§11.6: Absolute Convergence & the Ratio and Root Tests

Ch 11: Infinite Sequences and Series

Math 5B: Calculus II

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3. Absolute and Conditional Convergence
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6. Rearrangements

Guiding Questions for §11.6

Guiding Question(s)

1. How can we determine whether a series with irregular positive and negative terms converges or diverges?
2. What are absolutely convergent and conditionally convergent series?
3. What are the Ratio and Root Tests?
4. What can we say, if anything, about rearrangements of series?

Introduction

- We have studies series with only positive terms and series that alternate between positive and negative terms. But what about more irregular series with positive and negative terms without a clear alternating pattern?
- Example: $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}$

$$= \frac{\sin(1)}{1} + \frac{\sin(2)}{16} + \frac{\sin(3)}{81} + \frac{\sin(4)}{64} + \frac{\sin(5)}{125} + \frac{\sin(6)}{216} + \cdots$$
$$\approx 0.841 + 0.0568 + 0.00174 - 0.00296 - 0.00153 - 0.000216 + \cdots$$

Does it converge or diverge?

Absolute and Conditional Convergence

Definition 1: Absolute and Conditional Convergence

- A series $\sum a_n$ is called absolutely convergent if the corresponding series $\sum |a_n|$ of positive terms converges.
- A series $\sum a_n$ is called conditionally convergent if the corresponding series $\sum |a_n|$ of positive terms diverges BUT the original series converges.

Example 1:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is absolutely convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Absolute and Conditional Convergence

Activity 1:

Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+1}$ absolutely or conditionally convergent?

Theorem 1: AC implies C

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Absolute and Conditional Convergence

Absolute and Conditional Convergence

Activity 2:

Is $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}$ convergent or divergent?

The Ratio Test

Theorem 2: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be any series (positive or negative terms). Define

$$\boxed{L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.}$$

(i) If $L < 1$, then the series absolutely converges.

(ii) If $L > 1$ (including $L = \infty$), then the series diverges.

(iii) If $L = 1$ then the Ratio Test is inconclusive.

The Ratio Test

Remark on Case iii

- In Case (iii), anything can happen on a case-by-case basis.
- Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has $L = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$ and we know the original series converges absolutely (why?).
- Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ has $L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$ and we know the original series converges conditionally (why?).
- Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ has $L = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ and we know the original series diverges (why?).

The Ratio Test

The Ratio Test

The Ratio Test

Activity 3:

Test the series for absolute convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

The Ratio Test

The Ratio Test

Activity 4:

(a) Use the Ratio Test to test the series for absolute convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(b) Then use the Test for Divergence.

The Ratio Test

The Root Test

Theorem 3: Root Test

Let $\sum_{n=1}^{\infty} a_n$ be any series (positive or negative terms). Define

$$\boxed{L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.}$$

(i) If $L < 1$, then the series absolutely converges.

(ii) If $L > 1$ (including $L = \infty$), then the series diverges.

(iii) If $L = 1$ then the Ratio Test is inconclusive.

Remark on Case iii

- In Case (iii), anything can happen on a case-by-case basis.
- Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has $L = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 = 1$ and we know the original series converges absolutely (why?).
- Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ has $L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n+1}}{n}\right|} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$ and we know the original series is conditionally convergent (why?).
- Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ has $L = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$ and we know the original series is divergent (why?).

The Root Test

The Root Test

Activity 5:

Test the series for absolute convergence:

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+n-3n^3}{2n^3+5n-1}\right)^n$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^n}{8^{2+3n}}$

The Root Test

The Root Test

Activity 6:

Test the series for absolute convergence: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln(n)}{n}\right)^{n}$

The Root Test

Rearrangements

- In a finite sum, we can rearrange the terms in any way and get the same result (by associative and commutative laws)
- Infinite sums are much, much trickier.
- However:

  If $\sum a_n$ is absolutely convergent with $\sum a_n = S$, then any rearrangement of $\sum a_n$ is convergent with sum S.

- Also:

  If $\sum a_n$ is conditionally convergent, then for any real number $R \in \mathbb{R}$, there is a rearrangement of $\sum a_n$ which has sum R.

Example 2: Rearrangements

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely to $\frac{2}{3}$.

Hence, any rearrangement will sum to $\frac{2}{3}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally to $\ln(2)$ (see end of §11.5 notes).

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$$

The rearrangement in that geometric proof is:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 + \left[-\frac{1}{2} + \frac{1}{3}\right] + \left[-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}\right] \cdots = \ln(2)$$

Rearrangements

Example 3: Rearrangements

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally to $\ln(2)$ (see end of §11.5 notes).

However, if we consider the rearrangement:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \cdots$$

It can be shown (see book) that this rearrangement sums to $\dfrac{3}{2}\ln(2)$.