Section 8.3 - Polar form of Complex Numbers

Objectives:

- Graphing Complex Numbers
- Polar Form of Complex Numbers
- De Moivre's Theorem
- nth Roots of Complex Numbers

• Graphing Complex Numbers

Definition(s): The **imaginary unit** i is defined as $\boxed{i = \sqrt{-1}}$ (not a real #), or as the "positive" solution to the equation $x^2 + 1 = 0$. Of course, no such real number solves this equation.

$x^2 = -1 \xrightarrow{SRP} x = \pm\sqrt{-1}$ $\boxed{i^2 = -1}$

Complex numbers are numbers of the form $a + bi$, where $a, b \in \mathbb{R}$ and i is the imaginary unit. We say the number is in **standard form**.

$a + bi = a + ib$

$\mathbb{C}$

We denote the set of all complex numbers by $\mathbb{C}$. Notation: we usually use the letters z to denote a single complex number: if $z \in \mathbb{C}$ then

$z =$ $a + ib$ and $\mathbb{C} = \{ z \mid z = a + bi,\ a, b \in \mathbb{R} \}$

We call a the real part and b the imaginary part

Thus, there are two independent real numbers needed to create one complex number. This is why we can **graph** complex numbers on a plane. (two #)

We plot all the real parts on the horizontal axis by plotting $(a, 0)$.

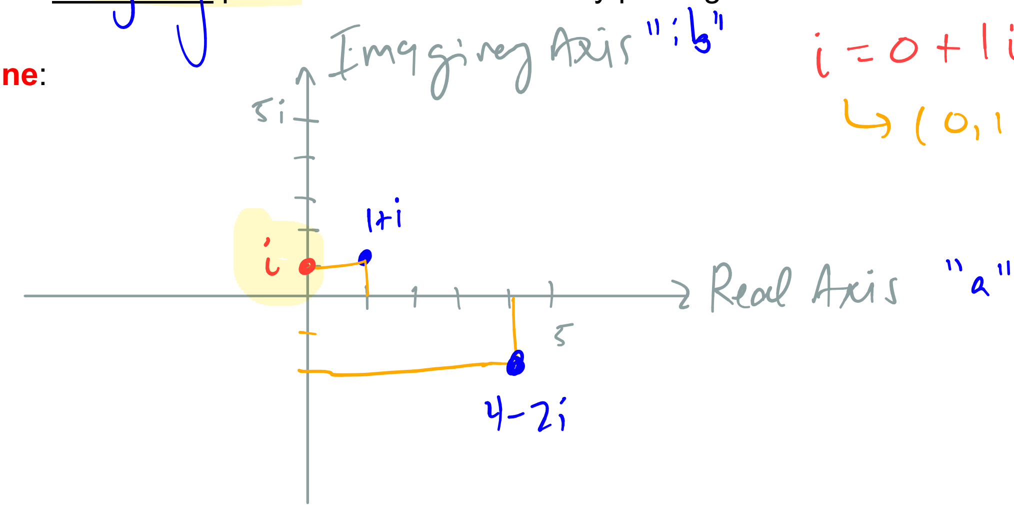
We plot all the imaginary parts on the vertical axis by plotting $(0, b)$.

The **Complex Plane**:

$i = 0 + 1i$
$\hookrightarrow (0, 1)$

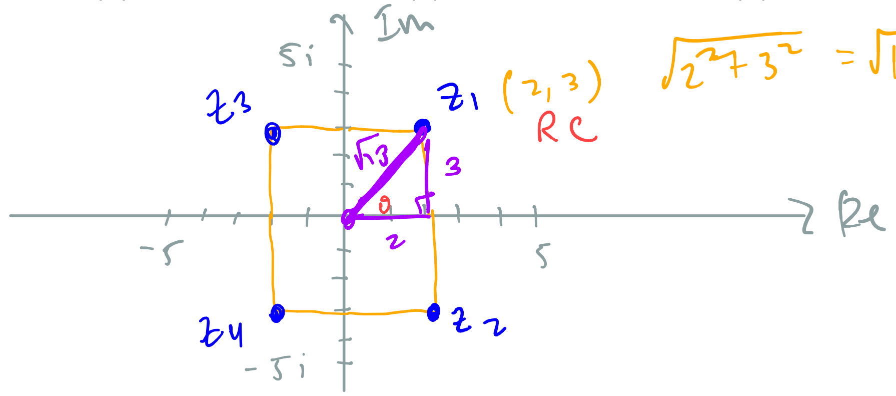
$z_1 = 1 + i$
$\hookrightarrow a = 1, b = 1$
plot $(1, 1)$

$z_2 = 4 - 2i$



Ex 1: Graph the complex numbers:

(a) $z_1 = 2 + 3i$ (b) $z_2 = 2 - 3i$ (c) $z_3 = -2 + 3i$ (d) $z_4 = -2 - 3i$



$\sqrt{2^2+3^2} = \sqrt{13}$

Definition: The **modulus** of a complex number $z = a + bi$ is defined to be real number $\sqrt{a^2+b^2}$. It is denoted by $|z|$.$z = a + bi$ is defined to be real number $\sqrt{a^2+b^2}$. It is denoted by $|z|$.

$|z| = \sqrt{a^2+b^2}$ (is a real #)

Interpretation: distance from z to origin in complex plane

Ex 2: Find the modulus of $z = -3 + 4i$.~~$z = -3 + 4i$~~.

$\to |-3+4i| = \sqrt{(-3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

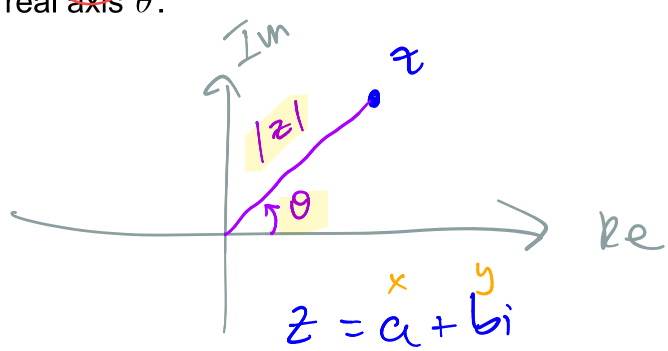
- **Polar Form of a Complex Number** (aka $\mathbb{C}$ in polar coordinates)

Given a complex number $z = a + bi$ we plot it in a plane in two ways. We can think of (a, b) as the rectangular coordinates of z but we can also write its polar coordinates (r, θ). Since the modulus of z is r and we can also find the angle with the real axis θ. ~~$z = a + bi$ we plot it in a plane in two ways. We can think of (a, b) as the rectangular coordinates of z but we can also write its polar coordinates (r, θ). Since the modulus of z is r and we can also find the angle with the real axis θ.~~ redundant

Use $r = |z|$ modulus of z

$\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}(b/a)$



Theorem: Given a complex number $z = a + bi$, we can write z in polar form as:

$z = a + bi$, we can write z in polar form as:

$$\boxed{z = (r\cos\theta) + (r\sin\theta)i} \text{ or } \boxed{z = |z|(\cos\theta + \sin\theta i)},$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

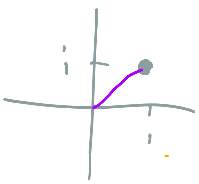
Definition: The **argument** of a complex number $z = a + bi$ is defined to be angle $\theta = \tan^{-1}(b/a)$. It is denoted by $Arg(z) = \theta$. $(r, \theta) \to$ (modulus, Argument)

Ex 3: Write each complex number in polar form:

Tip Draw the complex number first to make sure you select the correct argument. (θ)

(a) $z_1 = 1 + i$ $= |z_1|(\cos\theta + i\sin\theta)$

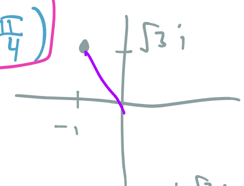
$= \boxed{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$



- $|z_1| = |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$
- $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \pi/4$

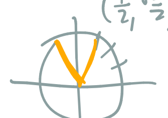
(b) $z_2 = -1 + \sqrt{3}i$

$= |z_2|(\cos\theta + i\sin\theta)$



$|z_2| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}(-\sqrt{3})$



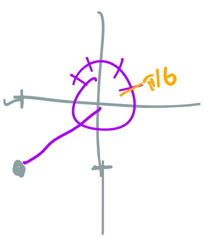
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$= 2\pi/3$

$\boxed{z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$

(c) $z_3 = -4\sqrt{3} - 4i$

$\boxed{z_3 = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)}$



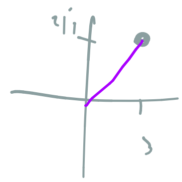
$|z_3| = \sqrt{(-4\sqrt{3})^2 + (-4)^2}$

$= \sqrt{48 + 16} = \sqrt{64} = 8$

$\theta = \tan^{-1}\left(\frac{-4}{-4\sqrt{3}}\right) = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{7\pi}{6}$ (sin, cos)

(d) $z_4 = 3 + 4i$

$\boxed{z_4 = 5(\cos 0.927 + i\sin 0.927)}$



$|z_4| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927$

radian

- **Arithmetic of Complex Number**

Recall we can add, subtract, multiply and divide complex numbers. (standard form)

- ADD: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- SUBTRACT: $(a + bi) - (c + di) = (a - c) + (b - d)i$
- MULTIPLY: $(a + bi) \cdot (c + di) = (ac - bd) + (ad - bc)i$
- DIVIDE: $\frac{a + bi}{c + di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i$

"group like terms"
- just like polynomials
- treat i like "x"

$\boxed{i^2 = -1}$

$(1+i)(3-2i) = 3 - 2i + 3i - 2i^2$

$= 3 + i + 2 = \boxed{5 + i}$

These formulas are straight-forward algebra. Basically, just do the same thing as you would with polynomials but remember to use $i^2 = -1$. You should have seen this in an intermediate algebra course.

Complex numbers have a beautiful geometry to them. We get a glimpse of this when we look at the multiplication and division formulas in polar form.

Note: add/subtraction formulas have a nice geometry too--geometry of parallelograms but we won't study this now (you'll study it again in Linear Algebra).

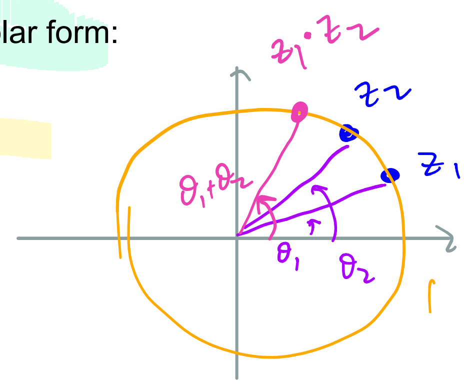
Geometry rotation

Theorem: Multiplication and Division of complex numbers in polar form:
Given two complex numbers z_1 and z_2 in polar form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then: the product $z_1 \cdot z_2$ and quotient $\frac{z_1}{z_2}$ are given by:

- $z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

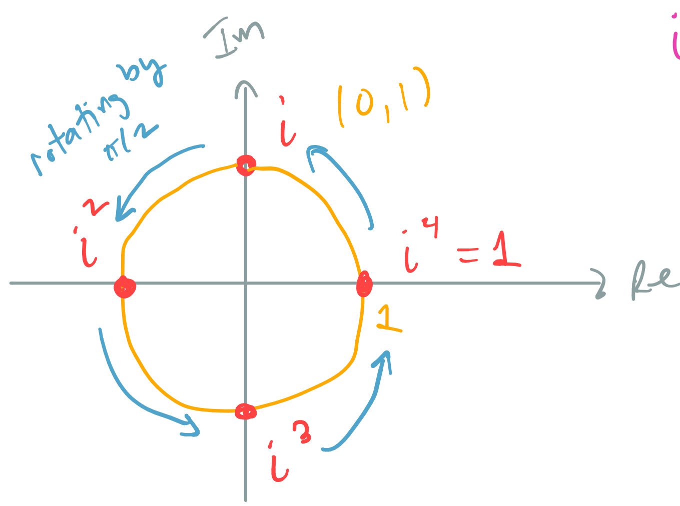


This theorem says:

- To multiply complex numbers, multiply the moduli and add the arguments
- To divide complex numbers,divide the moduli and subtract the arguments

Ex 4: Graph i, i^2, i^3, and i^4. Interpret multiplication by i geometrically. → rotation by $90°$ or $\pi/2$!

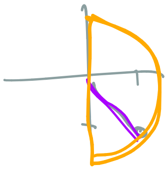
Geometry of i

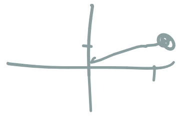


$i^2 = -1$ $(-1,0)$

$i^3 = i^2 \cdot i = -i$ $(0,-1)$

$i^4 = i^2 \cdot i^2 = (-1)^2 = 1$ $(1,0)$





Ex 5: Given $z = 1 - i$ and $w = \sqrt{3} + i$, find: (a) zw and (b) $\frac{z}{w}$. do this in polar form.

Polar form:

$z = 1 - i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{-1}{1}\right)$
$= \tan^{-1}(-1) = -\pi/4$

$w = \sqrt{3} + i$

$|w| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \text{Arg}(w) = \tan^{-1}\left(\overset{1/2}{1} / \underset{\sqrt{3}/2}{\sqrt{3}}\right) = \pi/6$

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ $\quad w = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

a) $z \cdot w = 2\sqrt{2}\left(\cos\left(\frac{-\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{-\pi}{4} + \frac{\pi}{6}\right)\right)$

$= \boxed{2\sqrt{2}\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right)}$

b) $\frac{z}{w} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{-\pi}{4} - \frac{\pi}{6}\right) + i \cdot \sin\left(-\frac{\pi}{4} - \frac{\pi}{6}\right)\right)$

$= \boxed{\frac{\sqrt{2}}{2}\left(\cos\left(\frac{-5}{12}\pi\right) + i\sin\left(\frac{-5\pi}{12}\right)\right)}$

- **DeMoivre's Theorem**

DeMoivre's Theorem is one of the most beautiful and elegant theorems in our course. And it is surprisingly simple and easy to remember! (And with this much pump up; you bet it will show up on the next test!). Here it is:

Theorem: DeMoivre's Theorem
If $z \in \mathbb{C}$ is written in polar form, $(z = r(\cos\theta + i\sin\theta))^n$, then we can find any integer power of z as follows:

$$z^n = \boxed{r^n\left(\cos(n\theta) + i\sin(n\theta)\right)}$$

for any $n \in \mathbb{Z}$.

Sketch *Proof:* Use Theorem about complex multiplication in polar form:

• $n > 0$: $\boxed{n = 2}$ $\quad z^2 = (r\cos\theta + ir\sin\theta)(r\cos\theta + ir\sin\theta)$
$= r^2(\cos(2\theta) + i\sin(2\theta))$

$z^3 = (r(\cos\theta + i\sin\theta))(r^2(\cos(2\theta) + i\sin(2\theta)))$
$= r^3(\cos(3\theta) + i\sin(3\theta))$

etc....

formally use mathematical induction.

$\square$

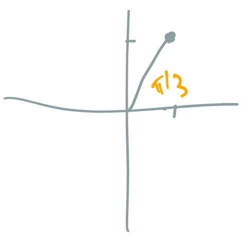
Ex 6: Let $z = 2 + i2\sqrt{3}$. Find z^{10} using DeMoivre's Theorem.
(a) Write your answer in polar form (b) Write your answer in standard form.

$z = 2 + i\,2\sqrt{3}$

$|z| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}(\sqrt{3})$ $\frac{\sqrt{3}/2}{1/2}$ $\frac{\sin}{\cos\theta}$

$= \pi/3$



$4^{10} = (2^2)^{10} = 2^{20}$

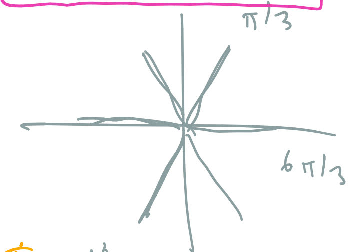
$z = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$$\boxed{z^{10} = 2^{20}\left(\cos\left(\frac{10\pi}{3}\right) + i\sin\left(\frac{10\pi}{3}\right)\right)}$$

$z^{10} = 2^{20}\left(\cos\left(\frac{10\pi}{3}\right) + i\sin\left(\frac{10\pi}{3}\right)\right)$

$= 2^{20}\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= \boxed{-2^{19} - i\,2^{19}\sqrt{3}}$



$\frac{4\pi}{3} = \frac{10\pi}{3}$

$\cos\left(\frac{4\pi}{3}\right) = -1/2$

$\sin\left(\frac{4\pi}{3}\right) = -\sqrt{3}/2$

- **Nth Roots of Complex Numbers**

don't always exist as $\mathbb{R}$!

Recall that an n^{th} root of a real number a is defined to be a real number b for which $b^n = a$. If b exits, then we write $b = \sqrt[n]{a}$ or $b = a^{1/n}$.

Essentially, roots are solutions to the equation $x^n = a$ or $x^n - a = 0$, where a is known.

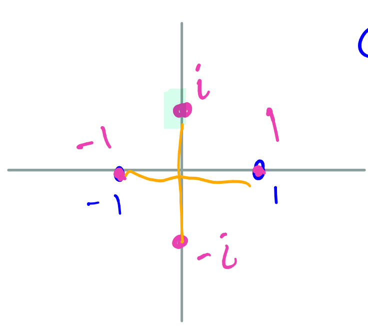
Ex 7: Find $\sqrt[4]{64}$. $= b \rightsquigarrow b^4 = 64 = 8^2 = (2\cdot 4)^2 = (2\cdot 2^2)^2 = 2^2\cdot 2^4 = ((\sqrt{2})^2)^2\cdot 2^4$

$n = 4$ $a = 64$ $\boxed{b = 2\sqrt{2}}$ any! $= (2\sqrt{2})^4$

Thanks to DeMoivre's Theorem we can find Nth roots to complex numbers. In fact, every complex number has Nth roots! This isn't true for real numbers!

Warm-up: Visual Roots. We want to find roots in $\mathbb{C}$.

Q: What are all the square roots of 1? Q: What are all the fourth roots of 1?



$\mathbb{C}$

$\sqrt[2]{1} = ?$ $1^2 = 1$ $(-1)^2 = 1$ (two)

$\sqrt[4]{1} = ?$ $?^4 = 1$

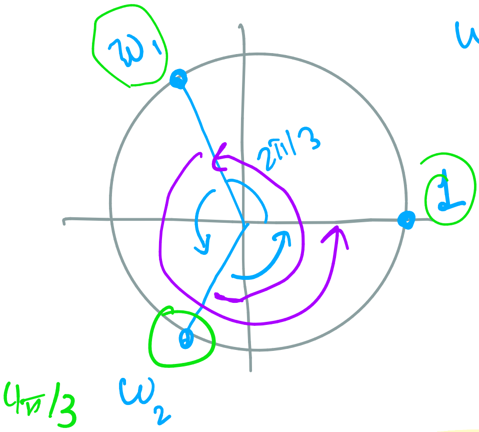
$(1)^4 = 1$ Also $i^4 = i^2\cdot i^2 = (-1)(-1) = 1$

$(-1)^4 = 1$ $(-i)^4 = (-1)^4\cdot(i)^4 = 1\cdot 1 = 1$

Q: What are all the cube roots of 1?

$\sqrt[3]{1} = ? \rightarrow ?^3 = 1$ Easy root: $\boxed{1^3 = 1}$

recall $z^3 = (r(\cos\theta + i\sin\theta))^3$
$= r^3(\cos(3\theta) + i\sin(3\theta))$
$= 1 \cdot \cos(3\theta) + i\sin(3\theta)$



$w_1^3 = 1$

$\hookrightarrow w_1 = 1 \cdot \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Check $\left(1 \cdot \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)\right)^3 = 1^3 \cdot \left(\cos(2\pi) + i\sin(2\pi)\right)$
$= 1(1 + i \cdot 0) = 1.$

Theorem: Nth Roots of Complex Numbers.

If $z \in \mathbb{C}$ is written in polar form, $z = r(\cos\theta + i\sin\theta)$, then for any integer $n \in \mathbb{Z}$ there are exactly n distinct Nth Root of z. They can be found as follows:

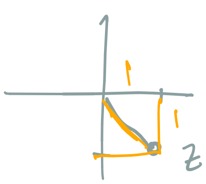
$$\boxed{w_k = \sqrt[n]{z} = z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)}$$

Where $k = 0, 1, 2, \ldots, n-1$.

That is, we write the distinct Nth roots as $w_0, w_1, w_2, \ldots, w_{n-1}$. Note that there are exactly n of them.

Ex 8: (a) Find all the **4th roots** of $1 - i$. (b) Graph the 4th roots in the complex plane.
(c) Verify that $w_0^4 = 1 - i$ and $w_1^4 = 1 - i$.
(d) Solve the equation $z^4 - 1 + i = 0$ for all $z \in \mathbb{C}$.

(a) $z = 1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ $r = \sqrt{2}$, $\theta = -\pi/4$



$w_k = (\sqrt{2})^{1/4}\left(\cos\left(\frac{-\pi/4 + 2\pi k}{4}\right) + i\sin\left(\frac{-\pi/4 + 2\pi k}{4}\right)\right)$, $k = \cancel{0}, 1, 2, 3$

$w_0 = 2^{1/8}\left(\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right)$ $(k=0)$

$w_1 = 2^{1/8}\left(\cos\left(\frac{7\pi}{16}\right) + i\sin\left(\frac{7\pi}{16}\right)\right)$ $(k=1)$ $\frac{-\pi/4 + 2\pi}{4} = \frac{\frac{7\pi}{4}}{4} = \frac{7\pi}{16}$

$w_2 = 2^{1/8}\left(\cos\left(\frac{15\pi}{16}\right) + i\sin\left(\frac{15\pi}{16}\right)\right)$ $(k=2)$ $\frac{-\pi/4 + 4\pi}{4} = \frac{15\pi/4}{4} = \frac{15\pi}{16}$

$w_3 = 2^{1/8}\left(\cos\left(\frac{23\pi}{16}\right) + i\sin\left(\frac{23\pi}{16}\right)\right)$ $(k=3)$ $\frac{-\pi/4 + 6\pi}{4} = \frac{23\pi/4}{4}$

The missing third root of 1

$\sqrt[3]{1} = w \longleftrightarrow w^3 = 1$

$w = r(\cos\theta + i\sin\theta) \rightsquigarrow w^3 = 1$

$$\left(r(\cos\theta + i\sin\theta)\right)^3 = 1$$

$$r^3\left(\cos(3\theta) + i\sin(3\theta)\right) = 1 = 1(\cos 0 + i\sin 0)$$

$$r^3 = 1 \longrightarrow r = \sqrt[3]{1} = 1. \quad (r \text{ is real } \#)$$

$$\cos(3\theta) = \cos 0 = 1$$

$$3\theta = 0 + 2\pi k, \quad k \in \mathbb{Z}$$

$$\theta = \frac{2\pi}{3}k, \quad k \in \mathbb{Z}$$

$$w_k = 1 \cdot \left(\cos\left(\frac{2\pi}{3}k\right) + i\sin\left(\frac{2\pi}{3}k\right)\right)$$

$k=0$ $\quad w_0 = \cos(0) + i\sin(0) = 1$

$k=1$ $\quad w_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$k=2$ $\quad w_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

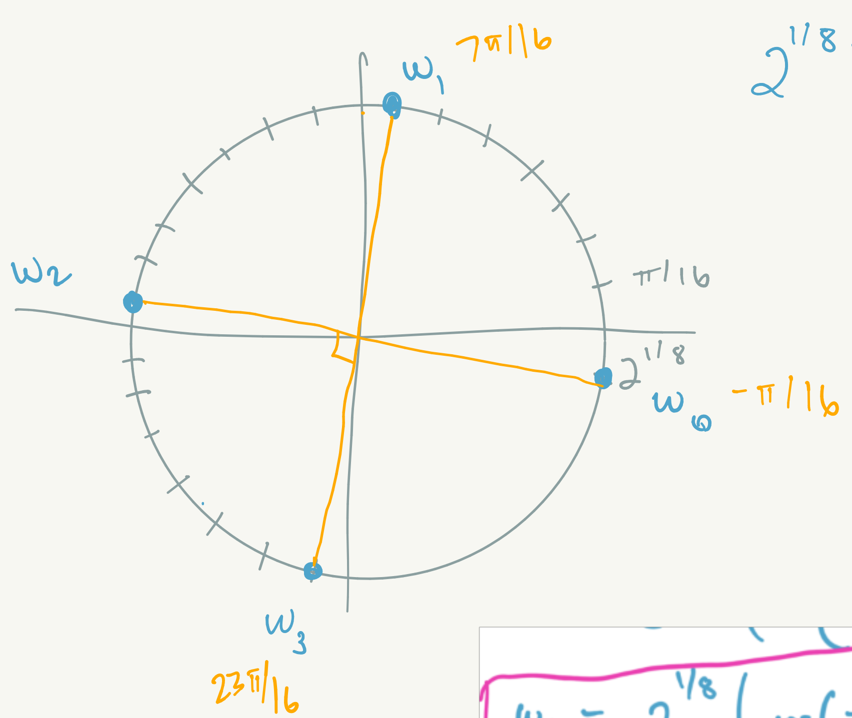
check $\quad w_2^3 = \cos\left(\cancel{3}\left(\frac{4\pi}{\cancel{3}}\right)\right) + i\sin\left(\cancel{3}\left(\frac{4\pi}{\cancel{3}}\right)\right)$ (DeMoivre's)

$= \cos(4\pi) + i\sin(4\pi)$

$= 1 + i\,0 = 1$ ✓

Ex 8 (b)

$2^{1/8} \cong 1.09$



$$\boxed{\begin{aligned}
w_0 &= 2^{1/8}\left(\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right) \\
w_1 &= 2^{1/8}\left(\cos\left(\frac{7\pi}{16}\right) + i\sin\left(\frac{7\pi}{16}\right)\right) \\
w_2 &= 2^{1/8}\left(\cos\left(\frac{15\pi}{16}\right) + i\sin\left(\frac{15\pi}{16}\right)\right) \\
w_3 &= 2^{1/8}\left(\cos\left(\frac{23\pi}{16}\right) + i\sin\left(\frac{23\pi}{16}\right)\right)
\end{aligned}}$$

(k =