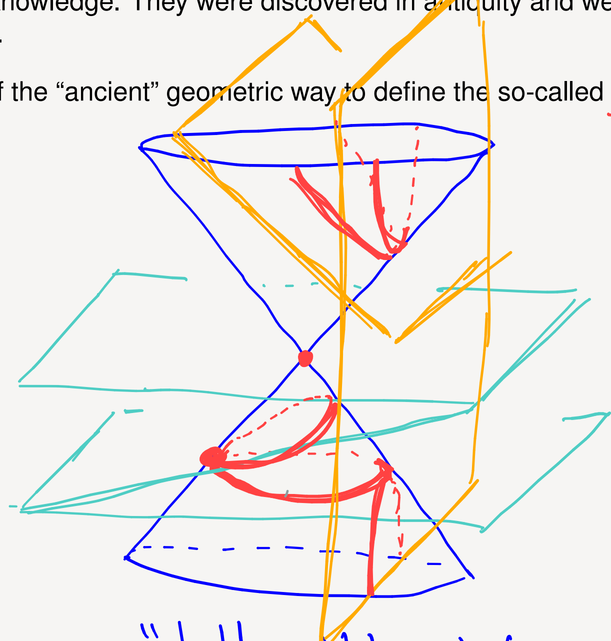
**Intro** When you first learn about a parabola it is when graphing functions, like $f(x) = x^2$. Then you learn that all quadratic function of the form $f(x) = ax^2 + bx + c$ look like parabolas opening up or down (they are functions and must pass the vertical line test after all).

$(a \neq 0)$ ← NOT DEF OF PARABOLA!

However, parabolas are ancient knowledge. They were discovered in antiquity and were perfectly understood by the time Euclid wrote *the Elements* ($\sim$ 500 BCE).

In this chapter you'll get a taste of the "ancient" geometric way to define the so-called **conic sections**.

↳ intersection of planes & a double-right angled cone.




Shapes possible:
1. point
2. circle (plane ∥ to base)
3. ellipse (plane w/ tilt small)
3. hyperbolas (plane ⊥ to base)
4. parabolas

## • Geometric Definition of a Parabola

**Defn 1 Geometric Definition of a Parabola**

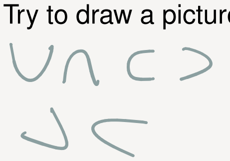
A **parabola** is the set of all points in the plane that are equidistant from a fixed point $F$ (called the **focus**) and a fixed line $L$ (called the **directrix**).

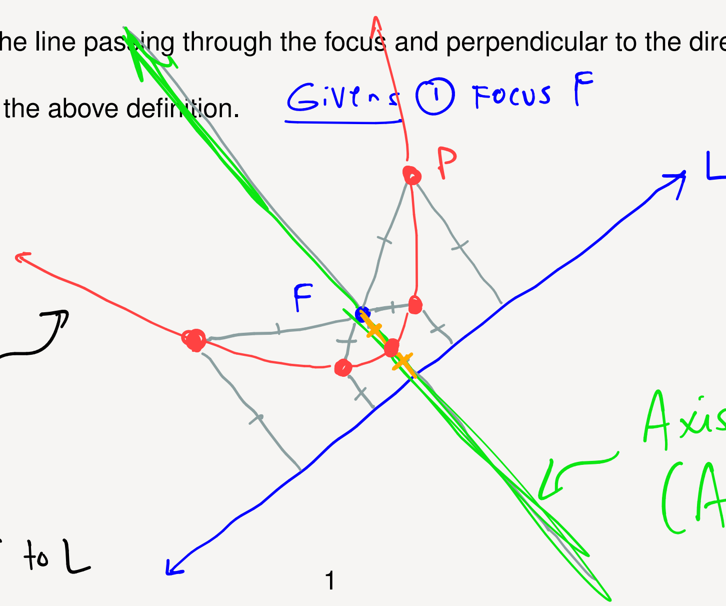
The **vertex**, $V$, of a parabola is defined as the point closest to the directrix. It lies half-way between the directrix and the focus.

The **axis of symmetry** (AOS), is the line passing through the focus and perpendicular to the directrix.

Try to draw a picture based purely on the above definition.

Givens ① Focus F ② Directrix L

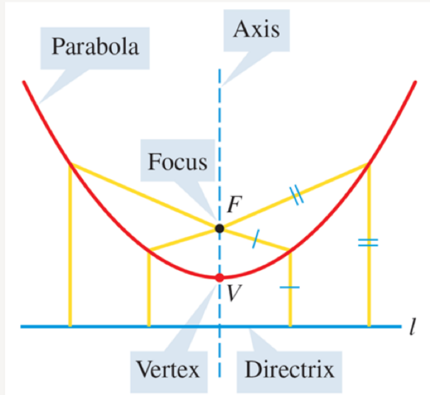





parabola = set of all points P in plane equidistant from F to L

Axis of Symmetry (AoS)

Here's a nice graphic of a parabola:



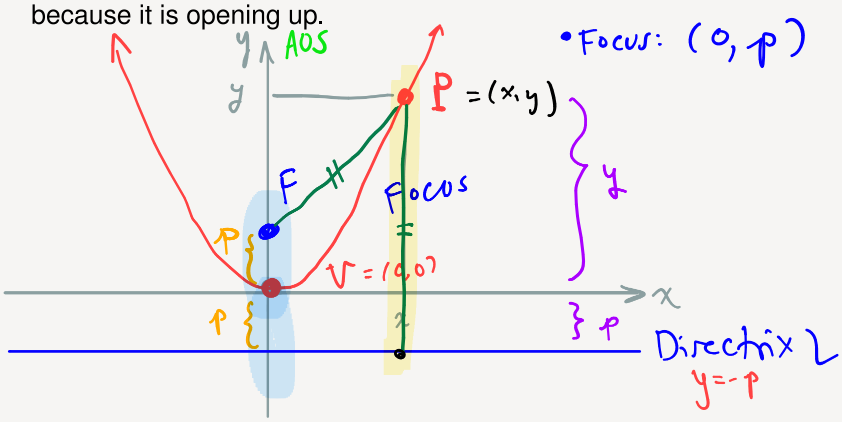

We want to discover the equations of a parabola from the given information: a focus point $F$ and the directrix $L$.

### • Derivation of the equation of a parabola

Starting only from the geometric definition, our goal to see what equation the set of points $(x, y)$ that lie on a parabola must satisfy.

We make some choices:

- Assume the parabola is opening up in the plane. That is, the directrix is a horizontal line.
- Choose the axes so that the origin passes through the vertex. That is, $V = (0, 0)$.
- Choose the $x$ axis so that it is parallel to the directrix and the $y$-axis so that is is parallel to the axis of symmetry.
- By the first assumption and our placement of the axes, the focus is on the $y$-axis. We will denote it by $F = (0, p)$, where $p > 0$ because it is opening up.




Next:

- Find the equation of the directrix $L$. $\boxed{y = -p}$

- For any point $P = (x, y)$ that is on the parabola, we know that

Def of Parabola $\Longrightarrow$ $\boxed{\text{dist}(P, F) = \text{dist}(P, L)}$

Use this information to arrive at the equation of a parabola.

$$\text{dist}(P, F) = \text{dist}(P, L)$$

$$\text{dist}((x,y), (0,p)) = y + p$$

$$\left(\sqrt{(x-0)^2 + (y-p)^2}\right)^2 = (y+p)^2$$

$$x^2 + (y-p)^2 = (y+p)^2$$

$$x^2 + \cancel{y^2} - 2yp + \cancel{p^2} = \cancel{y^2} + 2yp + \cancel{p^2}$$

$$-y^2 \quad -p^2 \qquad -y^2 \quad -p^2$$

$$x^2 - \cancel{2yp} = 2yp$$

$$+\cancel{2yp} \qquad +2yp$$

$$\Longrightarrow x^2 = 4yp$$

$$\boxed{x^2 = 4py}$$ EQ of Parabola (Opening Up)

•Focus: $(0, p)$

•Directrix: $y = -p$

$$y = \left(\frac{1}{4p}\right)x^2$$

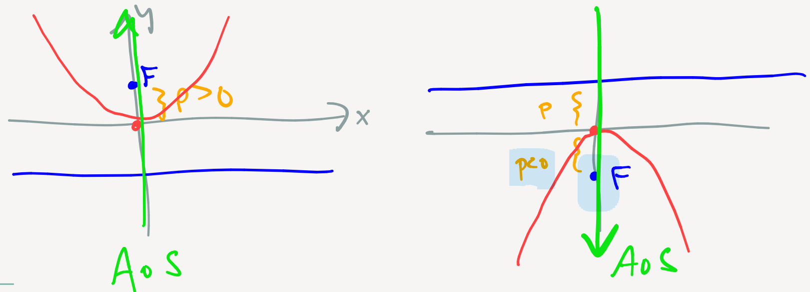
"$y = ax^2$"

## • Equations and Graphs of Parabolas

**Theorem 1 Equation of a Parabola with vertical AoS**

If a parabola is centered at the origin and has focus at $F = (0, p)$, then

- The equation of the directrix $L$ is: $\boxed{y = -p}$
- The equation of the parabola is: $\boxed{x^2 = 4py}$
- If $p > 0$ then it opens UP and if $p < 0$ then it opens DOWN.

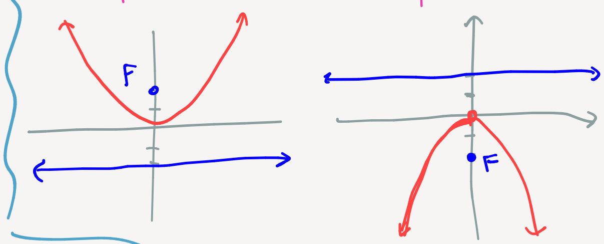



Ex $x^2 = 8y$ vs $x^2 = -8y$

$p = ?$ | $p = ?$

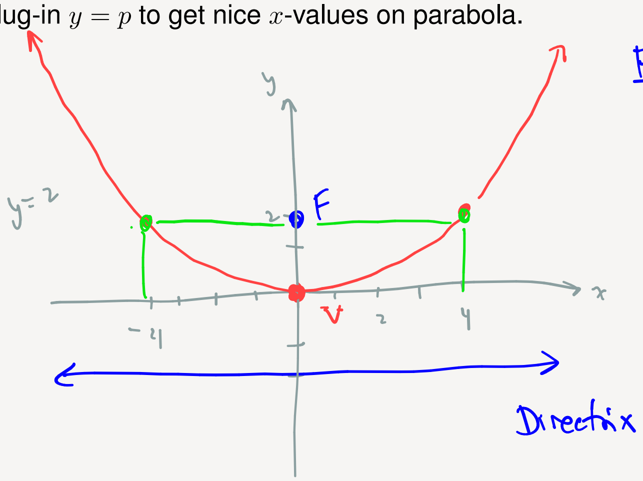
$4p = 8$ | $4p = -8$

$p = 2$ | $p = -2$




**Ex 1** Find the equation of the parabola with vertex $V = (0, 0)$ and focus $F = (0, 2)$. Does it open up or down? Sketch its graph.

**ProTip:** Plug-in $y = p$ to get nice $x$-values on parabola.




Focus $(0,2) \Rightarrow p = 2$

EQ parabola: $x^2 = 4py$

$\boxed{x^2 = 8y}$

To draw nicely follow ProTip:

$y = p = 2 \rightarrow y = 2$ & solve for $x$

$x^2 = 8 \cdot 2$

$x^2 = 16 \rightarrow x = \pm 4$

**Ex 2** Find the focus and directrix of $y = -x^2$, and sketch it's graph.

$y = -x^2$

put into correct form: $x^2 = \boxed{4p}y$

$x^2 = \boxed{-}y$

Compare $\boxed{-1 = 4p}$

$p = -1/4 \Rightarrow$ Focus: $\boxed{(0, -1/4)}$

$\Rightarrow$ Directrix: $y = -(-1/4)$

$\boxed{y = 1/4}$

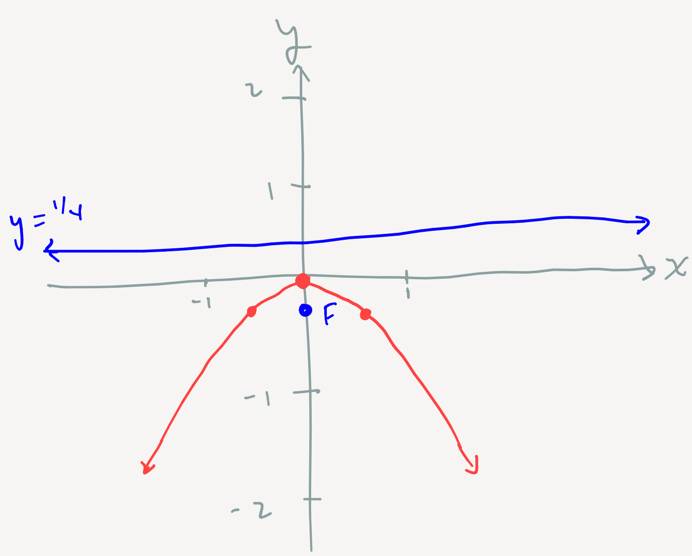
ProTip plug-in $y = p$

$x^2 = -y$

$x^2 = -(-1/4)$

$x^2 = 1/4$

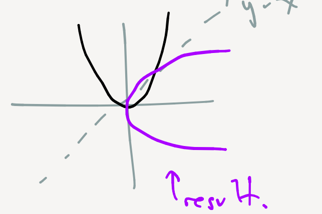
$x = \pm\sqrt{1/4} = \pm 1/2$

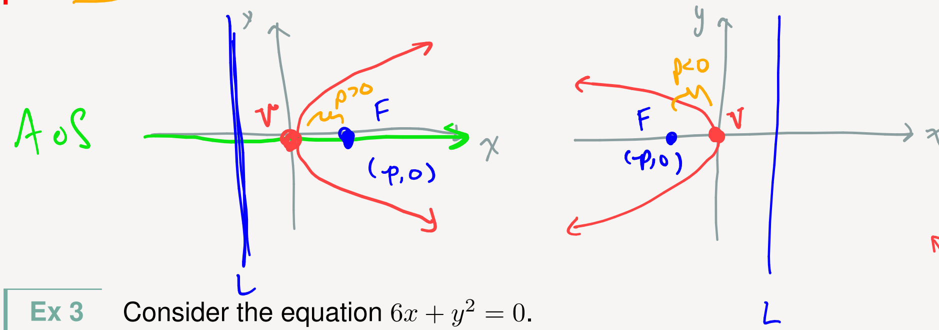


**Theorem 2 Equation of a Parabola with horizontal AoS**

If a parabola is centered at the origin and has focus at $F = (p, 0)$, then

- The equation of the directrix $L$ is: $\boxed{x = -p}$ → Pf interchange roles of $x$ & $y$ in $x^2 = 4py$.
- The equation of the parabola is: $\boxed{y^2 = 4px}$
- If $p > 0$ then it opens RIGHT and if $p < 0$ then it opens LEFT.




**Ex 3** Consider the equation $6x + y^2 = 0$.

(a) What type of curve is it?

(b) Determine the graph of the equation.

(c) Identify all distinguishing features associated with this curve.

a) $6x + y^2 = 0$

$y^2 = \boxed{-6}x$ → parabola opening $\boxed{\text{Left}}$/Right

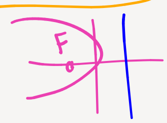
Form $y^2 = \boxed{4p}x$

Compare $-6 = 4p$

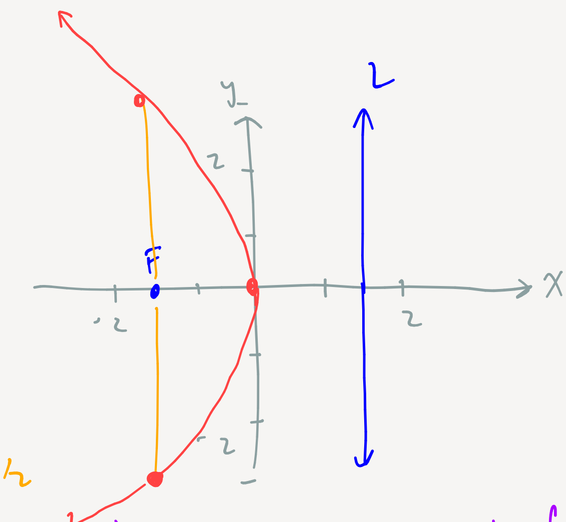
$\boxed{-\frac{3}{2} = p}$

Focus $(-3/2, 0)$

Directrix ( $x = -(-3/2) = 3/2$




a) parabola with vertical directrix & horizontal AoS.




ProTip plug-in $x = -3/2$ & solve for $y$

$y^2 = -6x \to y^2 = -6(-3/2)$

$y^2 = 9 \to y = \pm 3$

There are many interesting features of parabolas. Here are a couple.

**Defn 2**

The **latus rectum** is the line segment that passes through the focus, is parallel to the directrix (or perpendicular to the AoS), and has endpoints that meets the parabola.

The length of the latus rectum is called the **focal diameter**.

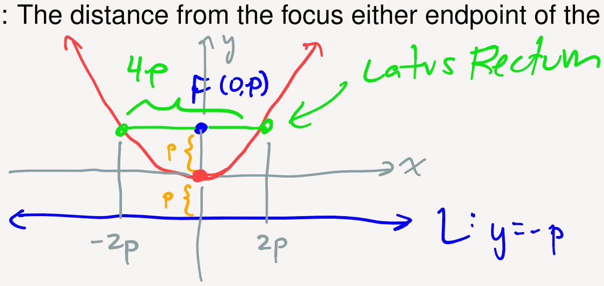
The focal diameter is excellent for helping determine the sketch of a parabola. This determines if it is “wide” of “narrow”. Fortunately, there’s an easy formula for it:

**Theorem 3 Focal Diameter Formula**

The focal diameter is: $\boxed{|4p|}$

Corollary: The distance from the focus either endpoint of the latus rectum is: $\boxed{|2p|}$ ie "ProTip"

p>0




$x^2 = 4py$

plug-in $y = p$

$x^2 = 4p^2$

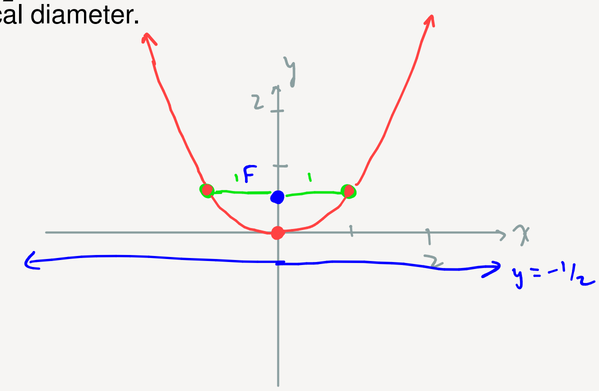
$x = \sqrt{4p^2}$

$= 2p.$

**Ex 4** Consider the equation: $y = \frac{1}{2}x^2$.

(a) Find the focus, directrix, and focal diameter.

(b) Sketch its graph.




Put into correct form:

$y = \frac{1}{2}x^2 \longrightarrow 2y = x^2$

compare:

$$\begin{cases} \boxed{x^2 = \boxed{2}y} \\ x^2 = \boxed{4p}y \end{cases}$$

$2 = 4p$

$p = 1/2$

$\begin{cases} x^2 = 2y & \cup \text{ or } \cap \\ p = 1/2 > 0 & \text{up} \end{cases}$

Focus $(0, p)$ $\boxed{(0, 1/2)}$

Focal Diameter $|4p| = |4(1/2)| = 2$

Directrix $y = -p$ $\boxed{y = -1/2}$

**Ex 5** Sketch the parabolas with the given focal points:

(a) $F_1 = (0, \frac{1}{8})$ $p = 1/8$ Focal Diameter $|4p|$ $4(1/8) = 1/2$

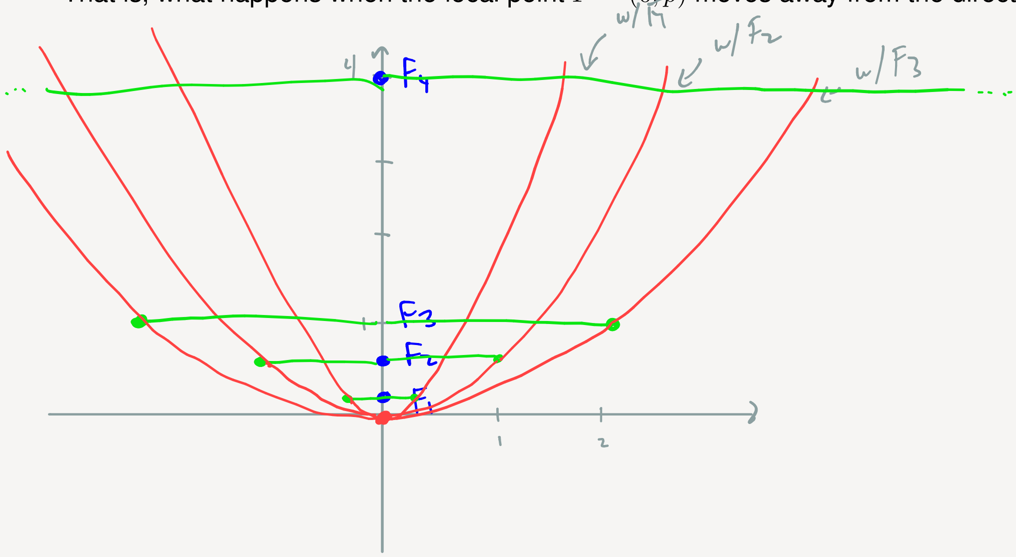
(b) $F_2 = (0, \frac{1}{2})$ $p = 1/2$ $4(1/2) = 2$

(c) $F_3 = (0, 1)$ $p = 1$ $4(1) = 4$

(d) $F_4 = (0, 4)$ $p = 4$ $4(4) = 16$

(e) Explain the effect of increasing $p$ on the graph $\boxed{4py = x^2}$.
That is, what happens when the focal point $F = (0, p)$ moves away from the directrix?

As $p$ increases the shape of parabola is that gets WIDER



## • Applications

Parabolas as reflectors for light, sound, or any electromagnetic radiation.
Mirrors were constructed for precisely this purpose. Newton wrote an entire book *Optiks* on the subject and would grind his own mirrors.

**Ex 6** A searchlight has a parabolic reflector that forms a ?bowl,? which is 12 in. wide from rim to rim and 8 in. deep, as shown in Figure 10. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?