§6.1: Inverse Functions

Ch 6: Exponentials, Logs, & Inverse Trig Functions
Math 5B: Calculus II

Dr. Jorge Eduardo Basilio

Department of Mathematics & Computer Science
Pasadena City College

Class Notes #1

February 19, 2019
Spring 2019

Outline

Introduction to Chapter 6

- Title: Chapter 6: Inverse Functions: Exponential, Logarithmic, and Inverse Trigonometric Functions
- In Calculus I (Ma 5A, Ch 1-5), you studied: limits, differentiation, integration of many functions.
  - Focused on "Algebraic functions": polynomials (e.g. $f(x) = x^n$), rational (e.g. $f(x) = \frac{1}{x^n}$), radical (e.g. $f(x) = \sqrt[n]{x} = x^{1/n}$)
  - And "trigonometric functions": $\sin(x)$, $\cos(x)$, $\tan(x)$, etc
  - Also: combinations of functions using addition, subtraction, multiplication, division, plus function composition.
- Example:

$$f(x) = \sin(\sqrt[3]{x^2+1}) + \frac{\tan(x)\sqrt{1-2x}}{x^2+4}$$

Guiding Questions for §6.1

Guiding Question(s)

1. If functions are input/output machines, which functions can we "undo"? For those which we can undo (called inverse functions), how can we find functional expressions for them?
2. How do the calculus concepts of continuity, differentiation and integrals apply to inverse functions?

Basics of Inverse Functions

Definition 1:

Recall that a function is an input/output "machine" given by a "rule" for which for each unique input there corresponds only one unique output.

$$\begin{tikzcd} x \arrow[r, "f"] & f(x) \arrow[l, bend right, dashed, "f^{-1}"'] \end{tikzcd}$$

The inverse function, f^{-1}, is a function that "undoes" the effects of f.

Example 1:

- $f(x) = 2x + 1$
- $f^{-1}(x) = \frac{x-1}{2}$

Example 2:

- $g(x) = \sqrt{x}$, $x \geq 0$
- $g^{-1}(x) = x^2$, $x \geq 0$

Basics of Inverse Functions

Definition 2:

Recall that a function is an input/output "machine" given by a "rule" for which for each unique input there corresponds only one unique output.

$$\begin{tikzcd} x \arrow[r, "f"] & f(x) \arrow[l, bend right, dashed, "f^{-1}"'] \end{tikzcd}$$

The inverse function, f^{-1}, is a function that "undoes" the effects of f.

CAUTION Do not mistake the "-1" in f^{-1} as an exponent! Thus, $f^{-1}(x)$ DOES NOT EQUAL $\dfrac{1}{f(x)}$.

Basics of Inverse Functions

Definition 3:

Recall that a function is an input/output "machine" given by a "rule" for which for each unique input there corresponds only one unique output.

$$\begin{tikzcd} x \arrow[r, "f"] & f(x) \arrow[l, bend right, dashed, "f^{-1}"'] \end{tikzcd}$$

The inverse function, f^{-1}, is a function that "undoes" the effects of f.

BUT! This only makes sense if we have a unique path backwards. We need a condition to guarantee that an inverse exists.

Basics of Inverse Functions

Definition 4:

A function is one-to-one if two different inputs gives two different outputs. That is, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. It is equivalent to prove: if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Looking at the graph of a one-to-one function shows that all horizontal lines intersect can intersect the graph of f at most once. This is called the Horizontal Line Test.

Example 3:

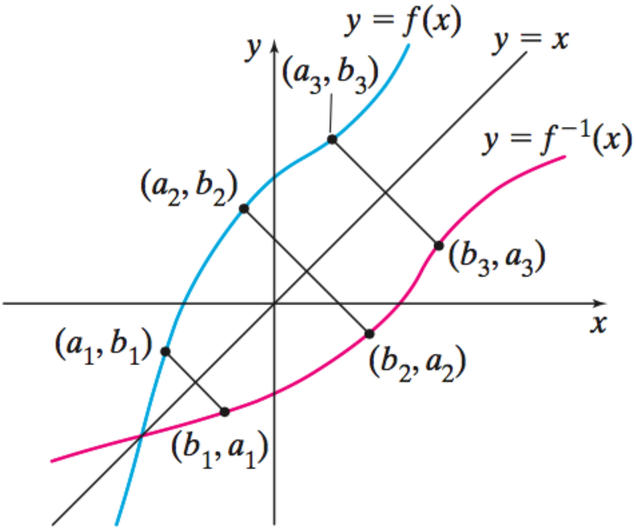
- Sketch: $f(x) = x^2$, $g(x) = x^3$. Which is one-to-one?
- Sketch: $h(x) = \sin(x)$ for $x \in [0, \pi]$. What about for $x \in [0, \pi/2]$?

Basics of Inverse Functions

Theorem 1: Properties of Inverse Functions

1. If f is one-to-one, then f^{-1} exists and is one-to-one
2. Inverse properties: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$
3. $D(f^{-1}) = R(f)$, i.e. Domain of f^{-1} = Range of f
4. $R(f^{-1}) = D(f)$, i.e. Range of f^{-1} = Domain of f
5. f and f^{-1} are symmetric across the line $y = x$
6. Finding an inverse algebraically:
   - STEP 1: replace $f(x)$ with y
   - STEP 2: interchange the roles of x and y
   - STEP 3: solve for y
   - STEP 4: replace y with $f^{-1}(x)$.

Basics of Inverse Functions



Basics of Inverse Functions

Activity 1:

For the following functions: show that $f(x)$ is one-to-one (hint: use the equivalent version). After this, find a formula for f^{-1} and determine the domain and range.

(a) $f(x) = x^3 + 5$

(b) $f(x) = \dfrac{1}{x+2}$

(c) $f(x) = \dfrac{x+3}{x-4}$

(d) $f(x) = \dfrac{2x-5}{3x+7}$

(e) $f(x) = \sqrt{3x-8}$

Activity 2:

Consider the function $f(x) = 3 - \sqrt{7-2x}$

(a) Sketch the graph and explain why its one-to-one.

(b) Use your graph to find the domain and the range of $f(x)$.

(c) Find a formula for $f^{-1}(x)$ and state its domain and range.

(d) Sketch the graph of $f^{-1}(x)$ along with the graph of $f(x)$.

Basics of Inverse Functions

Activity 3:

Consider the function $f(x) = 2x^2 - 12x + 23$.

(a) Sketch the graph and explain why its not one-to-one.

(b) Find the smallest possible value for a such that $f(x)$ is one-to-one on $[a, \infty)$.

(c) Sketch the graph of f on this restricted domain.

(d) Find a formula for $f^{-1}(x)$ and state its domain and range.

(e) Sketch the graph of $f^{-1}(x)$ along with the graph of $f(x)$.

Calculus of Inverse Functions

As a warm-up to the calculus ideas, let's work out the following example.

Example 4: test

- If $f(x) = 2x$, then $f'(x) = 2$
- So: $f^{-1}(x) = \frac{1}{2}x$. And $(f^{-1})'(x) = \frac{1}{2}$.
- Notice: $(f^{-1})'(x) = \frac{1}{2} = \frac{1}{f'(x)}$

Okay, so that was a really easy example. What about a more complicated situation?

- If $f(x) = x^3$, then $f'(x) = 3x^2$
- So: $f^{-1}(x) = x^{1/3}$. And $(f^{-1})'(x) = \frac{1}{3}x^{-2/3}$.
- Notice again: $(f^{-1})'(x) = \frac{1}{3(x^3)^2} = \frac{1}{f'(x)}$.

Is this always true?

Calculus of Inverse Functions

In our examples, all of the functions $f(x)$ were continuous and differentiable. By inspecting their graphs using the symmetry property, we see that the inverse are continuous and differentiable.

Theorem 2: Continuity of Inverses

If f is one-to-one and continuous on an interval I,
THEN its inverse function f^{-1} is also continuous.

Idea: a continuous graph remains continuous after reflection across the line $y = x$.

If you're curious about a rigorous proof, you can study the proof I provided at the end of the slides (and you can ask me questions).

Calculus of Inverse Functions

In our examples, all of the functions $f(x)$ were continuous and differentiable. By inspecting their graphs using the symmetry property, we see that the inverse are continuous and differentiable.

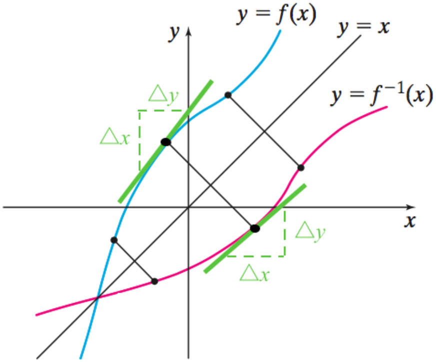
Theorem 3: Differentiability of Inverses

1. If f is one-to-one, differentiable, and $f'(b) \neq 0$ on an interval I (where $b = f^{-1}(a)$, or $a = f(b)$), THEN its inverse function f^{-1} is also differentiable.
2. Moreover, if a is in the domain of f^{-1} then the derivative is given by

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)} \tag{1}$$

3. In Liebniz notation: if $y = f(x)$, then $x = f^{-1}(y)$ and $\frac{dx}{dy} = 1/(\frac{dy}{dx})$.

Calculus of Inverse Functions



Ingredients:

- If f^{-1} is differentiable at a then it has a tangent line at a with some slope (in particular, it's slope can't be $\pm\infty$).
- Since f^{-1} is differentiable with slope $\neq \pm\infty$ then f has slope $\neq 0$ because of the symmetry across the line $y = x$
- If the slope of f^{-1} is approximately $\dfrac{\Delta y}{\Delta x}$, then the slope of f is approximately $\dfrac{\Delta x}{\Delta y}$

Proof: Differentiability of Inverse

- The proof of (1) is an easy application of the chain rule and implicit differentiation if we assume that f^{-1} is differentiable. The proof that f^{-1} is, indeed, differentiable is as the end of the slides.
- We start with the inverse property: $(f^{-1} \circ f)(x) = x$. We differentiate both sides with implicit differentiation:

$$\begin{aligned} \frac{d}{dx}\left[(f^{-1} \circ f)(x)\right] &= \frac{d}{dx}[x] \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'(f(x)) &= \frac{1}{f'(x)}. \end{aligned}$$

- If we set $a = f(x)$, then $x = f^{-1}(a)$ which is exactly the formula in (1).

Calculus of Inverse Functions

Activity 4:

Sove:

(a) If $f(0) = 4$ and $f'(0) = -2$, find $(f^{-1})'(4)$

(b) Given that $f(x) = \sqrt[3]{x} + 8$, compute: $(f^{-1})'(5)$

Activity 5:

Let's use the derivative formula for the inverse to find the derivatives of the inverse functions from Activity 11. Find $(f^{-1})'(x)$:

(a) $f(x) = x^3 + 5$

(b) $f(x) = \dfrac{1}{x+2}$

(c) $f(x) = \dfrac{x+3}{x-4}$

(d) $f(x) = \dfrac{2x-5}{3x+7}$

(e) $f(x) = \sqrt{3x-8}$

We recall the following useful fact from Calc 1:

Theorem 4: ID Test

1. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b)
2. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b)
3. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b)

This is a nice shortcut to showing a function is one-to-one!

Because if f is increasing on (a, b) then for $x_1 < x_2$ in (a, b) then $f(x_1) < f(x_2)$.

Calculus of Inverse Functions

Activity 6:

Consider the function $f(x) = x^3 + 5x - 3$.

(a) Use the ID Test to prove that $f(x)$ is one-to-one on its entire domain.

(b) By virtue of (a), we can construct the inverse function $f^{-1}(x)$. Without explicitly finding a formula for $f^{-1}(x)$, find the values of $f^{-1}(-9)$ and $f^{-1}(15)$. *(Hint: use rational roots theorem)*

(c) Use your answers in (b) and the derivative formula for $f^{-1}(x)$ to find the values of $(f^{-1})'(-9)$ and $(f^{-1})'(15)$.

Activity 7:

Consider the function $f(x) = 2\cos(x) - 5x$.

(a) Use the ID Test to prove that $f(x)$ is one-to-one on its entire domain.

(b) By virtue of (a), we can construct the inverse function $f^{-1}(x)$. Without explicitly finding a formula for $f^{-1}(x)$, find the values of $f^{-1}(5\pi/2)$ and $f^{-1}(-15\pi/2)$. *(Hint: try $x = \frac{\pi}{2}k$ and look for k)*

(c) Use your answers in (b) and the derivative formula for $f^{-1}(x)$ to find the values of $(f^{-1})'(5\pi/2)$ and $(f^{-1})'(-15\pi/2)$.

Proofs

We now provide proofs to the following:

- Proof that f^{-1} is continuous
- Proof that f^{-1} is differentiable

Proof that inverse is continuous

Ingredients:

- Definition of continuity at a point: f is continuous at $x = a$ if:
  1. $f(a)$ exists
  2. $\lim_{x \to a} f(x)$ exists
  3. $\lim_{x \to a} f(x) = f(a)$
- Definition of the limit: $\lim_{x \to a} f(x) = L$ means: for every $\epsilon > 0$, there exsits $\delta(\epsilon) > 0$ so that if x satisfies

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

- Intermediate Value Theorem: If f is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$ then for every k between $f(a)$ and $f(b)$ (i.e. $f(a) < k < f(b)$) there exists a $c \in (a, b)$ such that $f(c) = k$.

Proof that inverse is continuous

Assume that f is continuous and one-to-one on the open interval (a, b).

Lemma 1:

If f is one-to-one on $I = (a, b)$, then f is either increasing or decreasing on I.

Remark: Notice that we are not assuming anything about the differentiability of f. So the proof is a bit technical.

However, when we know that f is differentiable and $f'(x) > 0$ everywhere (or $f'(x) < 0$), the ID Test gives a super fast proof of this.

Proof that inverse is continuous

Proof: of Lemma

- We use proof by contradiction. Assume that f is not increasing nor decreasing. Then there must exists three numbers in I with $a < x_1 < x_2 < x_3 < b$ for which $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$.
- There's two possibilities: (1) $f(x_3)$ lies between $f(x_1)$ and $f(x_2)$, or (2) $f(x_1)$ lies between $f(x_2)$ and $f(x_3)$.
- Case (1): Because $f(x_3)$ is between $f(x_1)$ and $f(x_2)$ and f is continuous we canA apply the IVT to get a c between x_1 and x_2 so that $f(c) = f(x_3)$. But, notice that $c \neq x_3$ because $x_1 < c < x_2 < x_3$. This means f is not one-one-one contradicting our assumption.
- Case (2): Similarly, IVT says there's a c between x_2 and x_3 so that $f(c) = f(x_1)$ which contradicts that f is one-to-one since $x_1 < x_2 < c$ implies $c \neq x_1$. □

Proof that inverse is continuous

Proof: of Continuity Theorem-1

- By the lemma, we may assume f is increasing on (a, b).
- By the lemma, since f^{-1} is also one-to-one, it is also increasing (why?).
- Let y_0 and $x_0 \in (a, b)$ satisfy $f(x_0) = y_0$.
- We want to show that f^{-1} is continuous at y_0.
- Let $\epsilon > 0$ be given. We must find $\delta(\epsilon) > 0$ so that for all $0 < |y - y_0| < \delta$ implies $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$.
- Now, notice $f^{-1}(y_0) = x_0$ is in the open interval (a, b). By shrinking $\epsilon > 0$, if necessary, we can assume $a < x_0 - \epsilon < x_0 + \epsilon < b$.
- Since f is increasing $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. So we may pick a $\delta > 0$ so that
$$f(x_0 - \epsilon) < y_0 - \delta \quad \text{and} \quad y_0 + \delta < f(x_0 + \epsilon)$$

Proof that inverse is continuous

Proof: of Continuity Theorem-2

- Since f is increasing $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. So we may pick a $\delta > 0$ so that
$$f(x_0 - \epsilon) < y_0 - \delta \quad \text{and} \quad y_0 + \delta < f(x_0 + \epsilon)$$
(viewed geometrically: we can choose δ small enough so that the interval $(y_0 - \delta, y_0 + \delta)$ is inside $(f(x_0 - \epsilon), f(x_0 + \epsilon)))$
- Thus, if we have y between $0 < |y - y_0| < \delta$ then $-\delta < y - y_0 < \delta \implies y_0 - \delta < y < y_0 + \delta$. And so because it lies in the larger interval we also have $f(x_0 - \epsilon) < f^{-1}(y) < f(x_0 + \epsilon)$
- Next, we'll use the fact that f^{-1} is increasing!
$$\begin{aligned} f^{-1}(f(x_0 - \epsilon)) < f^{-1}(y) < f^{-1}(f(x_0 + \epsilon)) &\implies x_0 - \epsilon < f^{-1}(y) < x_0 + \epsilon \\ &\implies f^{-1}(y_0) - \epsilon < f^{-1}(y) < f^{-1}(y_0) + \epsilon \\ &\implies |f^{-1}(y) - f^{-1}(y_0)| < \epsilon \end{aligned}$$
- Done! □

Proof of Differentiability of Inverse

Proof: Differentiability of Inverse-1

- We now prove that f^{-1} is differentiable at $a = f(b)$ provided that f is one-to-one and differentiable on an interval I and $f'(b) \neq 0$.
- Using the definition of the derivative, we must show that:

$$f^{-1}(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}.$$

- Recall we set $b = f^{-1}(a)$. But because f^{-1} is one-to-one on I and $y = f^{-1}(x)$, we can solve for x uniquely using f: $x = f(y)$.

Proof of Differentiability of Inverse

Proof: Differentiability of Inverse-2

- So, making the substitutions

$$\begin{aligned} f^{-1}(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y)-f(b)}{y-b}} = \frac{1}{\lim_{y \to b} \frac{f(y)-f(b)}{y-b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))} \end{aligned}$$

- We were able to switch $x \to a$ with $y \to b$ because of the continuity of f^{-1} that we already proved (if $x \to a$ then $f^{-1}(x) \to f^{-1}(a)$ which is exactly $y \to b$).