§7.3: Trigonometric Substitution

Ch 7: Techniques of Integration
Math 5B: Calculus II

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Class #10 Notes

March 26, 2019
Spring 2019

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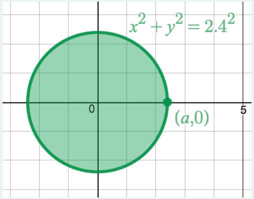
Guiding Questions for §7.3

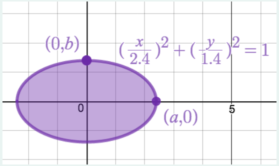
Guiding Question(s)

1. What is trigonometric substitution?
2. What are some blockbuster applications illustrating the technique?

Introduction

- The motivation is "classical" in that we would like to find the areas of some conic sections like circles, ellipses.
- A circle is given by $x^2 + y^2 = a^2$, $a > 0$, and an ellipse is given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $a, b > 0$.



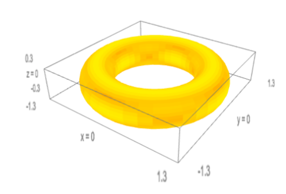


- These give rise to the corresponding integrals:

$$2\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx \quad \text{and} \quad 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

Introduction

- I think the blockbuster application is that we can find the volume of a torus.
- A torus is a surface of revolution given by rotating a circle about an axis:



- We'll work this out later so I wont spoil the surprise of the integral we get.
- Notice, we can already do other conics like parabolas ($y = ax^2$ by integrating and using the power rule) and hyperbolas ($y = 1/x$ by integrating using the natural logarithm).

Introduction

- I short, our goal is to learn to evaluate integrals that contain the expressions

$$\sqrt{a^2 - x^2} \quad \text{or} \quad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2}$$

  somewhere in the integrand.
- The technique we'll use is called trigonometric substitution.
- Recall this is technique #4 in the integration toolbox introduced in §7.1.

Trigonometric Substitution

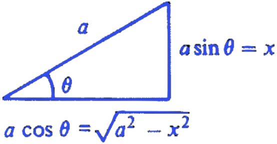
Definition 1: Trigonometric Substitution

- Integrals with $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin(\theta)$
  - Valid only if: $\theta \in [-\pi/2, \pi/2]$
  - Identity to use: $\cos^2(\theta) + \sin^2(\theta) = 1$
- Integrals with $\sqrt{a^2 + x^2}$ use the substitution $x = a\tan(\theta)$
  - Valid only if: $\theta \in (-\pi/2, \pi/2)$
  - Identity to use: $1 + \tan^2(\theta) = \sec^2(\theta)$
- Integrals with $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec(\theta)$
  - Valid only if: $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$
  - Identity to use: $\cos^2(\theta) + \sin^2(\theta) = 1$
- IMPORTANT POINT: GO BACK TO ORIGINAL VARIABLE AFTER YOU'VE INTEGRATED! Use the triangles for this.
- These are called inverse substitution since we'll need to solve for θ in the substitution equations.

Trigonometric Substitution

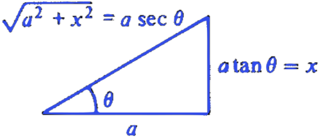
Case: $\sqrt{a^2 - x^2}$

$$\begin{cases} x &= a\sin(\theta) \\ dx &= a\cos(\theta)d\theta \end{cases}$$



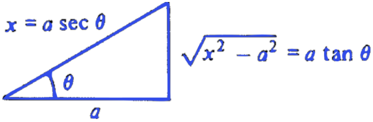
Case: $\sqrt{a^2 + x^2}$

$$\begin{cases} x &= a\tan(\theta) \\ dx &= a\sec^2(\theta)d\theta \end{cases}$$



Case: $\sqrt{x^2 - a^2}$

$$\begin{cases} x &= a\sec(\theta) \\ dx &= a\sec(\theta)\tan(\theta)d\theta \end{cases}$$



Why is works? For example,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin(\theta))^2} = \sqrt{a^2(1 - \sin^2(\theta))} = a\sqrt{\cos^2(\theta)} = a\cos(\theta)$$

since $a > 0$ and $\cos(\theta) > 0$ with the restrictions on θ. All the restrictions mentioned previously are so that we don't have to worry about absolute values!

Activity 1: Case: $\sqrt{a^2 - x^2}$

(a) Evaluate $\displaystyle\int \frac{1}{\sqrt{9 - x^2}}\, dx$ in two ways:

(i) using $\sin^{-1}(x)$ and it's DR/ADR, and (ii) using trig sub

(b) Find: $\displaystyle\int \sqrt{16 - x^2}\, dx$

Trig Sub: Case: $\sqrt{a^2 - x^2}$

Activity 2: Case: $\sqrt{a^2 - x^2}$

Find the area of a circle of radius $a > 0$ by setting up an appropriate definite integral and solving it with trig sub.

Trig Sub: Case: $\sqrt{a^2 - x^2}$

Activity 3: Case: $\sqrt{a^2+x^2}$

Evaluate:

(a) $\displaystyle\int \frac{1}{\sqrt{x^2+9}}\,dx$

(b) $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x+9)^{3/2}}\,dx$

(Example 6 in our book, this is a hard example. Hint: start with a sub $y = 2x$.)

Trig Sub: Case: $\sqrt{a^2 + x^2}$

Activity 4: Case: $\sqrt{x^2 - a^2}$

Evaluate: $\displaystyle\int \frac{1}{x^2\sqrt{4x^2-16}}\,dx$

Trig Sub: Case: $\sqrt{x^2 - a^2}$

Application to integrals with $ax^2 + bx + c$

To evaluate an integral with a general quadratic term, $ax^2 + bx + c$, try using complete the square and trig sub.

Activity 5: Application to integrals with $ax^2 + bx + c$

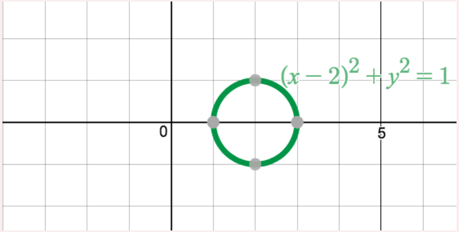
Evaluate: $\displaystyle\int \frac{1}{(x^2 - 6x + 11)^2}\,dx$

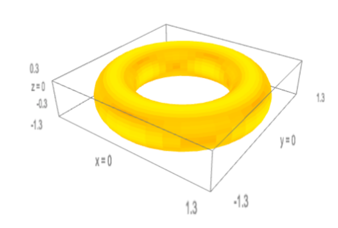
Application to integrals with $ax^2 + bx + c$

Application: Volume of a Doughnut

Activity 6: Application: Volume of a Doughnut

Find the volume of the "doughnut", that is, the inside of the surface of revolution obtained by rotating the circle $(x-2)^2 + y^2 = 1$ about the y-axis.





You may use either the Washer Method or the Shell Method.

Application: Volume of a Doughnut