§6.6: Inverse Trigonometric Functions

Ch 6: Exponentials, Logs, & Inverse Trig Functions

Math 5B: Calculus II

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Class #4 Notes

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Outline

Guiding Questions for §6.6

Guiding Question(s)

1. What are the derivatives of the inverse trigonometric functions?
2. What are the anti-derivative of the inverse trigonometric functions?

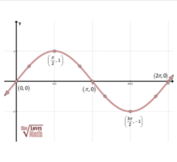
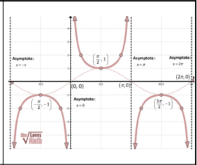
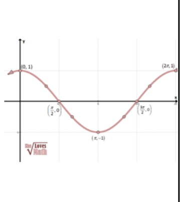
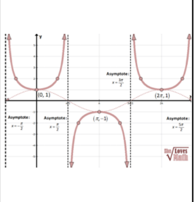
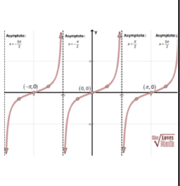
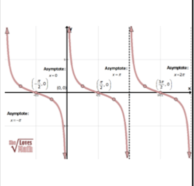
Review of the Trigonometric Function

Recall:

- $\sin(x)$ domain $= \mathbb{R}$
- $\cos(x)$ domain $= \mathbb{R}$
- $\tan(x)$ domain $= \cdots \cup (-\pi/2, \pi/2) \cup \cdots$

Restricted domain for inverse trig:

- $\sin(x)$ domain $= [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\cos(x)$ domain $= [0, \pi]$
- $\tan(x)$ domain $= (-\pi/2, \pi/2)$

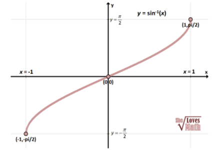
Parent Function	Graph	Parent Function	Graph
$y=\sin(x)$ Odd **Domain:** $(-\infty,\infty)$ **Range:** $[-1,1]$ **Period:** 2π **Zeros:** $(\pi k, 0)$, $k \in$ Integers		$y=\csc(x)$ Odd **Domain:** $x \neq \pi k$ **Range:** $(-\infty,-1] \cup [1,\infty)$ **Asymptotes:** $x=\pi k$ **Period:** 2π **Zeros:** None	
$y=\cos(x)$ Even **Domain:** $(-\infty,\infty)$ **Range:** $[-1,1]$ **Period:** 2π **Zeros:** $\left(\frac{\pi}{2}+\pi k, 0\right)$		$y=\sec(x)$ Even **Domain:** $x \neq \frac{\pi}{2}+\pi k$ **Range:** $(-\infty,-1] \cup [1,\infty)$ **Asymptotes:** $x=\frac{\pi}{2}+\pi k$ **Period:** 2π **Zeros:** None	
$y=\tan(x)$ Odd **Domain:** $x \neq \frac{\pi}{2}+\pi k$ **Range:** $(-\infty,\infty)$ **Asymptotes:** $x=\frac{\pi}{2}+\pi k$ **Period:** π **Zeros:** $(\pi k, 0)$		$y=\cot(x)$ Odd **Domain:** $x \neq \pi k$ **Range:** $(-\infty,\infty)$ **Asymptotes:** $x=\pi k$ **Period:** π **Zeros:** $\left(\frac{\pi}{2}+\pi k, 0\right)$	

Review of the Inverse Trigonometric Functions

Function and T-Chart	Graph

$y = \sin^{-1}(x)$ or $y = \arcsin(x)$

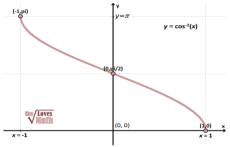
x	y
-1	$-\frac{\pi}{2}$
0	0
1	$\frac{\pi}{2}$



Domain: $[-1, 1]$ **Range:** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **Quadrants:** I and IV

$y = \cos^{-1}(x)$ or $y = \arccos(x)$

x	y
-1	π
0	$\frac{\pi}{2}$
1	0



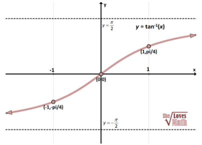
Domain: $[-1, 1]$ **Range:** $[0, \pi]$ **Quadrants:** I and II

Function and T-Chart	Graph

$y = \tan^{-1}(x)$ or $y = \arctan(x)$

x	y
und	$-\frac{\pi}{2}$
-1	$-\frac{\pi}{4}$
0	0
1	$\frac{\pi}{4}$
und	$\frac{\pi}{2}$

(und = undefined)

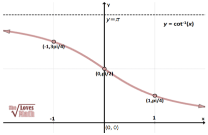


Domain: $(-\infty, \infty)$ **Range:** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Quadrants: I and IV **Asymptotes:** $y = -\frac{\pi}{2}, \frac{\pi}{2}$

$y = \cot^{-1}(x)$ or $y = \operatorname{arccot}(x)$

x	y
und	π
-1	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
1	$\frac{\pi}{4}$
und	0



Domain: $(-\infty, \infty)$ **Range:** $(0, \pi)$

Quadrants: I and II **Asymptotes:** $y = 0, \pi$

Review of the Inverse Trigonometric Functions

Definition 1: Arcsine = Inverse Sine

Let $f(x) = \sin(x)$ with restricted domain $D(f) = [-\frac{\pi}{2}, \frac{\pi}{2}]$. We defined the arcsine function to be $f^{-1}(x)$, that is, it is the inverse of the function $f(x) = \sin(x)$ on $D(f)$. We denote $f^{-1}(x)$ either by $\sin^{-1}(x)$ or $\arcsin(x)$.

- $D(f) = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $R(f) = [-1, 1]$
- $D(f^{-1}) = [-1, 1]$ and $R(f^{-1}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sin(x)$ with $x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Leftrightarrow \arcsin(x) = \sin^{-1}(x)$ with $x \in [-1, 1]$
- $y = \sin(x),\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Leftrightarrow x = \sin^{-1}(y),\ -1 \leq y \leq 1$
- $y = \sin(x),\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Leftrightarrow x = \arcsin(y),\ -1 \leq y \leq 1$

Example 1: Arcsine = Inverse Sine

(a) $\arcsin(-\frac{1}{2}) = ?$ $\quad\Leftrightarrow\quad$ $\sin(?) = -\frac{1}{2}$
Using Unit Circle approach, $\arcsin(-\frac{1}{2}) = -\pi/6$

(b) Find $\tan(\arcsin(\frac{1}{3}))$.
We start with $x = \arcsin(1/3)$ and re-write it as $\sin(x) = 1/3$. Draw a right-triangle with Hypothenuse $= 3$ and Opposite $= 1$ and angle x.
Then $\tan(x) =$O/A$=1/2\sqrt{2} = \sqrt{2}/2$.

Review of the Inverse Trigonometric Functions

Definition 2: Arccosine = Inverse Cosine

Let $f(x) = \cos(x)$ with restricted domain $D(f) = [0, \pi]$. We defined the arccosine function to be $f^{-1}(x)$, that is, it is the inverse of the function $f(x) = \cos(x)$ on $D(f)$. We denote $f^{-1}(x)$ either by $\cos^{-1}(x)$ or $\arccos(x)$.

- $D(f) = [0, \pi]$ and $R(f) = [-1, 1]$
- $D(f^{-1}) = [-1, 1]$ and $R(f^{-1}) = [0, \pi]$
- $\cos(x)$ with $x \in [0, \pi] \Leftrightarrow \arccos(x) = \cos^{-1}(x)$ with $x \in [-1, 1]$
- $y = \cos(x)$, $0 \leq x \leq \pi \Leftrightarrow x = \cos^{-1}(y)$, $-1 \leq y \leq 1$
- $y = \cos(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Leftrightarrow x = \arccos(y)$, $-1 \leq y \leq 1$

Review of the Inverse Trigonometric Functions

Definition 3: Arctangent = Inverse Tangent

Let $f(x) = \tan(x)$ with restricted domain $D(f) = (-\frac{\pi}{2}, \frac{\pi}{2})$. We defined the arctangent function to be $f^{-1}(x)$, that is, it is the inverse of the function $f(x) = \tan(x)$ on $D(f)$. We denote $f^{-1}(x)$ either by $\tan^{-1}(x)$ or $\arctan(x)$.

- $D(f) = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $R(f) = \mathbb{R} = (-\infty, \infty)$
- $D(f^{-1}) = \mathbb{R} = (-\infty, \infty)$ and $R(f^{-1}) = (-\frac{\pi}{2}, \frac{\pi}{2})$
- $\tan(x)$ with $x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Leftrightarrow \arctan(x) = \tan^{-1}(x)$ with $x \in \mathbb{R}$
- $y = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2} \Leftrightarrow x = \tan^{-1}(y)$, $-\infty < y < \infty$
- $y = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2} \Leftrightarrow x = \arctan(y)$, $-\infty < y < \infty$

Activity 1:

(a) Simplify the expression: $\cos(\arctan(x))$

(b) Evaluate: $\lim_{x\to -5^-} \arctan\left(\frac{1}{x+5}\right)$

Derivatives of the Inverse Trigonometric Functions

Keeping in mind the domains of each of the inverse trig functions and their graphs look continuous and differentiable, it's not surprising they have derivative rules.

Theorem 1: Derivatives of the Inverse Trig Functions

(DR 1) $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1$

(DR 2) $\frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}}, \qquad -1 < x < 1$

(DR 3) $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}, \qquad -\infty < x < \infty$

(DR 4) $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{x\sqrt{x^2-1}}, \qquad -1 < x < 1$

(DR 5) $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}, \qquad -1 < x < 1$

(DR 6) $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}, \qquad -\infty < x < \infty$

Derivatives of the Inverse Trigonometric Functions

We prove all of these with the same general idea: re-write the equation and use implicit differentiation. This is basically they same proof technique we used for the derivative of $\ln(x)$ in §6.4.

DR1: $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$

Proof: Proof of Derivative of Arcsine

- $y = \arcsin(x) \Leftrightarrow \sin(y) = x$.
- Implicitly Differentiate:

$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$$

$$\cos(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Derivatives of the Inverse Trigonometric Functions

Proof: Proof of Derivative of Arcsine

Cont.

- So, if we can replace $\cos(y)$ with the correct expression involving x, then we're done.
- Draw a right-triangle with angle y and side lengths O$=x$, H$=1$ using SOH. The missing adjacent side is then A$=\sqrt{1-x^2}$. So, using CAH, we have $\cos(y) = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$.
- Alternatively, we can use trig identities:
  $\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$ since $x = \sin(y)$.
- Done! $\square$

Derivatives of the Inverse Trigonometric Functions

Activity 2:

Prove the following formulas:

(a) $\frac{d}{dx}\left[\arccos(x)\right] = \frac{-1}{\sqrt{1-x^2}}, \qquad -1 < x < 1$

(b) $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}, \qquad -\infty < x < \infty$

Derivatives of the Inverse Trigonometric Functions

Activity 3:

Find the derivatives of the following functions:

(a) $L(x) = x^3 \arctan(x) + e^x \ln(x)$

(b) $P(t) = 2^{t \arcsin(t)}$

(c) $m(z) = \left(\sin^{-1}(5z) + \tan^{-1}(4 - z)\right)^{27}$

(d) $s(y) = \arctan(\log_5(1 + y^2))$

Anti-Derivatives and Definite Integrals of the Inverse Trigonometric Functions

Each of the formulas for the DRs of the inverse trig functions gives rise to its own anti-derivative formula. The two most important are:

Theorem 2: Anti-Derivative Rules for Inverse Trig

(ADR 1) $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C = \sin^{-1}(x) + C$$

(ADR 2) $$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C = \tan^{-1}(x) + C$$

Anti-Derivatives and Definite Integrals of the Inverse Trigonometric Functions

Activity 4:

Evaluate the following anti-derivatives and definite integrals:

(a) $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{t^2+a^2}\,dt$

(c) $\displaystyle\int \frac{1}{w^4+16}\,dw$