§11.10: Taylor and Maclaurin Series

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Outline

Guiding Question(s)

1. What are Taylor series?
2. How can we use Taylor series to evaluate transcendental functions via approximation?
3. What are the Taylor series for important functions?

Introduction

- Hold up!
- Why on earth would we take a perfectly good function like $f(x) = \frac{1}{1+x^2}$ and write it as a power series, especially since it only works for a restricted domain!?!?
- A few reasons:
  - Multiplying and adding on a computer is really fast and easy
  - Dividing by hand and on a computer is harder
  - We can use partial sums to approximate complicated functions with only adding and multiplying.
  - We can find derivatives of $f(x)$ easily using the power series
  - We can find anti-derivatives of $f(x)$, $\int \frac{1}{1+x^2}\,dx$, easily using power series
  - These are the heart of the chapter.

Introduction

- We've mentioned a few times now that this chapter is built on the foundation of the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad \text{for} \quad x \in (-1,1)$$

- Other important functions we found a PSR for:
  - $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad x \in (-1,1)$
  - $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for} \quad x \in (-1,1)$
- What about other important functions?

  $\sin(x)$, e^x, and others

Introduction

- We will develop a method to find power series expansions/representations for a wider range of functions and devise a method to identify the values of x for which the function equals the power series expansion. (This is not always the entire interval of convergence of the power series.)

Taylor Series

- Assume that a function $f(x)$ has a power series representation at $x = a$ with radius of convergence $R > 0$.

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

- Recall that by the "Just like Poly" theorem, we can find all the derivatives of $f(x)$ inside the interval of convergence, $(a - R, a + R)$.
- Then $f'(x) =$
- Then $f''(x) =$
- Then $f'''(x) =$
- Then $f^{(4)}(x) =$

Activity 1:

(a) Find a formula for the nth derivative of $f(x)$, $f^{(n)}(x) = \frac{d^n f}{dx^n}$.

(b) Find a formula for $f^{(n)}(a)$, for $n = 0, 1, 2, 3, 4$ and general n.

(c) Find a formula for c_n in terms of $f^{(n)}(a)$.

Taylor Series

Taylor Series

Definition 1: Taylor Series

Let $f(x)$ be a function that is infinitely differentiable at $x = a$.

- The Taylor series of $f(x)$ at $x = a$ is

$$T(x) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(a)}{n!} \right] (x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Recall: $f^{(0)}(x) = f(x)$, that is, the zeroth derivative is simply the function itself. Also, $0! = 1$ by definition.
- The Maclaurin series of $f(x)$ is a Taylor series at the origin, $x = 0$:

$$T(x) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(0)}{n!} \right] x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(1)}{2!}x^2 + \frac{f'''(2)}{3!}x^3 + \cdots$$

Taylor Series

Theorem 1: Taylor Series Part 1

- If $f(x)$ has a power series representation at $x = a$, then the Taylor series of f at $x = a$ is the same as the power series representation.
- That is, if $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ for $x \in (a-R, a+R)$ and

  $T_a(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, then

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}.$$

- When a function $f(x)$ is infinitely differentiable at $x = a$, then $T_a(x)$ exists and must be its PSR.
- We also have: $T_a'(a) = f'(a)$.

Taylor Series for e^x

Theorem 2: Taylor Series for e^x

The Taylor series for $f(x) = e^x$ at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

with radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

Remark: By plugging-in $x = 1$ into the formula above, we prove Definition 3 in our "Eight Definitions of e" handout:

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots}$$

Activity 2:

Prove Theorem 2: That is, find the Taylor series for e^x at $x = 0$ and it's interval of convergence.

Taylor Series for e^x

Taylor Series for $\sin(x)$

Theorem 3: Taylor Series for $\sin(x)$

The Taylor series for $f(x) = \sin(x)$ at $x = 0$ is

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

with radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

Remark: By plugging-in $x = 1$ into the formula above, we arrive at an answer to one of the questions from our introduction:

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} + \cdots \approx \frac{101}{120} = 0.841\overline{6}$$

Taylor Series for $\sin(x)$

Activity 3:

Prove Theorem 3: That is, find the Taylor series for $\sin(x)$ at $x = 0$ and it's interval of convergence.

Taylor Series for $\sin(x)$

Taylor Series for $\cos(x)$

Theorem 4: Taylor Series for $\cos(x)$

The Taylor series for $f(x) = \cos(x)$ at $x = 0$ is

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

with radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

Activity 4:

Prove Theorem 4: That is, find the Taylor series for $\cos(x)$ at $x = 0$ and it's interval of convergence.

Taylor Series for $\cos(x)$

Theorem 5: Taylor's Theorem

- Let $f(x)$ be an infinitely differentiable function at $x = a$.
- Let $T_a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ be the Taylor series of $f(x)$ at $x = a$ defined for $|x - a| < R$ for some $R > 0$.
- Let $T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$ – called the Nth Taylor Polynomial of $f(x)$.
- Let $R_N(x) = f(x) - T_N(x)$ – called the remainder term of the Nth Taylor Polynomial of $f(x)$. Note: $f(x) = T_N(x) + R_N(x)$.
- If $\lim_{N\to\infty} R_N(x) = 0$, for $|x - a| < R$, then $f(x)$ equals the sum of its Taylor series on the interval $|x - a| < R$.
- Moreover,

$$f(x) = \lim_{N\to\infty} T_N(x) = T_a(x).$$

Taylor series Part 2: Taylor's Theorem

Taylor series Part 2: Taylor's Theorem

The "hard part" of Taylor's Theorem is showing that $\lim_{N\to\infty} R_N(x) = 0$. This is done using:

Theorem 6: Taylor's Remainder Theorem (aka Taylor's Inequality)

- If $|f^{(N+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_N(x)$ of the Taylor series satisfies the inequality:

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1} \qquad \text{for} \qquad |x-a| \leq d$$

Taylor series Part 2: Taylor's Theorem

Taylor series Part 2: Taylor's Theorem

Taylor series Part 2: Taylor's Theorem

Example

- If $f(x) = \sin(x)$, then $f^{(N+1)}(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$. In either case, we have $|f^{(N+1)}(x)| \leq 1$ for all values of x.

  Therefore, $a = 0$, $M = 1$ and d (can be any positive number).

  Taylor's Remainder theorem then says: $|R_N(x)| \leq \dfrac{1}{(N+1)!}|x|^{N+1}$ for $|x| \leq d = \infty$, i.e. any number x.
- Now, we can show directly that: $\displaystyle\lim_{N\to\infty} \frac{|x|^{N+1}}{(N+1)!} = 0$, for any fixed x.
- Thus, by Taylor's Theorem, we conclude: that $\sin(x)$ is equal to the sum of its the Taylor series at $x = 0$.

Example

- If $f(x) = e^x$, then $f^{(N+1)}(x) = e^x$. Since e^x is increasing, we have $|f^{(N+1)}(x)| = |e^x| \leq e^d$ for $|x| < d$, ie $x \in (-d, d)$.

  Therefore, $a = 0$, $M = e^d$ and d (can be any positive number).

  Taylor's Remainder theorem then says: $|R_N(x)| \leq \dfrac{e^d}{(N+1)!}|x|^{N+1}$ for $|x| \leq d$.

- Now, we can show directly that: $\displaystyle\lim_{N\to\infty} \frac{e^d|x|^{N+1}}{(N+1)!} = 0$, for any fixed x, d.
- Thus, by Taylor's Theorem, we conclude: that e^x is equal to the sum of its the Taylor series at $x = 0$.

More Examples

Activity 5:

Find the Taylor series for $f(x) = x^4 e^{-3x^2}$ at $x = 0$ and the radius of convergence.

More Examples

More Examples

Activity 6:

Find the Taylor series for $f(x) = \sqrt{x}\sin(x^2)$ at $x = 0$ and the radius of convergence.

More Examples

More Examples

Activity 7:

Find the Taylor series for $f(x) = e^x$ at $x = -1$ and the radius of convergence.

More Examples

More Examples

Activity 8:

Find the Taylor series for $f(x) = \ln(x)$ at $x = 2$ and the radius of convergence.

More Examples

Activity 9:

Evaluate:

$$\lim_{x \to 0} \frac{\cos(x^5) - 1}{x^{10}}$$

Note: use power series. You can use L'Hôpital's Rule, but it is a very long calculation.

More Examples