§6.7: Hyperbolic Functions

Ch 6: Exponentials, Logs, & Inverse Trig Functions
Math 5B: Calculus II

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Class #4 Notes

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Outline

1. Guiding Questions
2. Hyperbolic Functions
3. Derivatives of the Hyperbolic Functions
4. Inverse Hyperbolic Functions

Guiding Questions for §6.7

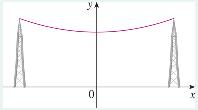
Guiding Question(s)

1. What are the hyperbolic functions?
2. What are their properties and where do they come from?
3. What are their derivative and anti-derivative rules for the hyperbolic functions?
4. What are the inverse hyperbolic functions?
5. What are the derivative and anti-derivative rules for the inverse hyperbolic functions?

Introducing the Hyperbolic Functions

What are they?

- Certain combinations of the two natural exponential functions: e^x and e^{-x}
- Show up in nature and engineering problems
- Claim to fame (i.e. THE main example): solution to the "hanging chain problem"



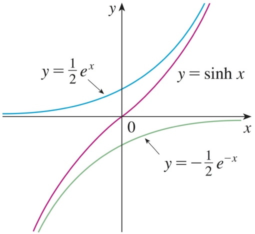
- Geometric derivation: intimately related to the hyperbola $x^2 - y^2 = 1$
- Just like the trigonometric (circular) functions are intimately related to the circle $x^2 + y^2 = 1$

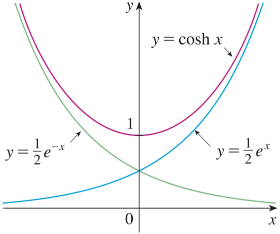
Introducing the Hyperbolic Functions

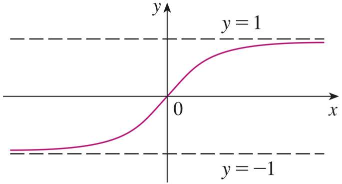
Definition 1: Hyperbolic Functions

- Hyperbolic Sine: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$
- Hyperbolic Cosine: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$
- Hyperbolic Tangent: $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$
- Hyperbolic Cosecant: $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$
- Hyperbolic Secant: $\sec(x) = \dfrac{1}{\cosh(x)}$
- Hyperbolic Cotangent: $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$

Introducing the Hyperbolic Functions







Properties of $\sinh(x)$:

- $D = \mathbb{R}$
- $R = \mathbb{R}$
- one-to-one

Properties of $\cosh(x)$:

- $D = \mathbb{R}$
- $R = [1, \infty)$
- one-to-one on $[0, \infty)$

Properties of $\tanh(x)$:

- $D = \mathbb{R}$
- $R = (-1, 1)$
- one-to-one

Introducing the Hyperbolic Functions

Theorem 1: Properties of Hyperbolic Functions

(1) $\sinh(-x) = -\sinh(x)$

(2) $\cosh(-x) = \cosh(x)$

(3) $\boxed{\cosh^2(x) - \sinh^2(x) = 1}$

(4) More in book...

Introducing the Hyperbolic Functions

Recall: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

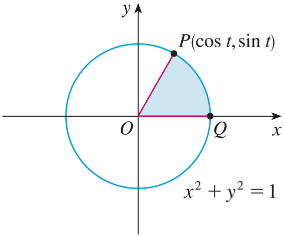
Activity 1:

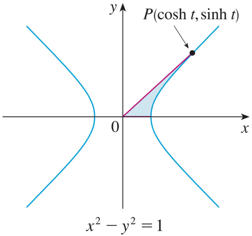
Verify the following properties of hyperbolic functions:

(a) $\cosh(-x) = \cosh(x)$

(b) $\cosh^2(x) - \sinh^2(x) = 1$

Geometry of the Hyperbolic Functions





Circle: $P = (x, y)$

- $x^2 + y^2 = 1$
- $\cos^2(t) + \sin^2(t) = 1$
- t radian measure
- $x = \cos(t)$, $y = \sin(t)$

Hyperbola: $P = (x, y)$

- $x^2 - y^2 = 1$
- $\cosh^2(t) - \sinh^2(t) = 1$
- $t = 2{\cdot}\text{Area(shaded)}$
- $x = \cosh(t)$, $y = \sinh(t)$

Derivative Rules of the Hyperbolic Functions

Because the hyperbolic functions are based on the exponential functions, and the exponential functions have easy derivatives, the following derivative rules are easy to verify:

Theorem 2: DRs of Hyperbolic Functions

(DR 1) $\dfrac{d}{dx}[\sinh(x)] = \cosh(x)$

(DR 2) $\dfrac{d}{dx}[\cosh(x)] = \sinh(x)$ BEWARE! Signs are different sometimes!

(DR 3) $\dfrac{d}{dx}[\tanh(x)] = \operatorname{sech}^2(x)$

(DR 4) $\dfrac{d}{dx}[\operatorname{csch}(x)] = -\operatorname{csch}(x)\coth(x)$

(DR 5) $\dfrac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$

(DR 6) $\dfrac{d}{dx}[\coth(x)] = -\operatorname{csch}^2(x)$

Derivative Rules of the Hyperbolic Functions

Activity 2: test

(a) Verify the DR for $\cosh(x)$, i.e. show from the definition that
$$\frac{d}{dx}[\cosh(x)] = \sinh(x)$$

(b) Verify the DR for $\tanh(x)$, i.e. show that $\frac{d}{dx}[\tanh(x)] = \text{sech}^2(x)$

Derivative Rules of the Hyperbolic Functions

Activity 3: test

(a) Find y' given that $y = e^x \tanh(x)$

(b) If $s(t) = \cosh(\ln(t))$, what is $\dfrac{ds}{dt}$?

Derivative Rules of the Hyperbolic Functions

Each DR has a corresponding ADR:

Theorem 3: ADRs of Hyperbolic Functions

(ADR 1) $$\int \sinh(x)\,dx = \cosh(x) + C$$

(ADR 2) $$\int \cosh(x)\,dx = \sinh(x) + C$$

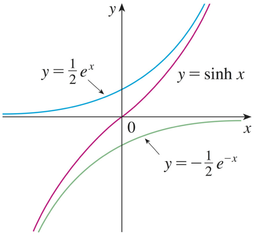
(ADR 3) $$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + C$$

(ADR 4) $$\int \operatorname{csch}(x)\coth(x)\,dx = -\operatorname{csch}(x) + C$$

(ADR 5) $$\int \operatorname{sech}(x)\tanh(x)\,dx = -\operatorname{sech}(x) + C$$

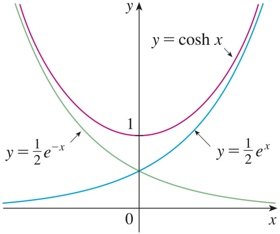
(ADR 6) $$\int \operatorname{csch}^2(x)\,dx = -\coth(x) + C$$

Inverse Hyperbolic Functions



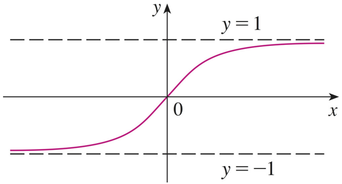
Properties of $\sinh(x)$:

- $D = \mathbb{R}$
- $R = \mathbb{R}$
- one-to-one
- Inverse exists!



Properties of $\cosh(x)$:

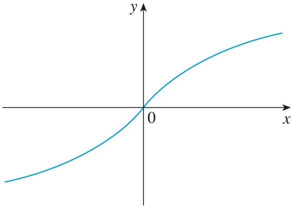
- $D = \mathbb{R}$
- $R = [1, \infty)$
- one-to-one on $[0, \infty)$
- Inverse exists on restriction!



Properties of $\tanh(x)$:

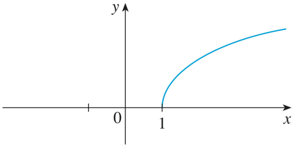
- $D = \mathbb{R}$
- $R = (-1, 1)$
- one-to-one
- Inverse exists!

Inverse Hyperbolic Functions



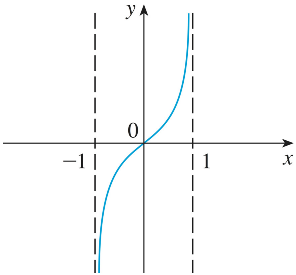
Properties of $\sinh(x)$:

- $D = \mathbb{R}$
- $R = \mathbb{R}$
- one-to-one
- Inverse exists!
- $\sinh^{-1}(x)$



Properties of $\cosh(x)$:

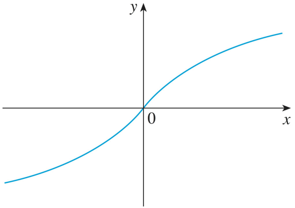
- $D = \mathbb{R}$
- $R = [1, \infty)$
- one-to-one on $[0, \infty)$
- Inverse exists on restriction!
- $\cosh^{-1}(x), x \geq 1$

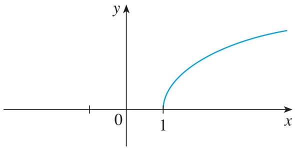


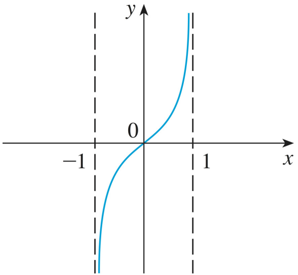
Properties of $\tanh(x)$:

- $D = \mathbb{R}$
- $R = (-1, 1)$
- one-to-one
- Inverse exists!
- $\tanh^{-1}(x)$

Inverse Hyperbolic Functions







Definition 2: test

- Inverse Hyperbolic Sine, denoted $\sinh^{-1}(x)$, is defined to be the inverse function of $\sinh(x)$.
- Inverse Hyperbolic Cosine, denoted $\cosh^{-1}(x)$, is defined to be the inverse function of $\cosh(x)$ restricted to $[0, \infty)$.
- Inverse Hyperbolic Tangent, denoted $\tanh^{-1}(x)$, is defined to be the inverse function of $\tanh(x)$.

Inverse Hyperbolic Functions

Since the hyperbolic functions are built from e^x and e^{-x}, it is not surprising that the inverse hyperbolic functions involve the natural log, $\ln(x)$

Theorem 4: Properties of Inverse Hyperbolic Functions

(a) $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in \mathbb{R}$

(b) $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right), \quad x \in [1, \infty)$

(c) $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad x \in (-1, 1)$

Activity 4:

Verify: $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \in [1, \infty)$

Inverse Hyperbolic Functions

The formulas for the inverse hyperbolic functions certainly look differentiable, and their graphs do as well. Indeed:

Theorem 5: DR/ADRs of Inverse Hyperbolic Functions

(DR 1) $$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{1}{\sqrt{1+x^2}}$$

(DR 2) $$\frac{d}{dx}\left[\cosh^{-1}(x)\right] = \frac{1}{\sqrt{x^2-1}}$$

(DR 3) $$\frac{d}{dx}\left[\tanh^{-1}(x)\right] = \frac{1}{1-x^2}$$

(ADR 1) $$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}(x) + C$$

(ADR 2) $$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x) + C$$

(ADR 3) $$\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x) + C$$

See the book for the others...

Inverse Hyperbolic Functions

Activity 5:

Verify DR1 $\dfrac{d}{dx}\left[\sinh^{-1}(x)\right] = \dfrac{1}{\sqrt{1+x^2}}$ in two ways:

(a) using "brute force" *(i.e. differentiate the formula given in Theorem 6 (a))*

(b) using an "elegant technique" *(i.e. switch* $y = \sinh^{-1}(x)$ *into the equivalent equation* $\sinh(y) = x$ *and use implicit differentiation)*

Inverse Hyperbolic Functions

Activity 6:

Evaluate:

(a) $\dfrac{d}{dx}\left[\ln(\tanh^{-1}(x))\right]$

(b) $\displaystyle\int \frac{1}{1-x^2}\,dx$

(c) $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$