§9.2: Direction Fields & Euler's Method

Ch 9: Differential Equations
Math 5B: Calculus II

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Class #7 Notes

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Outline

1. Guiding Questions
2. Direction Fields / Slope Fields
3. Euler's Method

Guiding Questions for §9.2

Guiding Question(s)

1. What are direction fields (or slope fields) and how can they help visualize solutions to differential equations?
2. What is Euler's Method and how does his method create approximate solutions to differential equations?

Direction Fields / Slope Fields

We'll study solutions to DEs that have a special form:

$$\frac{dy}{dx} = y' = F(x, y) \tag{1}$$

where $F(x, y)$ is some expression involving the variable, x, the unknown function, $y(x)$, and constants.

Examples:

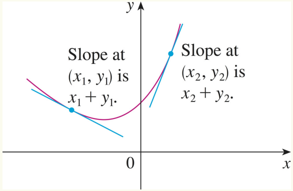
- $y' = x^2 + 1$
- $y' = 2y$
- $y' = x^2 + y^2$
- $xy' + x^2y^3 = 0$

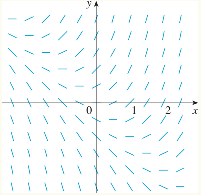
Direction Fields / Slope Fields

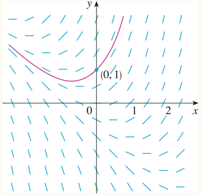
Definition 1: Direction Fields / Slope Fields

$$\frac{dy}{dx} = F(x, y)$$

- The direction field or slope field of the DE (1) is a graph that includes several short line segments at various (equally spaced) points where the slope of these line segments are given by $F(x, y)$
- Example: $y' = x + y$







Direction Fields / Slope Fields

Activity 1:

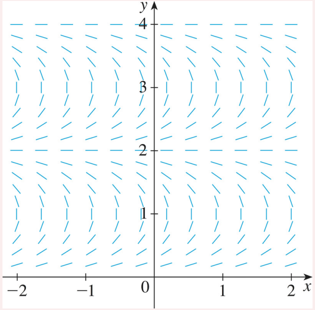
(a) Sketch the slope field of the DE: $y' = x^2$ with $x \in \{-2, -1, 0, 1, 2\}$ and $y \in \{-2, -1, 0, 1, 2\}$

(b) Use part (a), to sketch the graph of the particular solution passing through the origin.

Direction Fields / Slope Fields

Activity 2:

The slope field for the DE $y' = \tan\left(\frac{1}{2}\pi y\right)$ is given below:



(a) Sketch the graph of particular solution that satisfies the initial conditions $y(1) = 3$

(b) Identify all equilibrium solutions (i.e. constant solutions)

Euler's Method

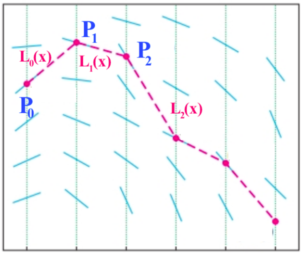
Recall: tangent line approximations

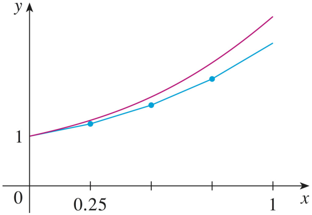
- First, recall that the tangent line of a function $f(x)$ at a point $P = (x_0, y_0)$ gives us the best linear approximation of f near P. The equation of a tangent line is: $y - y_0 = m_{\tan}(x_0)(x - x_0)$ where $m_{\tan}(x_0) = f'(x_0)$
- The function $L(x) = y_0 + f'(x_0)(x - x_0)$ is the tangent line of f at P
- Thus, $L(x)$ approximates $f(x)$ as long as $x \approx x_0$

Euler's Method

Idea of Euler's Method

- Approximate solutions to DE with line segments using the tangent line approximations where the slopes are given by $m = y' = F(x, y)$
- Starting at the initial conditions, $P_0 = (x_0, y(x_0))$, construct the tangent line $L_0(x)$ with point P_0 and slope $m = F(x_0, y_0)$
- We move along this line by taking a step of size h to arrive at a new point $P_1 = (x_1, y_1)$.
- Once we're at P_1 we use the slope $F(x_1, y_1)$ to construct a new tangent line $L_1(x)$
- Move along $L_1(x)$ by taking another step of size h to arrive at P_2
- Do this as many times as needed





Activity 3:

Consider the Initial Value Problem (IVP): $y' = y$; $y(0) = 1$

(a) Use Euler's Method with step size 0.25 to find the equations of the tangent lines $L_0(x)$ and $L_1(x)$. Then use these compute the approximate y-values: $y_1 = L_0(x_1)$ and $y_2 = L_1(x_2)$

(b) Keep going until you can approximate $y(1)$ where $y(x)$ represents the exact solution. That is, find $y_4 = L_3(x_4)$.

(c) What is the significance of y_4?

Euler's Method

Euler's Method

Definition 2: Euler's Method

To approximate a solution to $y' = F(x, y)$ using step size h, follow the following algorithm:

- Given the initial conditions: $P_0 = (x_0, y_0)$ (x_0 and $y_0 = y(x_0)$)
- To generate the next point $P_n = (x_n, y_n)$ (for $n = 1, 2, 3, \ldots$) follow:
  - $x_n = x_{n-1} + h = x_0 + nh$
  - $y_n = y_{n-1} + F(x_{n-1}, y_{n-1})h$

Euler's Method Calculator for some of the homework: http://bit.ly/EulerMethod

Euler's Method

$P_0 = (x_0, y_0)$ is given to start.

Step1: Find $L_0(x)$ at P_0.
Eq of L_0: $y - y_0 = m(x - x_0)$
$y = y_0 + m(x - x_0)$
$y = y_0 + F(x_0, y_0)(x - x_0)$

Step2: Apply $x_1 = x_0 + h$ to get y_1:
$y_1 = L_0(x_1) = y_0 + F(x_0, y_0)(x_1 - x_0)$
$y_1 = L_0(x_1) = y_0 + F(x_0, y_0)h$

Now have $P_1 = (x_1, y_1)$

Continue as many steps as needed

Step3: Find $L_1(x)$ at P_1.
Eq of L_1: $y - y_1 = m(x - x_1)$
$y = y_1 + m(x - x_1)$
$y = y_1 + F(x_1, y_1)(x - x_1)$

Step4: Apply $x_2 = x_1 + h$ $(= x_0 + 2h)$ to get y_2:
$y_2 = L_1(x_2) = y_1 + F(x_1, y_1)(x_2 - x_1)$
$y_2 = L_1(x_2) = y_1 + F(x_1, y_1)h$

Now we have $P_2 = (x_2, y_2)$

Euler's Method

Activity 4:

Consider the IVP: $y' = y - x$ with $y(0) = \frac{1}{2}$.

(a) Use Euler's Method with step size $h = 0.2$ to approximate $y(1)$ where $y(x)$ is the exact solution.

(b) Use the link above to approximate $y(1)$ with step size $h = 0.1$.

(c) Compare the errors from parts (a) and (b) to the exact solution $y(x) = 1 + x - \frac{1}{x}e^x$

Euler's Method