MATH 5C - Multivariable and Vector Calculus — Summer 2019

Complete Review

Chapters 12, 13, 14, 15, 16 — **Dr. Jorge Basilio**

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Notes

Chapter 12: Vectors and the Geometry of Space

- The length of a vector and the relationship to distances between points
- Addition, subtraction, and scalar multiplication of vectors, together with the geometric interpretations of these operations
- Basic properties of vector operations
- $\boxed{\text{The dot product}}$: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
- Basic algebraic properties
- The geometric meaning of the dot product in terms of lengths and angles: in particular the formula $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos(\theta)$
- Angle formula: $\theta = \cos^{-1}\left(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$
- $\boxed{\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}}$
- Vector projections: geometric meaning and formulas.
  Projection of $\vec{b}$ onto $\vec{a}$: $comp_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}$ this is just a length.
  There is also the vector version that points along the direction of $\vec{a}$:
  $proj_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2}\, \vec{a}$ or $proj_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\, \vec{a}$.
- The cross product: definition and basic properties
- The geometric meaning of the cross product: in particular $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$, with magnitude $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin(\theta)$, and direction given by the right-hand rule
- $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the volume of the parallelopiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.
- Tests for Orthogonality:
  • $\vec{v}$ and $\vec{w}$ are orthogonal $\iff$ $\vec{v} \cdot \vec{w} = 0$
  • $\vec{v}$ and $\vec{w}$ are parallel $\iff$ $\vec{v} \times \vec{w} = 0$
  • $\vec{u}, \vec{v}$ and $\vec{w}$ are coplanar $\iff$ $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$
- LINES AND PLANES WITH VECTORS
- Intrinsic description (vectors) vs. Extrinsic description (scalar equations)
- Lines: passage between a vector equation, parametric equations, and symmetric equations
- Vector Eq of a line: $\boxed{\vec{r} = \vec{P} + t\vec{v}}$ (in book $\vec{r}_0 = \vec{P}$)
- line segment between two points
- Planes: passage between a vector description (a point together with two direction vectors) and a scalar equation

- Vector Eq of a plane: $\boxed{\vec{n} \cdot \vec{v} = 0}$ (in book $\vec{r} - \vec{r}_0 = \vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$)
- Distance from point P and a plane $\mathcal{P} : ax + by + cz + d = 0$: $\boxed{D = comp_{\vec{n}}(\vec{PQ})}$, where Q is any point on $\mathcal{P}$, or $D = \frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$
- Using vector algebra to solve geometric problems about lines and planes–it is essential that you think geometrically and try to save the number crunching in components for the last moment.
- GEOMETRY OF SURFACES
- Cylinders: know how to spot a "free (missing) variable" to help sketch
- QUADRIC SURFACES: Spheres, Cones, Ellipsoids, Elliptic Paraboloid, Hyperboloid of 1-sheet, Hyperboloid of 2-sheets, Hyperbolic Paraboloid
- Be able to recognize the above either by memorizing their equations or by using intersection with planes as done in class

Chapter 13: Vectors Functions

- $\boxed{\text{Functions } f : X \to Y}$
  where set X is domain (=set of inputs), Y is the range (=set of outputs)
- We'll only worry about: $\boxed{f : \mathbb{R}^n \to \mathbb{R}^m}$ with $n, m \geq 1$
- $n = m = 1$: real-valued function of a real variable $f : \mathbb{R} \to \mathbb{R}$
  $x \in \mathbb{R}, y \in \mathbb{R}$, usually written $y = f(x)$
  Graph is a curve in the plane
- When $Y = \mathbb{R}$: scalar-valued functions
- When $X = \mathbb{R}$ and $Y = \mathbb{R}^2$: plane curves or vector-valued functions
  $t \in \mathbb{R}$, $f(t) \in \mathbb{R}^2$ usually written $f(t) = \vec{r}(t) = \langle f(t), g(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$
  Graph is a plane curve moving throughout 2D plane
- When $X = \mathbb{R}$ and $Y = \mathbb{R}^3$: space curves or vector-valued functions
  $t \in \mathbb{R}$, $f(t) \in \mathbb{R}^3$ usually written $f(t) = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$
  Graph is a space curve moving throughout 3D plane
- Line segment from a point P to Q: $\vec{\sigma}(t) = (1 - t)P + tQ$, $t \in [0, 1]$
- Sketching space curves, vector-valued functions
- Space Curves/VVFs: limits, continuity, differentiation rules (Theorem 3, p. 858), definite integral
- Example 4 on p. 858, know this proof
- Arclength = length of a curve; $\boxed{L = \int_a^b |\vec{r}\,'(t)| dt}$

  Alternatively, you can use: $L = \displaystyle\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$
- unit tangent vector: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
- Curvature = bending from flat; $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'(t)|^3}$
- TNB Frame: $\vec{T}$, $\vec{N}$, $\vec{B}$ all unit length and mutually orthogonal to each other. Hence, making a little "frame":
  $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
- Given a space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we call $\vec{r}(t)$ the position vector-valued function. The velocity vector-valued function is the derivative of the position function:$\vec{v}(t) = \vec{r}\,'(t)$ and it's speed is the length of the velocity vector: $|\vec{v}(t)|$. It's acceleration VVF is the derivative of the velocity: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.
- Newton's Second Law: $\vec{F} = m\vec{a}$.
- Vector Differential Equations; initial conditions

Chapter 14: Partial Derivatives

- Functions: $\boxed{f : \mathbb{R}^n \to \mathbb{R}^m}$ with $n, m \geq 1$
  Now, we will have $n > 1$: functions of several variables!

- $n = 2$, $m = 1$: Scalar-Valued function of TWO variables
  $(x, y) \in \mathbb{R}^2$, $f(x, y) \in \mathbb{R}$
  Graph is $\boxed{z = f(x, y)}$
  Graph is a surface in space
  Domain D is a subset of the plane $\mathbb{R}^2$
  Level Curves: $f(x, y) = k$ for k fixed are curves in plane with height fixed–"isotherms"

- $n > 3, m = 1$: SVFs of three or more variables
  $(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$, $f(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}$
  Graph: none! Instead need to use other techniques
  Level Surfaces: $f(x_1, x_2, x_3, \ldots, x_n) = k$ for k fixed

- Limits: $\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = L}$ means: "as (x, y) approaches (a, b) along any possible path, the values $f(x, y)$ approach the unique value L."

- Know how to compute limits and to show when limits DNE by using different paths

- Continuity: $\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)}$

- Partial Derivatives: Given $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y)$
  $\boxed{\frac{\partial f}{\partial x}(a, b) = \lim_{h to 0} \frac{f(a + h, b) - f(a, b)}{h}}$ the partial derivative of f with respect to x at the point (a, b)

  $\boxed{\frac{\partial f}{\partial y}(a, b) = \lim_{h to 0} \frac{f(a, b + h) - f(a, b)}{h}}$ the partial derivative of f with respect to y at the point (a, b)
  BUT: computing them is easy! Just: "pretend the other variable is constant"

- Know the geometry of the partial derivatives as slopes of the appropriate tangent lines

- Implicit Diff with partial derivatives

- Higher partial derivatives: $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, etc

- Clairaut's Theorem: equality of mixed partials is when the second-order partial derivatives are continuous functions

- Tangent Planes: Given $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y)$
  The tangent plane of f at $P = (a, b, f(a, b))$ is $\boxed{z = f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b)}$
  Know how this formula was derived in class with $\vec{n} = \langle -f_x, -f_y, 1\rangle$

- Linearization: $\boxed{L(x, y) = f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b)}$
  When (x, y) is close to (a, b), then $f(x, y) \approx L(x, y)$–that is the linearization is a good approximation of f near P

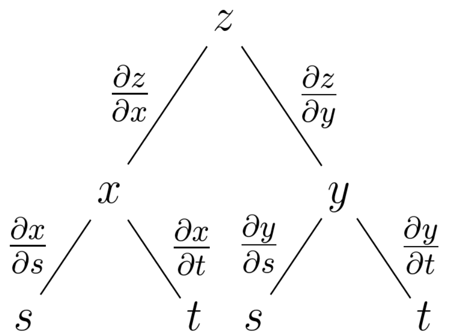
- f is differentiable at $P = (a, b, f(a, b))$ if the tangent plane exists at P.
  Notice: this is stronger than simply requiring that the partial derivatives f_x and f_y exist at P.
  Theorem: if f_x and f_y are continuous, then f is differentiable

- Differentials:
  dx and dy can be any real numbers (usually, $dx = \Delta x = x_2 - x_1$, $dy = \Delta y = y_2 - y_1$)
  Actual change in $z = f(x, y)$ from $P = (x_1, y_1)$ to $Q = (x_2, y_2)$ is: $\boxed{\Delta z = z_2 - z_1 = f(Q) - f(P)}$
  Approximate change is given by the differential dz: $\boxed{dz = f_x(a, b) \cdot dx + f_y(a, b) \cdot dy}$
  dz sometimes called the total differential
  Works for higher-dimensions too: $dz = f_{x_1} \cdot dx_1 + f_{x_2} \cdot dx_2 + \cdots + f_{x_n} \cdot dx_n$

- Chain Rule:
  Basic chain rule: $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x, yz)$, $g(t) : \mathbb{R} \to \mathbb{R}^3$ with $g(t) = \langle x(t), y(t), z(t)\rangle$, then the derivative of $(f \circ g)(t) : \mathbb{R} \to \mathbb{R}$ is

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

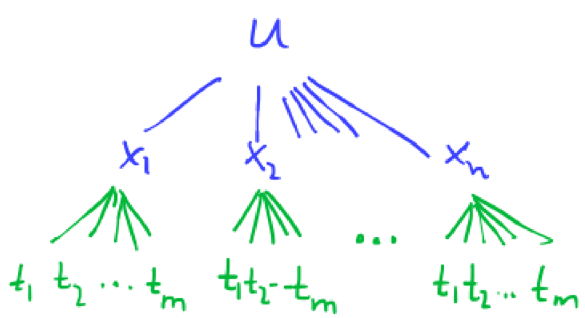
Tree diagrams are helpful for book-keeping:



- General Chain Rule:
  Assume $u : \mathbb{R}^n \to \mathbb{R}$ is a SVF of n variables written $u(x_1, x_2, \ldots, x_n)$ and each $x_i : \mathbb{R}^m \to \mathbb{R}$ is a SVF of m variables written $x_i(t_1, t_2, \ldots, t_m)$ for each $i = 1, 2, \ldots n$. Then

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

  Notice: in the above formula the t_j is the same, but we take all possible partial derivatives of u with respect to the x_i's as i ranges from 1 to n. The tree diagram is helpful:



- Gradient Vector: Given $f(x, y)$ or $f(x, y, z)$ the gradient collects all the partial derivatives into a vector:
  $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ or $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$
  Common notations: $\nabla f = \text{grad}(f) = \text{del}(f) = \partial(f)$
  This generalizes easily to higher dimensions

- Directional Derivative:
  The directional derivative of f in the direction of the unit vector $\vec{u} = \langle u_1, u_2\rangle$ (or $\vec{u} = \langle u_1, u_2, u_3\rangle$):
  $\boxed{D_{\vec{u}}(f) = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2}$ or $\boxed{D_{\vec{u}}(f) = f_x(a, b, c) \cdot u_1 + f_y(a, b, c) \cdot u_2 + f_z(a, b, c) \cdot u_3}$
  This generalizes easily to higher dimensions. We can write it compactly for all dimensions as: $\boxed{D_{\vec{u}}(f) = \nabla(f) \cdot \vec{u}}$

- Maximizing the Directional derivative:
  the maximum of $D_{\vec{u}}(f)$ at a point $P = (a, b)$ is given by $|\nabla f(a, b)|$ and occurs when $\vec{u}$ is in the same direction as $\nabla f(a, b)$.
  the minimum of $D_{\vec{u}}(f)$ at a point $P = (a, b)$ is given by $-|\nabla f(a, b)|$ and occurs when $\vec{u}$ is in the opposite direction as $\nabla f(a, b)$.

- Level Surfaces, Tangent Planes, and Gradients
  Given a function $F : \mathbb{R}^3 \to \mathbb{R}$. Consider it's level surface $S : F(x, y, z) = k$. Then the gradient of F is normal to the tangent plane at a point $P = (a, b, c)$ on the surface S (as long as it's not the zero vector), that is

$$(\nabla F)(a, b, c,) \cdot \vec{r}\,'(t_0) = 0$$

  for any space curve $\vec{r}(t)$ that travels inside the surface S and passes through P at t_0.
  We can use this to find the equation of the tangent plane: $(\nabla F)(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$.

- How is this related to the derivation of the tangent plane we learned earlier?
  Previously we started with $z = f(x, y)$ a function of two variables and its graph was a surface S.
  We can view it as a function of three variables $F(x, y, z) = z - f(x, y)$ and the surface S is the level surface of F with $k = 0$.
  From the gradient equation for $F(x, y, z) = z - f(x, y)$:

$$\begin{aligned}\nabla F(x, y, z) &= \langle \frac{\partial}{\partial x}(z - f(x, y)), \frac{\partial}{\partial y}(z - f(x, y)), \frac{\partial}{\partial z}(z - f(x, y))\rangle \\ &= \langle -f_x(x, y), -f_y(x, y), 1\rangle\end{aligned}$$

This was exactly what we got in section 14.4 where we used $\vec{n} = \vec{f_x} \times \vec{f_y} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle$.

- MAX & MIN VALUES: know the definitions of a local min/local max and global min/global max VALUES of a function f.
  Know the distinction between the min/max value of f and the point where it occurs.

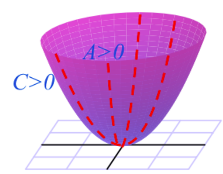
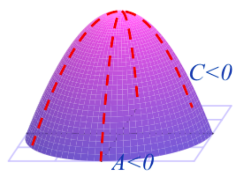
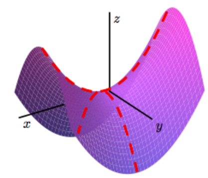
- Critical Points: $P = (a, b)$ is a critical point of f if $\nabla f(a, b) = 0$ or DNE. That is, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$; or if one of f_x or f_x DNE.

- "Fermat's Theoem:" If f has a local min/max at P and f is differentiable at P, then P is a critical point of f

- C^2 functions = second-order partial derivatives exist and are continuous

- Know: Let $A = f_{xx}(a, b)$, $C = f_{yy}(a, b)$, $B = f_{xy}(a, b)$.
  Let $\boxed{D = AC - B^2}$ called the discriminant.

- SDT: Second Derivative Test:
  Assume: f is C^2 and $P = (a, b)$ is a critical point of f.

	• if $D > 0$ and $A > 0$ then $f(a, b)$ is a local MIN value	if $D > 0$ and $A < 0$ then $f(a, b)$ is a local MAX value	if $D < 0$ then $f(a, b)$ is NOT an extremum (saddle point)	if $D = 0$ then test fails (anything can happen)
Second Derivative Test				

  Note: when $D > 0$, then $AC - B^2 > 0$ so $AC > B^2 > 0$. This implies that both A and C have the same sign. So either both $A > 0$ and $C > 0$ or both $A < 0$ and $C < 0$. This is why the bending in x and y directions make sense as in the figures above.

- Closed Subsets in the plane: a bounded set that contains all of its boundary points (the analogy of a closed interval in the line)

- Extreme Value Theorem: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continous and D is a closed subset of the plane, then f attains both an absolute minimum and absolute maximum value at points inside D.

- How to find Absolute Min/Max Values on a closed set D:
  Break up D into two parts, I = inside part (open set) of D, B = boundary curve
  Step 1: find critical points in I=inside D
  Step 2: find the points where f has extreme values in B
  To do this: parametrize the boundary curve (in pieces if necessary) with $(x(t), y(t))$, then find the extra of the one-variable function $f(t) = f(x(t), y(t))$ using Calc 1 techniques.
  Step 3: Evaluate f at points from Steps 1 and 2 and select the largest and smallest values.

- How to find Extrema on a closed set using Lagrange Multipliers:
  Let $f(x, y, z)$ and $g(x, y, z)$ be functions with continuous partial derivatives.
  To find the extremum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$, solve the equations:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$$

  for x, y, z, and λ. That is, we solve: $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, and $g = c$.

Chapter 15: Multiple Integrals

Summary:

- dA=infinitesimal unit of area:
  • Cartesian Coordinates in the plane: $dA = dxdy$
  • Polar Coordinates in the plane: $dA = rdrd\theta$

- dV=infinitesimal unit of volume:
  • Cartesian Coordinates in space: $dV = dxdydz$
  • Cylindrical Coordinates in space: $dV = rdrd\theta dz$
  • Spherical Coordinates in space: $dV = \rho^2 \sin(\phi)\, d\rho d\theta d\phi$

More details:

- Definition of a double integral as a limit
- Double Integrals of functions $f(x,y)$ over rectangles $R=[a,b]\times[c,d]$ as iterated integrals
- Geometric Interpretation of $\iint_D f(x,y)\,dA$: Volume under the graph of the surface $z=f(x,y)$ (when $f(x,y)\geq 0$) lying above the rectangle R in the plane.
- Fubini's Theorem:
  When integrating over a rectangle, you can do the integrals in any order!
  $$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$
- Area a domain D in the plane: $\text{Area}(D) = \iint_D 1\,dA$.
- Double Integrals over Elementary Domains D in the plane:
  • D is Type I:
  $$D: \begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases} \implies \iint_D f dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$
  • D is Type II:
  $$D: \begin{cases} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{cases} \implies \iint_D f dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$
- FACT: if f is continuous on the elementary region D, then the double integral over D exists.
- Be able to compute double integrals of Type I or II fully. But also be able to set-up the correct integrals. Given an integral, be able to read and sketch the domain and switch the order of integration.
- Double Integrals in Polar Coordinates:
  Given cartesian coordinates (x,y), the equations for polar coordinates are: $r^2=x^2+y^2$ and $\theta=\tan^{-1}(y/x)$.
  Given polar coordinates (r,θ), the equations for cartesian coordinates are: $x=r\cos(\theta)$ and $y=r\sin(\theta)$.
  The infinitesimal unit of area is: $\boxed{dA = r\,dr\,d\theta}$
  • When D can be easily described by polar coordinates as a sector (circles, quarter circles, annuli, etc):
  $$D: \begin{cases} a \leq r \leq b \\ \alpha \leq \theta \leq \beta \end{cases} \implies \iint_D f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\,r dr\,d\theta$$
  or $\int_a^b \int_\alpha^\beta f(r\cos(\theta), r\sin(\theta))\,r d\theta\,dr$ by Fubini's Theorem.
  • When D is a more general region in PC:
  When the "wobbly sector" i.e. $r=h_1(\theta)$ is a lower bound for r and $r=h_2(\theta)$ is an upper bound for r:
  $$D: \begin{cases} \alpha \leq \theta \leq \beta \\ h_1(\theta) \leq r \leq h_2(\theta) \end{cases} \implies \iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r dr\,d\theta$$
- Be able to find the area of regions described using PC
- Triple Integrals of $f(x,y,z)$ over boxes $B=[a,b]\times[c,d]\times[r,s]$ using iterated integrals
- Geometric Interpretation of $\iiint_E f(x,y,z)\,dV$: We can't visualize this! The units of this integral are 4-dimensional! It sums up the values of the function $f(x,y,z)$ times the infinitesimal volume dV as (x,y,z) ranges over the solid E in space.
  Best way to think of it: $T(x,y,z)$ is temperature at point (x,y,z) in the oven B then $\iiint_B T(x,y,z)\,dV$ is the total temperature inside B.
- Fubini's Theorem:
  When integrating over a box, you can do the integrals in any order!
  $$\iiint_B f(x,y,z)\,dV = \int_a^b \left[\int_c^d \left[\int_r^s f(x,y,z)\,dz\right] dy\right] dx = \int_a^b \left[\int_r^s \left[\int_c^d f(x,y,z)\,dy\right] dz\right] dx$$
  and equal to any of the other 4 possibilities.

- Volume of a region E in space: $\text{Vol}(E) = \iiint_E 1\, dV$.

- Triple Integrals over Elementary Regions E in space:
  • E is Type I:
$$E : \begin{cases} (x,y) \in D \\ u_1(x,y) \le z \le u_2(x,y) \end{cases} \quad \Longrightarrow \quad \iiint_E f dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$
  then depending on whether D is Type I or Type II:
$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dy \right] dx \qquad (D \text{ is Type I})$$
$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dx \right] dy \qquad (D \text{ is Type II})$$

  • E is Type II:
$$E : \begin{cases} (y,z) \in D \\ u_1(y,z) \le x \le u_2(y,z) \end{cases} \quad \Longrightarrow \quad \iiint_E f dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA$$
  then depending on whether D is Type I or Type II:
$$\iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dz \right] dy \qquad (D \text{ is Type I})$$
$$\iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA = \int_r^s \left[\int_{h_1(z)}^{h_2(z)} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dy \right] dz \qquad (D \text{ is Type II})$$

  • E is Type III:
$$E : \begin{cases} (x,z) \in D \\ u_1(x,z) \le y \le u_2(x,z) \end{cases} \quad \Longrightarrow \quad \iiint_E f dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA$$
  then depending on whether D is Type I or Type II:
$$\iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dz \right] dx \qquad (D \text{ is Type I})$$
$$\iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA = \int_r^s \left[\int_{h_1(z)}^{h_2(z)} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dx \right] dz \qquad (D \text{ is Type II})$$

- Important examples are to compute the volume of spheres using either Type I, II, or III triple integrals.

- Triple Integrals in Cylindrical Coordinates:
  Cylindrical coordinates: (r, θ, z)
  Given cartesian coordinates (x, y, z), the equations for cylindrical coordinates are: $x^2 + y^2 = r^2$, $\theta = \tan^{-1}(y/x)$, and $z = z$.
  Given cylindrical coordinates (r, θ, z), the equations for cartesian coordinates are: $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.
  The infinitesimal unit of volume is: $\boxed{dV = r\, dr\, d\theta\, dz}$
  • When E can be easily described by cylindrical coordinates as a cylinder (or part of):
$$E : \begin{cases} a \le r \le b \\ \alpha \le \theta \le \beta \\ r \le z \le s \end{cases} \quad \Longrightarrow \quad \iiint_E f(x,y,z) dV = \int_r^s \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta), z)\, r dr\, d\theta\, dz$$
  or in any of the other 5 possible orders of $dr, d\theta, dz$ by Fubini's Theorem.
  • When E is a more general region in CC:
  Besides cylinders know the equation of cone in CC: $z = r$. So you can describe regions like an "ice cream cone"

- Triple Integrals in Spherical Coordinates:
  Spherical coordinates: (ρ, θ, ϕ)
  Given cartesian coordinates (x, y, z), the equations for Spherical coordinates are: $\rho^2 = x^2 + y^2 + z^2$, $\theta = \tan^{-1}(y/x)$, and and $\phi = \cos^{-1}(z/\rho)$.

Given Spherical coordinates (ρ, θ, ϕ), the equations for cartesian coordinates are: $x = (\rho \sin(\phi)) \cos(\theta)$, $y = (\rho \sin(\phi)) \sin(\theta)$, and $z = \rho \cos(\phi)$.

The infinitesimal unit of volume is: $\boxed{dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi}$

• When E can be easily described by Spherical coordinates as a sphere (or part of):

$$E : \begin{cases} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ \delta \le \phi \le \gamma \end{cases} \qquad \Longrightarrow$$

$$\iiint_E f(x, y, z) dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))\, \rho^2 \sin(\phi) d\rho\, d\theta\, d\phi$$

or in any of the other 5 possible orders of $d\rho, d\theta, d\phi$ by Fubini's Theorem.

• When E is a more general region in SC:

Besides spheres know the equation of cone in CC: ϕ =constant. So you can describe regions like an "ice cream cone"

Chapter 16: Vector Calculus

- Vector Fields: a vector field $\vec{F}$ gives a vector (in plane or in space) at every point.
  More generally, vector fields are functions: $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$
  • VFs in the Plane: $\boxed{\vec{F} = \langle P, Q \rangle}$
  $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are SVFs.
  • VFs in Space: $\boxed{\vec{F} = \langle P, Q, R \rangle}$
  $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ where $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ are SVFs.
- Visualization of a vector field as a "field of arrows" and interpretation as a force field, or fluid flow
- Important examples: (a) "Explosion" $\vec{F}(x, y) = \langle x, y \rangle$; (b) "Implosion" $\vec{F}(x, y) = -\langle x, y \rangle$; (c) "Circulation" counter-clockwise $\vec{F}(x, y) = \langle -y, x \rangle$; (c) "Circulation" clockwise $\vec{F}(x, y) = \langle y, -x \rangle$
- Gradient Vector Fields: $\boxed{\nabla f = \langle f_x, f_y, f_z \rangle}$
- Recall: curves in the plane and in space:
  $\boxed{\vec{r}(t) = \langle x(t), y(t), z(t) \rangle}$ and $\boxed{ds = |\vec{r}'(t)|\, dt}$
  since $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = |\vec{r}'(t)| dt$.
  Infinitesimal unit of vector arclength: $d\vec{r} = \vec{T}(t) ds$.
  But this is a pain to compute, so instead we use: $\boxed{d\vec{r} = \vec{r}'(t)\, dt}$
- LINE INTEGRAL OF $\vec{F}$ ALONG A CURVE C: $\int_C \vec{F} \cdot d\vec{r}$.
  General: $\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt}$ (Notice: this uses the DOT product!)

  In the plane: $\boxed{\int_C \langle P, Q \rangle \cdot d\vec{r} = \int_C P dx + Q dy}$

  Notice: $\vec{F} = \langle P, Q \rangle$ and $d\vec{r} = \vec{r}'(t) dt = \langle x'(t), y'(t) \rangle dt$, so computing the dot product gives:
  $\vec{F} \cdot d\vec{r} = \langle P, Q \rangle \cdot \langle x'(t), y'(t) \rangle dt = P x'(t) dt + Q y'(t) dt = P dx + Q dy$
  since $\boxed{dx = x'(t) dt}$ and $\boxed{dy = y'(t) dt}$

  In space: $\boxed{\int_C \langle P, Q, R \rangle \cdot d\vec{r} = \int_C P dx + Q dy + R dz}$
- Geometric Meaning of a line integral of a vector field along a closed curve C: Circulation of $\vec{F}$ along the curve C
- Know how to parametrize curves: line segments, circles, ellipses, parabolas, squares, triangles, etc
- Properties of curves: orientation, $C_1 \cup C_2$, $-C$ etc
- Properties of Line integrals: $\int_{C_1 \cup C_2} \vec{F} = \int_{C_1} \vec{F} + \int_{C_2} \vec{F}$ and $\int_{-C} \vec{F} = -\int_C \vec{F}$.
- DEFINITIONS/TERMINOLOGY:
  Definition of $\vec{F}$ path independent
  Curves C: Closed, Simple

Domains D: Open, connected, simply connected

NOTATION: $\boxed{\partial D = C}$ is the notation for the boundary curve of D. It comes with orientation defined by: positive when traveling along the boundary curve, the domain D is on your left side. Negative when traveling along the boundary curve, the domain D is on your right side.

- CONSERVATIVE VECTOR FIELDS
  Definition of $\vec{F}$ conservative
  $\boxed{\text{THM}}$ $\vec{F}$ conservative $\iff \oint_C \vec{F} = 0$ for all closed loops
  $\boxed{\text{THM}}$ $\vec{F}$ conservative $\iff$ it is the gradient of some function, ie $\vec{F} = \nabla f$
  Note: f is called a Potential function. Know how to find f if given a conservative VF
  $\boxed{\text{THM}}$ (Fundamental Thm of Line Integrals): $\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(B) - f(A)$
  (where C a curve from A to B)
  $\boxed{\text{THM}}$ (Fundamental Theorem of Conservative VFs):
  Let D be a simply connected domain in the plane. Then
  $\vec{F} = \langle P, Q \rangle$ is conservative on $D \iff \boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$ on D

- GREEN'S THEOREM
  Assumptions needed:
  • D simply connected domain in the plane (=open+connected+no holes or punctures)
  • $\partial D = C$ the boundary curve is a simple, closed curve oriented positive sense (ie CCW)
  • $\vec{F} = \langle P, Q \rangle$ with P, Q continuous partial derivatives inside D and on ∂D

  $$\text{THEN } \boxed{\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}$$

  WARNING: $\vec{F}$ must be defined and differentiable inside D for you to apply Green's Theorem

- Scalar Curl: S.Curl$(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$
  Meaning: the infinitesimal circulation of $\vec{F}$ at the point (x, y)

- Vector Form of Green's Theorem: $\boxed{\oint_{\partial D} \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_D \text{S.Curl}(\vec{F})\, dA}$

GRADIENT OPERATOR, CURL, & DIVERGENCE

- Del Operators: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ in 2D and $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ in 3D

- CURL of F: $\boxed{\text{Curl}(\vec{F}) = \nabla \times \vec{F}}$ only for 3D $\vec{F} = \langle P, Q, R \rangle$

  $$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

  NOTE: Curl$(\vec{F})$ is clearly a vector!
  Geometric Meaning: the **circulation** at a point through a plane orthogonal to Curl$(\vec{F})$

- DIVERGENCE of F: $\boxed{\text{div}(\vec{F}) = \nabla \cdot \vec{F}}$

  div$(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle P, Q \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

  div$(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.
  Geometric Meaning: the contribution of $\vec{F}$ in the direction of the "explosion vector field" at a point. This is termed **"flux" or "divergence"** of the vector field.

INTEGRATION OVER SURFACES

- Recall Surfaces in space
  you can define a surface via a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $z = f(x, y)$
  you can define a surface implicitly via a function $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x, y, z) = c$ (think equation of sphere)

- Given a surface $S : z = f(x, y)$
  Infinitesimal piece of surface area: $dA = \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy$
  Normal vector to S at a point: $\vec{n} = \langle -f_x, -f_y, 1\rangle$ (outward pointing)
  Recall this comes from: $\vec{n} = \vec{f_x} \times \vec{f_y} = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle$
  Unit Normal: $\hat{n} = \dfrac{\vec{n}}{\|\vec{n}\|} = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1 + (f_x)^2 + (f_y)^2}}$
  Oriented infinitesimal area: $d\vec{A} = \hat{n}dA = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+(f_x)^2+(f_y)^2}} dA = \vec{n}dxdy$ so $\boxed{d\vec{A} = \vec{n}dxdy}$
  OR $\boxed{d\vec{A} = \langle -f_x, -f_y, 1\rangle\, dxdy}$

- SURFACE INTEGRAL OF $\vec{\Phi}$ ACROSS/THROUGH S: $\iint_S \vec{F} \cdot d\vec{A}$.
  General: $\boxed{\displaystyle\iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(x, y) \cdot \vec{n}\, dxdy}$
  $$\boxed{\iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(x, y) \cdot \langle -f_x, -f_y, 1\rangle\, dxdy}$$
  Alternate Form: $\boxed{\displaystyle\iint_S \langle P, Q, R\rangle \cdot d\vec{A} = \iint_D -Pf_x\, dx - Qf_y\, dy + R\, dz}$
  Geometric Meaning: "Flux/Divergence" of $\vec{F}$ across/through the surface S

STOKE'S THEOREM

- STOKE'S THEOREM
  Assumptions needed:
  • D and ∂D are planar domain and boundary curve that satisfy assumptions of Green's Theorem
  • S and ∂S is a surface in space of the form $z = f(x, y)$ over the domain D and $f(\partial D) = \partial S$ (this just says that the function f evaluated over the boundary curve in the plane gives the boundary curve ∂S of the surface S in space)
  • orientation ∂S is oriented in the positive sense (the surface is always on your left as you walk around the boundary)
  • orientation S is oriented in the positive sense (outward pointing normal vector)

  THEN $\boxed{\displaystyle\oint_{\partial S} \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_S \text{Curl}(\vec{F}) \cdot d\vec{A}}$

  Equivalently: $\boxed{\displaystyle\oint_{\partial S} \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{A}}$

  Or: $\boxed{\displaystyle\oint_{\partial S} Pdx + Qdy + Rdz = \iint_S -f_x(R_y - Q_z) - f_y, (P_z - R_x) + (Q_x - P_y)\, dxdy}$

  Geometric meaning: The "circulation/curl" of $\vec{F}$ along ∂S.

FLUX and DIVERGENCE

- FLUX of $\vec{F}$ ACCROSS C in the Plane: $\displaystyle\int_C \vec{F} \cdot \hat{n}ds$.
  Geometric meaning: the contribution of $\vec{F}$ across/through the curve C. The "flux/divergence" across C.

- Formula for $\hat{n}ds$:
  • parametrize C with $\vec{r}(t) = \langle x(t), y(t)\rangle$
  • ds=infinitesimal piece of arclength of the curve C: $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$
  • $\vec{n}$ = normal vector: outward pointing vector that is orthogonal to the tangent vector $\vec{r}'(t)$
  • $\boxed{\vec{n} = \langle \dfrac{dy}{dt}, -\dfrac{dx}{dt}\rangle}$
  • $\hat{n}$ = unit normal vector: $\hat{n} = \dfrac{\vec{n}}{\|\vec{n}\|} = \dfrac{\langle \frac{dy}{dt}, -\frac{dx}{dt}\rangle}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$
  • All of these simply to: $\boxed{\hat{n}ds = \langle \dfrac{dy}{dt}, -\dfrac{dx}{dt}\rangle dt}$

- Alternate form of flux using $F(x,y)=\langle P,Q\rangle$: $\int_C \vec{F}\cdot\hat{n}ds = \int_C -Qdx+Pdy$.

- GAUSS' DIVERGENCE THEOREM in the plane: $\boxed{\int_C \vec{F}\cdot\hat{n}ds = \iint_D (\nabla\cdot\vec{F})\,dxdy}$

- GAUSS' DIVERGENCE THEOREM in space: $\boxed{\iint_{\partial E} \vec{F}\cdot d\vec{A} = \iiint_E (\nabla\cdot\vec{F})\,dxdydz}$

  where E is a solid region in space and ∂E is the surface which is the boundary of E
  Note: $\mathrm{div}(\vec{F}) = \nabla\cdot\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle P,Q,R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.