§7.1: Integration by Parts

Ch 7: Techniques of Integration
Math 5B: Calculus II

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Outline

Guiding Questions for §7.1

Guiding Question(s)

1. What tools are in the integration toolbox?
2. What is integration by parts?
3. What is $2I$ trick?
4. What are the reduction formulas?

Integration Toolbox

Integration Toolbox

When confronted with an integral, $\int f(x)\,dx$, the main tools in your integration toolbox are:

1. know a lot of derivatives!
   If you can recognize DRs, use the corresponding ADRs!
   - General Theorems: power rule, sum/difference rule
   - DRs for basic functions: x^n, trig (sin, cos, ...), b^x, $\log_b(x)$, ...
   - DRs for more complex functions: $\tan^{-1}(x)$, $\sinh(x)$, ...
2. u-substitution (corresponds to the chain rule)
3. integration by parts
4. trigonometric substitution

You already know Tools 1 and 2. In this chapter, you'll learn many, many more techniques including Tool 3 (this section) and Tool 4 (later).

Integration Toolbox

Tool 1: summarizing integrals/ADRs up through Chapter 6:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

Integration by Parts

IBP is one of the most powerful tricks for evaluating integrals! In fact, it gets used all the time in advanced mathematics and engineering courses.

Definition 1: Integration by Parts

- Integration by Parts is a technique of integration derived from the product rule of differentiation. It states:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

or, equivalently,

$$\boxed{\int u\,dv = uv - \int v\,du}$$

where $u = f(x)$ and $dv = g'(x)dx$.

- When to use? When the integral on the RHS is easier!

Integration by Parts

Why is the formula true?

- Start with the product rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

- Integrate both sides:

$$\int \frac{d}{dx}[f(x) \cdot g(x)]\, dx = \int \left(f'(x)g(x) + f(x)g'(x)\right)\, dx$$

$$f(x) \cdot g(x) = \int g(x)f'(x)\, dx + \int f(x)g'(x)\, dx$$

- then re-arrange:

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

Integration by Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Advice:

- You are trying to make the integral $\int v\,du$ easier than $\int u\,dv$
- Since you can differentiate almost anything, picking the u and then finding the du is easy so don't worry about it yet.
- So, instead, decide on the dv FIRST and pick it so that you can integrate it by finding $\int dv = v$
- Another helpful tip: pick the u so that du is "simpler" than u.

Example 1: Integration by Parts

Consider the integral: $\displaystyle\int x\cos(x)\,dx$

Two choices for u and dv:

$$\begin{cases} u = x & dv = \cos(x)\,dx \\ du = dx & v = \sin(x) \end{cases} \qquad \begin{cases} u = \cos(x) & dv = x\,dx \\ du = -\sin(x)dx & v = \frac{x^2}{2} \end{cases}$$

Which is better?

$u = x$ and $dv = \cos(x)dx$ $\longrightarrow$

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)dx$$

$u = \cos(x)$ and $dv = xdx$ $\longrightarrow$

$$\int x\cos(x)\,dx = \cos(x)\left(\frac{x^2}{2}\right) - \int \cos(x)\left(\frac{x^2}{2}\right)dx$$

The first choice is better since it follows the advice on the previous page.

Activity 1:

Evaluate using IBP:

(a) $\displaystyle\int xe^x \, dx$

(b) $\displaystyle\int t^2 \sin(t) \, dt$

Integration by Parts

Integration by Parts with limits of integration

Definition 2: IBP with limits of integration

- Using integration by parts with limits of integration:

$$\boxed{\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du}$$

Activity 2:

Evaluate using IBP:

(a) $\displaystyle\int_1^3 \ln(x)\,dx$

(b) $\displaystyle\int_0^1 \tan^{-1}(x)\,dx$

Integration by Parts

Integration by Parts: $2I$ Trick

Activity 3:

Evaluate using IBP: $\displaystyle\int \cos(x)e^x\,dx$

In this activity, it feels like you go around in a circle.

You'll do IBPs twice and come back to the original integral. If we set $I = \int \cos(x)e^x\,dx$, then you can re-arrange to get $2I$ (after 2 IBPs).
So I call this the "$2I$-trick."

Integration by Parts: 2I Trick

Integration by Parts: Reduction Formulas

Theorem 1: Reduction Formulas

For any integer $n \geq 2$:

(a) $$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

(b) $$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

(c) $$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

(d) $$\int (\ln(x))^n\,dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\,dx$$

You can derive these formulas using IPBs with the $2I$ trick.

Integration by Parts: Reduction Formulas

We'll derive: $\displaystyle\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$

- We pick: $\begin{cases} u = \cos^{n-1}(x) & dv = \cos(x)dx \\ du = (n-1)\cos^{n-2}(x)(-\sin(x))dx & v = \sin(x) \end{cases}$
- $\displaystyle\int \cos^n(x)\,dx =$

$$\underbrace{\cos^{n-1}(x)}_{u}\underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v}\underbrace{(n-1)\cos^{n-2}(x)(-\sin(x))\,dx}_{dv}$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)(1-\cos^2(x))\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx$$

Integration by Parts: Reduction Formulas

We'll derive: $\displaystyle\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$

- $\displaystyle\int \cos^n(x)\,dx =$
  $$\cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx$$
- Setting $I = \cos^n(x)\,dx$ we can write:
  $$I = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)I$$
  $$nI = \cos^{n-1}(x)\sin(x) - (n-1)\int \cos^{n-2}(x)\,dx$$
- Then dividing by n gives us the reduction formula. Done :-) $\square$

Activity 4:

Use the reduction formula to evaluate: $\int \sin^3(x)\,dx$

Integration by Parts: Reduction Formulas

Activity 5:

Evaluate the following:

(a) $\displaystyle\int_0^{\pi/3} e^{2x}\cos(3x)\,dx$

(b) $\displaystyle\int x^7(x^4+1)^{2/3}\,dx$

(Hint: By taking x^3 and grouping it with the $(x^4+1)^{2/3}$ term we can pull off the integration using u-sub. So, choose $dv = (x^4+1)^{2/3}(4x^3)dx$ and $u = \frac{1}{4}x^4$)

(c) $\displaystyle\int \frac{x^3 e^{x^2}}{(x^2+1)^2}\,dx$

Integration by Parts: Some harder integrals