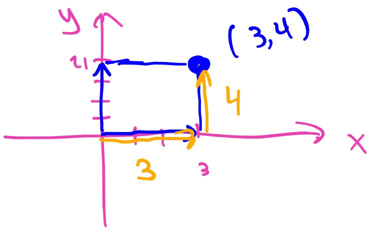
Recall Rectangular coordinates (Cartesian) RC



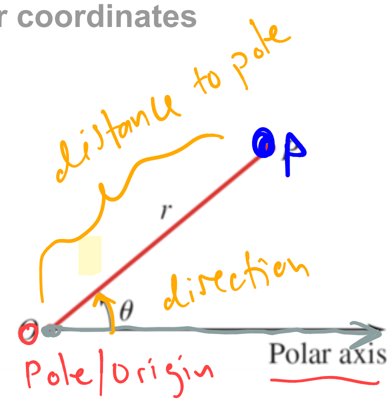
Section 8.1 - Polar Coordinates

Objectives:

- **Definition of polar coordinates**
- **Relationship between polar and rectangular coordinates**
- **Polar Equations**

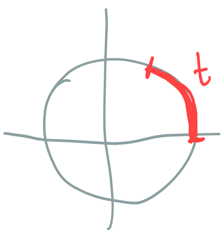
• Definition of Polar Coordinates (PC)

↳ "as the crow flies"



Definition(s): We need to understand the following key terms:

- Polar Coordinate System a line segment (half-line)
- Pole/Origin
- Polar Axis
- Describing a point P in polar coordinate system using r and θ:



  - r = distance from point P to Pole/Origin
  - θ = direction angle (in radian) between polar axis and line segment $\overline{OP}$
    measure length!
  - $P = (r, \theta)$ say that P is described using PC

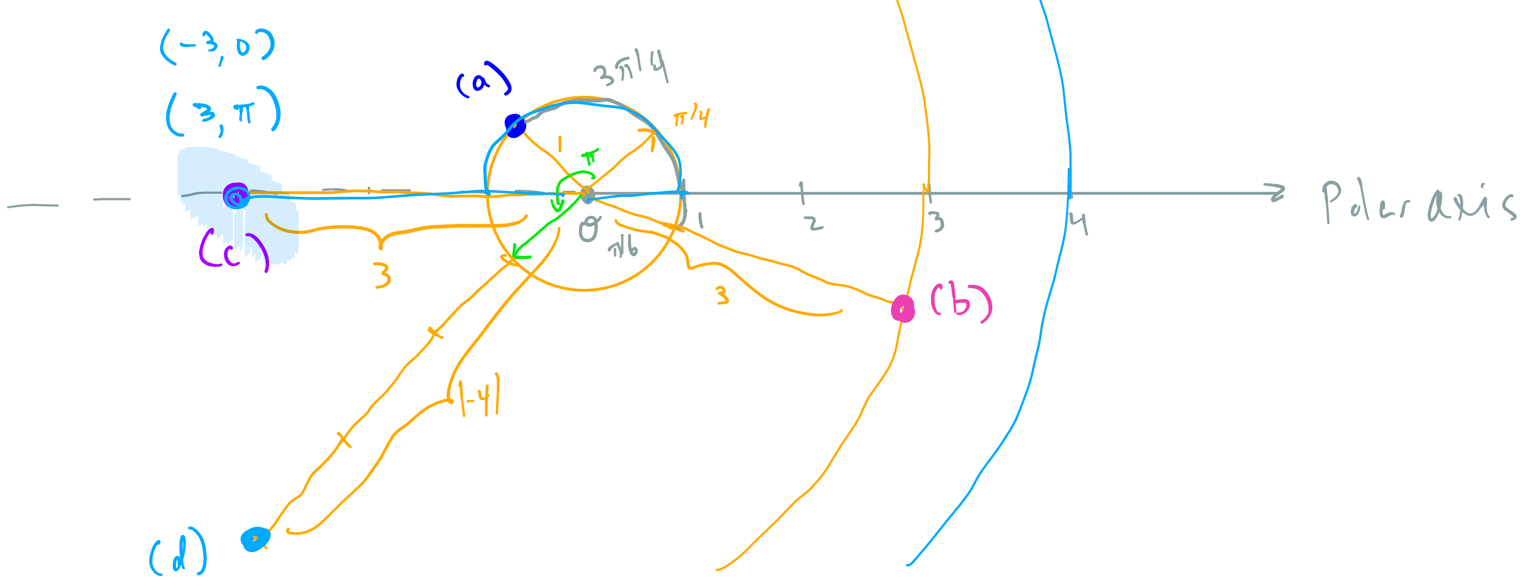
Ex $(4\pi, \pi)$ RC? PC?

negative r
$|r|$ & $\theta + \pi$
ie reflect $(|r|, \theta)$ thru origin

Ex 1: Plot the points whose coordinates are given in PC:

(a) $\left(1, \frac{3\pi}{4}\right)$ r θ (b) $\left(3, -\frac{\pi}{6}\right)$ r θ (c) $(3, 3\pi)$ r θ (d) $\left(-4, \frac{\pi}{4}\right)$ r θ

$|-4| = 4$



- **Relationship between Polar and Rectangular Coordinates**

Given a point P in the plane we can describe its location using rectangular coordinates (x,y) and also polar coordinates (r, θ)

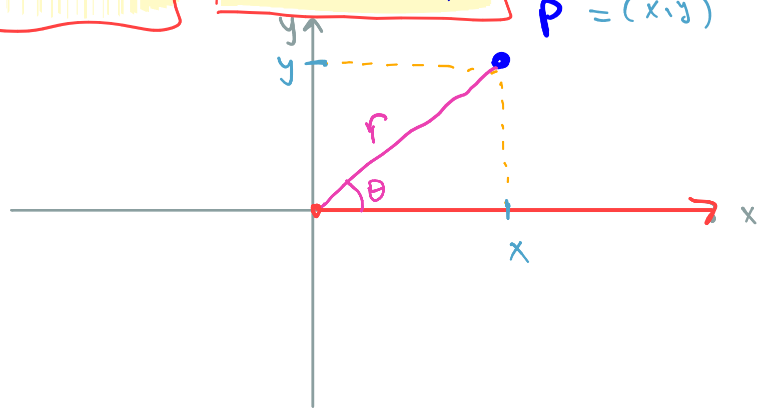
Theorem: Given a point $P = (x, y) = (r, \theta)$:

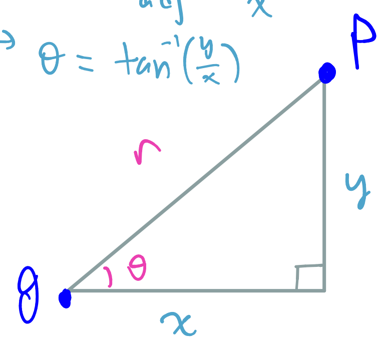
- If P is given in PC (r, θ) then the RC are (x, y) where:
  - M $\boxed{x = r\cos\theta}$ $\boxed{y = r\sin\theta}$
- If P is given in RC (x, y) then the PC are (r, θ) where:
  - M $\boxed{r = \pm\sqrt{x^2+y^2}}$ $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

- $x^2+y^2=r^2$ (Pythag)
  ↳ $r = \pm\sqrt{x^2+y^2}$
- $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$
  ↳ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Derivation:

convention: plot polar axis as +x-axis





- $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$
  ↳ $x = r\cos\theta$
- $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$
  ↳ $y = r\sin\theta$

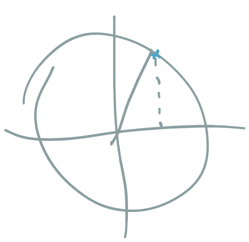
Ex 2: If $P = (4, \pi/3)$ is in PC, write P using RC.

Given: $r = 4$, $\theta = \pi/3$

Find x & y

$x = r\cos\theta = 4\cos(\pi/3) = 4(1/2) = 2$

$y = r\sin\theta = 4\sin(\pi/3) = 4(\sqrt{3}/2) = 2\sqrt{3}$

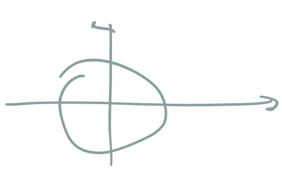


$$\boxed{P = (2, 2\sqrt{3})}$$

Question: Given $P = (x, y)$, are r and θ uniquely determined???? → in only one way

NO! NOT UNIQUE! $r \rightarrow$ two choices +/−
$\theta \rightarrow \infty$ly many choices! $(+2\pi k)$

Question: What are restriction on r and θ, if any?

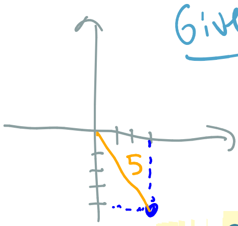


r – no restriction $r = 0$ is ok, ⊖ is ok

θ – direction (angle): when $r = 0$, θ is undefined.

Ex 3: Given $P=(3,-4)$ in RC, write P in PC.

$\boxed{P=(5,-0.927)}$



Given $x=3$, $y=-4$

Find r & θ

$r=\pm\sqrt{x^2+y^2} = +\sqrt{3^2+(-4)^2} = \sqrt{9+16} = 5$

$\theta = \tan^{-1}(y/x) = \tan^{-1}(-4/3) = -0.927 \approx -53.1°$

• Polar Equations

Using the theorem, we can convert equations in RC, x and y, into an equation in PC, r and θ; and vise-versa.

Key points:

- a complicated equation in RC, can be simpler PC (or at least recognizable)
  - Quick Ex: $\underbrace{\boxed{r=\theta}}_{\text{simple in PC}}$ is easier than $\underbrace{\underbrace{\pm\sqrt{x^2+y^2}}_{r}=\underbrace{\tan^{-1}(y/x)}_{\theta}}_{\text{complicated in RC}}$

- a complicated equation in PC, can be simpler RC (or at least recognizable)
  - Quick Ex: $\underbrace{\boxed{y=\ln(x)}}_{\text{simple in RC}}$ is easier than $\underbrace{\boxed{r\sin(\theta)=\ln(r)+\ln(\theta)}}_{\text{complicated in PC}}$

$\ln(x) = \ln(r\cos\theta) = \ln(r)+\ln(\cos\theta)$

Ex 4: Convert the equations given in PC as equations in RC.

(a) $r=2\sin(\theta)$

TRICK multiply both sides by r

$\underbrace{r^2}_{x^2+y^2} = 2\underbrace{r\sin\theta}_{y}$

$$\boxed{x^2+y^2=2y}$$

EQ circle

$x^2+y^2-2y=0$

$(x-0)^2+(y-1)^2-1=0$

$(x-0)^2+(y-1)^2=1$

Center: $(0,1)$
radius: $\sqrt{1}=1$

(b) $r=2+2\cos(\theta)$

same trick!

$r^2 = 2r+2r\cos\theta$

$x^2+y^2 = 2r+2x$

$x^2+y^2 = \pm2\sqrt{x^2+y^2}+2x$

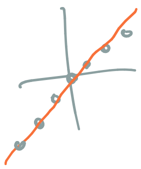
$-2x \qquad -2x$

$(x^2+y^2-2x)^2 = (\pm2\sqrt{x^2+y^2})^2$

$$\boxed{(x^2+y^2-2x)^2 = 4(x^2+y^2)}$$

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$\begin{cases} r=\pm\sqrt{x^2+y^2} \\ r^2=x^2+y^2 \\ \theta=\tan^{-1}(y/x) \end{cases}$



Do you remember when you first started drawing graphs? No doubt, at some point, you were given a simple formula like $y = 2x$, and were instructed to plot a bunch of points. You did so and you obtained a line!

You are now going to draw some polar graphs by plotting a bunch of polar points and seeing what happens. Plot *at least* 4 points and then guess the entire graph.

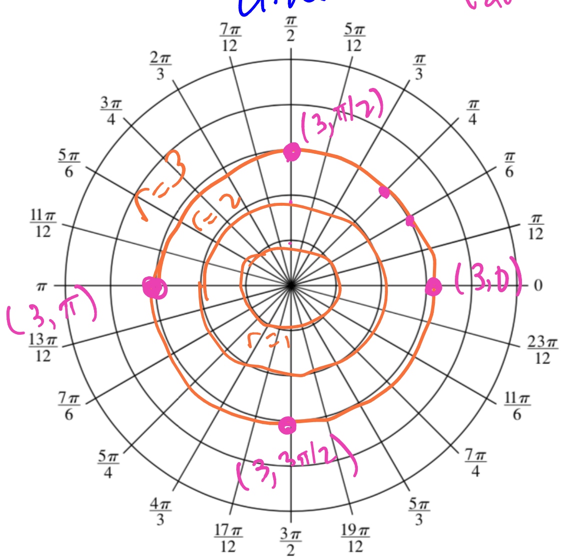
Ex 5: Plot the polar equations:

RC: $x^2+y^2=9$

PC: $r=3$

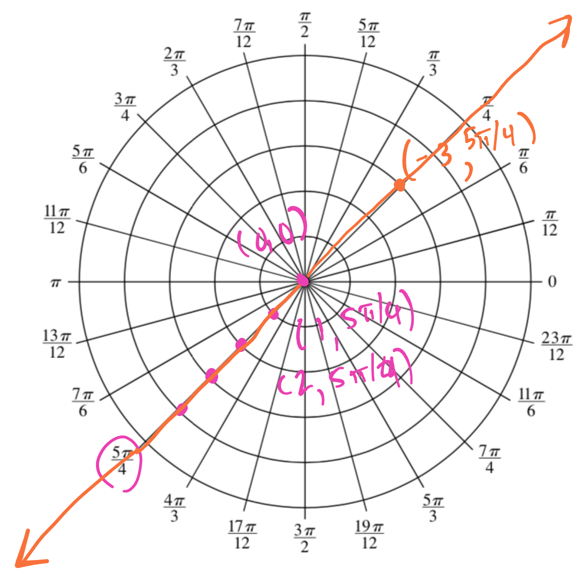
(a) $\boxed{r = 3.}$ → θ "free" free to be any value

circle



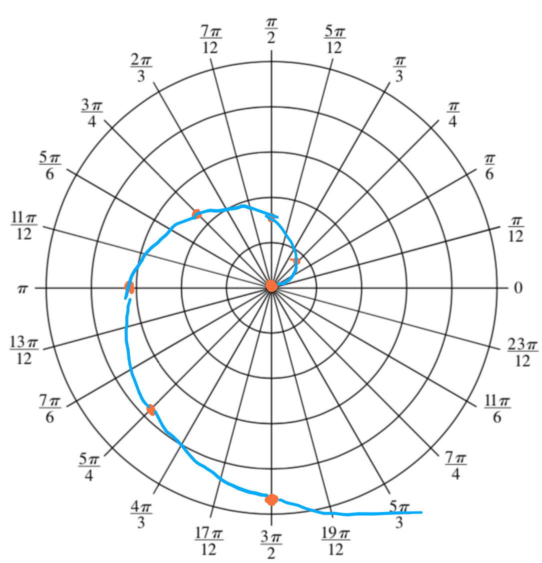
(b) $\boxed{\theta = 5\pi/4.}$ → r is "free"

line



r is negative: $\frac{5\pi}{4} + \pi = \frac{9\pi}{4} = \frac{\pi}{4}$

(c) $r = \theta.$



θ changes, graph r changing

$\theta = 0 \to r = 0$

$\theta = \pi/4 \to r = \pi/4 \approx 0.8$

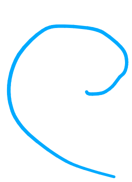
$\theta = \pi/2 \to r = \pi/2 \approx 1.6$

$\theta = 3\pi/4 \to r = 2.4$

$\theta = \pi \to r = 3.1$

$\theta = 5\pi/4 \to r = 3.9$

$\theta = 3\pi/2 \to r = 4.7$



Archimedean Spiral

#10 Convert $\cos(2\theta) = -1$ to RC

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\theta = \tan^{-1}(y/x)$

$$\cos^2\theta - \sin^2\theta = -1$$

multiply both sides r^2

$$r^2\cos^2\theta - r^2\sin^2\theta = -r^2$$

$$(\underbrace{r\cos\theta}_{x})^2 - (\underbrace{r\sin\theta}_{y})^2 = -r^2$$

$$x^2 - y^2 = -(x^2 + y^2)$$

$$2x^2 = 0$$

$$x^2 = 0$$

$$x = 0$$