§11.5: Alternating Series

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Class #19 Notes

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Outline

Guiding Questions for §11.5

Guiding Question(s)

1. What are alternating series?
2. How can we estimate the error of alternating series?

Introduction

- Finding the exact value (sum) of a general series is very difficult. We typically can only estimate the series with a partial sum, S_n.
- Still, we must know whether or not the series converges before we try to use the estimate (say, in solving an applied problem).
- Our goal in this section is to develop two more tools that helps us determine whether a series converges or diverges.
- In the previous sections, we've focused on series with positive terms. The most important are: geometric series, harmonic series, and p-series.
- The tools we learn in this section are based on series with both positive AND negative terms called alternating series.
- Alternating series come with their own: alternating series test.

Introduction

- Recall the important Leibniz series:
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$
- This is an example of an alternating series.

Alternating Series

Definition 1: Alternating Series

- An alternating series is a series that alternates between positive and negative successive terms.
- That is, a series $\sum a_n$ is alternating if it can be put in one of the forms:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - b_3 + \cdots, \qquad \text{for } b_n > 0$$

or

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^{n+1} b_n = -b_0 + b_1 - b_2 + b_3 - \cdots, \qquad \text{for } b_n > 0$$

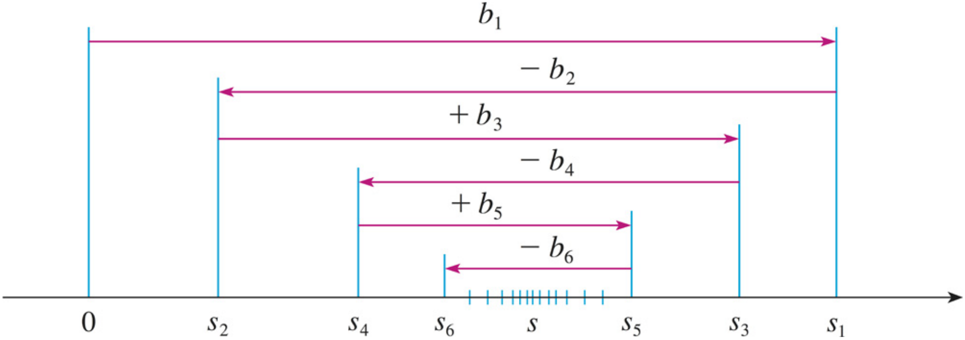
Theorem 1: Alternating Series Test

Let $\sum a_n = \sum (-1)^n b_n$ is an alternating series with $b_n > 0$. Assume:

(a) $\{b_n\}$ is decreasing: i.e. $b_{n+1} \leq b_n$ for all n

(b) $\lim_{n\to\infty} b_n = 0$

Then $\sum a_n$ converges.



Alternating Series

Activity 1:

Show that the alternating harmonic series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

Activity 2:

Determine whether or not the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n^2}{n^2+1}$

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n)}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{\pi}{n}\right)$

Alternating Series

Activity 3:

Determine whether the series converges or diverges:

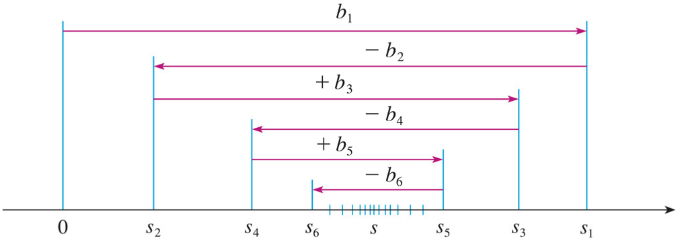
$$\sum_{j=1}^{\infty}(-1)^{j+1}\frac{j}{j^2+j+1}$$

Alternating Series

Theorem 2: Estimating Sums of Alternating Series

Let $S = \sum(-1)^n b_n$, where $b_n > 0$, be a convergent alternating series. Then

$$|R_n| = |S - S_n| \le b_{n+1}.$$



Activity 4:

Find the sum of the alternating harmonic series

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}$$

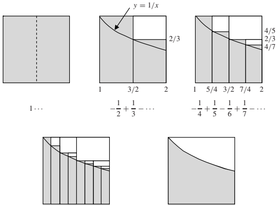
correct to 3 decimal places using Sage.

Estimating Sums of Alternating Series

Bonus: Alternating Harmonic Series Proof

Proof Without Words: The Alternating Harmonic Series Sums to ln 2

CLAIM. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = \ln 2.$



$-\frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \cdots \qquad = \int_1^2 \frac{1}{x}\,dx = \ln 2$

—Matt Hudelson
Washington State University
Pullman WA 99164

Summary We demonstrate graphically the result that the alternating harmonic series sums to the natural logarithm of two. This is accomplished through a sequence of strategic replacements of rectangles with others of lesser area. In the limit, we obtain the region beneath the curve $y = 1/x$ and above the x-axis between the values of one and two.

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