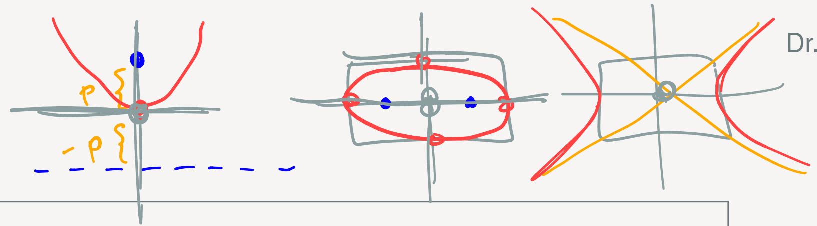
Section 12.4 Shifted Conics



Objectives
- Shifting the graphs of equations
- Shifted Ellipses
- Shifted Hyperbolas
- Shifted Parabolas
- The General Equation of a Shifted Conic

• Shifting the graphs of equations

In §2.6 we studies transformation of functions which included shifts (rigid motion) and dilations (stretches). We only study shifts of conic sections. We need to know how to shift equations instead of functions.

Theorem 1 Shifting the graphs of equations

Let $h, k \in \mathbb{R}$ both be positive: $h, k > 0$.

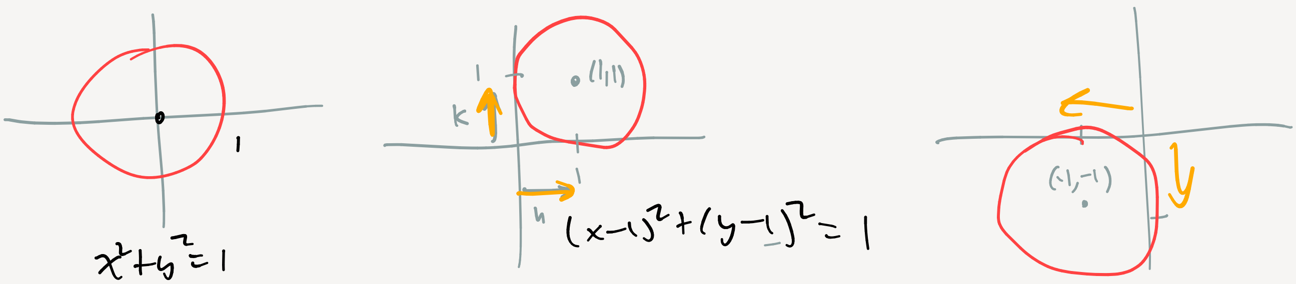
- **Horizontal Shifts** Replace x by $x - h$, or $x \to x - h$
  - **Left** $h < 0$
  - **Right** $h > 0$
  - **ProTip** Alternatively, you can think of it as $x + h$ and shift in the opposite direction as the sign of h
- **Vertical Shifts** Replace y by $y - k$, or $y \to y - k$
  - **Down** $k < 0$
  - **Up** $k > 0$
  - **ProTip** Remember it as $y + k$ and shift in the opposite direction as the sign of k

We write the transformations as $x - h$ and $y - k$ with a minus sign so that the a negative h or k corresponds to a shift in the negative x or y axis direction. Otherwise, the pro tip says to do it "in the opposite direction"

Ex 1 Shifted Circle I find that the easiest way to remember this is thinking about a circle.
- We all know that $x^2 + y^2 = 1$ is the unit circle centered at the origin, $(0, 0)$.
- Then $(x - 1)^2 + (y - 1)^2 = 1$ is the unit circle centered at $(1, 1)$.
- Then $(x + 1)^2 + (y + 1)^2 = 1$ is the unit circle centered at $(-1, -1)$.
- In general, $(x - h)^2 + (y - k)^2 = 1$ is a unit circle shifted to the center (h, k).
- In general, $(x + h)^2 + (y + k)^2 = 1$ is a unit circle shifted to the center $(-h, -k)$.

dist((x,y),(-1,-1))

$(x+1)^2+(y+1)^2=1$



dist((x,y),(1,1))

• Shifted Ellipses

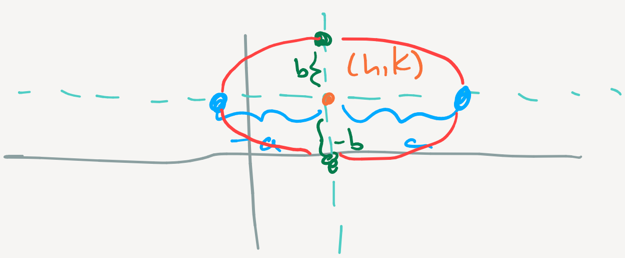
Theorem 2 Shifting Ellipses

The graph of

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

compare w/ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is an **ellipse** shifted horizontally by h and vertically by k. The center of the ellipse is at (h, k).



Ex 2 Shifted Ellipse Sketch the graph of the ellipse:

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

Also, determine the vertices, co-vertices, and foci.

$h = 1$

$k = -2$ $\quad y+2 = y-(-2)$

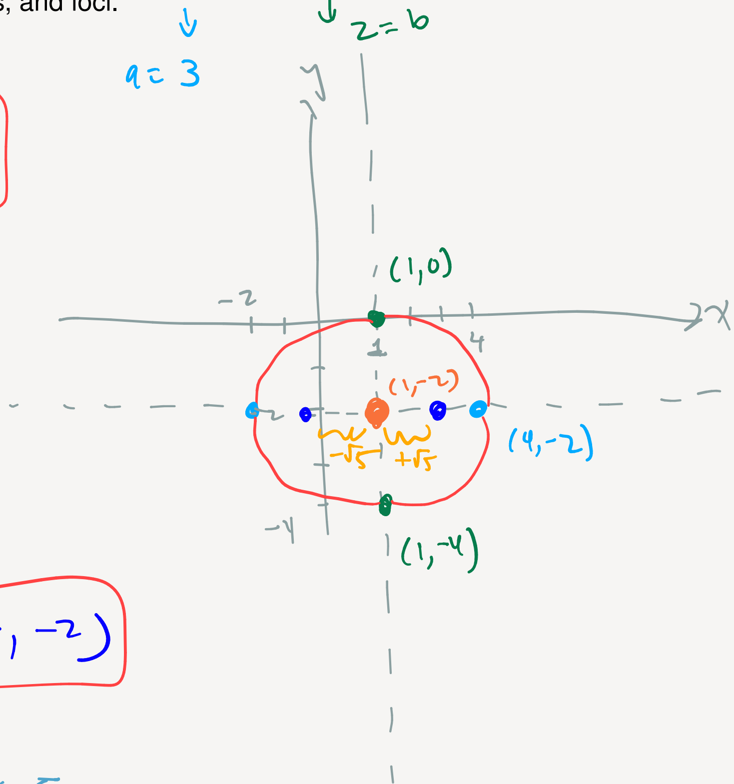
$a = 3$ $\quad 2 = b$

vertices $\boxed{(4,-2) \ \& \ (-2,-2)}$

"$h \pm a$"

Covertices $\boxed{(1,0) \ \& \ (1,-4)}$

"$k \pm b$"



Foci $\boxed{(1-\sqrt{5}, -2) \ \& \ (1+\sqrt{5}, -2)}$

Ellipse $b^2 = a^2 - c^2$

$c^2 = a^2 - b^2 = 9 - 4 = 5$

$c = \sqrt{5}$

• Shifted Hyperbolas

Theorem 3 Shifting Hyperbolas

- The graph of

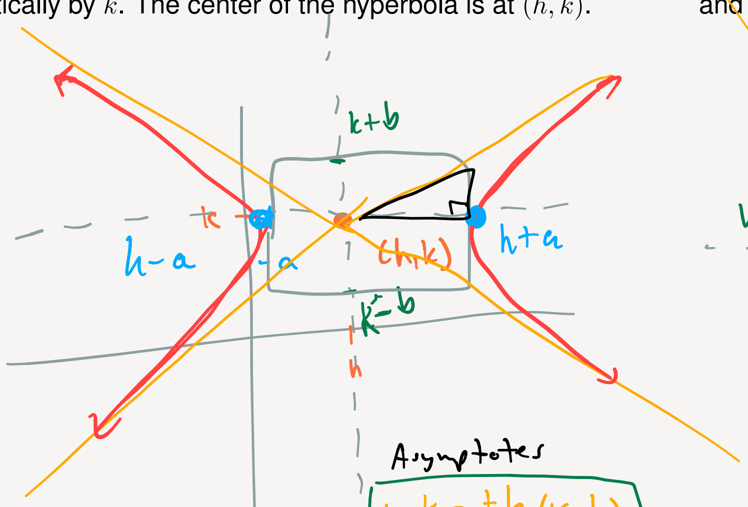
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a **hyperbola with foci on x-axis** shifted horizontally by h and vertically by k. The center of the hyperbola is at (h, k).

- The graph of

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

is a **hyperbola with foci on y-axis** shifted horizontally by h and vertically by k. The center of the hyperbola is at (h, k).

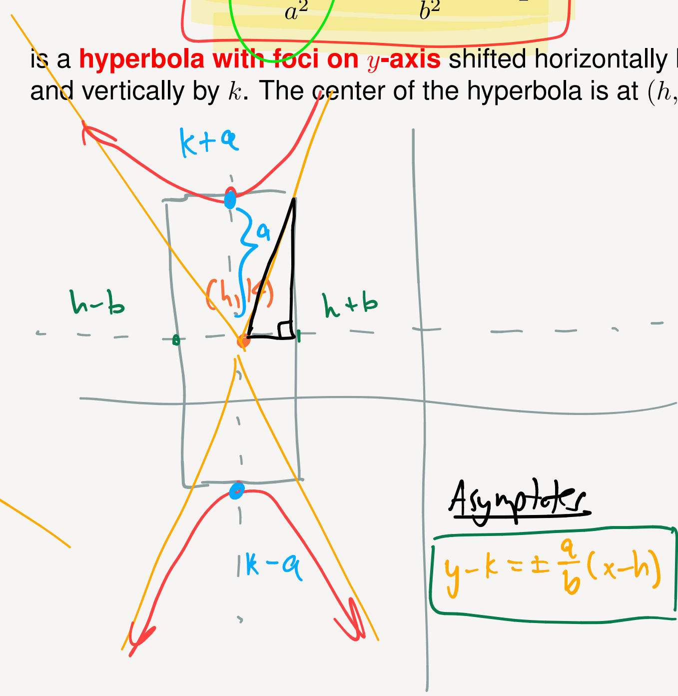


Asymptotes

$\boxed{y-k = \pm\frac{b}{a}(x-h)}$

$y - k = m(x-h)$

Eq Line passing (h,k)



Asymptotes

$\boxed{y-k = \pm\frac{a}{b}(x-h)}$

Ex 3 Shifted Hyperbola Sketch the graph of the hyperbola:

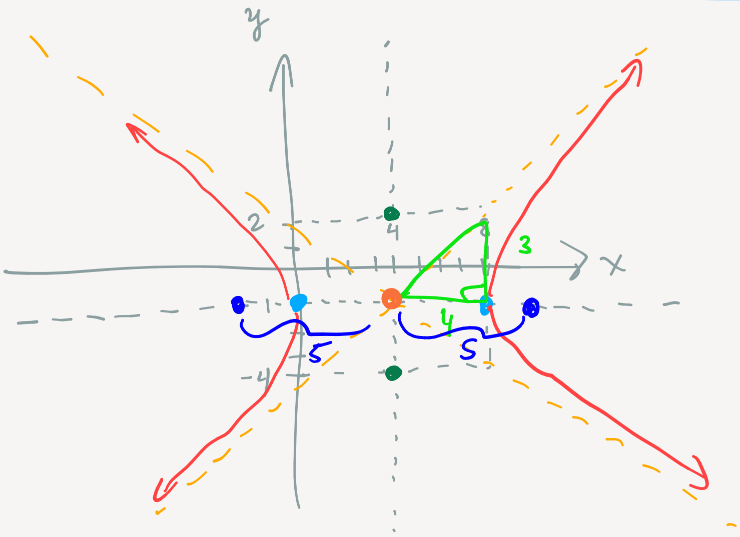
$$\frac{(x-4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

$a = 4$ $\quad$ $b = 3$

Also, determine the vertices, asymptotes and foci.

Shifts
$h = 4$
$k = -1$

Center
$(4, -1)$



Vertices $\boxed{(8,-1) \,\&\, (0,-1)}$

Asymptotes $y - k = m(x-h)$

$\boxed{y+1 = \pm\frac{3}{4}(x-4)}$

Foci $\boxed{(-1,-1) \,\&\, (9,-1)}$

$c^2 = a^2 + b^2$

$c^2 = 16 + 9 = 25$

$c = 5$

Ex 4 Shifted Hyperbola Sketch the graph of the hyperbola:

$$9x^2 - 72x - 16y^2 - 32y = 16$$

Also, determine the vertices, asymptotes and foci.

Put in graphing form:

Use Complete the Square (CtS) $\boxed{x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2}$

$9x^2 - 72x - 16y^2 - 32y = 16$

$9(x^2 - 8x) - 16(y^2 + 2y) = 16$

CtS $\downarrow B = -8$ $\qquad$ CtS $\downarrow B = 2$

$9\left[(x-4)^2 - \overset{16}{(-4)^2}\right] - 16\left[(y+1)^2 - \overset{1}{(1)^2}\right] = 16$

$9(x-4)^2 - 9 \cdot 16 - 16(y+1)^2 + \cancel{16} = \cancel{16}$

$\dfrac{9(x-4)^2 - 16(y+1)^2}{144} = \dfrac{144}{144}$

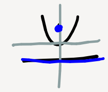
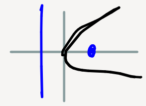
$$\boxed{\frac{(x-4)^2}{16} - \frac{(y+1)^2}{9} = 1}$$

$a = 4$ $\quad$ $b = 3$

Center
$(4, -1)$

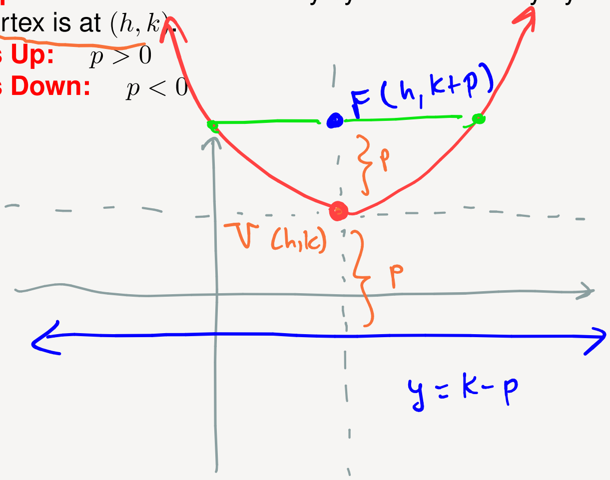
SAME AS PREVIOUS EX!

• Shifted Parabolas

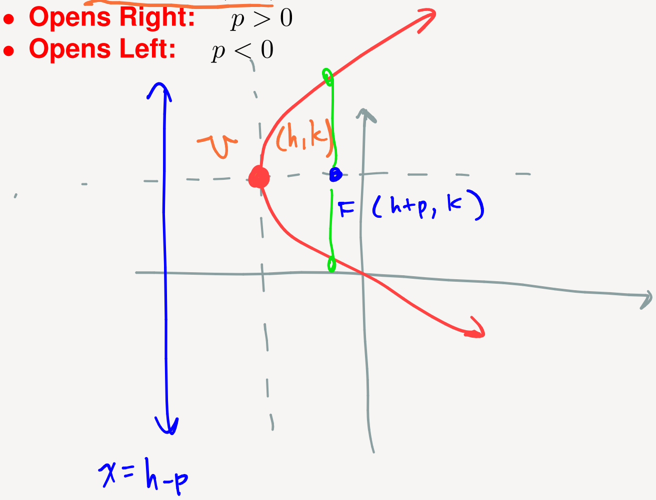
Recall $x^2 = 4py$  $y^2 = 4px$ 

Theorem 4 Shifting Parabolas

- The graph of
$$(x - h)^2 = 4p(y - k)$$
is a **hyperbola** shifted horizontally by h and vertically by k. The vertex is at (h, k).
- **Opens Up:** $p > 0$
- **Opens Down:** $p < 0$



- The graph of
$$(y - k)^2 = 4p(x - h)$$
is a **hyperbola** shifted horizontally by h and vertically by k. The vertex is at (h, k).
- **Opens Right:** $p > 0$
- **Opens Left:** $p < 0$



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Ex 5 Shifted Parabola Sketch the graph of the parabola:

In graphing form → $$(x-2)^2 = 8(y-3)$$ vs $(x-2)^2 = 8y - 24$

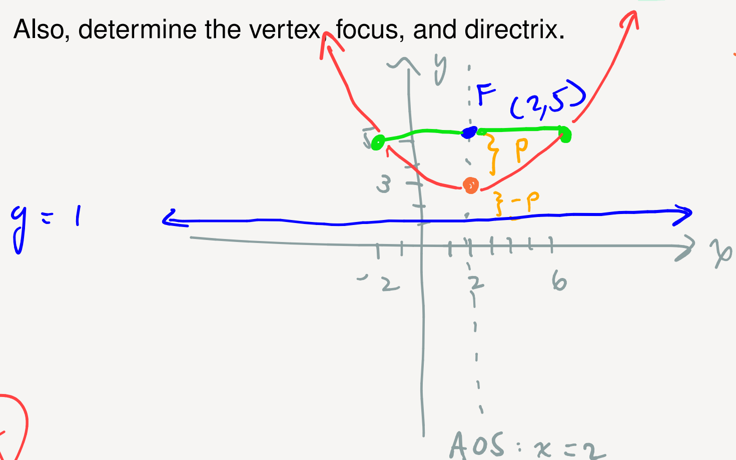
Also, determine the vertex, focus, and directrix.

Vertex $\boxed{(2,3)}$

open up/down: b/c x^2

open up: $p > 0$ $(8 = 4p)$

$\hookrightarrow p = 2$



Foci $(h, k+p)$

$(2, 3+2) = \boxed{(2,5)}$

Directrix $y = k - p$

$y = 3 - 2 = 1$

$\boxed{y = 1}$

PROTIP

LATUS RECTUM

"plug-in $y = p$"

Now $y = k + p$

EQ $(x-2)^2 = 8(5-3)$ $y = 5$

$(x-2)^2 = 16$

$x - 2 = \pm\sqrt{16} = \pm 4$

$x = 2 \pm 4 = 6, -2$

GQT

Ex 6 Shifted Parabola Sketch the graph of the parabola:

$$x^2 - 4x = 8y - 28$$

Also, determine the vertex, focus, and directrix.

Not in graphing form → put into graphing form: $(x-2)^2 = 8(y-3)$

$x^2 - 4x = 8y - 28$

$B = -4$ ↓ CtS

$[(x-2)^2 - (-2)^2] = 8y - 28$

$(x-2)^2 - 4 = 8y - 28$

$(x-2)^2 = 8y - 24$

$= 8(y-3)$

• The General Equation of a Shifted Conic

One of the major accomplishments of analytic geometry (also called algebraic geometry) is the following theorem:

Theorem 5 General Equation of Conic Sections

Assume at least one of A, B, C, D is not 0.

The graph of an equation of the form:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

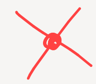
is a **conic section**.

- When $A = B = 0$ it is a **line**.
- When A or B equal 0, but not both, it is a **parabola**.
- When $A = B$ (either both positive or both negative), it is a **circle**.
- When $A \neq B$ and have the same sign (either both positive or both negative), it is an **ellipse**.
- When $A \neq B$ and have the opposite sign (one is positive and the other is negative), it is a **hyperbola**.
- Note that this general form also includes **points**.

Lines and points are called **degenerate conics**.

$(x-1)^2 + (y-1)^2 = 0. \rightarrow$ point $(1,1)$.

$(x-1)^2 - (y-1)^2 = 0 \rightarrow x - 1 = \pm (y-1)$ pair of lines



How to determine the conic section If an equation of the form $Ax^2 + By^2 + Cx + Dy + E = 0$ is given,

- use the above theorem when A or B are zero;
- when $A \neq 0$ and $B \neq 0$, use **complete the square** TWICE (once in each variable) and see which type it is.

Recall: **complete the square** $x^2 + Bx = (x + \frac{B}{2})^2 - (\frac{B}{2})^2$.

Ex 7 General Conic Sketch the graph of $9x^2 - y^2 + 18x + 6y = 0$.

$A = 9 \quad B = -1 \longrightarrow$ Hyperbola

Put into graphing form:

$$9x^2 + 18x - y^2 + 6y = 0$$

$$9\left[\underbrace{x^2 + 2x}_{}\right] - \left[\underbrace{y^2 - 6y}_{}\right] = 0$$

$B = 2 \downarrow$ CtS $\qquad B = -6 \downarrow$ CtS

$$9\left[(x+1)^2 - 1^2\right] - \left[(y-3)^2 - (-3)^2\right] = 0$$

$$9(x+1)^2 - \cancel{9} - (y-3)^2 + \cancel{9} = 0$$

$$9(x+1)^2 - (y-3)^2 = 0$$

$$9(x+1)^2 = (y-3)^2$$

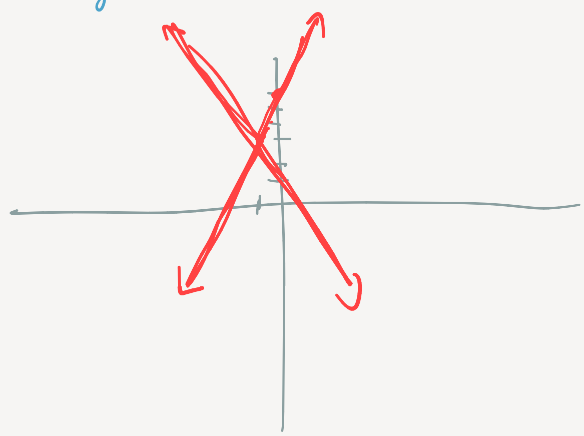
Degenerate conic

Solve for y:

$$(y-3)^2 = 9(x+1)^2$$

$$y - 3 = \pm\sqrt{9(x+1)^2}$$

$$y = 3 \pm 3|x+1|$$



$y = 3 + 3|x+1|$