Section 7.4 → EQ

7.1 Trig ID

7.2 Sum / Difference

7.3 Double-Angle, ...

EQ vs ID

EQ ↓ some / no solutions

ID ↓ true for all values

Objectives

- Solving Basic Trigonometric Equations
- Solving Trigonometric Equations by Factoring

Basic Trigonometric Equations

Basic Trigonometric Equations (1 of 3)

sin(x), cos(x), tan(x), ... θ, x, t variable

An equation that contains trigonometric functions is called a **trigonometric equation**.

For example, the following are trigonometric equations:

M ID EQ EQ

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \qquad \boxed{2\sin\theta - 1 = 0} \qquad \boxed{\tan 2\theta - 1 = 0}$$

The first equation is an *identity*—that is, it is **true for every value of the variable *θ***.

The other two equations are true only for certain values of θ.

To solve a trigonometric equation, we find all the values of the variable that make the equation true.

Basic Trigonometric Equations (2 of 3)

Solving any trigonometric equation always reduces to solving a **basic trigonometric equation**—an equation of the form $T(\theta) = c$, where T is a trigonometric function and c is a constant.

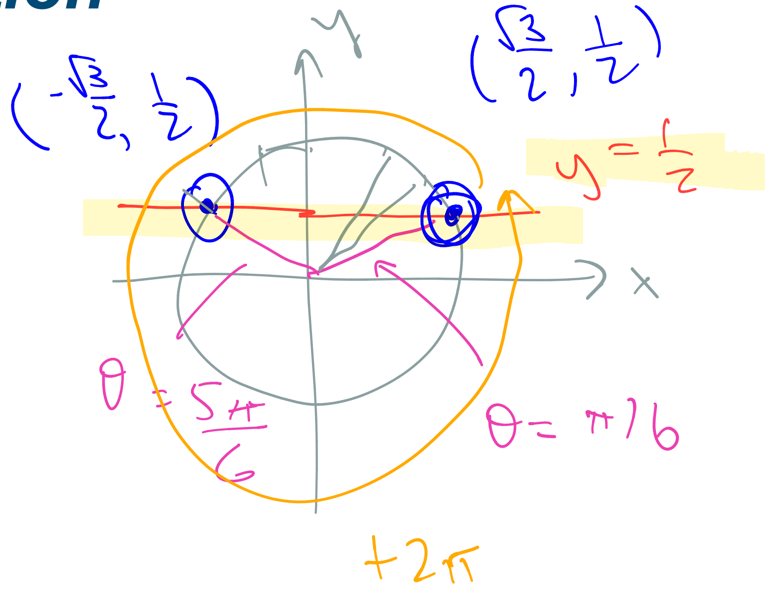
In the next examples we solve such basic equations.

$\cos(\theta) = 4/5$

$T(\theta) = c$

Example 1 – *Solving a Basic Trigonometric Equation*

Solve the equation $\boxed{\sin\theta = \frac{1}{2}}$.



Solution:

$\theta = \frac{\pi}{6}$

$\theta = \frac{5\pi}{6}$

two solution in $[0, 2\pi)$

To get infinitely many solutions:

$\theta = \frac{\pi}{6} + 2\pi, \ \frac{\pi}{6} + 4\pi, \ \ldots \ \frac{\pi}{6} + 2\pi k$ where k is an integer

$\mathbb{Z}$ integers

$k: \ldots, -2, -1, 0, 1, 2, 3, \ldots$

Final answer:

$$\boxed{\theta = \frac{\pi}{6} + 2\pi k, \quad \theta = \frac{5\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}}$$

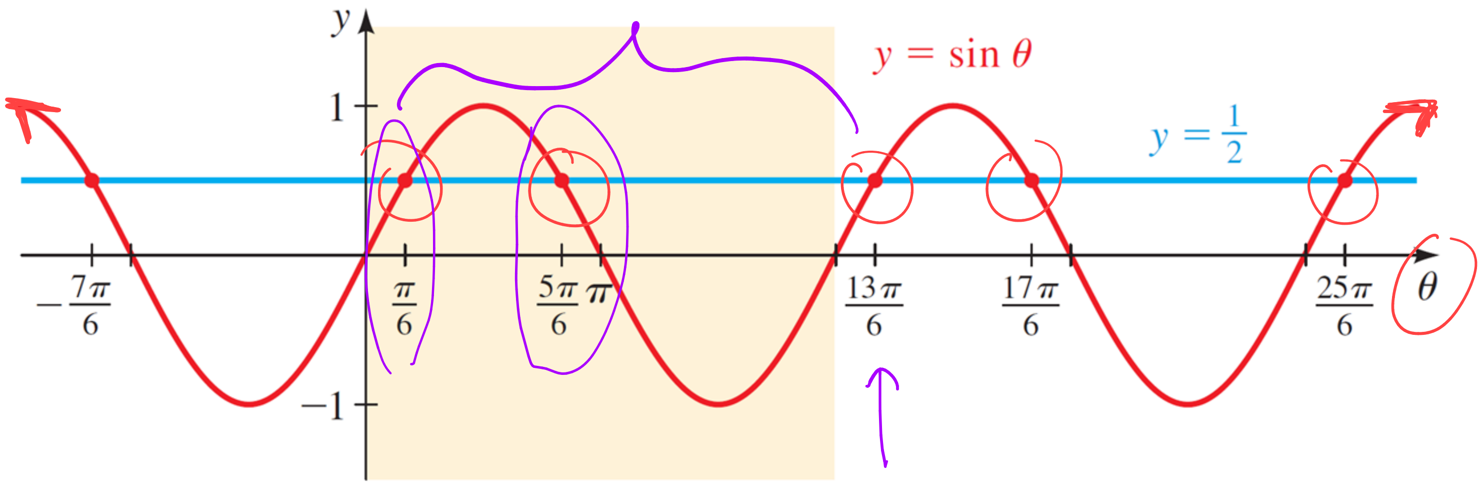
Example 1 – *Solution*

Figure 2 gives a graphical representation of the solutions.

$y = \sin(\theta)$ $\sin\theta = \frac{1}{2}$

$y = \frac{1}{2}$

2π



How many solutions?

infinitely many!

Figure 2

$\frac{\pi}{6} + 2\pi = \frac{\pi + 12\pi}{6} = \frac{13\pi}{6}$

CENGAGE

Example 1 – *Solving a Basic Trigonometric Equation*

Solve the equation $\sin\theta = \frac{1}{2}$.

Solution:

Find the solutions in one period.

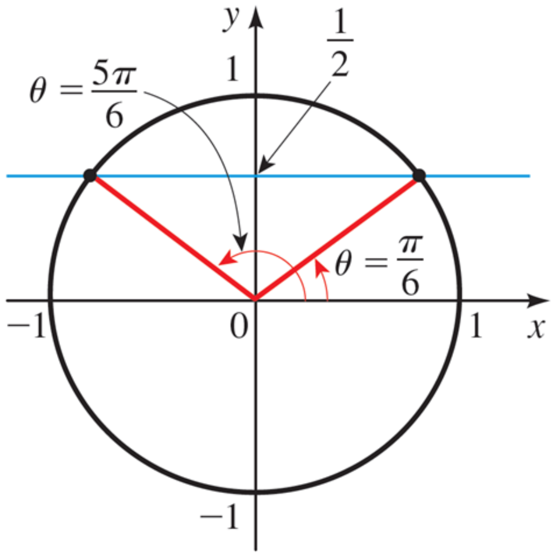


Figure 1

Example 1 – *Solving a Basic Trigonometric Equation*

Solve the equation $\sin\theta = \frac{1}{2}$.

Solution:

Find the solutions in one period.

Because sine has period 2π, we first find the solutions in any interval of length 2π. To find these solutions, we look at the unit circle in Figure 1.

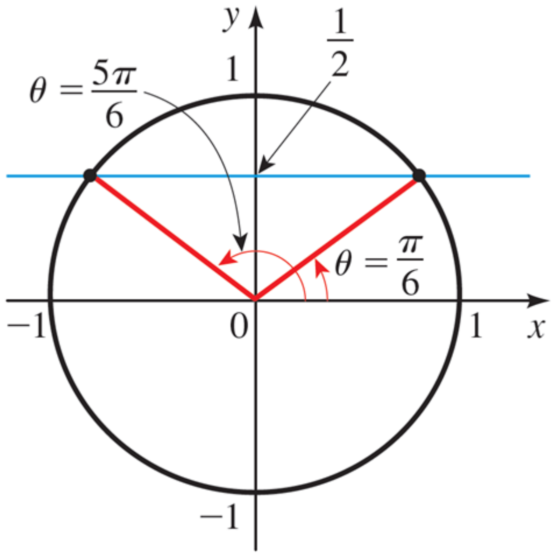


Figure 1

Example 1 – *Solution* (1 of 2)

We see that $\sin\theta = \frac{1}{2}$ in Quadrants I and II, so the solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{\pi}{6} \qquad \theta = \frac{5\pi}{6}$$

Find all solutions.

Because the sine function repeats its values every 2π units, we get all solutions of the equation by adding integer multiples of 2π to these solutions:

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where k is any integer.

Example 4 – *Solving a Basic Trigonometric Equation*

no restrictions!

Solve the equation tan θ = 2.

$y = \tan\theta$

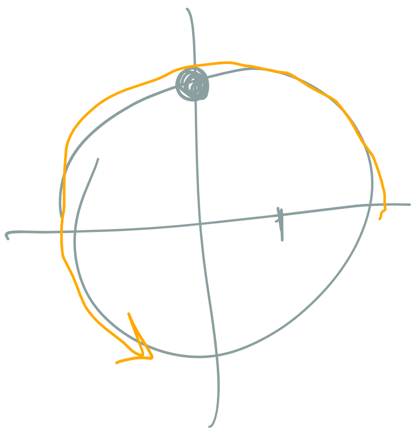
Solution:

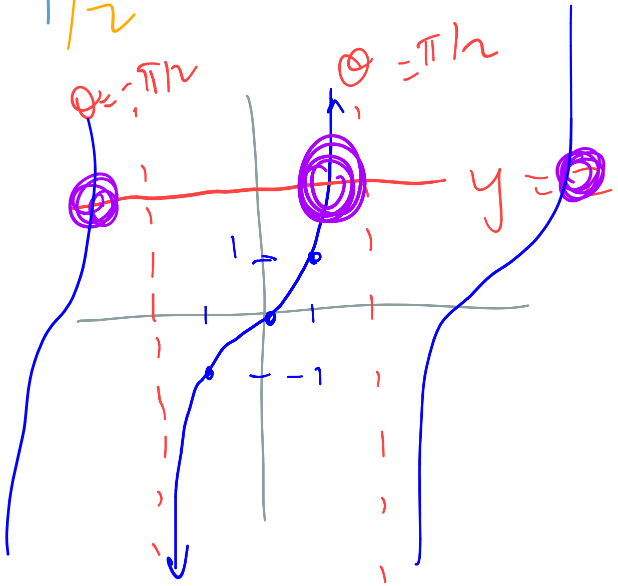
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2/2}{1/2}$ → θ not standard angle

$\tan\theta = 2 \leftrightarrow \theta = \tan^{-1}(2)$

Use Calculator!

Warning: use Radians!





$\theta = \tan^{-1}(2) \approx 1.10714 87\ldots$

$\theta = 1.107$

$\boxed{\theta = 1.107 + \pi k, \; k \in \mathbb{Z}}$

integer

π

Example 4 – *Solution*

A graphical representation of the solutions is shown in Figure 6.

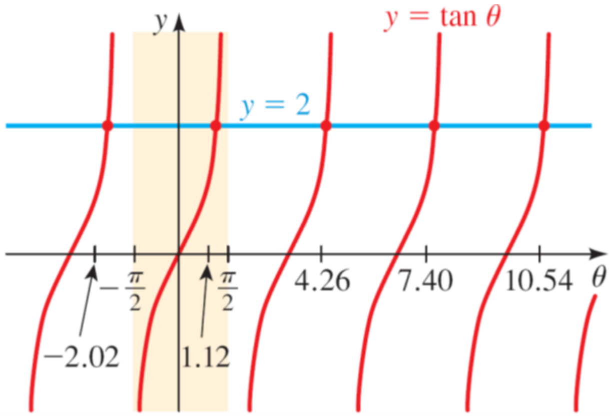


Figure 6

You can check that the solutions shown in the graph correspond to $k = -1, 0, 1, 2, 3$.

Example 4 – *Solving a Basic Trigonometric Equation*

Solve the equation $\tan \theta = 2$.

Solution:

We start with the standard value

$\tan 0 = 0$

$$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

CENGAGE

Example 4 – *Solving a Basic Trigonometric Equation*

Solve the equation $\tan \theta = 2$.

Solution:

Find the solutions in one period.

We first find one solution by taking $\tan^{-1}$ of each side of the equation.

$\tan \theta = 2$	Given equation
$\theta = \tan^{-1}(2)$	Take $\tan^{-1}$ of each side
$\theta \approx \cancel{1.12}$	Calculator (in radian mode)

$1.107 + \pi k$

Example 4 – *Solution* (1 of 2)

By the definition of $\tan^{-1}$ the solution that we obtained is the only solution in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (which is an interval of length π).

Find all solutions.

Since tangent has period π, we get all solutions of the equation by adding integer multiples of π:

1.07

$$\boxed{\theta \approx \cancel{1.12} + k\pi}$$

where k is any integer.

Example 4 – *Solution*

A graphical representation of the solutions is shown in Figure 6.

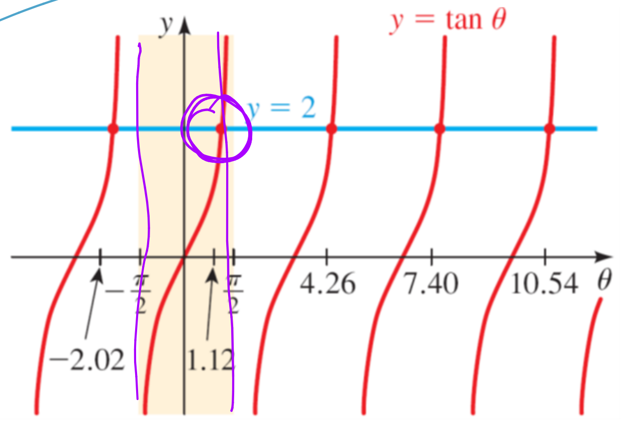


Figure 6

Compare w/

$x^2 = 16$

$x = \pm\sqrt{16} = \pm 4$

$(\pm 4)^2 = 16$

$x = \sqrt{16} = 4$

You can check that the solutions shown in the graph correspond to $k = -1, 0, 1, 2, 3$.

Basic Trigonometric Equations (3 of 3)

In the next example we solve trigonometric equations that are algebraically equivalent to basic trigonometric equations.

Example 5 – *Solving Trigonometric Equations*

Find all solutions of the equation.

$T(\theta) = c$

(a) $2 \sin \theta - 1 = 0$

(b) $\tan^2\theta - 3 = 0$

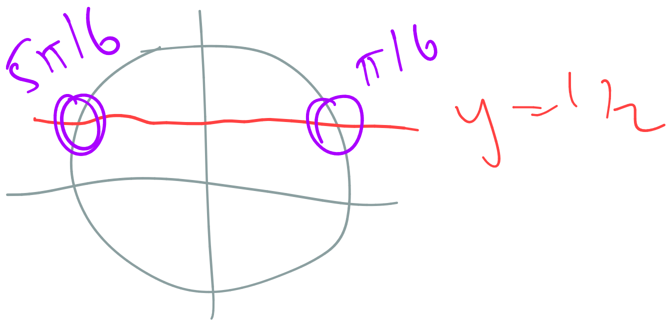
Solution:

(a)

$$2 \sin \theta - 1 = 0$$
$$+1 \quad +1$$
$$\frac{2 \sin \theta}{2} = \frac{1}{2}$$
$$\sin \theta = \frac{1}{2}$$

$\sin \theta = \frac{1}{2}$ ⟶ Example 1

$$\boxed{\begin{aligned} \theta &= \frac{\pi}{6} + 2\pi k \\ \theta &= \frac{5\pi}{6} + 2\pi k \\ k &\in \mathbb{Z} \end{aligned}}$$



Example 5 – *Solving Trigonometric Equations*

key trick (reverse cancelling)

Find all solutions of the equation.

(a) $2 \sin \theta - 1 = 0$

(b) $\tan^2\theta - 3 = 0$

Solution:

(b) $\tan^2\theta - 3 = 0$

$+3 \quad +3$

$\tan^2\theta = 3$

↓ SRP

$\tan\theta = \pm\sqrt{3}$

$T(\theta) = c$

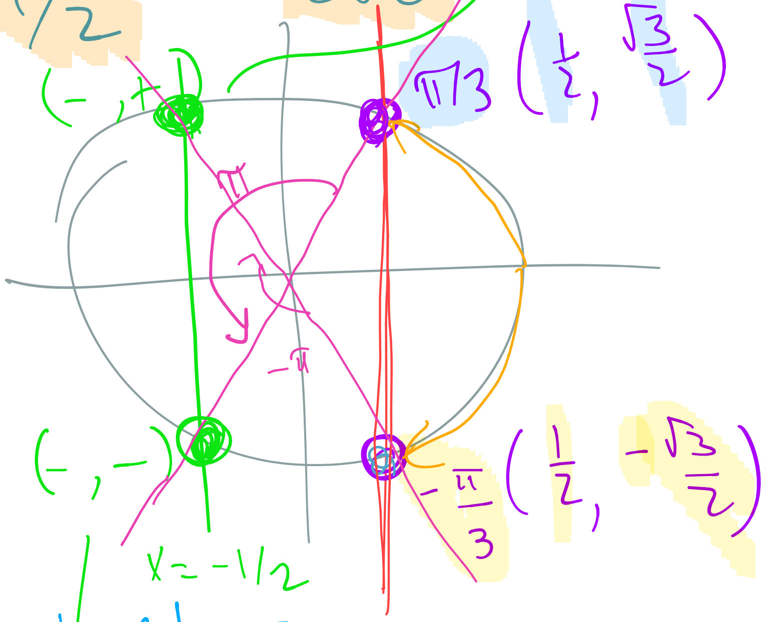
$\tan\theta = \pm\sqrt{3} = \dfrac{\pm\sqrt{3}/2}{1/2} = \dfrac{\sin\theta}{\cos\theta}$

$\theta = \dfrac{\pi}{3}$

$\theta = -\dfrac{\pi}{3} = 2\pi - \dfrac{\pi}{3}$

$= \dfrac{6\pi - \pi}{3} = \dfrac{5\pi}{3}$

Inside $[0, 2\pi)$



period tangent $= \pi$

$\dfrac{\pi}{3} + \pi = \dfrac{4\pi}{3}$

$\dfrac{5\pi}{3} - \pi = \dfrac{2\pi}{3}$

$$\boxed{\begin{aligned}\theta &= \frac{\pi}{3} + \pi k \\ \theta &= \frac{5\pi}{3} + \pi k\end{aligned}}$$

$k \in \mathbb{Z}$

$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Example 5 – *Solving Trigonometric Equations*

Find all solutions of the equation.

(a) $2 \sin \theta - 1 = 0$

(b) $\tan^2 \theta - 3 = 0$

Solution:

(a) We start by isolating $\sin \theta$.

$2 \sin \theta - 1 = 0$	Given equation
$2 \sin \theta = 1$	Add 1
$\sin \theta = \dfrac{1}{2}$	Divide by 2

Example 5 – *Solution* (1 of 2)

This last equation is the same as that in Example 1. The solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where *k* is any integer.

(b) We start by isolating tan θ.

$\tan^2\theta - 3 = 0$	Given equation
$\tan^2\theta = 3$	Add 3
$\tan\theta = \pm\sqrt{3}$	Take the square root

Example 5 – *Solution* (2 of 2)

Because tangent has period π, we first find the solutions in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of length π. In the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the solutions are $\theta = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{3}$.

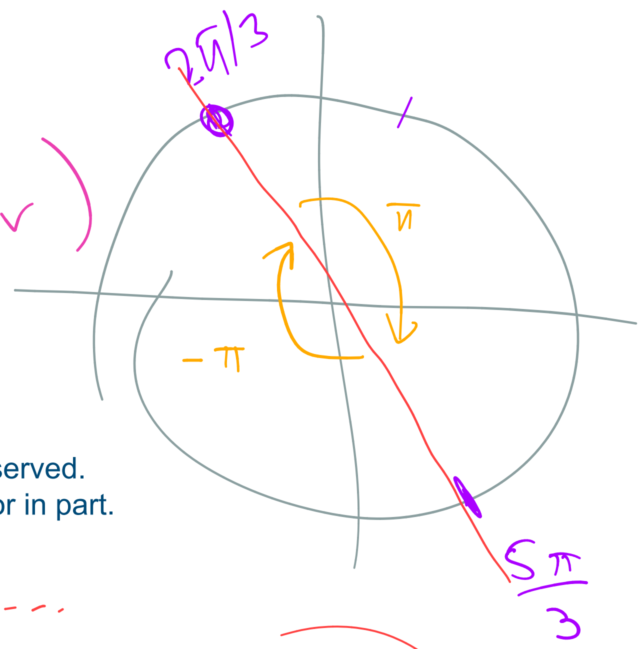
To get all solutions, we add integer multiples of π to these solutions:

$$\theta = \frac{\pi}{3} + k\pi \qquad \theta = -\frac{\pi}{3} + k\pi$$

where k is any integer.

$\theta = \frac{5\pi}{3} + k\pi$ (prefer)

$\theta = \frac{2\pi}{3} + k\pi$



CENGAGE

∞

$\theta = \frac{\pi}{3} + \pi k$ $\quad k = \ldots, -2, -1, 0, 1, 2, \ldots$

$k = 12 \qquad \theta = \frac{\pi}{3} + \pi(12) = \frac{37\pi}{3}$

Solving Trigonometric Equations by Factoring

Solving Trigonometric Equations by Factoring

Factoring is one of the most useful techniques for solving equations, including trigonometric equations.

The idea is to move all terms to one side of the equation, factor, and then use the Zero-Factors Property (ZFP).

$$x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

$x+1=0$ OR $x-1=0$ (or both)

$x=-1$ $\quad$ $x=1$

Example 6 – *A Trigonometric Equation of Quadratic Type*

u-substitution : $u = \cos\theta$

Solve the equation $2\cos^2\theta - 7\cos\theta + 3 = 0$.

Solution:

replace w/ u

$$2 \cdot u^2 - 7u + 3 = 0$$

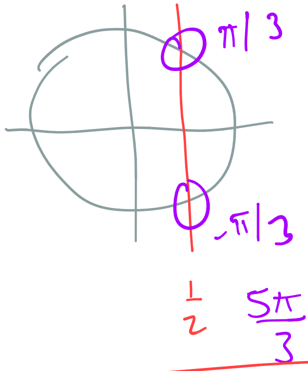
$$(2u - 1)(u - 3) = 0$$

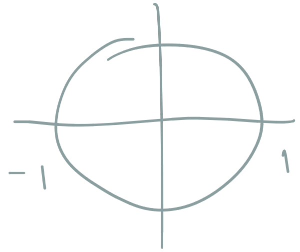
ZFP

$2u - 1 = 0$ OR $u - 3 = 0$

$u = 1/2$ $\qquad$ $u = 3$

$\cos\theta = 1/2$ or $\cos\theta = 3$





No solutions

$\theta = \frac{\pi}{3} + 2\pi k$ $\qquad$ $k \in \mathbb{Z}$

$\theta = \frac{5\pi}{3} + 2\pi k$

$\mathbb{Z}$ integer

can also write as

$\theta = -\frac{\pi}{3} + 2\pi k$

CENGAGE

Example 6 – *A Trigonometric Equation of Quadratic Type*

Solve the equation $2\cos^2\theta - 7\cos\theta + 3 = 0$.

Solution:

Example 6 – *A Trigonometric Equation of Quadratic Type*

Solve the equation $2\cos^2\theta - 7\cos\theta + 3 = 0$.

Solution:

We factor the left-hand side of the equation.

$2\cos^2\theta - 7\cos\theta + 3 = 0$	Given equation
$(2\cos\theta - 1)(\cos\theta - 3) = 0$	Factor
$2\cos\theta - 1 = 0$ or $\cos\theta - 3 = 0$	Set each factor equal to 0
$\cos\theta = \frac{1}{2}$ or $\cos\theta = 3$	Solve for cos θ

Example 6 – *Solution* (1 of 2)

Because cosine has period 2π, we first find the solutions in the interval $[0, 2\pi)$.

For the first equation $\cos\theta = \frac{1}{2}$ the solutions are

$$\theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}$$ (see Figure 7).

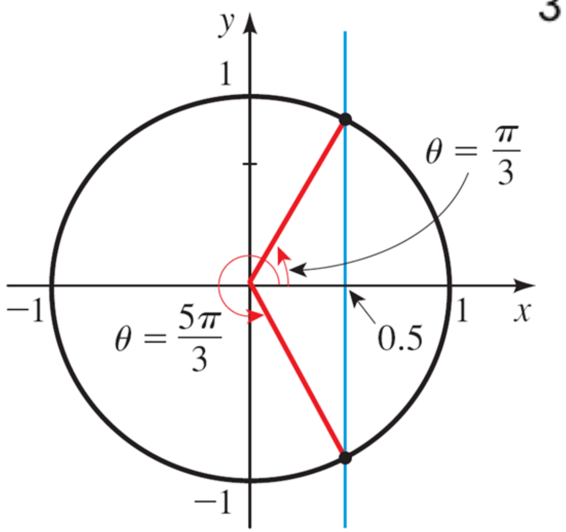


Figure 7

Example 6 – *Solution* (2 of 2)

$$\cos\theta = 3$$

The second equation has **no solution** because cos *θ* is never greater than 1.

Thus the solutions are

$$\theta = \frac{\pi}{3} + 2k\pi \qquad \theta = \frac{5\pi}{3} + 2k\pi$$

where *k* is any integer.

Example 7 – *Solving a Trigonometric Equation by Factoring*

Solve the equation 5 sin θ cos θ + 4 cos θ = 0.

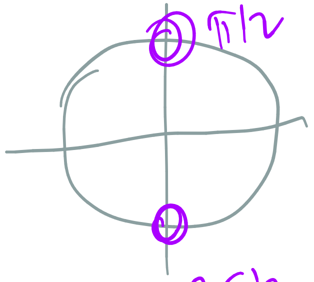
Solution:

$\cos\theta\,(5\sin\theta + 4) = 0$

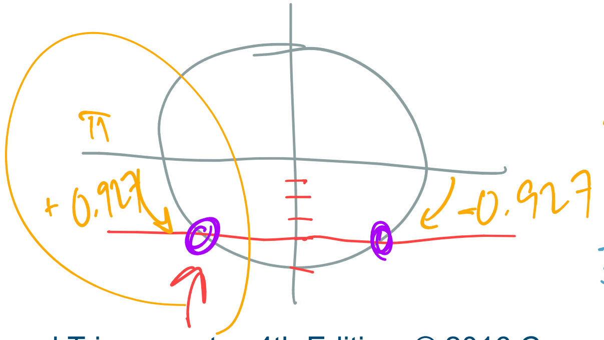
ZFP

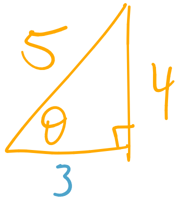
$\cos\theta = 0$ OR $5\sin\theta + 4 = 0$

$\sin\theta = -4/5$



$\theta = \pi/2,\ 3\pi/2$





$3 = \sqrt{5^2 - 4^2} = \sqrt{25-16} = \sqrt{9}$

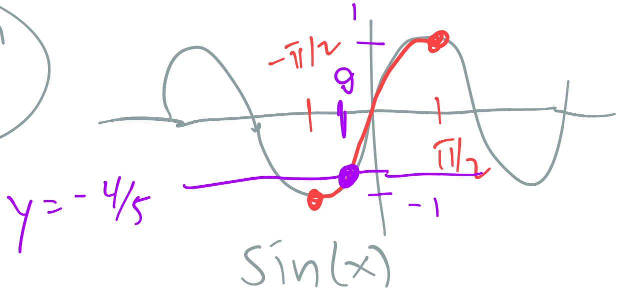
$\theta = \sin^{-1}(-4/5)$

$\theta \approx -0.92729....$

$\theta = -0.927$

$\theta = 4.069$

$\sin^{-1}$



$\pi + 0.927 \approx 4.069$

Example 7 – *Solving a Trigonometric Equation by Factoring*

Solve the equation 5 sin θ cos θ + 4 cos θ = 0.

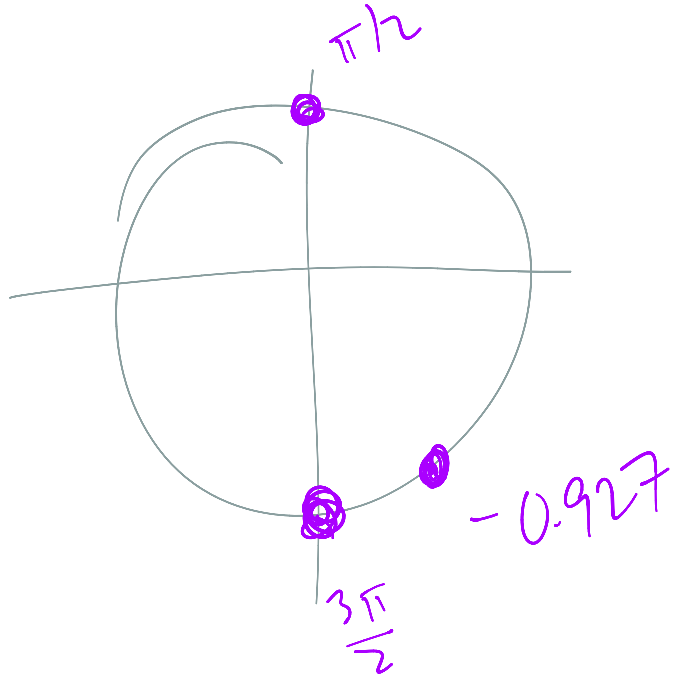
Solution:

$\theta = \pi/2, \ 3\pi/2$

$\theta = -0.927$

$\theta = 4.069$

$$\boxed{\begin{aligned} \theta &= \frac{\pi}{2} + 2\pi k \\ \theta &= \frac{3\pi}{2} + 2\pi k \\ \theta &= -0.927 + 2\pi k \\ \theta &= 4.069 + 2\pi k \end{aligned}}$$



Example 7 – *Solving a Trigonometric Equation by Factoring*

Solve the equation $5 \sin \theta \cos \theta + 4 \cos \theta = 0$.

Solution:

We factor the left-hand side of the equation.

$5 \sin \theta \cos \theta + 2 \cos \theta = 0$	Given equation
$\cos \theta \, (5 \sin \theta + 2) = 0$	Factor
$\cos\theta = 0 \quad \text{or} \quad 5\sin\theta + 4 = 0$	Set each factor equal to 0
$\sin \theta = -0.8$	Solve for $\sin \theta$

CENGAGE

Example 7 – *Solution* (1 of 3)

Because sine and cosine have period 2π, we first find the solutions of these equations in an interval of length 2π.

For the first equation the solutions in the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

To solve the second equation, we take $\sin^{-1}$ of each side.

$\sin\theta = -0.80$	Second equation
$\theta = \sin^{-1}(-0.80)$	Take $\sin^{-1}$ of each side

Example 7 – *Solution* (2 of 3)

$\theta \approx -0.93$ Calculator (in radian mode)

So the solutions in an interval of lengt

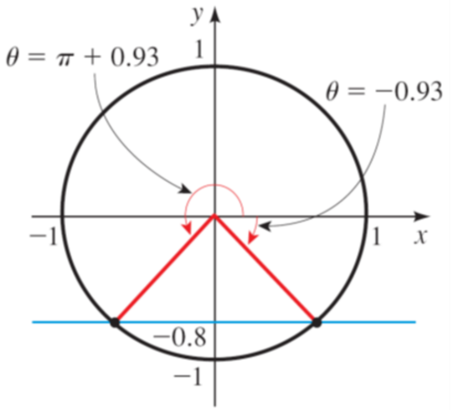


Figure 8

Example 7 – *Solution* (3 of 3)

We get all the solutions of the equation by adding integer multiples of 2π to these solutions.

$$\theta = \frac{\pi}{2} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

$$\theta \approx -0.93 + 2k\pi \qquad \theta \approx 4.07 + 2k\pi$$

where k is any integer.

Example 8 – *Solving a Trigonometric Equation by Factoring*

Solve the equation $5 \sin \theta \cos \theta + 4$ (

Example 8 – *Solving a Trigonometric Equation by Factoring*

Solve the equation $5 \sin \theta \cos \theta + 4$

In the previous example, we found all the solutions of the equation:

$$\theta = \frac{\pi}{2} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

$$\theta \approx -0.93 + 2k\pi \qquad \theta \approx 4.07 + 2k\pi$$

where *k* is any integer.

Example 8 – *Solving a Trigonometric Equation by Factoring*

Solve the equation 5 sin θ cos θ + 4 c

Now, use your calculator to find all so

Round answers to the nearest *thousa*

Solution: We need to keep *at least 4*

$$\theta = \frac{\pi}{2} + 2\pi k \approx 1.5708 + 2\pi k \text{ and } \theta =$$

CENGAGE

Example 8 – *Solving a Trigonometric Equation by Factoring*

Solve the equation 5 sin θ cos θ + 4 c

Now, use your calculator to find all sc

Round answers to the nearest *thousa*

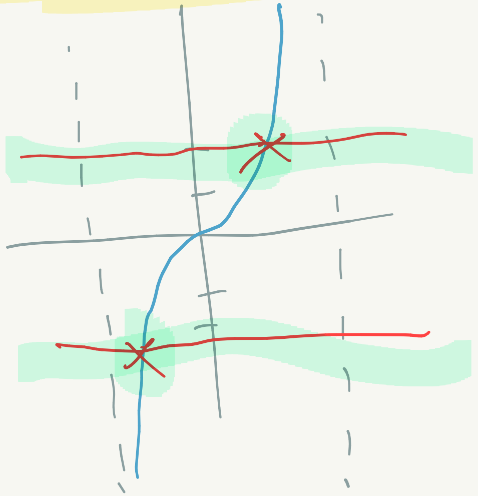
Solution: We need to keep *at least 4*

CENGAGE

#39

$\tan^2\theta = 4$

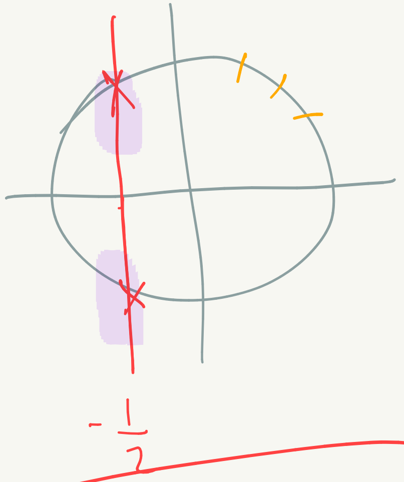
$\tan\theta = \pm 2$



$\theta = \tan^{-1}(2)$

$\boxed{\begin{aligned}\theta &= 1.11 + \pi k \\ \theta &= -1.11 + \pi k\end{aligned}}$

$\cos\theta = -\frac{1}{2}$



$\boxed{\begin{aligned}\theta &= \frac{2\pi}{3} + 2\pi k \\ \theta &= \frac{4\pi}{3} + 2\pi k\end{aligned}}$

$k \in \mathbb{Z}$

(k is an integer)

$\tan\theta = \pm 1$ @ $\pm\pi/4$

$\pm\sqrt{3}$ @ $\pi/6$

$\pm 1/\sqrt{3}$ @ $\pi/3$

$(0, 1)$ undef.

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\quad \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ ...

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ ....

$(1, 0)$ $\quad 0$