Chapter 5: Probability

Section 5.5: Counting Techniques

This section is optional. We will only briefly discuss the main ideas but I will not test you on these types of problems.

Counting Techniques

1. Fundamental Principle of Counting (aka "Multiplicative Principle" or "slot method")
2. Factorials
3. Permutations
4. Combinations

Fundamental Principle of Counting

When we have independence and use **the Fundamental Counting Principle (FPoC)**, we can think of each level of choice as a column in a slot machine. Each category fits in its own column in the slot machine and the choices can revolve independently. Or we think of it as filling in the blanks at each step with the number of choices:

$$\underset{\text{\# choices of item 1}}{\underline{\qquad\qquad}} \times \underset{\text{\# choices of item 2}}{\underline{\qquad\qquad}} \times \underset{\text{\# choices of item 3}}{\underline{\qquad\qquad}} \times \cdots \times \underset{\text{\# choices of item n}}{\underline{\qquad\qquad}} = \underset{\text{TOTAL \# of ways to combine}}{\underline{\qquad\qquad}}$$

Key Point: the FPoC works *with or without* replacing the items.

Ex 1: How many passwords of length four can you make

(a) using digits?

$0, 1, 2, \ldots, 9$

$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10}$

$10^4 = 10{,}000$

(b) using digits that don't start with 0?

$\underline{1} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10}$

10^3

(c) using digits or letters?

Factorials

The **factorial symbol** of a positive whole number *n* is defined to be the product of *n* with all the numbers decreasing by one down to 1. It is denoted by $n!$ for brevity. This means:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

We also define *0!* to be *1*, that is $0! = 1$.

By the Fundamental Principle of Counting, $n!$ is the number of ways of arranging *n* objects in a line without replacement.

PERMUTATIONS → order matters ABC BAC

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. *This is without repeating the objects!* Notice also that because we are arranging the objects in a line the ORDER MATTERS!

Since there are n$ ways of choosing the 1st object, and after this is done, n-1 ways of choosing the 2nd object, … , and finally n-(r-1) ways of choosing the r-th object, it follows by the fundamental principle of counting that the number of different arrangements, or **permutations** as they are often called, is given by

$$nPr = n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1))$$

The notation nPr is the number of different permutations of n objects taken r at a time.

When r = n, that is, when we choose n objects and arrange all of them, there are n! ways to do this.

Theorem: $nPr = \frac{n!}{(n-r)!}$

Ex2: In how many ways can 10 people be seated on a bench if only 4 seats are available?

COMBINATIONS → order doesn't matter

In a permutation we are interested in the order of arrangement of the objects. For example, ABC is a different permutation from BCA. In many problems, however, we are interested only in selecting or choosing objects *without regard to order AND without replacement*. Such selections are called **combinations**. For example, ABC and BCA are the same combination but different permutations. Think of combinations are groupings. It doesn't matter that your order the individuals in the group, just which group you are assigned to.

The total number of **combinations** of r objects selected from n (also called the combinations of n things taken r at a time), without regard to order, is denoted by nCr.

Theorem: $nCr = \frac{n!}{r!(n-r)!} = \frac{{}_nP_r}{r!}$

Ex3: I have 50 math books in my collection and want to donate 5 to the local library. In how many ways can I do this?

Summary

We learned 3 different techniques to help us count the number of ways: Fundamental Principle of Counting, Permutations, and Combinations. This flow chart helps you know how to choose the right technique: