Section 13.6 The Binomial Theorem

Objectives
- Expanding $(a+b)^n$
- The Binomial Coefficients
- The Binomial Theorem
- Proof of the Binomial Theorem

• Expanding $(a+b)^n$

sum or difference of two terms

An expression with two terms is called **binomial**, such as $a+b$ or $4x+x^3$.

Compute:

- $(a+b)^1 = a + b$
- $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$
- $(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b5$

Know these!

There is a simple pattern that emerges in the coefficients (called **binomial coefficients**) of the expansion of $(a+b)^n$ (called the **binomial expansion**).

1. There are $n+1$ terms; the first being a^n and the last being b^n.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

2. As we move to the right, the exponents of a **decrease** by 1 and the exponents of b increase.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

3. The sum of the exponents of a and b in each term is n.

$n = 5$ 5 to 0   0 to 5

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

4. There's tons of "hidden pattens" in the binomial coefficients! Can you discover any?

Question What about higher powers? The above 3 patterns can help us write:

$$(a+b)^8 = a^8 + \mathbf{?}a^7b + \mathbf{?}a^6b^2 + \mathbf{?}a^5b^3 + \mathbf{?}a^4b^4 + \mathbf{?}a^3b^5 + \mathbf{?}a^2b^6 + \mathbf{?}ab^7 + b^8$$

Question How do we determine the coefficients?

Of course, we can just expand $(a+b)^8$ the LONG way. But, is there a **shortcut**?

• Pascal's Triangle

Yes, and it's thanks to a French mathematician named Blaise Pascal.

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
$(a+b)^6$	1 6 15 20 15 6 1

1
1 1
1 2 1
1 3 3 1

Key Property of Pascal's Triangle

Every entry (other than the 1 in the first row) is the SUM of the TWO entries DIAGONALLY ABOVE IT.

From the Key Property, it is easy to generate Pascal's triangle quickly.

Ex 1 Generating Pascal's Triangle Starting from

$n=0$ 1
$n=1$ 1   1

generate the first 9 rows (up to $n=8$) of Pascal's triangle.

$n=2$ 1   2   1
$n=3$ 1   3   3   1
$n=4$ 1   4   6   4   1
$n=5$ 1   5   10   10   5   1
$n=6$ 1   6   15   20   15   6   1
$n=7$ 1   7   21   35   35   21   7   1
$n=8$ 1   8   28   56   70   56   28   8   1

Ex 2 Explaining Pascal's Triangle Expand $(a+b)^8$ using Pascal's triangle.

ProTip You can check your work using https://www.wolframalpha.com/ or https://www.symbolab.com/

$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Ex 3 Explaining Pascal's Triangle Let's look at the sixth, seventh, and eighth rows of Pascal's triangle and compare them to $(a+b)^5$, $(a+b)^6$, and $(a+b)^7$:

$(a+b)^5$			1		5		10		10		5		1		
$(a+b)^6$		1		6		15		20		15		6		1	
$(a+b)^7$	1		7		21		35		35		21		7		1

To see why this holds, we look at the expansions of $(a+b)^5$ and $(a+b)^6$:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + \boxed{15a^2b^4} + 6ab^5 + b^6$$

$= (a+b)(a+b)^5 = (a+b)(\ldots\ldots\ldots)$

On the other hand, we arrive at the expansion of $(a+b)^6$ by multiplying $(a+b)$ and $(a+b)^5$, i.e.

$$(a+b)(a^5 + 5a^4b + 10a^3b^2 + \underline{10a^2b^3} + \underline{5ab^4} + b^5).$$

Notice that the circled term in the expansion of $(a+b)^6$ is obtained via this multiplication from the two circles above it.

Remarks on Pascal's Triangle Remarks on Pascal's Triangle.

- **GOOD:** it is easy to memorize how to generate! I would definitely practice until you can write it for $n=4$, $n=5$.
- **GOOD:** it is **recursive**, meaning you get a new row from the previous row.
- **BAD:** It is not practical for large values of n. Would you use it to find $(a+b)^{100}$? The fact that it is recursive means that we would need to find $(a+b)^n$ for $n = 0, 1, 2, 3, \ldots, 99$ first and then we can find $(a+b)^{100}$. Ouch.

• The Binomial Coefficients

We want a formula for the coefficients of a binomial expansion.
We review what a **factorial** is: the factorial of an integer n is the product of n with all successively decreasing values until 1. Namely,

$$\boxed{n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1}$$

$3! = 3\cdot 2\cdot 1 = 6$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

We also define $\boxed{0! = 1}$ out of convenience (and because it makes the formulas true for $n=0$).

Defn 1 Binomial Coefficients
Let $n, r \in \mathbb{N}$ with $r \le n$. A **binomial coefficient** is

$$\boxed{\binom{n}{r} = \frac{n!}{r!\,(n-r)!}}$$

"n choose r"

· Also called "combination" $_nC_r$
"n choose r"

· $\binom{n}{r} = {}_nC_r$ = the # of ways of selecting r things from a collection of n things without respect to order

Ex 4 Test Title Compute:
(a) $\binom{10}{4}$ (b) $\binom{10}{6}$ (c) $\binom{100}{97}$ (d) $\binom{100}{3}$

Do part (a) by hand and the rest with your calculator.

a) $\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10!}{4!\,6!} = \frac{10\cdot \cancel{9}^{3}\cdot\cancel{8}\cdot 7\cdot(\cancel{6!})}{(\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)(\cancel{6!})} = 10\cdot 3\cdot 7 = \boxed{210}$

b) $\binom{10}{6} = \boxed{210}$ c) $\binom{100}{97} = \boxed{161700}$ d) $\binom{100}{3} = \boxed{161700}$

Notice any patterns?
- What do you think is the pattern between $\binom{n}{r}$ and $\binom{n}{n-r}$ ⟶ always same? a conjecture!
- Is $\binom{n}{r}$ always an integer?

↳ they should be if they are truly a binomial coeff!

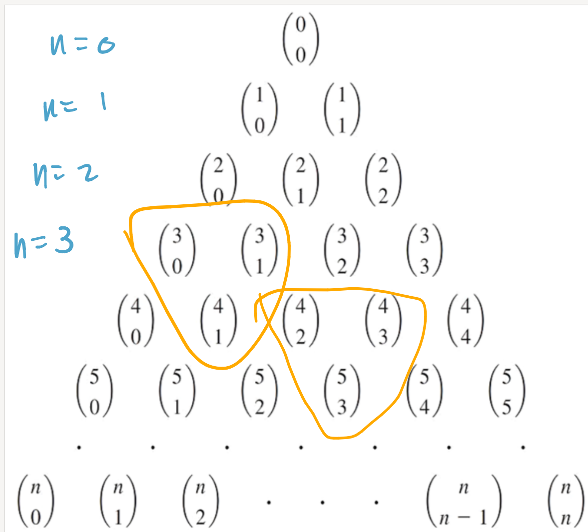
There's a connection between the binomial coefficients and the binomial expansions (duh! what's in a name, anyways?):

$$\binom{5}{0}=1 \quad \binom{5}{1}=5 \quad \binom{5}{2}=10 \quad \binom{5}{3}=10 \quad \binom{5}{4}=5 \quad \binom{5}{5}=1$$

and compare that to:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

So, because the coefficients of Pascal's triangle correspond to the coefficients of a binomial expansion, we get:



Patterns

- rows: top # always same ↳ n
- rows: bottom # it increases from 0 to n

Of course, we need to prove this (exercise using mathematical induction)!

Theorem 1 Key Property of Binomial Coefficients

Let $n, r \in \mathbb{N}$ with $r \leq n$. Then

$$\boxed{\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}}$$

Notice that the two terms on the left-hand side of this equation are adjacent entries in the rth row of Pascal?s triangle and the term on the right-hand side is the entry diagonally below them, in the $(r+1)$st row.

Thus this equation is a restatement of the key property of Pascal's triangle in terms of the binomial coefficients.

• The Binomial Theorem

This was originally discovered and proved by Newton (yes, THAT Newton)–one of the inventors of calculus.

Theorem 2 The Binomial Theorem

Let $n, r \in \mathbb{N}$ with $r \leq n$. A **binomial coefficient** is

$$\boxed{(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n}$$

Remarks Some helpful **ProTips**.

- Keep in mind the 3 properties for binomial expansion discussed earlier.
- The top numbers in each binomial coefficient is n.
- The bottom numbers in each binomial coefficient are increasing.
- The bottom numbers in each binomial coefficient match the exponent of b for that term.
- **Finding a specific term**: If we want the term with b^r, then using the previous remark, we see that it is $\boxed{\binom{n}{r}a^{n-r}b^r}$ The exponent of a is $n-r$ because it needs to sum to n.
- **Finding a specific term**: If we want the term with a^r, then using the previous remark, we see that it is $\boxed{\binom{n}{n-r}a^rb^{n-r}}$ using the symmetry of Pascal's triangle.

Ex 5 Binomial Expansion Expand $(x+y)^4$ using the Binomial Theorem.

Sol $(x+y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4$

$= 1\cdot x^4\cdot y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1\cdot x^0y^4$

$= \boxed{x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}$

check Pascal's Triangle

0					1					
1				1		1				
2			1		2		1			
3		1		3		3		1		
4	1		4		6		4		1	

$\binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{4!}{1\cdot 4!} = 1$

6R

Ex 6 Binomial Expansion Expand $(x-2y)^6$ using the Binomial Theorem.

Sol $(\overset{a}{\boxed{x}}\overset{b}{\boxed{-2y}})^6 = \overset{1}{\binom{6}{0}}x^6(-2y)^0 + \overset{6}{\binom{6}{1}}x^5(-2y)^1 + \overset{15}{\binom{6}{2}}x^4(-2y)^2 + \overset{20}{\binom{6}{3}}x^3(-2y)^3 + \overset{15}{\binom{6}{4}}x^2(-2y)^4$

$+ \overset{6}{\binom{6}{5}}x^1(-2y)^5 + \overset{1}{\binom{6}{6}}x^0(-2y)^6$

$= 1\cdot x^6\cdot 1 + 6\cdot x^5\cdot(-2)y + 15\cdot x^4\cdot(4)y^2 + 20\cdot x^3\cdot(-8)y^3 + 15\cdot x^2(16)y^4$

$+ 6\cdot x(-32)y^5 + 1\cdot 1\cdot(64)y^6$

$= \boxed{x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6}$

Ex 7 Finding a single term Find the term in the expansion of $(2x+y)^{20}$ that contains x^5.

Hint: Use $\binom{n}{n-r}(2x)^r(y)^{n-r}$.

$(2x+y)^{20} = \binom{20}{0}(2x)^{20}(y)^0 + \binom{20}{1}(2x)^{19}(y)^1 + \cdots + \boxed{\binom{20}{15}(2x)^5(y)^{15}} + \cdots$

$\binom{20}{15}(2x)^5(y)^{15} = 15504\cdot 32\cdot x^5\cdot y^{15}$

$= \boxed{496128x^5y^{15}}$

Ex 8 Binomial Expansion Expand $(4-\sqrt{x})^5$ using the Binomial Theorem.

Sol $$(4-\sqrt{x})^5 = \overset{1}{\binom{5}{0}}4^5(-\sqrt{x})^0 + \overset{5}{\binom{5}{1}}4^4(-\sqrt{x})^1 + \overset{10}{\binom{5}{2}}4^3(-\sqrt{x})^2 + \overset{10}{\binom{5}{3}}4^2(-\sqrt{x})^3 + \overset{5}{\binom{5}{4}}4^1(-\sqrt{x})^4 + \overset{1}{\binom{5}{5}}4^0(-\sqrt{x})^5$$

$$= (1\cdot 1024\cdot 1) + (5\cdot 256\cdot(-\sqrt{x})) + (10\cdot 64\cdot x) + (10\cdot 16\cdot -(\sqrt{x})^3) + (5\cdot 4\cdot x^2) + (1\cdot 1\cdot -(\sqrt{x})^5)$$

$$= 1024 - 1280\sqrt{x} + 640x - 160(\sqrt{x})^3 + 20x^2 - (\sqrt{x})^5$$

$$= \boxed{1024 - 1280x^{1/2} + 640x - 160x^{3/2} + 20x^2 - x^{5/2}}$$

GTG!

Ex 9 Binomial Expansion Expand $\left(x+\dfrac{1}{x}\right)^7$ using the Binomial Theorem.

Sol $$\left(x+\frac{1}{x}\right)^7 = \overset{1}{\binom{7}{0}}x^7\left(\frac{1}{x}\right)^0 + \overset{7}{\binom{7}{1}}x^6\left(\frac{1}{x}\right)^1 + \overset{21}{\binom{7}{2}}x^5\left(\frac{1}{x}\right)^2 + \overset{35}{\binom{7}{3}}x^4\left(\frac{1}{x}\right)^3 + \overset{35}{\binom{7}{4}}x^3\left(\frac{1}{x}\right)^4 + \overset{21}{\binom{7}{5}}x^2\left(\frac{1}{x}\right)^5$$

$$+ \overset{7}{\binom{7}{6}}x^1\left(\frac{1}{x}\right)^6 + \overset{1}{\binom{7}{7}}x^0\left(\frac{1}{x}\right)^7$$

$$= (1\cdot x^7\cdot 1) + \left(7\cdot \overset{5}{\cancel{x^6}}\cdot \cancel{\frac{1}{x}}\right) + \left(21\cdot \overset{3}{\cancel{x^5}}\cdot \cancel{\frac{1}{x^2}}\right) + \left(35\cdot \overset{1}{\cancel{x^4}}\cdot \cancel{\frac{1}{x^3}}\right) + \left(35\cdot \cancel{x^3}\cdot \frac{1}{\cancel{x^4}_{1}}\right) + \left(21\cdot \cancel{x^2}\cdot \frac{1}{\cancel{x^5}_{3}}\right)$$

$$+ \left(7\cdot \cancel{x}\cdot \frac{1}{\cancel{x^6}_{5}}\right) + \left(1\cdot 1\cdot \frac{1}{x^7}\right)$$

$$= \boxed{x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}}$$

• Proof of the Binomial Theorem

We now prove the Binomial Theorem using The Principle of Mathematical Induction.

Proof: (By Induction)

Let $P(n)$: $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$

Base Step:

For $n=1$, $P(1)$ says: $RHS = \binom{1}{0}a^1 b^0 + \binom{1}{1}a^{1-1}b^1 = 1\cdot a\cdot 1 + 1\cdot a^0\cdot b = a+b = (a+b)^1 = LHS$ ✓

Inductive Step:

(IH) Assume $P(k)$ is true: $(a+b)^k = \binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b^1 + \binom{k}{2}a^{k-2}b^2 + \dots + \binom{k}{k-1}a^1 b^{k-1} + \binom{k}{k}a^0 b^k \quad (*)$

NTS $P(k+1)$ true: $(a+b)^{k+1} = \binom{k+1}{0}a^{k+1}b^0 + \binom{k+1}{1}a^k b + \dots + \binom{k+1}{k}a b^k + \binom{k+1}{k+1}a^0 b^{k+1} \quad (**)$

We have:

$$(a+b)^{k+1} = (a+b)(a+b)^k$$

$$= (a+b)\left[\binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b^1 + \dots + \binom{k}{k-1}a^1 b^{k-1} + \binom{k}{k}a^0 b^k\right] \quad \text{(by } (*) \text{) (IH)}$$

$$= \left.\begin{array}{l} \binom{k}{0}a^{k+1}b^0 + \binom{k}{1}a^k b^1 + \dots + \binom{k}{k-1}a^2 b^{k-1} + \binom{k}{k}a b^k \\ \binom{k}{0}a^k b^1 + \binom{k}{1}a^{k-1}b^2 + \dots + \binom{k}{k-1}a^1 b^k + \binom{k}{k}a^0 b^{k+1} \end{array}\right\} \text{(foil)}$$

$$\left.\begin{array}{l} = \binom{k}{0}a^{k+1}b^0 + \left[\binom{k}{0}+\binom{k}{1}\right]a^k b^1 + \left[\binom{k}{1}+\binom{k}{2}\right]a^{k-1}b^2 \\ \quad + \dots + \left[\binom{k}{k-1}+\binom{k}{k}\right]a^1 b^k + \binom{k}{k}a^0 b^{k+1} \end{array}\right\} \text{(group like terms)}$$

$$\left.\begin{array}{l} = \binom{k+1}{0}a^{k+1}\cdot b^0 + \left[\binom{k}{0}+\binom{k}{1}\right]a^k b^1 + \left[\binom{k}{1}+\binom{k}{2}\right]a^{k-1}b^2 + \\ \quad + \dots + \left[\binom{k}{k-1}+\binom{k}{k}\right]a^1 b^k + \binom{k+1}{k+1}a^0 b^{k+1} \end{array}\right\} \begin{array}{l} \binom{k}{0}=1 \\ \& \binom{k+1}{0}=1 \\ \binom{k}{k}=1 \ \& \binom{k+1}{k+1}=1 \end{array}$$

Now we use the $\boxed{\text{KEY PROPERTY}}$ $\binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$

$$= \binom{k+1}{0}a^{k+1}b^0 + \binom{k+1}{1}a^k b^1 + \binom{k+1}{2}a^{k-1}b^2 + \dots + \binom{k+1}{k}a^1 b^k + \binom{k+1}{k+1}a^0 b^{k+1}.$$

"So, $P(k+1)$ follows from $P(k)$."
"Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$." □