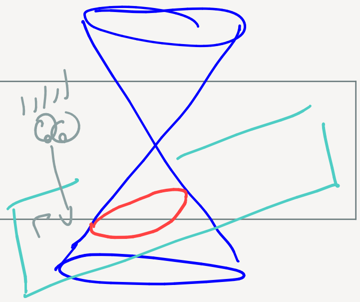
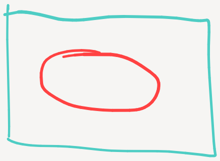
Section 12.2 Ellipses

Objectives
- Geometric Definition of an Ellipse
- Equations and Graphs of Ellipses
- Eccentricity of an Ellipse





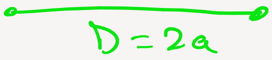
• Geometric Definition of a Parabola

The next conic section we study are "squished circles" called ellipses.

string length

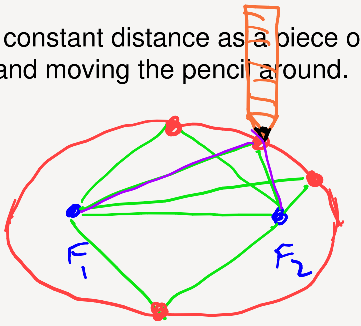
Defn 1 Geometric Definition of an Ellipse

> An **ellipse** is the set of all points in the plane the **sum** of whose distances from two fixed points F_1 and F_2 is a constant.
>
> The two fixed points F_1 and F_2 are called the **foci** (plural of focus) of the ellipse.



Try to draw a picture based purely on the above definition.

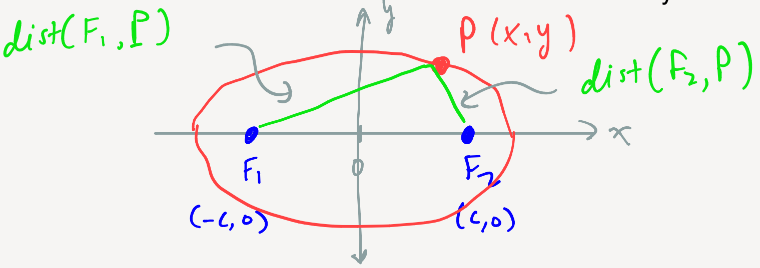
A great way is to think about the constant distance as a piece of string and attach it to the two foci then the ellipse is the curve that develops from pulling the string and moving the pencil around.



• Derivation of the equation of an ellipse

Starting only from the geometric definition, our goal to see what equation the set of points (x, y) that lie on an ellipse must satisfy.

We make some choices:
- Assume the two foci are on the x-axis.
- Position the origin so that one focus F_1 is on the negative side and the other, F_2, is on the positive side of the x-axis.
- Position the y-axis so that it is half-way between F_1 and F_2.
- By these assumptions we have: $F_1 = (-c, 0)$ and $F_2 = (0, c)$ for some constant $c > 0$.
- **Notation** The sum of the distances that's constant will be denoted by $D = 2a$. "string dist"



Next:
- Let $P = (x, y)$ denote a point on the ellipse.

- For any point $P = (x, y)$ that is on the ellipse, the geometric definition says that

$$\text{dist}(P, F_1) + \text{dist}(P, F_2) = D = 2a$$

Use this information to arrive at the equation of a parabola.

$$\text{dist}\big((x,y), (-c,0)\big) + \text{dist}\big((x,y), (c,0)\big) = 2a$$

Use this information to arrive at the equation of a parabola.

Some tips:
- Eliminate one of the radicals first, then do the second one.
- Why is $a > c$? Start with $2a > 2c$.
- We define $b^2 = a^2 - c^2$ and define $b > 0$ to be the positive root.
- Notice $b < a$.

$$\text{dist}((x,y),(-c,0)) + \text{dist}((x,y),(c,0)) = 2a$$

$$\sqrt{(x+c)^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} = 2a$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2 = \left(2a - \sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + \left[(x-c)^2+y^2\right]$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2 - \left((x+c)^2+y^2\right) + (x-c)^2+y^2$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2 - (x^2+2xc+c^2) \cancel{-y^2} + (x^2-2xc+c^2) \cancel{+y^2}$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2 \cancel{-x^2} - 2xc \cancel{-c^2} \cancel{+x^2} - 2xc \cancel{+c^2}$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2 - 4xc$$

$$\left(a\sqrt{(x-c)^2+y^2}\right)^2 = (a^2 - xc)^2$$

$$a^2\left((x-c)^2+y^2\right) = a^4 - 2a^2xc + x^2c^2$$

$$a^2\underline{(x-c)^2} + a^2y^2 = a^4 - 2a^2xc + x^2c^2$$

$$a^2(x^2 - 2xc + c^2) + a^2y^2 = a^4 - 2a^2xc + x^2c^2$$

$$a^2x^2 \boxed{\cancel{-2a^2xc}} + a^2c^2 + a^2y^2 = a^4 \boxed{\cancel{-2a^2xc}} + x^2c^2$$

$$a^2x^2 - x^2c^2 + a^2y^2 = a^4 - a^2c^2$$

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$$

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$$

Divide by a^2 on both sides:

$$\frac{a^2-c^2}{a^2}\cdot x^2 + y^2 = a^2-c^2$$

Divide by a^2-c^2 on both sides:

$$\frac{1}{a^2}\cdot x^2 + \frac{1}{a^2-c^2}\cdot y^2 = 1$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1}$$

$$b^2 = a^2 - c^2$$

$$\begin{cases} b = \sqrt{a^2-c^2} \\ b > 0 \end{cases}$$

Defn 2

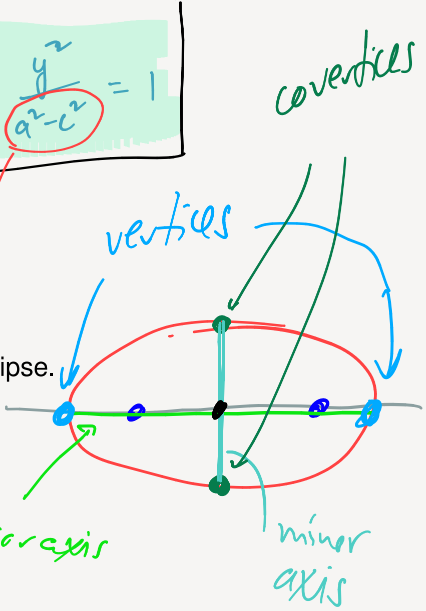
The pairs of points on the ellipse that are farthest apart are called the **vertices** of the ellipse.

The **major axis** is defined to be the axis which contains the vertices. It's length is clearly $2a$.

The pairs of points on the ellipse that are closest together are called the **co-vertices** of the ellipse.

The **minor axis** is defined to be the axis which contains the co-vertices.

The **center** of an ellipse is defined to be where the major and minor axis intersect.



Notice that the major axis is always greater than the minor axis for an ellipse.

• Equations and Graphs of an Ellipse

DESMOS: https://www.desmos.com/calculator/tbqesfvecz

Theorem 1 Equation of an Ellipse

Horizontal Major Axis

If an ellipse has foci along the x-axis, at $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ with $c > 0$, and the sum of the distances is $D = 2a > 0$, then

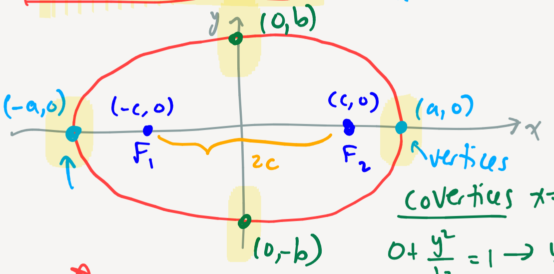
- The equation of the ellipse in **graphing form** is:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

vertices $y = 0$

$\frac{x^2}{a^2} + 0 = 1 \rightarrow x^2 = a^2$

$x = \pm a$



covertices $x = 0$

$0 + \frac{y^2}{b^2} = 1 \rightarrow y^2 = b^2 \rightarrow y = \pm b$

- We define $\boxed{b^2 = a^2 - c^2}$ and take $b > 0$. Note that $b < a$.
- The foci are: $(c, 0)$ & $(-c, 0)$
- The major axis is: x-axis length is $2a = D$
- The minor axis is: y-axis length is $2b$
- The vertices are: $(\pm a, 0)$
- The co-vertices are: $(0, \pm b)$

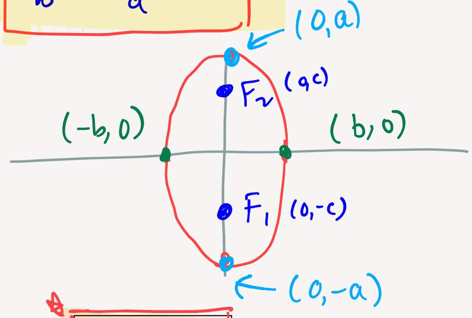
Vertical Major Axis

If an ellipse has foci along the y-axis, at $F_1 = (0, -c)$ and $F_2 = (0, c)$ with $c > 0$, and the sum of the distances is $D = 2a > 0$, then

- The equation of the ellipse in **graphing form** is:

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

$a > b$



- We define $\boxed{b^2 = a^2 - c^2}$ and take $b > 0$. Note that $\boxed{b < a.}$
- The foci are: $(0, c)$ & $(0, -c)$
- The major axis is: y-axis length $= 2a$
- The minor axis is: x-axis length $= 2b$
- The vertices are: $(0, \pm a)$
- The co-vertices are: $(\pm b, 0)$

How to Find Vertices and Co-Vertices To find vertices and co-vertices is easy:

- Vertices: If the major axis is the x-axis then plug-in $y = 0$ and solve for x: $\boxed{(\pm a, 0)}$.
- Co-Vertices: If the minor axis is the y-axis then plug-in $x = 0$ and solve for y: $\boxed{(0, \pm b)}$.

Ex 1 What we call graphing form is also called **standard form**.

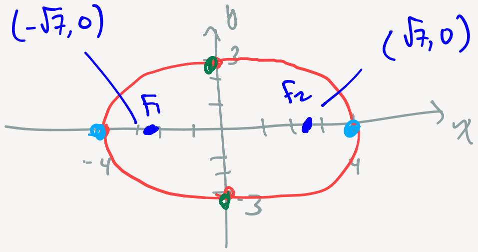
(a) Write the equation of the ellipse in graphing form: $9x^2 + 16y^2 = 144$, and sketch its graph.

(b) Find the vertices, co-vertices and foci.

$$\frac{9x^2 + 16y^2}{9} = \frac{144}{9}$$

$$\frac{x^2 + \frac{16}{9}y^2}{16} = \frac{16}{16}$$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{9} = 1}$$



$a = ?$ ~~$a = 16$~~ $a = \sqrt{16} = 4$

$b = ?$ ~~$b = 9$~~ $b = \sqrt{9} = 3$

Vertices: $(-4, 0)$ & $(4, 0)$

covertices: $(0, 3)$ & $(0, -3)$

Foci $(\pm c, 0)$

$b^2 = a^2 - c^2$

$c^2 = a^2 - b^2$

$c^2 = 16 - 9 =$

$c = \sqrt{7} \simeq 2.6$

TIP $4=\frac{1}{1/4}$

Ex 2 What we call graphing form is also called **standard form**.

(a) Write the equation of the ellipse in graphing form: $25x^2+4y^2=1$, and sketch its graph.

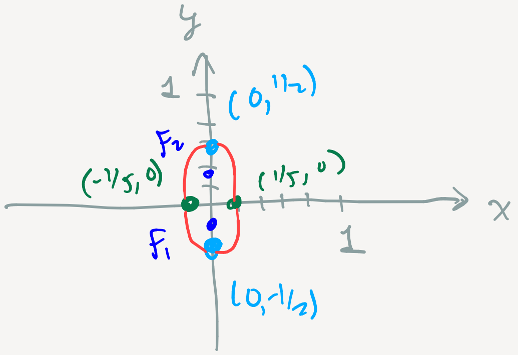
(b) Find the vertices, co-vertices and foci.

a) $25x^2+4y^2=1$

$$\frac{x^2}{\frac{1}{25}}+\frac{y^2}{\frac{1}{4}}=1$$

$a=\sqrt{1/4}=1/2$

$b=\sqrt{1/25}=1/5$



Foci $\boxed{\left(0,\pm\frac{\sqrt{21}}{10}\right)}$

Vertices $\boxed{(0,\pm 1/2)}$

Covertices $\boxed{(\pm 1/5, 0)}$

Foci $(0,\pm c)$

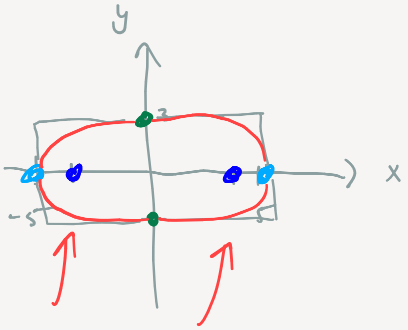
$$\begin{cases} b^2=a^2-c^2 \\ c^2=a^2-b^2 \end{cases}$$

$$c^2=\frac{25}{25}\cdot\frac{1}{4}-\frac{1}{25}\cdot\frac{4}{4}$$

$$=\frac{25-4}{100}=\frac{21}{100}$$

$$c=\sqrt{\frac{21}{100}}=\frac{\sqrt{21}}{10}$$

Ex 3 The vertices of an ellipse are $(\overset{a}{\pm 5},0)$ and the foci are $(\overset{c}{\pm 4},0)$. Find the equation of the ellipse, and sketch its graph.



Foci x-axis

a & c

EQ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$b^2=a^2-c^2$

$b^2=25-16=9$

$b=\sqrt{9}=3$

$$\frac{x^2}{25}+\frac{y^2}{b^2}=1$$

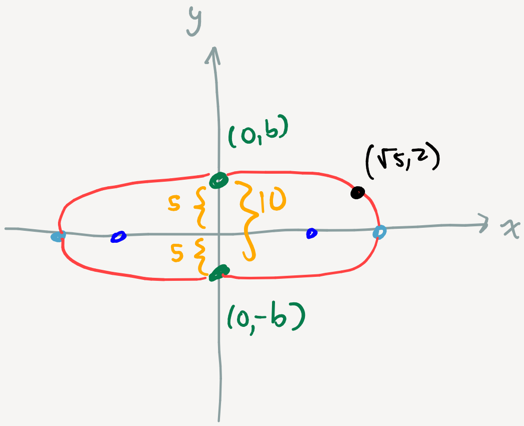
$$\boxed{\frac{x^2}{25}+\frac{y^2}{9}=1}$$

Ex 4 The length of the minor axis is 10, the foci are on the x-axis (major axis), and the ellipse passes through the point $(\overset{x}{\sqrt{5}},\overset{y}{2})$. Find the equation of the ellipse.

↳ $b=5$

EQ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ✓ OR ~~$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$~~

Length of minor axis $=2b=10 \Rightarrow b=5$



$$\frac{x^2}{a^2}+\frac{y^2}{25}=1$$

$$\frac{(\sqrt{5})^2}{a^2}+\frac{(2)^2}{25}=1$$

$$\frac{5}{a^2}+\frac{4}{25}=1$$

$$\frac{5}{a^2}=1-\frac{4}{25}$$

$$\frac{5}{a^2}\times\frac{21}{25}$$

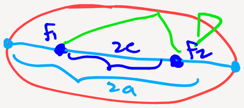
$$\boxed{a^2=\frac{125}{21}}$$

$$a=\sqrt{\frac{125}{21}}=\frac{5\sqrt{5}}{\sqrt{21}}$$

EQ: $\frac{x^2}{\left(\frac{125}{21}\right)}+\frac{y^2}{25}=1$

$$\boxed{\frac{21x^2}{125}+\frac{y^2}{25}=1}$$

• Eccentricity of an Ellipse



$0 < e = \frac{c}{a} < 1$

What gives an ellipse its shape? If we start with a circle and squish it we get an ellipse. But how are the major and minor axes determined by the relationship between the foci and the "string constant" $D = 2a$?

Geometrically, this is determined by the values of $2a$ (length of major axis) and $2c$ (distance between foci). We could, then, consider the number $2a - 2c$, or perhaps instead look at $b = \sqrt{a^2 - c^2}$.

Turns out the best number to measure the "deviation from a circle" is the ratio between a and c. This is called the "eccentricity."

Defn 3

The **eccentricity** of an ellipse is the ratio of c to a and it is denoted by e. Thus, $\boxed{e = \frac{c}{a}}$

Recall that $b^2 = a^2 - c^2$. Then $c = \sqrt{a^2 - b^2}$ and $e = \frac{\sqrt{a^2-b^2}}{a}$.

Theorem 2 Eccentricity

If e is the eccentricity of an ellipse then

- $\boxed{e = \frac{\sqrt{a^2 - b^2}}{a}}$.
  Recall that $b^2 = a^2 - c^2$, then $c = \sqrt{a^2 - b^2}$.
- We always have that $\boxed{0 < e < 1}$.
- As e approaches 1, then the ellipse becomes more elongated, i.e. the distance between the vertices and the foci increases.
  → along major axis
- As e approaches 0, then the ellipse becomes more circular, i.e. the minor axis length increases to the major axis length.

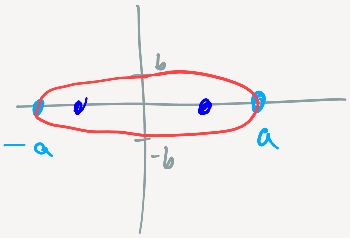
↳ = 0 it is a circle!

$e \approx 1$

$\frac{\sqrt{a^2-b^2}}{a} \approx 1$

$\sqrt{a^2-b^2} \approx a$

↳ $b \approx 0$

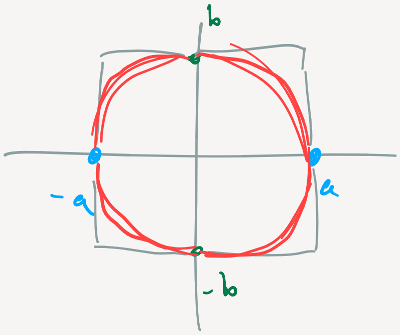


$e \approx 0$

$\frac{\sqrt{a^2-b^2}}{a} \approx 0$

$\sqrt{a^2-b^2} \approx 0$

$a \approx b$



Ex 5 Find the equation of the ellipse with foci $(0, \pm 8)$ and eccentricity $e = 4/5$, and sketch its graph.

Given: Foci $(0, \pm c)$ ⟶ major axis is y-axis $\boxed{c = 8}$

EQ $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$b^2 = a^2 - c^2$

$e = \frac{c}{a}$

$\frac{4}{5} = \frac{8}{a}$

$\boxed{a = 10}$

$b^2 = a^2 - c^2$

$= 100 - 64$

$= 36$

$b = 6$

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$\boxed{\frac{x^2}{36} + \frac{y^2}{100} = 1}$