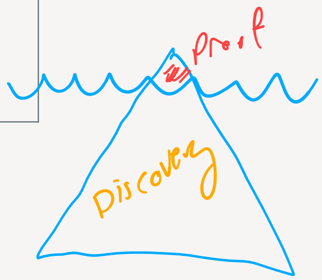
# Section 13.5 Mathematical Induction

**Objectives**
- Conjecture and Proof
- Principle of Mathematical Induction
- A **False** proof using Mathematical Induction




## • Conjecture and Proof

There are two aspects of mathematics–**discovery** and **proof**–and they are of equal importance.

We must discover something before we can attempt to prove it, and we cannot be certain of its truth until we prove it.

Discovery is the "playful" side of mathematics. Sadly, in current math courses, you don't get to discover any results because you are learning what others already discovered. However, this can be fun.

**Ex 1** Let's try a simple experiment. Add odd numbers together.

sum 1 odd → $1 = 1$
sum of first 2 odd #s → $1 + 3 = 4$
sum of first 3 odd → $1 + 3 + 5 = 9$
4 → $1 + 3 + 5 + 7 = 16$
5 → $1 + 3 + 5 + 7 + 9 = 25$

**What do you notice about the sums?** That is, what do you notice about the numbers on the right-hand side of these equations?

They are: Perfect Squares!

Now, make a guess about the sum of the first 10 odd numbers without actually adding them all up! 100

100 10000

Now, make a claim about the sum of the first $n$ odd numbers, for any natural number $n$. This is called a **conjecture**.

Your conjecture in words:

The sum of the first n odd numbers is $n^2$.

Now, we try to write this in mathematical notation.

Step 1: How do we write the $n$th odd number? For example, if $n = 5$ we should get $9$.

1 2 3 4 5 6 7 8 9 10
1 2 3 4 5

$n^{th}$ odd # $= 2n-1$ or $2n+1$

· if n starts at 1, use $2n-1$
· if n starts at 0, use $2n+1$

to get even #s: use $2n$

Step 2: How do we write the sum of the first $n$ odd numbers?

$1 + 3 + 5 + \cdots + (2n-1)$ "the sum of first n odd #s"

Step 3: What is our conjecture about this sum? Can you write it in mathematical notation?

$1 + 3 + 5 + \cdots + (2n-1) = n^2$

Final Step: State Your conjecture in mathematical notation:

"For any natural number $n$, $1+3+5+\cdots+(2n-1) = n^2$."

This is still a conjecture and just because we wrote the equality doesn't mean it is magically true. If I write $2 + 2 = 5$ it doesn't make it true. We have to **prove** it is true.

A **proof** is a clear and convincing mathematical argument using logic and deduction that demonstrates the truth of a statement beyond doubt.

Methods of Proof

There are entire books written about the different methods of proof. My favorite is called *The Book of Proof* written by Richard Hammack. It is available for free! Here's the website if you are interested: https://www.people.vcu.edu/ rhammack/BookOfProof/.

We will only study the method of proof called "the principle of mathematical induction." A long name. Moreover, a fun fact is that this is studied in Chapter 10 of Hammack's book!

## • Principle of Mathematical Induction


# EVERYBODY – DO THE WAVE!

If done properly, everyone will *eventually* end up joining in. Watch: wave gif and wave gif 2

Why is that?
- Someone (me) has to get it started.
- Then as soon as the person before you does the wave, you start doing the wave.

This is essentially the same as the **Principle of Mathematical Induction**.

Here's how it works: Suppose we have a statement that says something about all natural numbers $n$.

**Notation** Let us write $\mathbb{N} = \{1, 2, 3, \ldots\}$. That is, $\mathbb{N}$ denotes the set of all natural numbers. Then we can write $n \in \mathbb{N}$ as short-hand for $n = 1$ or $2$ or $3 \ldots$

For example, for all natural numbers $n$, let $P(n)$ denote the following statement:

Conjecture: $P(n)$: " the sum of the first $n$ ODD natural numbers is $n^2$ "

Notice this is a statement about all natural numbers–so it is, in fact, infinitely many statements that we are making!

sum of first 3 odd is 9

$$\begin{aligned} &\to P(1): \quad 1 = 1 \\ &\to P(2): \quad 1+3 = 4 \\ &\to P(3): \quad 1+3+5 = 9 \\ &\quad \vdots \end{aligned}$$

*The crux of the idea is this:*
Suppose we can prove that whenever one of these statements is true, then the one following it in the list is also true.

In other words,

For ~~every~~ Some $k$, Assume $P(k)$ is true, then next $P(k+1)$ is true. "If.... then...."

This is called the **induction step** because it leads us from the truth of one statement to the truth of the next.

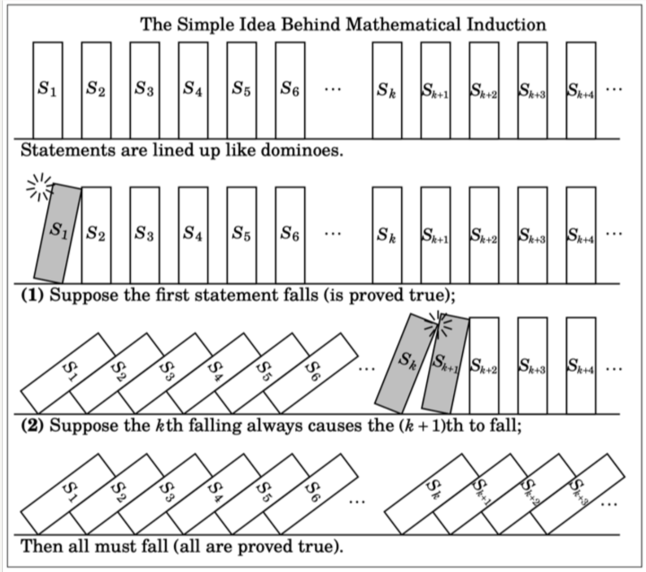
Now suppose that we can also prove that $P(1)$ is true.

**Together** the induction step **AND** the fact that $P(1)$ is true lead us to the following chain of "inductions":

Since $P(1)$ is true, by the induction step, $P(2)$ is also true.
Since $P(2)$ is true, by the induction step, $P(3)$ is also true.
Since $P(3)$ is true, by the induction step, $P(4)$ is also true.
and so on . . .

Modus Ponens
P True
"If P then Q" True
∴ Q True

The induction step and the fact that $P(1)$ is true leads us through the following chain of statements as visualize by a set of dominos:




← P(1) true
actually knock over first domino

← Inductive Step
can always knock over "next" domino

**Principle of Mathematical Induction**

If both the induction step and $P(1)$ are proved, then statement $P(n)$ is proved for all $n$.  PMI

Here is a summary of this important method of proof.

For each natural number $n \in \mathbb{N}$, let $P(n)$ be a statement depending on $n$.

Suppose that the following two conditions are satisfied.

1. **Base Step:** $P(1)$ is **true**.
2. **Inductive Step:** For every natural number $k$, "**IF** $\boxed{P(k) \text{ is true}}$ **THEN** $P(k+1)$ is **true**." (Assume: IH)

**THEN**, by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \in \mathbb{N}$.

The assumption that $P(k)$ is true is called the **Induction Hypothesis (IH)**.

**How to prove a statement using Math Induction**

1. Prove the **Base Step**, i.e. show that $P(1)$ is true.
2. Prove the **Inductive Step**, i.e. Assume that $P(k)$ is true. Prove that $P(k+1)$ is true.

Here's a template for writing proofs using mathematical induction.

**Proof: (By Induction)**

"Let $P(n)$ : (( ADD statement ))."  $1+3+5+\cdots+(2n-1) = n^2$.

**Base Step:**
(( ADD proof that $P(1)$ is true. ))  $P(1)$ is true b/c $1 = 1^2 = 1$.

**Inductive Step:**
"**(IH)** Assume that (( ADD statement $P(k)$ )) is true, for some $k \in \mathbb{N}$."  $1+3+5+\cdots+(2k-1) = k^2$ ← Assume
"**NTS** (( ADD statement $P(k+1)$ )) is also true."  NTS $\underline{1+3+5+\cdots+(2k-1)}+(2(k+1)-1) = (k+1)^2$
(( ADD proof that $P(k+1)$ is true USING the statement $P(k)$ ))

"So, $P(k+1)$ follows from $P(k)$."
"Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$." □

**Remarks**

- IH stands for "induction hypothesis" and NTS stands for "need to show"
- Why write $P(k)$ again if it essentially the same thing as re-writing $P(n)$ but with $k$ replacing $n$? Because I said so is why! Just kidding, but more seriously, I think it is good "proof writing" form and it really helps! Ditto for $P(k+1)$! It is helpful to know what you are trying to verify!

**Ex 2 Prove by Math Induction** Prove that for all natural numbers $n$,

$$1+3+5+\cdots+(2n-1)=n^2.$$

Here's a template for writing proofs using mathematical induction.

> **Proof: (By Induction)**
>
> "Let $P(n)$ : (( ADD statement ))." $1+3+5+\cdots+(2n-1)=n^2.$
>
> **Base Step:**
> (( ADD proof that $P(1)$ is true. )) $P(1)$ is true b/c $1=1^2=1.$
>
> **Inductive Step:**
> "(IH) Assume that (( ADD statement $P(k)$ )) is true, for some $k\in\mathbb{N}$." $1+3+5+\cdots+(2k-1)=k^2$ ← Assume
> "**NTS** (( ADD statement $P(k+1)$ )) is also true." NTS $1+3+5+\cdots+(2k-1)+(2(k+1)-1)=(k+1)^2$
> (( ADD proof that $P(k+1)$ is true USING the statement $P(k)$ ))
>
> "So, $P(k+1)$ follows from $P(k)$."
> "Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n\in\mathbb{N}$." □

Remarks

Proof (By Induction)

Let $P(n):\ 1+3+5+\cdots+(2n-1)=n^2.$

Base Step $P(1)$ is true because $1=1^2.$

Inductive Step

(IH) Assume that $P(k)$ is true. That is, assume $1+3+5+\cdots+(2k-1)=k^2.$

NTS $P(k+1)$ is true: $\boxed{1+3+5+\cdots+(2(k+1)-1)=(k+1)^2}$

We have:

$$LHS = 1+3+5+\cdots+(2(k+1)-1) \qquad \text{(sum of first } k+1 \text{ odd \#s)}$$

$$= \underbrace{[1+3+5+\cdots+(2k-1)]}_{\text{first } k \text{ odd \#s (IH)}} + (2(k+1)-1)$$

$$= k^2 + (2(k+1)-1) \qquad \text{(by the IH)}$$

$$= k^2+2k+2-1$$

$$= k^2+2k+1$$

$$= (k+1)^2 \qquad \text{(factor).}$$

$$= RHS.$$

So $P(k+1)$ follows from $P(k)$.

So, by PMI, $P(n)$ is true for all $n\in\mathbb{N}$. □

## Discovery

**Ex 3 A new conjecture** Consider the sequence of natural numbers: $1, 2, 3, \ldots$.

Make a list of the SUM of the first 1 numbers, the first two numbers, the first three numbers, the first four numbers, and the first five numbers.

1 $\quad 1 = 1 \qquad \frac{1\cdot(1+1)}{2} = \frac{2}{2} = 1$

2 $\quad 1+2 = 3 \qquad \frac{2(2+1)}{2} = 3$

3 $\quad 1+2+3 = 6$

4 $\quad 1+2+3+4 = 10$

5 $\quad 1+2+3+4+5 = 15 \qquad \frac{5(5+1)}{2} = \frac{5\cdot 6}{2} = \frac{30}{2} = 15$

$\vdots \qquad\qquad \vdots$

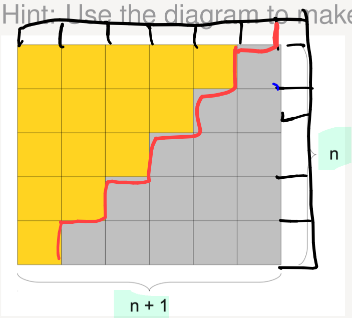
n $\quad 1+2+3+\cdots+n = \ ? \qquad \frac{n(n+1)}{2}$

Make a conjecture about the the sum of the first $n$ numbers.

Your conjecture:

$$\boxed{1+2+3+\cdots+n = \frac{n(n+1)}{2}}$$

Hint: Use the diagram to make a conjecture about $P(n)$ where $P(n)$: $1+2+3+\cdots+n = \square$




• total # blocks in the rectangle is $n(n+1)$.

• this is twice as much as what we want!

• so: $\frac{n(n+1)}{2}$

Hint #2: If we write $1+2+3+\cdots+n = S(n)$, where $S(n)$ is the sum we are looking for, try listing the left-hand side in two ways: first in ascending order and then in descending order. Then add the two equations together. Can you solve for $S(n)$?:

Gauss

$$\begin{array}{rl} & 1+2+3+\cdots+98+99+100 = S \\ + & 100+99+98+\cdots+3+2+1 = S \\ \hline & \underbrace{101+101+101+\cdots+101+101+101}_{\text{How many are added? 100 of them!}} = 2S \end{array}$$

$100\cdot 101 = 2S \longrightarrow S = \frac{100\cdot 101}{2} = \frac{10100}{2} = \boxed{5050}$

$$\begin{array}{rl} & 1+2+3+\cdots+(n-2)+(n-1)+n = S \\ + & n+(n-1)+(n-2)+\cdots+3+2+1 = S \\ \hline & \underbrace{(n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1)}_{n \text{ of them}} = 2S \end{array}$$

$\frac{n(n+1)}{2} = \frac{2S}{2}$

$S = \frac{n(n+1)}{2}$.

**Ex 4 Prove by Math Induction** Prove that your conjecture in Example 2 is true for all natural numbers $n$ using the principle of mathematical induction.

template

**Proof: (By Induction)**

"Let $P(n)$ : (( ADD statement ))."

**Base Step:**
(( ADD proof that $P(1)$ is true. ))

**Inductive Step:**
"(IH) Assume that (( ADD statement $P(k)$ )) is true, for some $k \in \mathbb{N}$."
"**NTS** (( ADD statement $P(k+1)$ )) is also true."
(( ADD proof that $P(k+1)$ is true USING the statement $P(k)$ ))

"So, $P(k+1)$ follows from $P(k)$."
"Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$." □

Proof (By Induction).

Let $P(n)$: $1+2+3+\dots+n = \frac{n(n+1)}{2}$.

Base Step For $n=1$, $P(1)$: $1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ✓ So $P(1)$ is true.

Inductive Step

(IH) Assume that $1+2+3+\dots+k = \frac{k(k+1)}{2}$ is true.

NTS $1+2+3+\dots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$.

We have:

$$\text{LHS} = \underbrace{1+2+3+\dots+k}_{\text{IH}} + (k+1) = \frac{k(k+1)}{2} + (k+1)\frac{2}{2}$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2} = \text{RHS}.$$

So $P(k+1)$ follows from $P(k)$.

Therefore, by PMI, we've prove $P(n)$ is true for all $n \in \mathbb{N}$. □

## Common Sums

In Examples 3 and 4, we found and then proved the formula for the sum of the first $n$ numbers using mathematical induction.

Not surprisingly, others have already discovered the formulas for the sums of squares and the sums of cubes. They are given in the theorem below. They can all be proved by using mathematical induction.

These formulas are important in calculus.

sum of first n #s

sum of squares of first n #s

GTQ GTQ

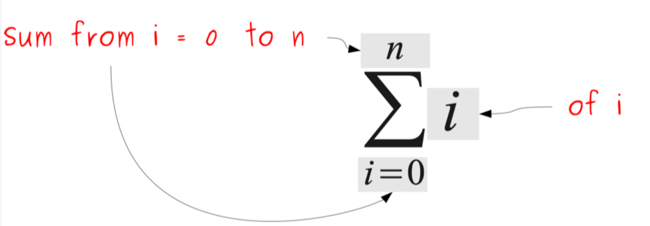
**Theorem 1 Common Sum Formulas**

Let $k, n \in \mathbb{N}$ with $k \le n$.

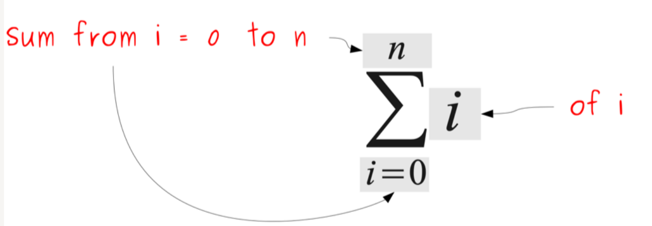
(1) $\sum_{k=1}^{n} 1 = n$ (2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (3) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (4) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

**Remarks** This is called **Sigma notation**. Sigma, $\Sigma$, is the capital Greek letter for "S"–which makes sense since this notation is short for "Sum."

- Instead of writing 1 + 2 + 3 + … + n, we write




- Instead of writing 1 + 2 + 3 + … + n, we write



### Modified Induction

You can start induction at any value that is bigger than or equal to one, $k \geq 1$. For example, let's say we have a statement $P(n)$ that is not true for $n = 1, 2, 3, 4$. But it is true for all $n \geq 5$. Then we can let $k = 5$ and say that $P(n)$ is true for $n \geq k$.

In the next example the statement $P(n)$ is not an equation but an inequality. Other examples are found in the homework.

**Ex 5 Prove by Math Induction** Prove that for all natural numbers $n \geq 5$,

$$4n < 2^n.$$

$n=1$: $4 < 2$ False
$n=2$: $8 < 4$ False
$n=4$: $16 < 2^4 = 16$ False

**Remarks** This is saying something interesting: that "exponential growth" $(2^n)$ *eventually* (after $n = 4$) is bigger than "linear growth" $(4n)$.

Proof (By Induction). Let $P(n): 4n < 2^n$, for $n \geq 5$.

Base Step For $n=5$, $P(5)$: $4(5) = 20$ & $2^5 = 32$ so $20 < 32$ so $P(5)$ is true.

Inductive Step IH: Assume $4k < 2^k$ for some $k \geq 5$.

NTS $4(k+1) < 2^{k+1}$

We have: $4(k+1) = \underbrace{4k}_{IH} + 4 < 2^k + 4 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$, (b/c $k \geq 5$, $2^k \geq 64$)

So $P(k+1)$ follows from $P(k)$.

Therefore, by PMI, $P(n)$ is true for all $n \geq 5$. $\square$

**Ex 6 Prove by Math Induction** Prove: For all $n \in \mathbb{N}$,

$$\frac{(2n)!}{2^n \cdot n!} \in \mathbb{Z}.$$

That is, $\frac{(2n)!}{2^n \cdot n!}$ is an integer and is not a fraction.

"Scratch work" vs "formal proof". This time we'll just do scratch work (show Inductive Step only) and leave it as an exercise to write out a full, formal proof.

## • A **False** proof using Principle of Mathematical Induction

**Theorem 2**

For any $n \in \mathbb{N}$: $1+2+3+\cdots+n = \frac{1}{2}\left(n+\frac{1}{2}\right)^2$. $\stackrel{?}{=} \frac{n(n+1)}{2}$

$\frac{1}{2}\frac{9}{4} = 9/8$

$\cancel{\frac{1}{2}}\left(n+\frac{1}{2}\right)^2 = \frac{n(n+1)}{\cancel{2}}$

$\left(n+\frac{1}{2}\right)^2 = n(n+1)$

$n^2+n+\frac{1}{4} = n^2+n$

$\frac{1}{4} = 0$ False!

**Proof:**

Let $P(n):\ 1+2+3+\cdots+n = \frac{1}{2}\left(n+\frac{1}{2}\right)^2$.

Now, assume that for some $k$, $P(k)$ holds, so $P(k): 1+2+\cdots+k = \frac{1}{2}\left(k+\frac{1}{2}\right)^2$.
We want to show that $P(k+1)$ is true, which means we need to show that

$$1+2+\cdots+k+(k+1) = \frac{1}{2}\left((k+1)+\frac{1}{2}\right)^2.$$

Inductive Step is ok!

We have

$$\begin{aligned}
1+2+\cdots+k+(k+1) &= (1+2+\cdots+k)+(k+1) \\
&= \left(\frac{1}{2}\left(k+\frac{1}{2}\right)^2\right)+(k+1) && \text{(by Induction Hyp.)} \\
&= \left(\frac{1}{2}\left(k+\frac{1}{2}\right)^2\right)+\frac{2(k+1)}{2} && \text{(by algebra)} \\
&= \frac{\left(k+\frac{1}{2}\right)^2+2(k+1)}{2} && \text{(by algebra)} \\
&= \frac{(k^2+k+\frac{1}{4})+2k+2}{2} && \text{(expand)} \\
&= \frac{k^2+3k+\frac{9}{4}}{2} && \text{(simplify)} \\
&= \frac{\left(k+\frac{3}{2}\right)^2}{2} && \text{(factor)} \\
&= \frac{\left((k+1)+\frac{1}{2}\right)^2}{2} && \text{(algebra)} \\
&= \frac{1}{2}\left((k+1)+\frac{1}{2}\right)^2 && \text{(algebra)}
\end{aligned}$$

So, $P(k+1)$ follows from $P(k)$. So, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

### What went wrong????? MISSING: BASE STEP!

### Can you find the mistake??????

**Moral of the story**:

Don't forget to prove **BOTH STEPS**:

**Base Case**

**AND**

**Inductive Case**