§11.9: Representation of Functions as Power Series

Ch 11: Infinite Sequences and Series

Math 5B: Calculus II

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Outline

Guiding Questions for §11.9

Guiding Question(s)

1. How can we express regular functions as power series?

Introduction

- We've mentioned a few times now that this chapter is built on the foundation of the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad \text{when} \quad |x| < 1$$

- By manipulating this series, we can find other power series and their associated functions (though on restricted domains):
  - $\displaystyle \frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1} = x + x^2 + x^3 + \cdots \quad \text{when } |x| < 1$
  - $\displaystyle \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots \quad \text{when } |x| < 1$
- Thus, for example, the function $f(x) = \frac{1}{1+x^2}$ can be represented using the above power series whenever $|x| < 1$.

Introduction

- Thus, for example, the function $f(x) = \frac{1}{1+x^2}$ can be represented using the above power series whenever $|x| < 1$.
- But there's some serious questions we need to answer.
- $f(x) = \frac{1}{1+x^2}$ is actually defined for all real numbers $x \in (-\infty, \infty)$.
- But the power series representation only works for $x \in (-1, 1)$.
- This will be resolved in the next section.

Introduction

- Hold up!
- Why on earth would we take a perfectly good function like $f(x) = \frac{1}{1+x^2}$ and write it as a power series, especially since it only works for a restricted domain!?!?
- A few reasons:
  - Multiplying and adding on a computer is really fast and easy
  - Dividing by hand and on a computer is harder
  - We can use partial sums to approximate complicated functions with only adding and multiplying.
  - We can find derivatives of $f(x)$ easily using the power series
  - We can find anti-derivatives of $f(x)$, $\int \frac{1}{1+x^2}\,dx$, easily using power series
  - These are the heart of the chapter.

Power Series Representation

Definition 1: Power Series Representation

- A power series representation (PSR) of a function $f(x)$ is any power series $\sum c_n(x-a)^n$ that agrees with $f(x)$ on some open interval.
- Warning! There can be more than one power series representation of the same function, but it will be on different intervals of convergence.
- When asked to find the PSR, you must also give the interval of convergence.

Power Series Representation

Activity 1:

Find the **power series representation** of

$$f(x) = \frac{1}{1+x^7}$$

and include the **interval of convergence**.

Power Series Representation

Activity 2:

Find the **power series representation** of

$$f(x) = \frac{2x}{1+x}$$

and include the **interval of convergence**.

Power Series Representation

Power Series Representation

Activity 3:

Find the **power series representation** of

$$f(x) = \frac{x^2}{4+3x}$$

and include the **interval of convergence**.

Power Series Representation

Just like Poly

- Power Series are "essentially infinite" polynomials
- Polynomials, thanks to the power rule, can be differentiated and anti-differentiated as many times as we want.
- Inside the interval of convergence, we can differentiate and integrate (anti-differentiate) power series as many times as we want!

Just like Poly

Theorem 1: Just like Poly

Assume that $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$. Then, if we let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$,

(i) $f(x)$ is infinitely differentiable inside the interval of convergence $(a-R, a+R)$

(ii) $f(x)$ is integrable inside the interval of convergence $(a-R, a+R)$

Warning! Though the radius of convergence is the same, and we can integrate and differentiate **inside** the interval of convergence, the endpoints might not behave as the original function.

This is called term-by-term differentiation or integration.

Just like Poly

By the Just Like Poly Theorem:

- $$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$
  $$= \frac{d}{dx}\left[c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots\right]$$
  $$= c_1 + 2\cdot c_2(x-a) + 3\cdot c_3(x-a)^2 + 4\cdot c_4(x-a)^3 + \cdots$$
- $$f''(x) = \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=2}^{\infty} n\cdot(n-1)c_n(x-a)^{n-2}$$
  $$= \frac{d}{dx}\left[c_1 + 2\cdot c_2(x-a) + 3\cdot c_3(x-a)^2 + 4\cdot c_4(x-a)^3 + \cdots\right]$$
  $$= 2\cdot c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \cdots$$
- Etc.

Just like Poly

By the Just Like Poly Theorem:

- $$\int f(x)dx = \int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = C + \sum_{n=1}^{\infty} \frac{1}{n+1} c_n(x-a)^{n+1}$$
  $$= \int \left[c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots\right] dx$$
  $$= C + c_0 x + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \frac{1}{4}\cdot c_3(x-a)^4 + \cdots$$
- Etc.

Activity 4:

Find the **power series representation** of $f(x) = \frac{2}{(1+x)^2}$ using the PSR of $\frac{2x}{1+x}$ from Activity 2.

Just like Poly

PSR for Natural Log

Theorem 2: PSR for Natural Log

The power series for the natural logarithm is

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad \text{for} \qquad |x| < 1$$

PSR for Natural Log

PSR for Natural Log

Activity 5:

Approximate $\ln(1.1)$ with an error less than 10^{-5}.

PSR for Natural Log

PSR for Arctangent

Activity 6:

(a) Find the **power series representation** of $\arctan(x) = \tan^{-1}(x)$.

(b) Use part (a), to derive Leibniz' formula given in the Chapter 11 intro:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

PSR for Arctangent

PSR for Arctangent

Activity 7:

Use the series in part (b) of Activity 5 to approximate π by using 5 terms.

PSR for Arctangent

Approximating Integrals with PSR

Activity 8:

Approximate

$$\int_0^{0.1} \frac{1}{1+x^7}\,dx$$

with an error less than 10^{-10}.

Approximating Integrals with PSR