§11.11: Applications of Taylor Series

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Class #25 Notes

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Outline

Guiding Questions

Intro

Geometry of Taylor Series

Approximations with Taylor Series

Application: Integrals

Application: Solving DEs with PS

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1. Guiding Questions
2. Introduction
3. Geometry of Taylor Series
4. Approximations with Taylor Series
5. Application: Integrals
6. Application: Solving Differential Equationss with Power Series

Guiding Questions for §11.11

Guiding Question(s)

1. What are important applications of Taylor series?
2. What's the geometry of Taylor series?
3. How can power series be used to evaluate integrals?
4. How can power series be used to solve differential equations?

Introduction

- Armed with Taylor series, we can solve lots of difficult problems:
  - We can use partial sums to approximate complicated functions with only adding and multiplying using $T_N(x)$ for various values of N.
  - We can find derivatives of $f(x)$ easily using term-by-term differentiation.
  - We can find anti-derivatives of $f(x)$, $\int \frac{1}{1+x^2}\,dx$, easily using term-by-term integration.
  - We can solve differential equations with power series!

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Introduction

- PSRs we've found so far (memorize):
  - $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad \text{for} \quad x \in (-1,1)$
  - $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad x \in (-1,1)$
  - $\arctan(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for} \quad x \in (-1,1)$
  - $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for} \quad x \in (-\infty,\infty)$
  - $\sin(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for} \quad x \in (-\infty,\infty)$
  - $\cos(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for} \quad x \in (-\infty,\infty)$

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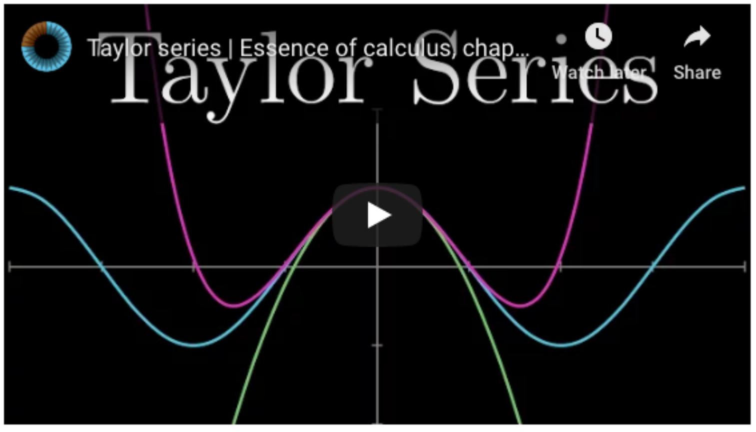
Geometry of Taylor Series

- Watch the video: https://www.youtube.com/watch?v=3d6DsjIBzJ4



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Geometry of Taylor Series

- In Calc 1, one of the most important results is that the tangent line approximation is the best linear approximation of $f(x)$ at $x = a$.
$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ near } a$$
- In Calc 1, we also learned that the second derivative controls the curvature (bending) of the function. Thus, the best quadratic approximation of $f(x)$ is a parabola where the second derivative of $f(x)$ at $x = a$ plays a role.
- On can show (as in the video) that the best quadratic approximation of $f(x)$ is $f(x) \approx f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2 \quad \text{for } x \text{ near } a$
- The best Nth degree polynomial approximation of $f(x)$ is $f(x) \approx$
$$T_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Geometry of Taylor Series

- The higher degree N of the Taylor polynomial, $T_N(x)$, the better the approximation of $f(x)$ for x that can be farther away from a.
- A Taylor series, then, can be viewed as the best "infinite" polynomial approximation of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots$$

- When you get lucky, the radius of convergence is $R = \infty$ and you can use the same Taylor series everywhere! However, as a general rule:
- The father away x is from a, the higher degree N you need to get $T_N(x)$ close to $f(x)$.

Approximations with Taylor Series

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- A Taylor polynomial $T_N(x)$ can be used to approximate $f(x)$.
- In estimating the error (recall: error = exact - approx):
  - If the Taylor series is alternating, use the error estimate for Alternating series:
    $$|Error| \leq b_{n+1}$$
  - Otherwise, use Taylor's Remainder Theorem: If $|f^{(N+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_N(x)$:
    $$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1} \qquad \text{for} \qquad |x-a| \leq d$$
  - Or, use technology like SAGE.

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Activity 1:

For $\cos(x)$ centered at $x = 0$:

(a) Find the fourth degree Taylor polynomial.

(b) Find the interval around 0 for which the Taylor polynomial is accurate to within 0.005.

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Activity 2:

(a) Use power series to evaluate: $\displaystyle\int \frac{\sin(x)}{x}\,dx$

(b) Write the first three, non-zero terms of your answer from part (a).

Application: Integrals

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Activity 3:

Consider $y' - y = 0$.

(a) Use power series to find the general solution.

(b) Use part (a), to find the particular solution when $y(0) = 1$.

Application: Solving Diff Eqs with Power Series

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Activity 4:

Consider $y' = x^2 y$.

(a) Use power series to find the general solution.

(b) Use part (a), to find the particular solution when $y(0) = 1$.

Application: Solving Diff Eqs with Power Series

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