MATH 110B - Precalculus II Spring 2020

Exam 3

Ch 8, 9

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April_27

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Solutions! Solutions! Solutions!

Directions

1. Please hand-write the statement provided below in quotes; print your name; and sign your name below it that acknowledges the **honor code**:

   "On my honor, by printing and signing my name, I vow to neither receive nor give any unauthorized assistance on this examination. I understand what my professor has deemed appropriate and inappropriate for this test and vow to follow these rules."

2. The exam is written to last 1 hour and 25 minutes from 9:10 am to 10:35 am, however, you have until 11:05 am to submit this exam without penalty.

3. How to submit: upload a **single PDF file** of your solutions to Canvas no later than 11:05 am to avoid penalties.

4. You will need to be logged in to class via Zoom. You will also need to have your camera and microphone both functioning and turned ON for the duration of the test. Ask family members, roommates, to not disrupt you during the test. Please disable any virtual backgrounds.

5. Write your solutions to the exam on one side of the page (front side only, do not write on the back of the page).

6. Work problems out in the order provided! Also, carefully indicate the problem and part that you are working on. Failure to do so will result in significant point deductions.

7. Make sure to ALWAYS SHOW YOUR WORK; you will not receive any partial credits unless work is clearly shown unless told otherwise. *If in doubt, ask for clarification.* Correct answers with little to no work will receive no points. Students might be randomly selected to have a 1-1 conference where you are asked to defend your work and explain to me all your steps on certain questions or problems that are similar to a test question.

8. **Penalties for late submissions:** Exams received between 11:06 and 11:15 am will have 10 points deducted from their score. Exams received between 11:16 and 11:25 am, I will deduct an additional point from your score for each additional minute the exam is late. Exams received after 11:26 am will not be graded and be given a score of 0.

9. **Allowed Materials**
   - Scientific calculator with no graphing features.
   - Blank pieces of paper to write your solutions. Writing utensils, erasers, etc
   - Physical copy of the textbook. No e-books. No solutions manuals.
   - Class notes (you can have your hand-written class notes)

10. **Materials NOT Allowed**
    - Do not use your cell phone (for any reason: do not send or receive texts or calls or use the internet, etc)
    - Do not use digital or printed out notes: the slides, the study guides, etc (only your textbook and hand-written class notes)
    - Do not consult your HW or ICAs
    - Do not give or receive any outside help (no getting help from a family member, friend, or any person either in person, via chat, message board, text message or any form of communication–again–you will be on camera the entire time so I will be looking for suspicious behavior)
    - Do not use your computer to look up anything using the internet (don't google; don't consult homework help websites, etc)

    Continued on the next page...

Directions Continued:

11. The exam totals **102 points** but will be scored out of 100.
12. There are 7 problems (plus one extra-credit problem at the end), many of them with multiple parts.
13. Handwriting should be neat and legible. If I cannot read your writing, zero points will be given.
14. Some questions contain multiple-parts which you must do individually and the parts are denoted by (a), (b), (c), etc. Some questions are multiple-choice and the choices are denoted with (A), (B), (C), (D), and (E). For True/False questions, you must spell out the entire word "true" or "false" in your answer.
15. Leave answers in exact form (as simplified as possible), unless told otherwise.
16. Put a $\boxed{\text{box around your final answer}}$ where applicable.
17. **PLEASE CHECK YOUR WORK!!!**
18. GOOD LUCK!!!!

Problem 1: 20 pts (2 pts each)

Fill-in the blank: *(No work needed)*

(a) A **polar equation**, is an equation in the variables r and θ.

(b) The **equation in polar coordinates** $r = \theta$ describes what shape: (Archimedean) Spiral.

(c) Given a point $P = (x, y)$ in rectangular coordinates, to find the **radius** r we must use the formula $r = \pm\sqrt{x^2+y^2}$.

(d) If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number in polar form,

then **De Moivre's Theorem** says $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

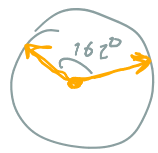
(e) The DEFINITION of the **dot product** between $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v_1, v_2\rangle$ is $\vec{u} \bullet \vec{v} = (u_1)(v_1) + (u_2)(v_2)$.

TRUE or **FALSE** (please spell out/write the entire word for credit). *(No work needed)*

(a) FALSE **Parametric equations** will ~~always~~ rarely trace out curves that pass the vertical line test.

(b) TRUE The **dot product** between two vectors $\vec{u}$ and $\vec{v}$ is a scalar.

(c) FALSE If the **angle** between two unit vectors $\vec{u}$ and $\vec{v}$ is 162°, then $\vec{u} \bullet \vec{v}$ is ~~positive~~ neg.



(d) FALSE The term **orthogonal** means the that two vectors are ~~parallel~~. perpendicular

(e) TRUE In **polar coordinates**, the points $P = (r, \theta)$ and $Q = (-r, \theta + \pi)$ correspond to the same point.

Problem 2: 2 pts

Please give **exact values!** You must show work/formulas used to receive full credit.

The point $P = (8, 8)$ is given in rectangular coordinates, express P in **polar coordinates**.

Problem 3: 8 pts

Solve the following. Please give **exact values!** You must show work/formulas used to receive full credit.

(a) **Convert** the polar equations into equations in rectangular coordinates: $r = 2\cos(5\theta)$ (too hard — everyone was given full points for this part (part b))

(b) Provide a **graph** of the polar equation in part (a). Label at least 4 distinct points on your graph.

Problem 4: 4 pts

Sketch two full periods of the cycloid:

$$\begin{cases} x &= 3(\theta - \sin(\theta)) \\ y &= 3(1 - \cos(\theta)) \end{cases}$$

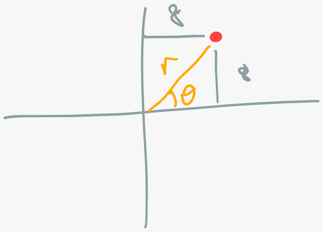
Use arrows to indicate the direction of the curve as θ increases.

$\boxed{\text{Problem 2}}$ Write $P=(8,8)$ in polar coord.

Sol

$x=8$
$y=8$

$$\begin{cases} r^2 = x^2+y^2 \\ \theta = \tan^{-1}(y/x) \end{cases}$$



- $r^2 = x^2+y^2$
  $r^2 = 8^2+8^2$
  $r^2 = 2\cdot 64$
  $r = \pm\sqrt{2\cdot 64} = \sqrt{2}\cdot 8 = 8\sqrt{2}.$

  pick +

- $\theta = \tan^{-1}(y/x)$
  $= \tan^{-1}(8/8) = \tan^{-1}(1) = \pi/4$

$$\boxed{P = (8\sqrt{2}, \pi/4)}$$ in polar coord.

$\boxed{\text{Problem 3}}$ a) Convert $r = 2\cos(5\theta)$ into RC. (too hard - full credit given)

b) Graph eq in part (a). Graph at least 4 points.

Sol a) $r = 2\cos(5\theta)$

recall $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $x^2+y^2=r^2$

- $\cos(3\theta) = \cos(2\theta+\theta)$
  $= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$
  $= [\cos^2\theta - \sin^2\theta]\cos\theta - [2\sin\theta\cos\theta]\sin\theta$
  $= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\cos\theta$
  $= \cos^3\theta - 3\sin^2\theta\cos\theta$

- $\sin(3\theta) = \sin(2\theta+\theta)$
  $= \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta$
  $= [2\sin\theta\cos\theta]\cos\theta + [\cos^2\theta - \sin^2\theta]\sin\theta$
  $= 3\cos^2\theta\sin\theta - \sin^3\theta$

- $\cos(5\theta) = \cos(2\theta+3\theta)$
  $= \cos(2\theta)\cos(3\theta) - \sin(2\theta)\sin(3\theta)$
  $= [\cos^2\theta - \sin^2\theta]\cos(3\theta) - [2\sin\theta\cos\theta]\sin(3\theta)$
  $= [\cos^2\theta - \sin^2\theta](\cos^3\theta - 3\sin^2\theta\cos\theta)$
  $- [2\sin\theta\cos\theta](3\cos^3\theta\sin\theta - \sin^3\theta)$
  $= \cos^5\theta - 3\sin^2\theta\cos^3\theta - \cos^3\theta\sin^2\theta$
  $+ 3\sin^4\theta\cos\theta - 6\cos^3\theta\sin^2\theta$
  $+ 2\sin^4\theta\cos\theta$
  $= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

- so: $r = 2\cos(5\theta)$

$\Rightarrow r = 2(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$

Multiply both sides by r^5:

$$r^6 = 2r^5(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$$

Problem 3

(a) Cont:

$$r^6 = 2\left((r\cos\theta)^5 - 10(r\cos\theta)^3(r\sin\theta)^2 + 5(r\cos\theta)(r\sin\theta)^4\right)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$

$$\boxed{(x^2+y^2)^3 = 2(x^5 - 10x^3y + 5xy^4)}$$

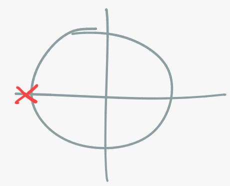
check this on Desmos & see if it agrees w/ Rose below! :)

Sol (b) Graph is a rose w/ 5 petals ($n=5$ odd).

Since $|\cos(5\theta)| = 1$ we graph $|r| = |2\cos(5\theta)| = 2$.

We want to find θ when $\cos(5\theta) = 1$ & -1.

$\cos(5\theta) = 1$ & $\cos(5\theta) = -1$.



$5\theta = 0 + 2\pi k$

$\theta = \frac{2\pi}{5}k, \ k \in \mathbb{Z}$

$k = 0, 1, 2, \ldots$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \frac{10\pi}{5}$

$5\theta = \pi + 2\pi k$

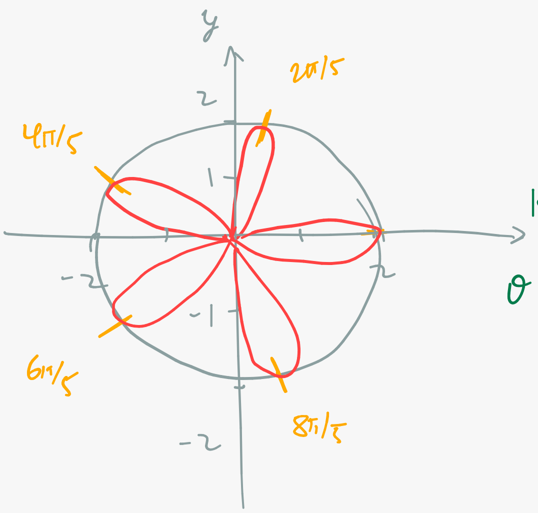
$\theta = \frac{\pi}{5} + \frac{2\pi}{5}k$

$k = 0, 1, 2, \ldots$

$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$

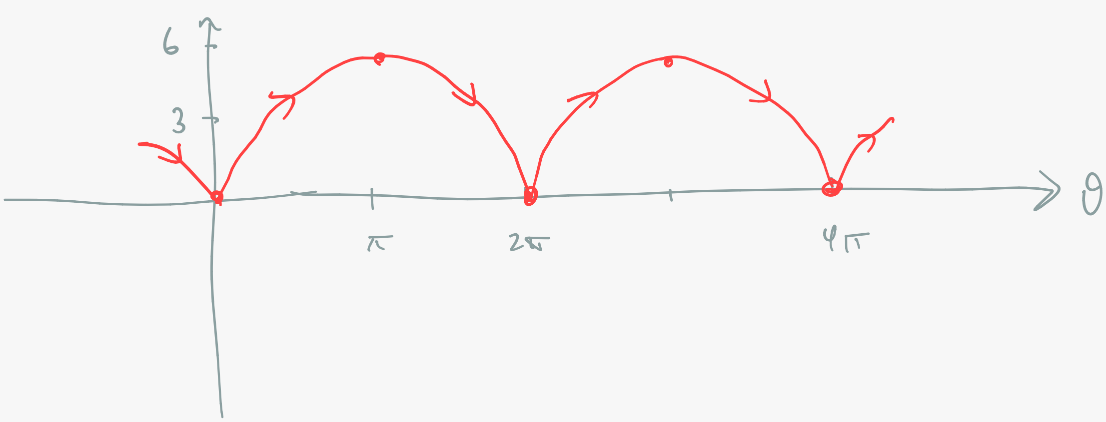
Since $r = -2$ here need to add π to it!

$6\pi/5, \ 8\pi/5, \ 2\pi, \ \frac{12\pi}{5} = \frac{2\pi}{5}, \ \frac{14\pi}{5}$



Problem 4 sketch two full period of cycloid

$$\begin{cases} x = 3(\theta - \sin\theta) \\ y = 3(1 - \cos\theta) \end{cases}$$



Problem 5: 24 pts – 3 pts each

Solve the following. Please give **exact values!** You must show work/formulas used to receive full credit.
Let z_1 and z_2 be two complex numbers:

$$z_1 = 1 + \sqrt{3}i \qquad \text{and} \quad z_2 = -2 - 2i$$

(a) Find the **modulus** of z_1: $|z_1|$

(b) Find the **argument** of z_1: θ_1

(c) Express z_1 in **polar form**

(d) Find the **modulus** of z_2: $|z_2|$

(e) Find the **argument** of z_2: θ_2

(f) Express z_2 in **polar form**

(g) **Compute:** $(z_1)^3$ and express your answer in polar form.

(h) **Compute:** $(z_2)^8$ and express your answer in polar form.

Problem 6: 20 pts

(a) Consider the following **parametric equations**:

$$\begin{cases} x &= \frac{1}{\sqrt{t}} \\ y &= t+1 \end{cases}$$

  (i) Find the **rectangular-coordinate equation** for the curve above by **eliminating the parameter**.

  (ii) **Sketch** the following **parametric equations**. Use arrows to indicate the direction of the curve as t increases.

(b) Consider the following **parametric equations**:

$$\begin{cases} x &= \cos(t) + 1 \\ y &= 4\sin(t) - 1 \end{cases}$$

  (i) Find the **rectangular-coordinate equation** for the curve above by **eliminating the parameter**.

  (ii) **Sketch** the following **parametric equations**. Use arrows to indicate the direction of the curve as t increases.

Problem 7: 24 pts

Let $\vec{u} = \langle 1, 2 \rangle$ and $\vec{v} = \langle 5, -3 \rangle$ be two vectors in the plane.

(a) **Compute:** $3\vec{u} - 4\vec{v}$

(b) **Compute:** $\vec{u} \bullet \vec{v}$

(c) Are $\vec{u}$ and $\vec{v}$ **orthogonal**?

(d) Find the **angle** between $\vec{u}$ and $\vec{v}$

(e) **Compute:** $|\vec{u} + \vec{v}|$

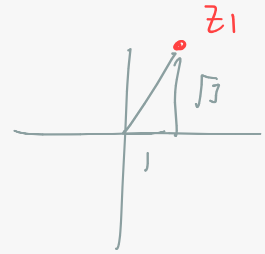
(f) Find the **component** (i.e. scalar projection) of $\vec{u}$ onto $\vec{v}$

(g) Find the **projection** of $\vec{u}$ onto $\vec{v}$

(h) **Resolve** $\vec{u}$ into two vectors $\vec{u}_1$ and $\vec{u}_2$, where $\vec{u}_1$ is **parallel** to $\vec{v}$ and $\vec{u}_2$ is **orthogonal** to $\vec{v}$.

$\boxed{\text{Problem 5}}$ $\quad z_1 = 1+\sqrt{3}\, i \qquad z_2 = -2-2i$

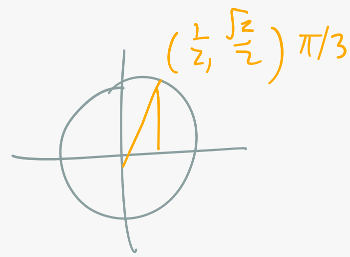
a) Find $|z_1|$



$$|z_1| = |1+\sqrt{3}\, i| = \sqrt{(1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$$

So $\boxed{|z_1| = 2}$

b) Find $\text{Arg}(z_1) = \theta_1$



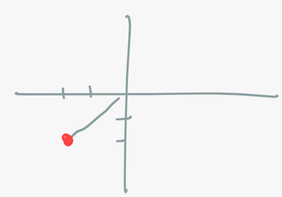
$$\theta_1 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \boxed{\frac{\pi}{3}}$$

c) Polar form z_1: $\quad z_1 = r(\cos\theta_1 + i\sin\theta_1)$

$$\boxed{z_1 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)}$$

d) Find $|z_2| = |-2-2i| = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4} = \boxed{2\sqrt{2}}.$

e) Find $\text{Arg}(z_2) = \theta_2$:



$$\theta_2 = \tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1) = \boxed{\frac{5\pi}{4}}$$

f) Polar form z_2: $\quad \boxed{z_2 = 2\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)}$

(g) $(z_1)^3 = 2^3\left(\cos\left(3\left(\frac{\pi}{3}\right)\right) + i\sin\left(3\left(\frac{\pi}{3}\right)\right)\right) = 8(\overset{-1}{\cos(\pi)} + i\,\overset{0}{\sin(\pi)})$

$$= \boxed{-8}$$

Problem 5 (h) $(z_2)^8 = (2\sqrt{2})^8\left(\cos\left(8\left(\frac{5\pi}{4}\right)\right) + i\sin\left(8\left(\frac{5\pi}{4}\right)\right)\right)$

$$= 4096\left(\cos(\underset{2\pi}{\overset{1}{10\pi}}) + i\sin(\underset{2\pi}{\overset{0}{10\pi}})\right)$$

$$= \boxed{4096}$$

Problem 6

i) Eliminate parameter

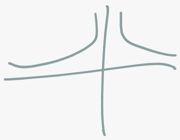
ii) Sketch.

a) $\begin{cases} x = \frac{1}{\sqrt{t}} \\ y = t+1 \end{cases}$

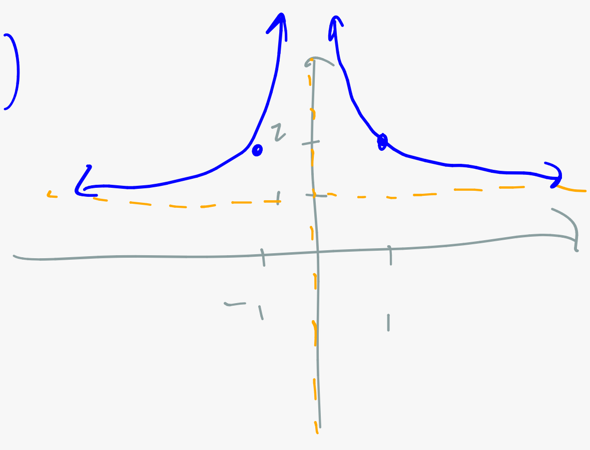
Sol

i) $y = t+1 \rightarrow t = y-1 \Rightarrow x = \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{y-1}} \Rightarrow x = \frac{1}{\sqrt{y-1}}$

$\Rightarrow x^2 = \frac{1}{y-1} \Rightarrow y-1 = \frac{1}{x^2} \Rightarrow \boxed{y = \frac{1}{x^2} + 1}$

↳ standard shape: 

ii)



Problem 6 (b) $\begin{cases} x = \cos(t) + 1 \\ y = 4\sin(t) - 1 \end{cases}$

Sol (i)

$x = \cos(t) + 1$, $y = 4\sin(t) - 1$ $\rightarrow$ $x - 1 = \cos(t)$, $y + 1 = 4\sin(t)$ $\rightarrow$ $x - 1 = \cos(t)$, $\frac{y+1}{4} = \sin(t)$

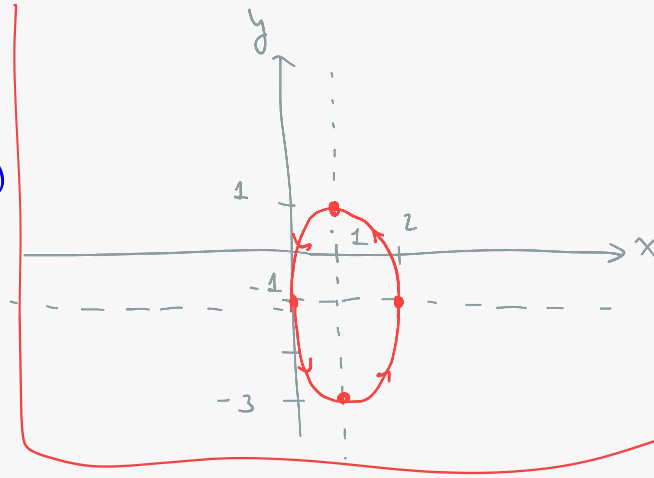
Use Pythag

$(x-1)^2 + \left(\frac{y+1}{4}\right)^2 = 1$

$$\boxed{(x-1)^2 + \frac{(y+1)^2}{4} = 1}$$

(ii) This is an ellipse!

- Center $(1,-1)$
- major axis: y-axis $(a=2)$
- minor axis: x-axis $(b=1)$



Problem 7 $\vec{u} = \langle 1,2 \rangle$ & $\vec{v} = \langle 5,-3 \rangle$.

a) $3\vec{u} - 4\vec{v} = 3\langle 1,2 \rangle - 4\langle 5,-3 \rangle = \langle 3,6 \rangle + \langle -20, 12 \rangle = \langle 3-20, 6+12 \rangle = \boxed{\langle -17, 18 \rangle}$

b) $\vec{u} \cdot \vec{v} = \langle 1,2 \rangle \cdot \langle 5,-3 \rangle = (1)(5) + (2)(-3) = 5 - 6 = \boxed{-1}$

c) $\vec{u}$ & $\vec{v}$ are NOT orthogonal since $\vec{u} \cdot \vec{v} \neq 0$. $\boxed{\text{NO}}$

d) Angle b/w $\vec{u}$ & $\vec{v}$: $\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5} \cdot \sqrt{34}}\right) = \boxed{1.65 \text{ radians}}$ $= \boxed{94.4° / \text{degrees}}$

- $|\vec{u}| = \sqrt{(1)^2 + (2)^2} = \sqrt{1+4} = \sqrt{5}$
- $|\vec{v}| = \sqrt{(5)^2 + (-3)^2} = \sqrt{25+9} = \sqrt{34}$

e) $|\vec{u} + \vec{v}| = |\langle 1,2 \rangle + \langle 5,-3 \rangle| = |\langle 6,-1 \rangle| = \sqrt{6^2 + (-1)^2} = \boxed{\sqrt{37}}$

$\boxed{\text{Problem 7}}$ (Cont.)

(f) $\vec{u} = \langle 1, 2 \rangle$ component of $\vec{u}$ onto $\vec{v}$:

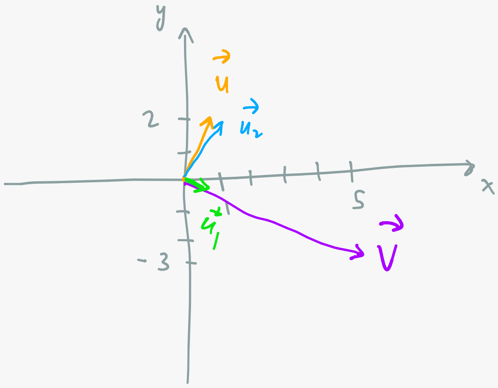
$\vec{v} = \langle 5, -3 \rangle$.

$$\text{comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \boxed{\frac{-1}{\sqrt{34}}}$$

(g) projection of $\vec{u}$ onto $\vec{v}$:

$$\vec{\text{proj}}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{-1}{(\sqrt{34})^2}\langle 5,-3\rangle = \frac{-1}{34}\langle 5,-3\rangle = \boxed{\left\langle \frac{-5}{34}, \frac{3}{34}\right\rangle}$$

(h) resolve $\vec{u}$ into $\vec{u}_1$ & $\vec{u}_2$ where $\vec{u}_1 \parallel \vec{v}$ & $\vec{u}_2 \perp \vec{v}$.



$$\vec{u}_1 = \text{proj}_{\vec{v}}(\vec{u}) = \boxed{\left\langle \frac{-5}{34}, \frac{3}{34}\right\rangle = \vec{u}_1}$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 = \langle 1, 2\rangle - \left\langle \frac{-5}{34}, \frac{3}{34}\right\rangle$$

$$= \left\langle 1 + \frac{5}{34}, 2 - \frac{3}{34}\right\rangle$$

$$\boxed{\vec{u}_2 = \left\langle \frac{39}{34}, \frac{65}{34}\right\rangle}$$

Extra Credit: This was "free points"

It was a GTQ from §9.2. Proof in book & in ~~notes~~!

Problem 8: Extra-credit: 3 pts (Optional)

Let $\vec{v} = \langle v_1, v_2 \rangle$ be any vector in the plane. **Prove:** $|\vec{v}|^2 = \vec{v} \bullet \vec{v}$.