Section 7.5 - More Trigonometric Equations

Objectives:

- **Solving Trig Eqs using IDs**
- **Eqs with multiple functions of Multiple Angles**

• Solving Trig Eqs using IDs

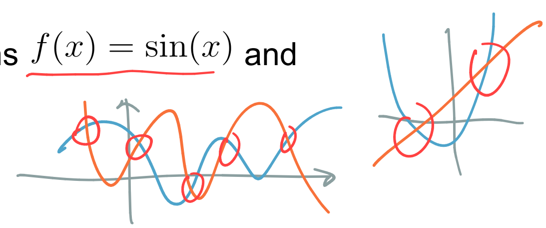
We focus on solving equations using all the identities we learned in the previous sections.

Ex 1: Find all solutions to the equation: $1 + \sin\theta = \cos^2\theta$.
To solve this, we will need to convert one of the trig functions into the other.

Ex 2: Solve: $\cos\theta + 1 = \sin\theta$ in the interval $[0, 2\pi)$.
To solve this, square both sides.

Remember: when you do this you can introduce extraneous or "false roots" so you must check all your answers in the original equation.
As a quick example: x=2 has only one solution. But squaring both sides gives $x^2 = 4$ which has two solutions x=2,-2. Of course, x=-2 is a "false root".

Ex 3: Find all the points for which the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ intersect.



Hint: Divide both sides by cosine. Why is this allowed?
Check your answer in Desmos.
Note: Another way to solve this is by squaring both sides and solving. But this is much harder!

Ex 1: Find all solutions to the equation: $1 + \sin\theta = \cos^2\theta$.
To solve this, we will need to convert one of the trig functions into the other.

Sol $\quad 1 + \sin\theta = \cos^2\theta \qquad$ Pythagorean $\cos^2\theta + \sin^2\theta = 1$

$1 + \sin\theta = 1 - \sin^2\theta \qquad$ use u-sub : $u = \sin\theta$

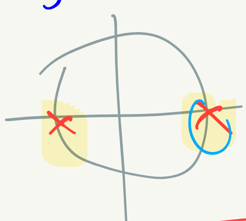
$\cancel{1} + u = \cancel{1} - u^2$

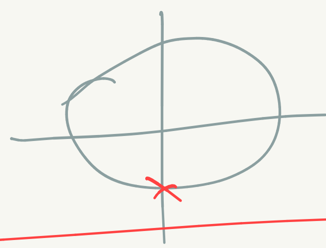
$u^2 + u = 0$

$u(u+1) = 0$

ZPP: $\quad u = 0$ or $u + 1 = 0$

$\sin\theta = 0 \qquad\qquad \sin\theta = -1$





$$\boxed{\begin{cases}\theta = 0 + 2\pi k \\ \theta = \pi + 2\pi k\end{cases} \qquad \theta = \frac{3\pi}{2} + 2\pi k \qquad k \in \mathbb{Z}}$$

combine into one

$\theta = 0 + \pi k$

$\boxed{\theta = \pi k}$

Ex 2: Solve: $\cos\theta + 1 = \sin\theta$ in the interval $[0, 2\pi)$.
To solve this, square both sides. → Moral: check answers! w/ original eq!

Remember: when you do this you can introduce extraneous or "false roots" so you must check all your answers in the original equation.
As a quick example: x=2 has only one solution. But squaring both sides gives $x^2 = 4$ which has two solutions x=2,-2. Of course, x=-2 is a "false root".

Sol

$$(\cos\theta + 1)^2 = (\sin\theta)^2$$

$$\cos^2\theta + 2\cdot\cos\theta + 1 = \sin^2\theta \qquad \text{(rewrite using Pythagorean)}$$

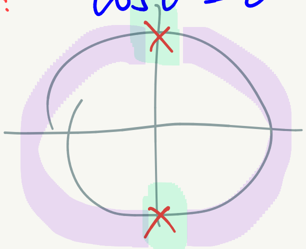
$$\cos^2\theta + 2\cos\theta + \cancel{1} = \cancel{1} - \cos^2\theta$$

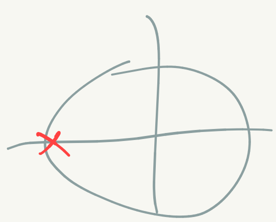
$$2\cos^2\theta + 2\cos\theta = 0 \qquad \text{(divide by 2)}$$

$$\cos^2\theta + \cos\theta = 0$$

$$\cos\theta(\cos\theta + 1) = 0$$

ZFP: $\cos\theta = 0$ or $\cos\theta = -1$





$$\theta = \frac{\pi}{2} + 2\pi k$$

$$\theta = \frac{3\pi}{2} + 2\pi k$$

$$\boxed{\theta = \pi + 2\pi k}$$

combine into one:

$$\boxed{\theta = \frac{\pi}{2} + \pi k}$$

$$\boxed{k \in \mathbb{Z}}$$

GTQ

Ex 3: Find all the points for which the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ intersect.

points $\to (x, y)$

Hint: Divide both sides by cosine. Why is this allowed?
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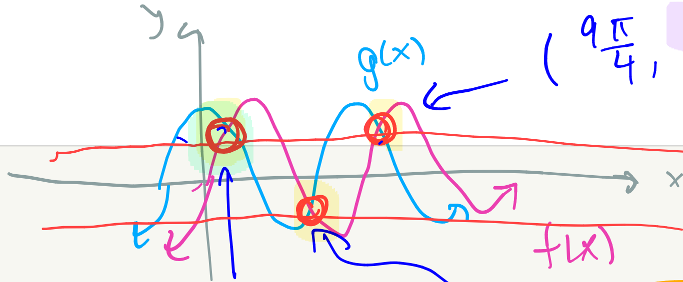
Note: Another way to solve this is by squaring both sides and solving. But this is much harder!

Sketch



$\left(\frac{9\pi}{4}, 0.707\right)$

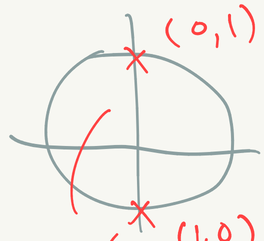
$\left(\frac{\pi}{4}, 0.707\right)$ $\left(\frac{5\pi}{4}, -0.707\right)$

using Desmos

Two graphs intersect when $f(x) = g(x)$ & solve for x.

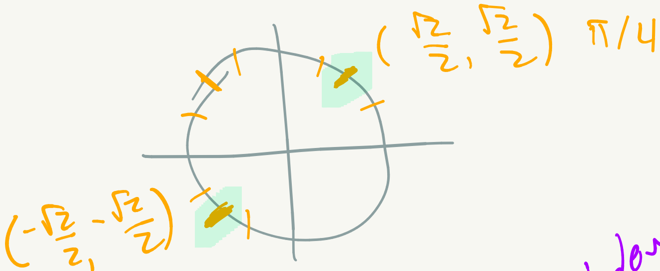
$$\boxed{\text{Solve: } \frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)}}$$

(Divide by cosine)
↳ why is this ok?



Cosine = 0
here: $\cos(x) \neq \sin(x)$
$0 \neq \pm 1$

$\tan(x) = 1$



Need to find the points (x, y)

$x = \frac{\pi}{4} + \pi k$

$x = \frac{5\pi}{4} + \pi k$

$k \in \mathbb{Z}$

not done

$f(x) = g(x) = \cos(x)$

@ $\frac{\pi}{4}$: $\cos\left(\frac{\pi}{4}\right) = \sqrt{2}/2$ (≈ 0.707)

@ $\frac{5\pi}{4}$: $\cos\left(\frac{5\pi}{4}\right) = -\sqrt{2}/2$

Answers

$\left(\frac{\pi}{4} + \pi k, \frac{\sqrt{2}}{2}\right)$ & $\left(\frac{5\pi}{4} + \pi k, -\frac{\sqrt{2}}{2}\right)$, $k \in \mathbb{Z}$

- **Eqs with multiple functions of Multiple Angles**

When solving equations with “multiple angles” like $\sin(n\theta)$.

The trick is to use **“u-substitution”**. Substitute $u = n\theta$ and solve the problem using the techniques of previous sections.

Ex 4: Consider the equation: $2\sin(3\theta) - 1 = 0$.

(a) Find all solutions in $\mathbb{R}$.

(b) Find all solutions in the interval $[0, 2\pi)$.

Ex 5: Consider the equation: $\sqrt{3}\tan\left(\frac{\theta}{2}\right) - 1 = 0$.

(c) Find all solutions in $\mathbb{R}$.

(d) Find all solutions in the interval $[0, 4\pi)$.

The trick is to use **"u-substitution"**. Substitute $u = n\theta$ and solve the problem using the techniques of previous sections.

Ex 4: Consider the equation: $2\sin(3\theta) - 1 = 0$.

(a) Find all solutions in $\mathbb{R}$.

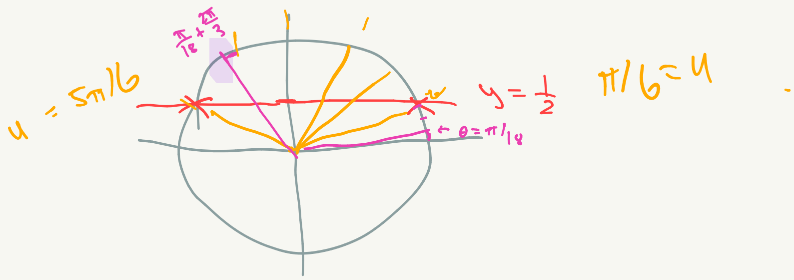
(b) Find all solutions in the interval $[0, 2\pi)$.

a) $2\sin(3\theta) - 1 = 0$ $\quad (T(\theta) = c)$

$\sin(3\theta) = \frac{1}{2}$

$u = 3\theta$

$\sin(u) = \frac{1}{2}$



$u = \frac{\pi}{6} + 2\pi k$

$u = \frac{5\pi}{6} + 2\pi k$

$\frac{3\theta}{3} = \frac{\frac{\pi}{6} + 2\pi k}{3} \rightsquigarrow \theta = \frac{\pi}{18} + \frac{2\pi}{3}k$

$\frac{3\theta}{3} = \frac{\frac{5\pi}{6} + 2\pi k}{3} \rightsquigarrow \theta = \frac{5\pi}{18} + \frac{2\pi}{3}k$

$k \in \mathbb{Z}$

Common denom:

$\frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{6}{6} = \frac{12\pi}{18}$

(b) Need answers in $[0, 2\pi)$

Key convert 2π into fraction w/ denom 18

$2\pi \cdot \frac{18}{18} = \frac{36\pi}{18}$

- $\frac{\pi}{18} + \frac{12\pi}{18}k$: $k = 0, 1, 2, \ldots$

$k=0$: $\boxed{\frac{\pi}{18}}$ $\quad k=1$: $\frac{\pi}{18} + \frac{12\pi}{18} \cdot 1 = \boxed{\frac{13\pi}{18}}$

$k=2$: $\frac{\pi}{18} + \frac{12\pi}{18} \cdot 2 = \boxed{\frac{25\pi}{18}}$

$k=3$: $\cancel{\frac{37\pi}{18}}$

- $\frac{5\pi}{18} + \frac{12\pi}{18}k$: $k = 0, 1, 2, \ldots$

$k=0$: $\boxed{\frac{5\pi}{18}}$

$k=1$: $\frac{5\pi}{18} + \frac{12\pi}{18} = \boxed{\frac{17\pi}{18}}$

$k=2$: $\frac{5\pi}{18} + \frac{12\pi}{18} \cdot 2 = \boxed{\frac{29\pi}{18}}$

$k=3$: $\frac{5\pi}{18} + \frac{12\pi}{18} \cdot 3 = \cancel{\frac{41\pi}{18}}$

- Note: there's 6 solutions to part (b)

Ex 5: Consider the equation: $\sqrt{3}\tan\left(\frac{\theta}{2}\right) - 1 = 0$.

(c) Find all solutions in $\mathbb{R}$.

(d) Find all solutions in the interval $[0, 4\pi)$.

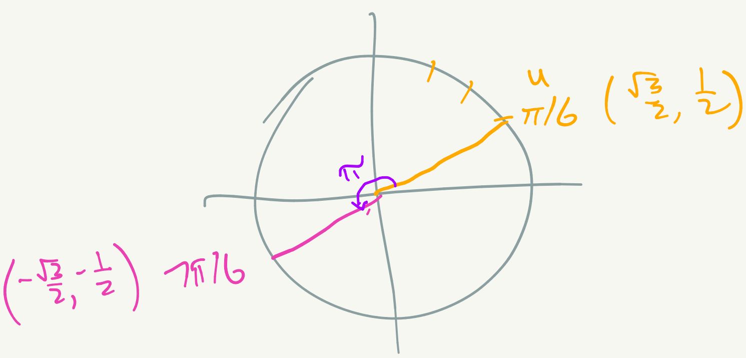
(a) $\sqrt{3}\tan\left(\frac{\theta}{2}\right) - 1 = 0$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 \cdot 1/2}{\sqrt{3} \cdot 1/2} = \frac{1/2}{\sqrt{3}/2}$$

$4\pi = \frac{12\pi}{3}$

$u = \theta/2$

$\tan(u) = \frac{1/2}{\sqrt{3}/2}$ ← sin ← cos

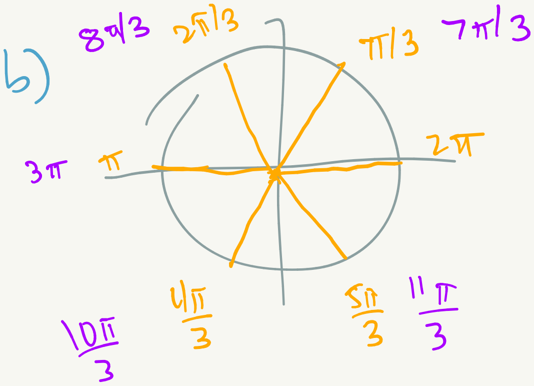


$2 \cdot \frac{\theta}{2} = u = \left(\frac{\pi}{6} + \pi k\right) \cdot 2$

$\theta = \frac{2\pi}{6} + 2\pi k$

$$\boxed{\theta = \frac{\pi}{3} + 2\pi k, \quad k \in \mathbb{Z}}$$

b)



$$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, \frac{8\pi}{3}, 3\pi, \frac{10\pi}{3}, \frac{11\pi}{3}}$$