§6.5: Exponential Growth & Decay

Ch 6: Exponentials, Logs, & Inverse Trig Functions
Math 5B: Calculus II

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Class #5 Notes

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Outline

Guiding Questions for §6.7

Guiding Question(s)

1. How can we apply the derivative to study problems from the physics, chemistry, biology, economics, and other sciences?
2. How can we use derivatives to realistically model the growth of a population?
3. How can we use derivatives to realistically model the decay of a radioactive substance?
4. What is Newton's Law of Cooling?
5. How can we compound interest instantaneously?

Introduction to Mathematical Modeling

What is mathematical modeling?

- Mathematical modeling is a process using various mathematical structure (equations, functions, graphs, etc) to represent, describe, or predict real world situations. The aim is to reduce a (usually very complex) problem to a few essential characteristics.
- We'll explore how the derivative is a good tool for modeling many phenomena in the sciences.

The Derivative...

The derivative has many interpretations. You've already studied the three main ones in Calc I:

- Slope of the tangent line at a point, $m_{\tan}(P)$
- Instantaneous velocity of an object, $v(t) = \frac{ds}{dt} = \dot{s}$
- Instantaneous rate of change of a quantity, $f'(x) = \frac{df}{dx}$

In the history of calculus lecture, the common answer to the question, What is Calculus?, was "it's the study of change."

- Given a function, where does the change come in?
- It comes from our perspective.
- If x changes from x_1 to x_2, then the function changes from $f(x_1)$ to $f(x_2)$.

The Derivative...

What does the derivative have to do with this?

- If we denote the change in x by $\Delta x = x_2 - x_1$ and the change in our function f by $\Delta f = f(x_2) - f(x_1)$ then the **average rate of change** of f over the interval $[x_1, x_2]$ is given by

$$AROC = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

- The derivative is the limit of the average rate of change as the interval shrinks to zero, or as $x_2 \to x_1$:

$$\left.\frac{df}{dx}\right|_{x_1} = \lim_{x_2 \to x_1} \frac{\Delta f}{\Delta x}.$$

The Derivative...

What does the derivative have to do with this?

- Now, if x_2 is close to x_1 then the average rate of change is approximately the derivative at x_1, so:

$$\left.\frac{df}{dx}\right|_{x_1} \approx \frac{\Delta f}{\Delta x} \qquad (\text{for } x_2 \approx x_1)$$

- Another useful fact is that if the derivative is positive at $x = x_1$, then the numerator of the AROC is positive so $f(x_1) < f(x_2)$. Thus f is increasing for points close to x_1!
- Similarly, if the derivative is negative at $x = x_1$, then f is decreasing for points close to x_1!

Exponential Growth & Decay

- Assume that $f(t)$ is the population of a certain species at a time t.
- Then it seems reasonable from experience that the higher the population, the faster the population grows. That is, the rate of change of the population should be bigger provided the population is bigger.
- This suggests that the derivative $\frac{df}{dt}$ (rate of growth) is proportional to the population $f(t)$:

$$\frac{df}{dt} = kf(t), \tag{1}$$

  for a constant k.
- If $k > 0$ then the derivative will be positive and so $f(t)$ will increase (grow).
- If $k < 0$ then the derivative will be negative and so $f(t)$ will decrease (decay).

Exponential Growth & Decay

- Population growth: the derivative $\frac{df}{dt}$ is proportional to the population $f(t)$:

$$\frac{df}{dt} = kf(t),$$

for a constant k.

- This is an example of a Differential Equation (DE). A DE is an equation which involves a function $f(t)$ and its derivative, and the goal is to find function(s) that satisfy the equation. In other words, the goal in "solving the differential equation" is to "produce a function, or functions, that satisfy the equation."

Exponential Growth & Decay

Definition 1: Law of Natural Growth & Decay

Equation (1)

$$\frac{df}{dt} = kf(t)$$

or, setting $y = y(t) = f(t)$, it is also written as

$$\frac{dy}{dt} = ky$$

- If $k > 0$, it is called the law of natural growth
- If $k < 0$, it is called the law of natural decay

Exponential Growth & Decay

- It's not hard to guess the solution to the differential equation given in (1): $\frac{df}{dt} = kf(t)$.
- It says that the derivative of f is itself times a constant. We've encountered this already!
- Remember that the derivative of e^x is itself. Well, if we need the constant k to come out in front, then we can instead consider e^{kx} since, thanks to the chain rule, $\frac{d}{dx}[e^{kx}] = ke^{kx}$.
- So, a solution to (1) is $f(t) = e^{kt}$ since

$$\frac{df}{dt} = \qquad\qquad = kf.$$

- But, notice that $f(t) = 5e^{kt}$ is also a solution to (1) since

$$\frac{df}{dt} = \frac{d}{dt}[5e^{kt}] = \qquad\qquad = kf(t).$$

Exponential Growth & Decay

There's nothing special about the 5, any constant multiple of e^{kt} will also solve the DE (1). We've (almost) proven:

Theorem 1: Exponential Growth & Decay Equation

The only solutions to the "natural growth/decay equation," $\frac{df}{dt} = kf(t)$, for a constant $k \neq 0$, are of the form:

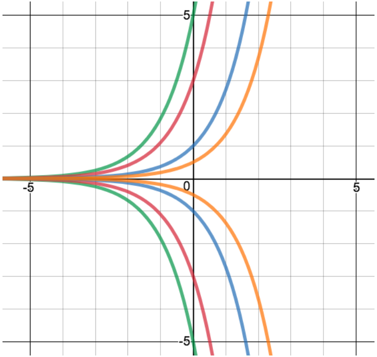
$$f(t) = Ce^{kt}. \tag{2}$$

The constant $C = f(0) = C_0$ is called the initial condition.

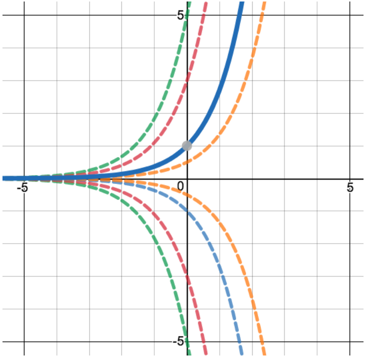
- The solutions are $f(t) = Ce^{kt}$ for all constant $C \neq 0$ are called general solutions to the DE.
- A particular solution corresponds to a specific initial population at time $t = 0$.

A proof will be given later in §9.3.

Exponential Growth & Decay



General Solutions: $f(t) = Ce^{kt}$



Particular Solution: $f(t) = C_0 e^{kt}$

Example 1:

- As an example, the equation $\frac{df}{dt} = 2f(t)$ tells us the rate of change is double the quantity of $f(t)$.
  - The solutions are $f(t) = Ce^{2t}$ for any constant C, called general solutions to the DE.
  - The solution $f(t) = 5e^{2t}$ is called a particular solution–it corresponds to an initial population of 5 since $f(0) = 5e^0 = 5$.
- If t is measured in years (for example), then population growth at a relative rate of 200% (doubles in size).

Activity 1: Bacteria

A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour, the population has increased to 420.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 hours.

(c) Find the rate of growth after 3 hours.

(d) When will the population reach 10,000?

(e) The doubling-time T_D is defined to be the time it takes a population to double in size, that is: $P(T_D) = 2C = 2P(0)$. Find the doubling-time for the bacteria.

The half-life T_H is defined to be the time it takes a population/substance to cut its size in half, that is: $P(T_H) = \frac{1}{2}C = \frac{1}{2}P(0)$.

Activity 2: Radioactive Strontium

Strontium-90 decreases at a rate proportional to its mass. Strontium-90 has a half-life of 28 days.

(a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.

(b) Find the mass remaining after 40 days.

(c) How long does it take the sample to decay to a mass of 2 mg?

Newton's Law of Cooling

- Imagine we're trying to find a model for the temperature $T(t)$ of a hot object put into a large room with a stable temperature of T_S (the "S" is for surroundings)
- If the temperature of the surroundings is lower than the initial temperature of the object, then the temperature, $T(t)$, will decrease over time.
- Newton's Law of Cooling states that under these conditions the temperature is proportional to the temperature difference between the object and its surroundings, provided this difference is not too large:

$$\frac{dT}{dt} = k(T - T_S), \tag{3}$$

where k is a constant.

Activity 3:

Find the solutions to the DE: $\frac{dT}{dt} = k(T - T_S)$, where k and T_S are constants, by solving:

(a) Make the substitution: $y(t) = T(t) - T_S$. What does the new DE look like?

(b) Solve the new DE from part (a)

(c) Solve for $T(t)$

Activity 4:

A thermometer reading $70^\circ F$ is taken outside where the ambient temperature is $22^\circ F$. Four minutes later the reading is $32^\circ F$.

(a) Write the differential equation (DE) that models the temperature $T = T(t)$ of the thermometer at time t.

(b) Find the general solution of the differential equation (i.e. all of the solutions with C).

(c) Find the particular solution to the differential equation, using the initial condition that when $t = 0\,min$, then $T = T(0) = 70^\circ F$.

(d) Find the thermometer reading $7\,min$ after the thermometer was brought outside.

(e) Find the time it takes for the reading to change from $70^\circ F$ to within $0.5^\circ F$ of the air temperature.

Continuous Compound Interest

- Recall that in the "Eight Definitions of e" hand-out, we introduced the number e from compounding interest. If an investment of P dollars (called the principal) is compounded at a rate of $r\%$ a total of n times per year for t years, then you have $A(t)$ dollars in the bank account. The formula for $A(t)$ is given by:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

- When the number of times you compound interest in a year goes to infinity, that is, $n \to \infty$, then we imagine that this is compounding interest instantaneously, or continuously. We call this continuously compounding interest and the formula is:

$$A(t) = Pe^{rt}$$

- Notice how much simpler the formula is!

Continuous Compound Interest

- We call this continuously compounding interest and the formula is:

$$A(t) = Pe^{rt}$$

- Since

$$\frac{dA}{dt} = \frac{d}{dt}\left[Pe^{rt}\right] = Pre^{rt} = rA(t)$$

  the amount in the bank undergoing continuous compound interest grows proportional to its size.
- Continuous compound interest satisfies the law of natural exponential growth
- The constant of proportionality is the interest rate, r.

Continuous Compound Interest

Proof: Formula for Continuous Compound Interest

Why is this true? It's tricky:

$$\begin{aligned}
A(t) &= \lim_{n\to\infty} P\left(1+\tfrac{r}{n}\right)^{nt} \\
&= \lim_{n\to\infty} P\left[\left(1+\tfrac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^{n/r}\right]^{rt} && \text{(because } P, t, r \text{ don't depend on } n\text{)} \\
&= P\left[\lim_{m\to\infty}\left(1+\tfrac{1}{m}\right)^{m}\right]^{rt} && \text{(use substitution } m = n/r\text{)} \\
&= P[e]^{rt}
\end{aligned}$$

$\square$

Continuous Compound Interest

Proof: Formula for Continuous Compound Interest

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&= P\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^{n/r}\right]^{rt} && \text{(because } P, t, r \text{ don't depend on } n\text{)} \\
&= P\left[\lim_{m\to\infty}\left(1+\tfrac{1}{m}\right)^{m}\right]^{rt} && \text{(use substitution } m = n/r\text{)} \\
&= P[e]^{rt}
\end{aligned}$$

The last step is true because it was one of our definitions of e, and proved in §6.4 in fact. $\square$

Continuous Compound Interest

Activity 5:

In this activity, we attempt to answer the question asked by many investors: "How long is it going to take for me to double my money?"

(a) Consider an investment of \$100 invested at 5%, compounded continuously. How long would it take for the investor to have \$200?

(b) What would the doubling-time be if the initial investment were \$1,000? \$10,000? What effect does changing the principal have on the doubling time, and why?

Activity 6: Continued.

One of the first things that is taught in an economics class is the Rule of 72. It can be summarized thusly:

> "The number of years it takes an investment to double is equal to 72 divided by the annual percentage interest rate."

(a) What would the Rule of 72 say the doubling time of a 5% investment is? Is it a good estimate?

(b) Repeat Parts (a) and (c) for investments of 3%, 8%, 12% and 18%. What can you say about the accuracy of the Rule of 72?

(c) Derive a precise formula for the time T to double an initial investment.

Continuous Compound Interest

Activity 7: Continued.

One of the first things that is taught in an economics class is the Rule of 72. It can be summarized thusly:

> "The number of years it takes an investment to double is equal to 72 divided by the annual percentage interest rate."

(a) There is an integer that gives a more accurate answer for continuous or nearly continuous compounding than the Rule of 72. What is this number? Check your answer by using it to estimate the doubling time of a 5% investment.

(b) It turns out that there is a reason that we use the number 72 in the Rule. It has to do with one of the assumptions we made. Why do economists use the Rule of 72?