§6.2: Exponential Functions & Their Derivatives

Ch 6: Exponentials, Logs, & Inverse Trig Functions

Math 5B: Calculus II

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Outline

1. Guiding Questions
2. Basics of Exponential Functions
3. Derivatives of Exponential Functions
4. Eight Definitions of "e"
5. Derivative of the Natural Exponential Function
6. Integration of the Natural Exponential Function
7. Applications of Exponential Functions

Guiding Questions for §6.2

Guiding Question(s)

1. What exponential functions? Are they one-to-one? If so, what are their inverses?
2. Are exponential functions differentiable? If so, what is the derivative rule for computing their derivatives?
3. What are some important applications of exponential functions?

Basics of Exponential Functions

Definition 1:

An exponential function is a function of the form:

$$f(x) = b^x \tag{1}$$

where b is a real number (briefly, $b \in \mathbb{R}$) satisfying

$$b > 0 \qquad \text{and} \qquad b \neq 1 \tag{2}$$

Graph on Desmos:

Example 1:

- $f(x) = 2^x$
- $f(x) = 5.89^x$
- $f(x) = \pi^x$

Example 2:

- $g(x) = \left(\frac{1}{2}\right)^x$
- $g(x) = 0.15^x$
- $g(x) = 3^{-x}$

Basics of Exponential Functions

But what does this mean??? $f(x) = b^x$

Using algebra we know:

- Integers: $x \in \mathbb{Z}$
  - $x = 0, 1, 2, 3, \ldots$ repeated multiplication $b^3 = b \cdot b \cdot b$
  - $x = -1, -2, -3, \ldots$ use exponent rules $b^{-3} = \dfrac{1}{b^3}$
- Fractions: $x \in \mathbb{Q}$
  - $x = p/q$ two steps $b^{p/q} = \left(\sqrt[q]{b}\right)^p = \sqrt[q]{b^p}$
- Rest of the Real Numbers: $x \in \mathbb{R}$?
  - How do we compute $2^{\sqrt{3}}$?

Basics of Exponential Functions

- To compute: $2^{\sqrt{3}}$
  - Since $\sqrt{3}$ is irrational, we can approximate it with rationals
  - Example: $1.732 = \frac{1732}{1000}$ so $2^{1.732} = 2^{1732/1000} = \sqrt[1000]{2^{1732}}$

$$
\begin{array}{ccc}
1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \quad\quad \vdots & & \vdots \quad\quad \vdots
\end{array}
$$

- So we get a sequence of numbers that can be computed from above and below which "squeeze" the ideal value $2^{\sqrt{3}}$ in the limit.

Basics of Exponential Functions

Definition 2:

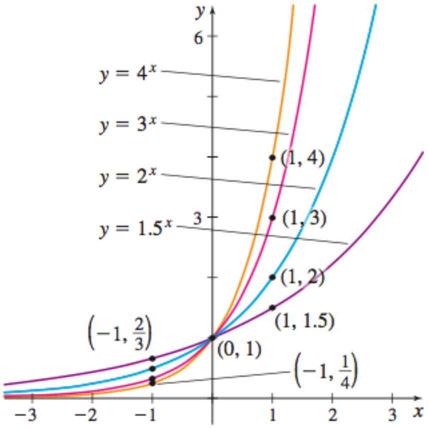
Using limits, we can define b^x rigorously using:

$$b^x = \lim_{r \to x} b^r \tag{3}$$

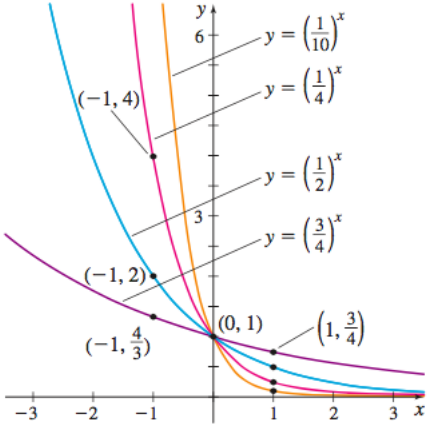
where $r \to x$ means "we choose a sequence of rational numbers r that approaches x."

For more details, read the book carefully.

Basics of Exponential Functions



Case $b > 1$



Case $0 < b < 1$

Basics of Exponential Functions

We summarize some important properties of exponential functions:

Theorem 1: Properties of Exponential Functions

Let $f(x) = b^x$, with $b > 0$, $b \neq 1$.

1. f is continuous & one-to-one (hence, it's inverse exists!)
2. $D(f) = (-\infty, +\infty)$
3. $R(f) = (0, +\infty)$ (USEFUL!! $b^x > 0$ for all x)
4. the line $y = 0$ is a horizontal asymptote as $x \to -\infty$
5. Graphing Properties: pass point $(0, 1)$ and

Case: $b > 1$

- increasing
- $\lim_{x \to -\infty} b^x = 0$
- $\lim_{x \to +\infty} b^x = +\infty$

Case: $0 < b < 1$

- decreasing
- $\lim_{x \to -\infty} b^x = +\infty$
- $\lim_{x \to +\infty} b^x = 0$

Basics of Exponential Functions

The exponential rules are also important:

Theorem 2: Laws of Exponents (or Exponent Rules)

Let $f(x) = b^x$, with $b > 0$, $b \neq 1$.

1. $b^{x+y} = b^x \cdot b^y$
2. $b^{x-y} = \frac{b^x}{b^y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x \cdot b^x$

These can be rigorously proved using the limits theorems and Definition 7.

Basics of Exponential Functions

Activity 1:

(a) Sove for x: $3^x = 0$

(b) $\lim_{x \to \infty} (5^{-x} - 1)$

(c) Sketch: $y = 3^{-x} + 1$

Let's compute the derivative of $f(x) = b^x$:

$$\frac{d}{dx}\left[b^x\right] =$$

Derivatives of Exponential Functions

With $f(x) = b^x$

Notice: $f'(x) = f'(0)f(x)$

This says:
rate of change of an exponential function is PROPORTIONAL to the function itself!

Geometrically:
The slope of an exponential function at a point P is PROPORTIONAL to the height of the point P (y-coordinate)

Derivatives of Exponential Functions

With $f(x) = b^x$:

$$\begin{aligned} \frac{d}{dx}\left[b^x\right] &= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \\ &= b^x \lim_{h\to 0} \frac{b^h - 1}{h} \\ &= Cb^x \end{aligned}$$

where

$$C = \lim_{h\to 0} \frac{b^h - 1}{h}. \tag{4}$$

Now what? What is the value of C? It is a constant that depends only on the base of the exponent b.

Eight Definitions of "e"

What is the value of C? As we mentioned, it is a constant that depends only on the base of the exponent b.

- One answer is to just set this constant equal to 1 and find the base b that makes this true. That is, define the number e to be the the unique real number for which

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h} \tag{5}$$

- We can interpret this geometrically as follows: we define the number e to be the the unique base of the exponential function whose tangent line has slope 1 at $x = 0$.

These are equivalent since the second definition says $f'(0) = 1$ which is equivalent to $\frac{d}{dx}[b^x]_{x=0} = Cb^0 = 1$ which is equivalent to $C = 1$ in (4), or (5)

Eight Definitions of "e"

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h}$$

- It's really hard to figure out how to compute the value of e from this definition.
- There are many other ways we can define "e"
- Alternative definition: Compound Interest version 1:

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \tag{6}$$

- Alternative definition: Compound Interest version 2:

$$e = \lim_{h \to 0} (1 + h)^{1/h} \tag{7}$$

Eight Definitions of "e"

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.7182818284590\ldots$$

- The compound interest (version 1) is the easiest formula to use to actually compute (an approximate value of) e.
- Notice how "slow" it is to get close to it's true value

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
10	2.59374
20	2.65329
100	2.70481
1000	2.71692

Eight Definitions of "e"

- You promised us "eight defintions!"
- See my **hand-out**: "Eight Definiton of e" on my website
- e is an important constant to many areas of study and so it was (re-)discovered in many different ways by many different people
- First discovered in finance! Compound Interest

Definition 3:

The definition of e we'll assume is that e is the base of the exponential function that gives $C = 1$, i.e.

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h}$$

The function $f(x) = e^x$ is called the natural exponential function.

Theorem 3: Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left[e^x\right] = e^x \tag{8}$$

Formula (8) is one of my favorite formulas! It says: the natural exponential function is its own derivative!

Derivative of the Natural Exponential Function

Activity 2:

Find: $\displaystyle\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1}$

Derivative of the Natural Exponential Function

Activity 3:

Given that $y = e^{x^3}$, what is the equation of the tangent line at $P = (0, 1)$?

Derivative of the Natural Exponential Function

Activity 4:

If $y = e^{-4x}\sin(5x)$, what is y'?

Derivative of the Natural Exponential Function

Activity 5:

What is the absolute maximum of $f(x) = xe^{-x}$?

Derivative of the Natural Exponential Function

Activity 6:

Use the "curve sketching information" (CSI Lines) of f' and f'' to sketch the graph of $f(x) = e^{1/x}$?

Since e^x is its own derivative, it has an equally simple anti-derivative: $\int e^x dx = e^x + C$. So, let's note this awesome fact as a theorem.

Theorem 4: Integration of Natural Exponential Functions

$$\int e^x\,dx = e^x + C \tag{9}$$

Integrals of the Natural Exponential Function

Activity 7:

(a) Evaluate: $\int x^2 e^{x^3}\, dx$

(b) Find the area under the curve $y = e^{-3x}$ from $x = 0$ to $x = 1$.

Wrap-up: final questions

So, you've convinced me that e^x is totally awesome because it has really easy derivative and anti-derivative rules.

What about other bases b? For $b > 0$, $b \neq 1$, $b \neq e$,

- Do we have $\frac{d}{dx}[b^x] = b^x$?
- Do we have $\int b^x \, dx = b^x + C$?

We'll find out later in the chapter.

Applications of Exponential Function

There are so many applications of exponential functions that we'll study them in detail in §6.5. For now, we'll just list a few:

- Population Growth and Decay
- Compound Interest in Fincance
- Radioactive Carbon Dating
- And much, much more