§11.2: Series

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Outline

1. Guiding Questions
2. Introduction
3. Series
4. Decimal Expansions
5. Geometric Series
6. Harmonic Series
7. Some Theorem on Series

Guiding Questions for §11.2

Guiding Question(s)

1. What are series?
2. What are decimal expansions (notation)?
3. What are geometric series?
4. What are harmonic series?
5. What are some theorems about series?

Introduction

- In the last section we introduced some interesting discoveries:
  - Leibniz: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
  - Euler: $\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$
  - $e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$
- What do these really mean?

Introduction

- Recall what decimal notation (or "expansion") means:

$$-7.65791 = -7 + \frac{6}{10} + \frac{5}{100} + \frac{7}{1000} + \frac{9}{10000} + \frac{1}{10^5}$$

- Rational numbers = numbers with finite decimal expansions (like example above) or infinite decimal expansions provided the infinite part is a repeating pattern. Ex:

$$\frac{1}{3} = 0.33333\ldots$$

- So we can make the intuitive definition of real numbers as any "infinite decimal expansion" whether or not we have a repeating pattern.
- This opens pandoras box.

Introduction

- Another motivation for studying infinite sums: integration!
- Recall the definition of a definite integral using Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)\Delta x_i$$

- So, all integrals are "infinite sums"
- Recall also the improper integral (type I):

$$\int_1^\infty f(x)\,dx = \lim_{R\to\infty} \int_1^R f(x)\,dx$$

- By analogy to definite integrals, we can define any "infinite sum" of any sequence of numbers by taking a limit of finite sums.

Definition 1: Series

Let $\{a_i\}_{i=0}^{\infty}$ be a sequence.

- The expresion $\displaystyle\sum_{i=0}^{\infty} a_i$ is called a series (associated to $\{a_i\}_{i=0}^{\infty}$).
- Partial sums: $S_n = \displaystyle\sum_{i=0}^{n} a_i$ is called a partial sum of the sequence $\{a_i\}_{i=0}^{\infty}$.
- The partial sums give us another sequence $\{S_n\}_{n=0}^{\infty}$. If the limit of the sequence of partial sums exits (and equals S), that is, converges, then we say the series converges to S. Otherwise, we say the series diverges.

$$\boxed{\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i.}$$

Example 1:

Some wickedly cool series that converge are:

(a) Leibniz: $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1}$$

(b) Euler: $$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k^2}$$

(c) Euler: $$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Recall that this is Definition 3 in our "Eight Definitions of e" handout.
We haven't proved any of these are true yet. And will do so much later.

Activity 1:

Let $\{a_i\}_{i=0}^{\infty}$ be a sequence whose partial sums are $S_n = \frac{4n^2-3n-7}{1-6n-8n^2}$. What is $\sum_{i=0}^{\infty} a_i$?

Activity 2:

Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge or diverge? If it converges, find its sum.
(Hint: use partial fractions to express $a_n = \frac{1}{n(n+1)}$ *as a difference and look for a clever trick for* S_n*.)*

Series

Decimal Expansions

Activity 3:

(a) Write $\frac{13}{51}$ in decimal notation. (Long Division)

(b) Write $7.1\overline{24}$ as a fraction. (Tens Trick)

Geometric Series

Example 2:

Think of a geometric picture that convinces you that

$$\sum_{i=0}^{\infty} \frac{1}{2^i}$$

converges. Use either lines or squares.

Geometric Series

Definition 2: test

A geometric series is any series of the form: $\sum_{n=0}^{\infty} a \cdot r^n$.

For any constant $a \in \mathbb{R}$.

Theorem 1: Geometric Series

A geometric series converges for any r satisfying $|r| < 1$, ie. $-1 < r < 1$. In fact, it sums to:

$$\boxed{\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r}}$$

A geometric series diverges for any r satisfying $|r| \geq 1$.

Geometric Series

Geometric Series

Geometric Series

Activity 4:

Converge or Diverge?

(a) $\displaystyle\sum_{n=0}^{\infty} 5^{-n}$

(b) $\displaystyle\sum_{i=0}^{\infty} 2^{2i}3^{1-i}$

(c) $\displaystyle\sum_{k=0}^{\infty} -7\left(-\frac{3}{4}\right)^{k}$

Geometric Series

Geometric Series

Activity 5:

If $x \in (-1, 1)$, does

$$\sum_{n=0}^{\infty} x^n$$

converge or diverge? If it converges, what does it sum to?

Harmonic Series

Definition 3: test

- The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Note: visualization can be deceiving!

Theorem 2: Harmonic Series

The harmonic series diverges!

Harmonic Series

Harmonic Series

Theorem 3: Arithmetic of Series

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent series. Then:

(a) $\displaystyle\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n)$

(b) $\displaystyle\sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n - b_n)$

(c) $\displaystyle\sum_{n=0}^{\infty} (c \cdot a_n) = c \sum_{n=0}^{\infty} a_n$ (for any constant $c \in \mathbb{R}$)

Note: multiplication and division work, but it is much trickier to write the formulas and we will not need to do this.

Theorem 4: Some Theorems on Series

(a) If $\sum_{n=0}^{\infty} a_n$ converges. Then: $\lim_{n\to\infty} a_n = 0$. That is, the "tail of the series" must decrease to zero. (Think about it!)

Warning! The "converse of this is false!" That is, there are series whose tails go to zero but do not converge! Main example: harmonic series.

(b) Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$ (or DNE), then the series $\sum_{n=0}^{\infty} a_n$ diverges.

Some Theorem on Series

Some Theorem on Series

Activity 6:

(a) Evaluate the series: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2+3^n}{5^n}-3\frac{1}{2^n}\right)$

(b) Show that $\displaystyle\sum_{j=0}^{\infty}\frac{3j^2}{j^2+j+1}$ diverges.

Some Theorem on Series

Some Theorem on Series