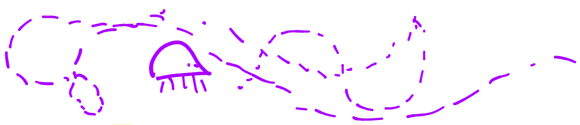
Section 8.4 - Plane Curves and Parametric Equations

Objectives:
- **Plane Curves and Parametric Equations**
- **Eliminating the Parameter**
- **Finding parametric equations for a curve**
- **Using Desmos to graph parametric equations**
- **Polar Equations in Parametric Form**

• Plane Curves and Parametric Equations

Imagine trying to write down the path of a bug crawling on a table. It might look like this:

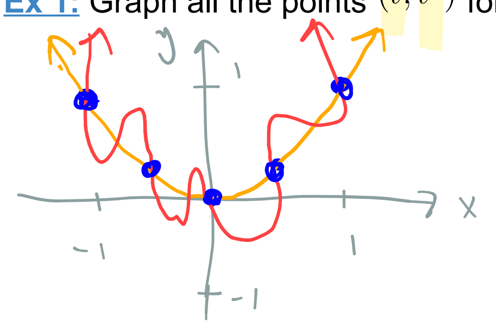


(paths / plane curve)

Using a single function is not going to work since it will fail the vertical line test. We've studied polar coordinates and functions in terms of polar coordinates which allowed for some interesting paths a bug might make. However, now we study this in more generality. But, first, a simple example:

Ex 1: Graph all the points (t, t^2) for $t \in [-1, 1]$.

(x, x^2)
$\uparrow$
y
$y = x^2$



t	x	y
$t = -1$	-1	1
$t = -1/2$	$-1/2$	$1/4$
$t = 0$	0	0
$t = 1/2$	$1/2$	$1/4$
$t = 1$	1	1

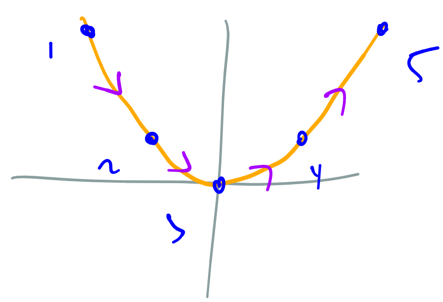
$(-1, 1)$
etc
$(1, 1)$

Desmos demos: https://www.desmos.com/calculator/zbhfylhjnw

Definition(s): Let $f(t)$ and $g(t)$ be two functions defined on an interval I. A **plane curve** is defined as the set of all points $(f(t), g(t))$ in the plane. Then, the x-coordinates must be $x = f(t)$ and the y-coordinates must be $y = g(t)$. We call these the **parametric equations** for the plane curve with **parameter** t.

Parametric Equations: $\boxed{\begin{cases} x = f(t) \\ y = g(t) \end{cases}}$

Ex1

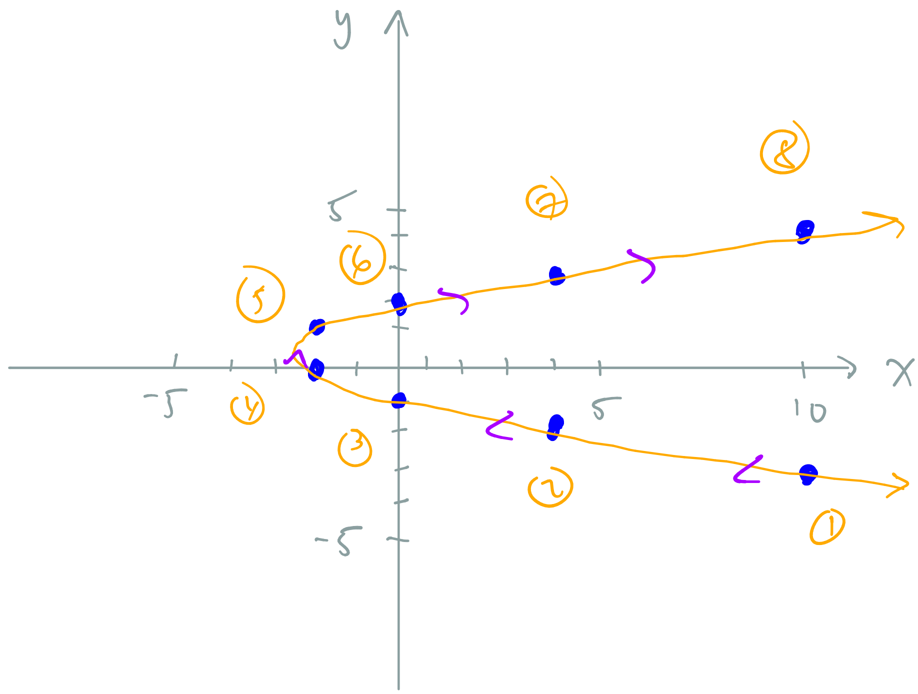


Note: we draw arrows to indicate the direction of motion.

Ex 2: The paths of comets follow parabolic paths. If the parametric equation for the path of Basilio's Comet are given by: $\begin{cases} x &= t^2 - 3t \\ y &= t - 1 \end{cases}$ for $t \in \mathbb{R}$. sketch graph plane curve.

Tip: plot points for $t = -2, -1, 0, 1, 2, 3, 4, 5$. Make a table with t, x, y.

	t	x	y
1	-2	10	-3
2	-1	4	-2
3	0	0	-1
4	1	-2	0
5	2	-2	1
6	3	0	2
7	4	4	3
8	5	10	4



Ex 3: Replace t by $-t$ in the equations of Example 2. Write the new parametric equations. Use Desmos to plot the plane curve. What do you observe?

moves along same path but in reverse direction (backwards)

Ex 4: Replace t by $2t$ in the equations of Example 2. Write the new parametric equations. Use Desmos to plot the plane curve. What do you observe?

moves along same path but twice as fast!

Important Note: the parametric equations contain more information than simply the shape of the plane curve. It also indicated the speed and the direction of an object traveling along the path of the plane curve.

- ## Eliminating the Parameter

It is time consuming to plot a lot of points and guess the shape of the plane curve. A technique for writing the equation of the shape of the curve is called **eliminating the parameter**. To do this, solve one equation for t, then substitute into the other equation.

Ex 5: Use the elimination of parameter technique to find an equation for Basilio's Comet in Example 2.

$$\begin{cases} x = t^2 - 3t \\ y = t - 1 \longrightarrow t = y+1 \end{cases}$$

$x = (y+1)^2 - 3(y+1)$

$x = (y+1)^2 - 3(y+1)$

$x = y^2 + 2y + 1 - 3y - 3$

$\boxed{x = y^2 - y - 2} \longrightarrow$ parabola opening

$x = (+)y^2$

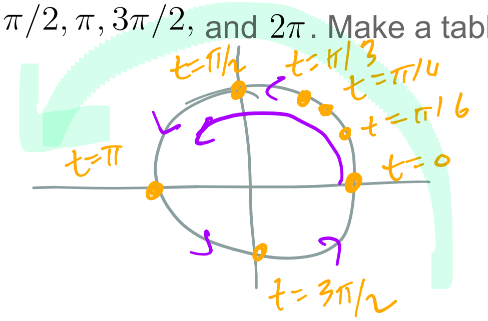
- ## Finding Parametric EQs for a curve
- ### Circles

We know the equation of the unit circle is: $x^2 + y^2 = 1$. Can we describe the circle as a path using parametric equations? Yes! The trigonometric functions we learned in Ch 6 do exactly this!

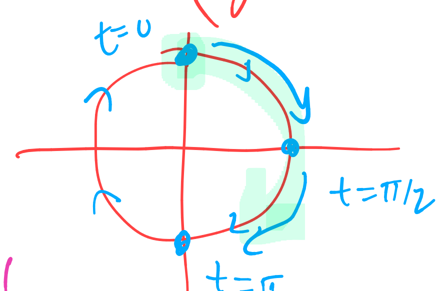
ie Definitions of sine & cosine

Ex 6: Sketch the plane curve determined by $\begin{cases} x &= \cos(t) \\ y &= \sin(t) \end{cases}$ for $t \in [0, 2\pi]$.

Tip: plot points for $t = 0, \pi/2, \pi, 3\pi/2,$ and 2π. Make a table with t, x, y.



Bonus $\begin{cases} x = \sin(t) \\ y = \cos(t) \end{cases}$



Wrong way to eliminate the parameter:

$$\begin{cases} x = \cos t \longrightarrow t = \cos^{-1}(x) \\ y = \sin t \end{cases}$$

$\boxed{y = \sin(\cos^{-1}(x))}$ not useful!

Instead of solving for t as above, we can use Pythagorean ID to recover the equation for the circle.

square EQ

$x^2 = \cos^2 t \Rightarrow x^2 + y^2 = \underbrace{\cos^2 t + \sin^2 t}_{=1 \text{ always}} \Rightarrow \boxed{x^2 + y^2 = 1}$

$y^2 = \sin^2 t$

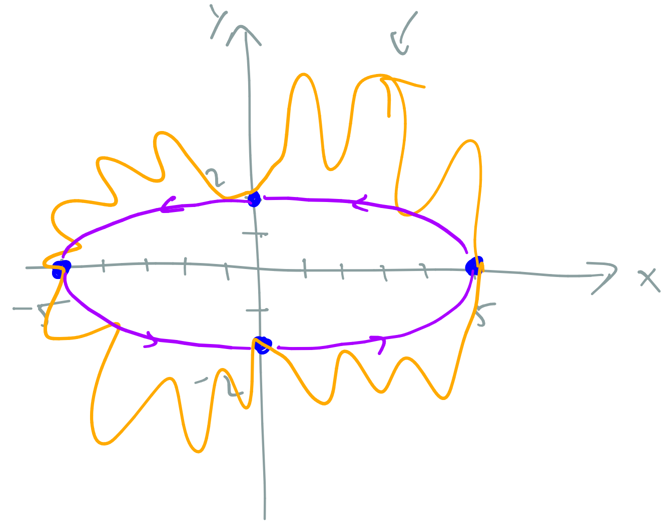
Ex 7: Sketch the plane curve determined by $\begin{cases} x &= 5\cos(t) \\ y &= 2\sin(t) \end{cases}$ for $t \in [0, 2\pi]$.

Tip: plot points for $t = 0, \pi/2, \pi, 3\pi/2,$ and 2π. Make a table with t, x, y.

t	x	y
0	$5\cos(0) = 5$	0
$\pi/2$	0	2
π	-5	0
$3\pi/2$	0	-2
2π	5	0

Ellipse

one possible path



Theorem: Circles

The parametric equations for a circle centered at (a, b) with radius r moving in counter-clockwise direction are: $\boxed{\begin{cases} x &= a + r\cos(t) \\ y &= b + r\sin(t) \end{cases}}$ for $t \in [0, 2\pi]$.

$x - a = r\cos(t)$

$y - b = r\sin(t)$

$\left(\frac{x-a}{r}\right)^2 = (\cos t)^2$

$\left(\frac{y-b}{r}\right)^2 = (\sin t)^2$

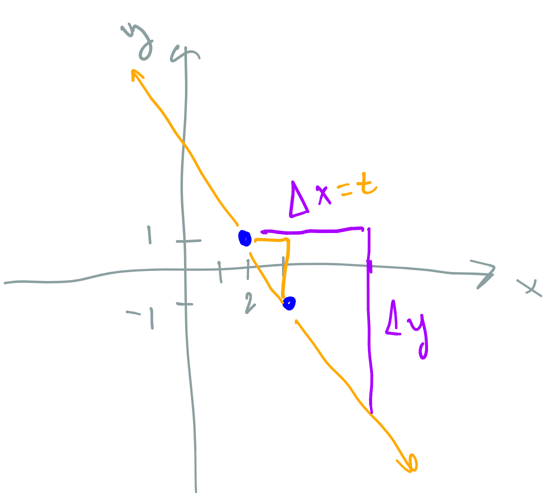
$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1$

$\boxed{(x-a)^2 + (y-b)^2 = r^2}$

- **Lines**

Let's find parametric equations for lines next.

Ex 8: Find the parametric equations for a line passing through the point $(2, 1)$ with slope -2. Check your answer is correct by eliminating the parameters.



$t=0$ $\begin{cases} x = 2 \\ y = 1 \end{cases}$ $\quad$ $t=1$ $\begin{cases} x = 3 = 2+1 \\ y = -1 = 1-2(1) \end{cases}$

$t = \#$ $\begin{cases} x = 2 + \# \\ y = 1 + (-2)\# \end{cases}$

Parametric EQ

$\boxed{\begin{cases} x = 2 + t \\ y = 1 + (-2)t \end{cases}}$

Check

$x - 2 = t$

$y = 1 + -2(x-2)$

$y - 1 = -2(x-2)$

EQ line in point slope

$y - y_1 = m(x - x_1)$

Slope $= \frac{\Delta y}{\Delta x} = -2 \rightarrow$ $\Delta y = -2\Delta x = -2t$

Theorem: Lines

The parametric equations for a line passing through the point (a,b) with slope m are:

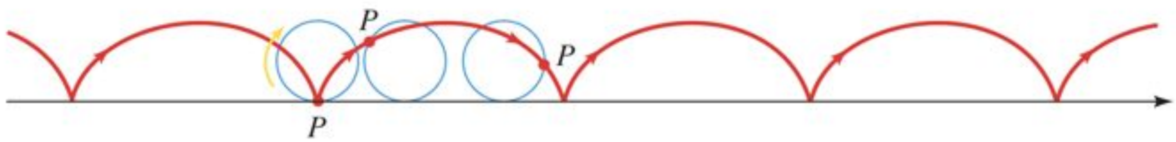
$$\begin{cases} x &= a + t \\ y &= b + mt \end{cases} \text{ for } t \in (-\infty, \infty).$$

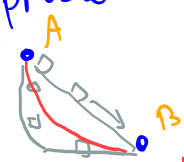
Excercise check EQ line using elimination of parameter.

- **Cycloid**

A circle rolls along a straight line. The curve traced out by a fixed point P on the circumference of the circle is called a cycloid.

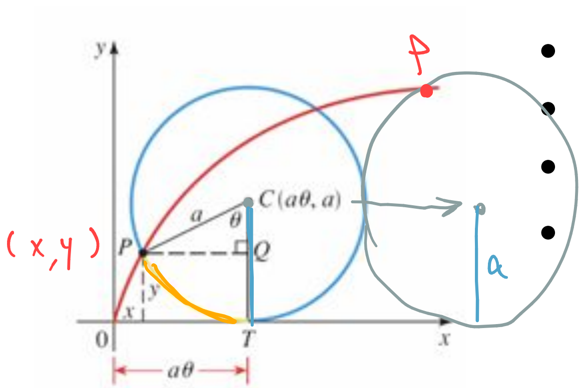
Cycloids solve Brachistochrone problem



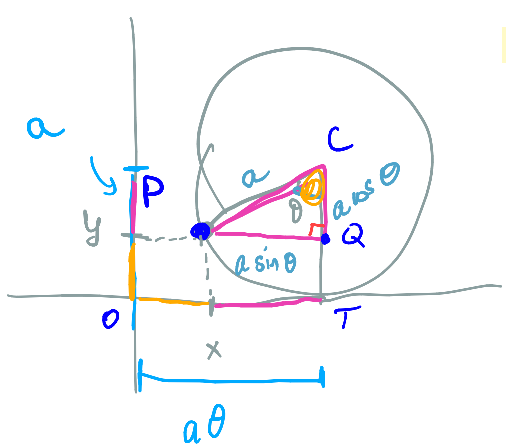


upside down cycloid

Ex 9: Find the parametric equations for the cycloid using θ as the parameter.
Hint: The key is in the following diagram: a = radius of circle



- Distance dist(O,T) = $a \cdot \theta$
- Arc length arc(PT) = $a \cdot \theta$ arclength
- Center of circle = $(x, y) = (a \cdot \theta, a)$
- Coordinates of P = (x, y):
  - x = dist(O,T) - dist(P,Q) = $a\theta - a\sin\theta$
  - y = dist(C,T) - dist(C,Q) = $a - a\cos\theta$



Parametric EQ of Cycloid radius a using parameter θ

$$\boxed{\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}}$$

• Using Desmos to graph Parametric Equations

Desmos can plot parametric equations easily as shown in the demos linked above. You must put it in the form $(f(t), g(t))$.

Ex 10: Plot the following in Desmos:

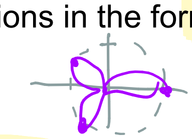
(a) $\begin{cases} x &= \sin(2t) \\ y &= 3\cos(t) \end{cases}$ (b) $\begin{cases} x &= 2\sin(t) \\ y &= \cos(3t) \end{cases}$ (c) $\begin{cases} x &= \sin(2t) \\ y &= \cos(3t) \end{cases}$

(a) $-1 \le x \le 1$, $-3 \le y \le 3$ (b) $-2 \le x \le 2$, $-1 \le y \le 1$ (c) $-1 \le x \le 1$, $-1 \le y \le 1$

For each example, determine the "viewing rectangle". That is, find intervals $a \le x \le b$ and $c \le y \le d$ for which the points (x, y) of the plane curve must lie in. These are examples of **Lissajous figures**.

• Polar Equations in Parametric Form

In section 8.1 we studied polar coordinates, then in section 8.2 we studied polar equations and their graphs. We observed that the graphs didn't need to pass the vertical line test but were graphs of polar functions in the form $r = f(\theta)$. For example, we have the rose with 3 petals: $r = \cos(3\theta)$.



$n = 3$ odd
3 petals

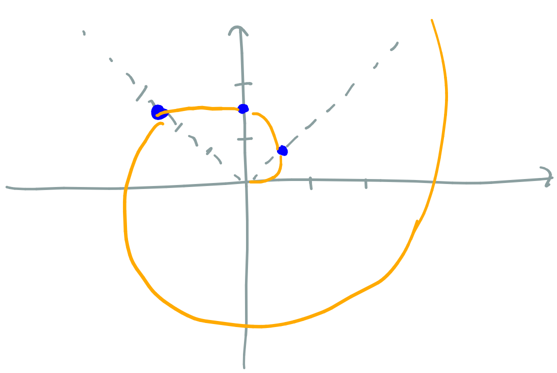
We also have the important equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ that connects the rectangular coordinates with polar coordinates. To arrive at parametric equations, we need only replace θ by t and r by $f(\theta)$. We've shown the following

Theorem: Polar Equations in Parametric Form
The graph of the polar equation $r = f(\theta)$ can be described by the parametric equations $\boxed{\begin{cases} x &= f(t)\cos(t) \\ y &= f(t)\sin(t) \end{cases}}$ where t is the parameter.

Ex 11: Write the parametric equations for the Archimedean Spiral: $r = \theta$.

$\boxed{\begin{cases} x = t\cos(t) \\ y = t\sin(t) \end{cases}}$

OR



$\theta = \pi/4 = 0.78$
$\theta = \pi/2 = 1.8$
$\theta = 3\pi/4 = 2.4$

$\theta = t$
$r = \theta$ ⟸

$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\boxed{\begin{cases} x = \theta\cos\theta \\ y = \theta\sin\theta \end{cases}}$

Activity: Use Desmos for these investigations while working in groups.

(a) Try to come up with the most interesting looking parametric curve.

(b) Plot $(\cos(et), \sin(\sqrt{3}t))$ over the intervals $[0, n]$ for $n = 1, 2, 3, 5, 10, 20, 50, 100, 200$, etc. Describe what happens when n grows arbitrarily large.

(c) Convert the polar function $$r = e^{\sin\theta} + \sin\left(\left(-\frac{\pi}{24} + \frac{\theta}{12}\right)^5\right) - 2\cos(4\theta)$$ into parametric equations and plot over the interval $[0, n]$ for $n = 10, 50, 100$.