§9.4: Models for Population Growth

Ch 9: Differential Equations
Math 5B: Calculus II

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Class #8 Notes

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Outline

1. Guiding Questions
2. Law of Natural Growth & Decay

Guiding Questions for §9.4

Guiding Question(s)

1. Recall that previously we introduced two models for population growth: (1) the law of natural growth/decay and (2) the Logistic Equation. How do we find their solutions and prove our results?

Law of Natural Growth & Decay

Recall that in §6.5 we introduced the Law of Natural Growth & Decay: the rate of change of a population is proportional to the population size. If $P(t)$ denotes our population at time t, the DE reads:

$$\frac{dP}{dt} = kP.$$

Theorem 1: Exponential Growth & Decay Equation

The only solutions to the law of natural growth/decay, $\frac{dP}{dt} = kP(t)$, for a constant $k \neq 0$, with initial conditions $P_0 = P(0)$, are of the form:

$$\boxed{P(t) = P_0 e^{kt}.} \tag{1}$$

We already showed that the functions $P(t) = P_0 e^{kt}$ do indeed solve the DE. So, there remains to prove these are the only solutions now.

Law of Natural Growth & Decay

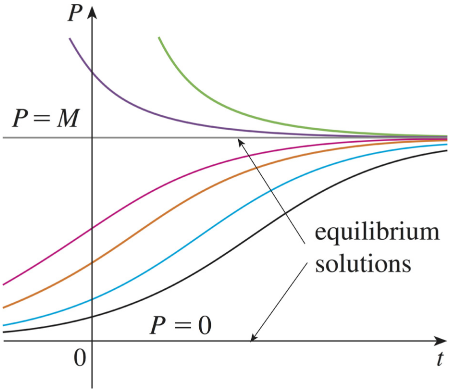
Activity 1:

Prove that the only solutions to the law of natural growth/decay, $\frac{dP}{dt} = kP$, are of the form $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$.

Law of Natural Growth & Decay

Logistic Differential Equation

Recall:



Logistic Differential Equation

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

- If to start, $P(0)$ lies between 0 and M then $P'(t) > 0$ and $P(t)$ increases
- If $P(t)$ exceeds carrying capacity, $P(t) > M$, then $P'(t) < 0$
- Both cases: $P(t) \to M$ as $t \to \infty$ so $P'(t) \to 0$ as $t \to \infty$

Logistic Differential Equation

Theorem 2: Logistic Differential Equation

The solutions to the logistic equation, $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$, for a constant $k \neq 0$, with initial conditions $P_0 = P(0)$ and carrying capacity M, are of the form:

$$\boxed{P(t) = \frac{M}{1 + Ae^{-kt}}}, \quad \text{where} \quad A = \frac{M - P_0}{P_0} \tag{2}$$

Logistic Differential Equation

Before we study the proof, let's look at some examples.

Activity 2:

We consider the DE: $\dfrac{dP}{dt} = 0.3P(4-P)$

(a) What is the k and the carrying capacity M?

(b) What are the general solutions?

(c) If the initial conditions are $P(0) = 1$, predict the population size when $t = 3$.

Logistic Differential Equation

Logistic Differential Equation

Activity 3:

A deer population grows logistically with growth constant $k = 0.4$ (units are $year^{-1}$) in a forrest with carrying capacity of 1000 deer.

(a) Find the population of deer after t years if the initial population is 100 deer.

(b) How long does it take for the deer population to reach 500?

Logistic Differential Equation

Logistic Differential Equation

Proof: Part-1

- We can solve: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ using separation of variables–keeping in mind that k and M are constants and $P(t)$ is the unknown function of t.
- We separate the variables and get:

$$\frac{dP}{P(1 - P/M)} = kdt$$
$$\int \frac{dP}{P(1 - P/M)} = \int kdt$$

- The right-hand side (RHS) is a really integral but the left-hand side (LHS) is trickier but do-able.
- In the next page, we tackle the LHS.

Logistic Differential Equation

Proof: Part-2

- We want to evaluate: $\displaystyle\int \frac{dP}{P(1-P/M)}$.
- To do this, we need a clever algebra trick to split it into two pieces:

$$\begin{aligned}\frac{1}{P(1-P/M)} &= \frac{M}{P(M-P)} \qquad \textit{(multiply by } M/M\textit{)}\\ &= \frac{1}{P} + \frac{1}{M-P} \qquad \textit{(check that this works)}\end{aligned}$$

- So, now we can integrate each piece and get:

$$\begin{aligned}\int \frac{dP}{P(1-P/M)} &= \int \frac{1}{P}dP + \int \frac{1}{M-P}dP\\ &= \ln|P| - \ln|M-P|\end{aligned}$$

- And since the RHS is $\int kdt = kt + C$, we get:

Logistic Differential Equation

Proof: Part-3

- We have:

$$\int \frac{dP}{P(1-P/M)} = \int k dt$$

$$\ln|P| - \ln|M-P| = kt + C$$

- We can solve for P using the log properties:

$$\ln|P| - \ln|M-P| = kt + C$$

$$\ln|M-P| - \ln|P| = -kt - C \quad \textit{(multiply by } -1 \textit{ just for fun)}$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C}$$

$$\frac{M-P}{P} = \pm e^{-C}e^{-kt} \quad \textit{(since both sides need to be positive)}$$

Logistic Differential Equation

Proof: Part-4

- Almost done:

$$\frac{M-P}{P} = \pm e^{-C} e^{-kt}$$
$$\frac{M-P}{P} = Ae^{-kt} \qquad (\textit{set } A = \pm e^{-C})$$
$$M - P = APe^{-kt}$$
$$M = APe^{-kt} + P = P(Ae^{-kt} + 1)$$

- Finally solve for P:

$$P = \frac{M}{1 + Ae^{-kt}}$$

- To find the equation for A, we simply solve the equation for $t = 0$:

$$P_0 = \frac{M}{1+A} \implies A = \frac{M - P_0}{P_0}. \quad \square$$