MATH 460: Probability & Statistics — September 2018

Chapter 1: Basic Probability

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Guiding Question(s)

(1) What does it mean for events to be "certain" or "uncertain"?

(2) How can we systematically study uncertainty?

(3) How can we apply this to our everyday lives?

Sets

To begin our study of certainty, we first need to discuss the basic language it is written in–mathematical set theory.

Definition 1: Set Theory

A **set** is a collection of things, or **elements**. Elements of a set can be anything, such as numbers or people. We denote a set with curly brackets $\{$ and $\}$. So if we want to talk about the set of even numbers from 1 to 10, we can write it like this: $\{2, 10, 8, 6, 4\}$–note that we separate the elements by a comma and that the order doesn't matter.
Here are some important terms and notations:

- $\emptyset$ denotes the empty set or the set that contains no elements. We also write this $\{\}$.
- Capital letters denote sets: A, B, C etc. We also use Greek letters like "Omega" Ω.
- S denotes the universe of all possible elements in consideration. We call it the **sample space**.
- We denote $A \subset B$ to say that A **is a subset of** B. This means that every element of A is also an element of set B.
- A' (or A^c) denotes the elements of the universe S that are NOT in A. We call this the **complement of A**.
- $A \cup B$ is the set consisting of all elements in the set A combined with all the elements in set B. We call this A **union** B.
- $A \cap B$ is the set consisting of all elements that are in both A and B. We call this A **intersect** B.
- We write $A - B$ to mean the set containing elements that are in A and not in B. Notice that $A' = S - A$.
- We say that two sets are **disjoint** if they have no elements in common. In other words, A and B are disjoint if and only if $A \cap B = \emptyset$.

Activity 1: Venn Diagrams

What is a Venn Diagram? If you know what it is, use them to illustrate the all of the definitions from Definition 1.

Activity 2: Set Theory

Consider the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$ where $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Compute each of the following sets:

(a) $A \cup B$

(b) $A \cap B$

(c) A'

(d) B'

(e) $B - A$

(f) $A - B$

(g) $(A \cup B)'$

(h) $A' \cup B'$

(i) $(B - A') \cap (A \cap B)'$

Activity 3: Set Theory

Consider the sets $A = \{4, 5, 6, 7, 9, 13, 16\}$ and $B = \{3, 6, 9, 12, 15\}$ where the sample space S consists of all positive integers less than or equal to 16. Find the following:

(a) $A \cup B$

(b) $A \cap B$

(c) A'

(d) $(A \cap B)'$

Random Experiments

An **experiment** is any activity or situation in which two or more outcomes will result. A **random experiment** is an experiment where there is uncertainty in which outcome will occur.

Example 1: Random Experiments

(a) If we toss a die, the result of the experiment is that it will come up with one of the numbers in the set $\{1, 2, 3, 4, 5, 6\}$. Is it random?

(b) If we toss a coin twice, there are four results possible, as indicated by $\{HH, HT, TH, TT\}$, i.e., both heads, heads on first and tails on second, etc. Is it random?

Sample Space

A set S that consists of all possible outcomes of a random experiment is called a **sample space**, and each outcome is called a **sample point**. Often there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.

Example 2: Sample Space

(a) If we toss a die, one sample space, or set of all possible outcomes, is given by $S = \{1, 2, 3, 4, 5, 6\}$ while another is $S = \{odd, even\}$. It is clear, however, that the latter would not be adequate to determine, for example, whether an outcome is divisible by 3.

(b) If we flip a coin the sample space is $S = \{H, T\}$ or $S = \{1, 0\}$ where 1 represents a H and 0 represents T.

(c) If we draw a card the sample space has 52 items $\{AH, 10K, 4D, \ldots\}$. Notice we use dots to denote more elements are there but are not completely listed. This can be either for finite or infinite sets.

(d) Buying a car: One possible sample space is $S = \{$(male, hybrid), (female, hybrid), (male, traditional), (female, traditional)$\}$

Activity 4: Sample Space/Outcomes

List all possible outcomes for the following.

(a) Flipping a coin 4 times.

(b) Rolling 2 six-sided dice at the same time.

Events

An **event** is a subset A of the sample space S, i.e., it is a set of possible outcomes. If the outcome of an experiment is an element of A, we say that the event A has occurred. An event consisting of a single point of S is often called a **simple** or **elementary event**.

Example 3: Events

(a) If we toss a coin, the result of the experiment is that it will either come up "tails," symbolized by T (or 0), or "heads," symbolized by H (or 1), i.e., one of the elements of the set $\{H, T\}$ (or $\{0, 1\}$).

(b) If we toss a die, and the result of the experiment is that a 5 comes up, then 5 is a simple event.

Definition 2: Events and Sets

We can use the set operations on events in S to obtain other events in S. The empty-set, $\emptyset$, is called an impossible event because it cannot occur. If A and B are events in S, then

(a) $A \cup B$ is the event that "either A or B or both" occur.

(b) $A \cap B$ is the event that "both A and B" occur.

(c) A' is the event "not in A"

(d) $A - B$ is the even "in A but not in B"

(e) If the sets corresponding to events A and B are disjoint, i.e., $A \cap B = \emptyset$, we often say that the events are **mutually exclusive**. This means that they cannot both occur.

Activity 5: Events

Let S be the sample space of flipping a coin twice. Let A b the event "at least one head occurs" and B be the event "the second toss results in a tail." Express A and B using the H and T notation and find:

(a) $A \cup B$

(b) $A \cap B$

(c) A'

(d) $A - B$

Permutations

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the 1st object, and after this is done, $n-1$ ways of choosing the 2nd object, ..., and finally $n-(r-1)$ ways of choosing the rth object, it follows by the fundamental principle of counting that the number of different arrangements, or **permutations** as they are often called, is given by

$$_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1)) \tag{1}$$

The notation $_nP_r$ is the number of different permutations of n objects taken r at a time.
When $r = n$, that is when we choose n objects and arrange them then

$$_nP_n = n \cdot (n-1) \cdot (n-2) \cdots (n-(n-1)) = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n! \tag{2}$$

The exclamation mark is called the **factorial symbol**.
We define $0!$ to be 1, that is $0! = 1$.

Theorem 1: Permutations

When selecting n objects and order them r at a time, that is, permutation of n objects of length r, we have

$$_nP_r = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1)) \tag{3}$$

Example 4: Permutations

The number of different arrangements, or permutations, consisting of 3 letters each that can be formed from the 7 letters A, B, C, D, E, F, G is

$$_7P_3 = 7 \cdot 6 \cdot 5 = 210$$

or we can calculate it using the formula from the theorem:

$$_7P_3 = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

Activity 6: Permutations

Calculate the following:

(a) $10!$

(b) $_8P_5$

(c) $_4P_4$

Activity 7: Permutations and Combinations

(a) In how many ways can 10 people be seated on a bench if only 4 seats are available?

(b) Castel and Joe are planning trips to three countries this year. There are 7 countries they would like to visit. One trip will be one week long, another two days, and the other two weeks. How many possibilities are there?

Combinations

In a permutation we are interested in the order of arrangement of the objects. For example, abc is a different permutation from bca. In many problems, however, we are interested only in selecting or choosing objects *without regard to order*. Such selections are called **combinations**. For example, abc and bca are the same combination.
The total number of combinations of r objects selected from n (also called the combinations of n things taken r at a time) is denoted by ${}_nC_r$ or $\binom{n}{r}$.

Theorem 2: Combinations

When computing the n combinations of r objects, we have

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{{}_nP_r}{r!} \tag{4} $$

Example 5: Combinations

The number of ways in which 3 cards can be chosen or selected from a total of 8 different cards is

$$ {}_8C_3 = \binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56. $$

Activity 8: Combinations

Calculate the following:

(a) ${}_{12}C_{10}$

(b) ${}_7C_7$

Activity 9: Permutations and Combinations

(a) In how many ways can 10 objects be split into two groups containing 4 and 6 objects, respectively?

(b) In how many ways can a team of 17 softball players choose three players to refill the water cooler?