# Section 11.1 Matrices and Systems of $\boxed{\text{Linear}}$ Equations

Goal: Solve linear SOE

But in a systematic way that can help us solve EQs w/ many variables

**Objectives**
- Augmented Matrix of a Linear SOE
- Matrices
- Elementary Row Operations
- Gaussian Elimination and Row-Echelon Form (REF)
- Gauss-Jordan Elimination and Reduced Row-Echelon Form (RREF)
- Inconsistent and Dependent Systems

## • Augmented Matrix of a Linear SOE

**Intro** If we want to solve large SOEs, where "large" is 3 or more variables, we need a more systematic approach which we will learn in this section.

Let's consider a simple SOE:

$$\begin{cases} 1x + 4y &= 14 \\ 3x - 2y &= 0 \end{cases}$$

2 EQs in 2 unknowns

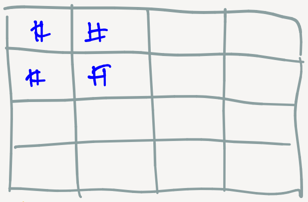
We can arrange the same information concisely as:

$$\begin{array}{ccc} x & y & = \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right]$$

We decide to order the variables as $x$ is the first and $y$ is the second, so the first column corresponds to the coefficients in front of the $x$ variable.

We put vertical bars to separate the last column to help us remember that the last column corresponds to the numbers on the right side of the equal sign.

**Defn 1**
This is an example of a **matrix**; which is a rectangular array of numbers.
The matrix associated to the system of equations (SOE) is called an **augmented matrix**.



**Ex 1** Write the augmented matrix of each SOE:

(a) $\begin{cases} 3x - 4y = -6 \\ 2x - 3y = -5 \end{cases}$

Augmented matrix

$$\begin{array}{ccc} x & y & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 3 & -4 & -6 \\ 2 & -3 & -5 \end{array}\right]$$

(b) $\begin{cases} 2x - y + z = 0 \\ x + z - 1 = 0 \\ x + 2y - 8 = 0 \end{cases}$ → wrong order!

$$\begin{array}{ccc} x & y & z \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 8 \end{array}\right]$$

$x + 0y + z = 1$

KEY
- remember correct order
- remember to put 0s for missing variables

**Ex 2** We can, of course, recover the original SOE from an augmented matrix. Write the SOE.

$$\begin{array}{cccc} x & y & z & = \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ 2 & 0 & 2 & 8 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{cases} 3x - y - z = 7 \\ 2x + 2z = 8 \\ y + z = 0 \end{cases}$$

**Note on Missing Variables** **Reminder** If a variable is missing, you must put a 0 in the correct spot.

## • Matrices

**Defn 2**

A SOE of $m$ equations in $n$ variables is written as:

$$\begin{array}{l} EQ1 \\ EQ2 \\ \vdots \\ EQm \end{array} \quad \underbrace{\phantom{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n}}_{n \text{ variables}}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2,n-1}x_{n-1} + a_{2,n}x_n & = b_2 \\ & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n-1}x_{n-1} + a_{m,n}x_n & = b_m \end{cases}$$

$x_1$ $x_2$ $x_n$ $b$

with corresponding **augmented matrix**:

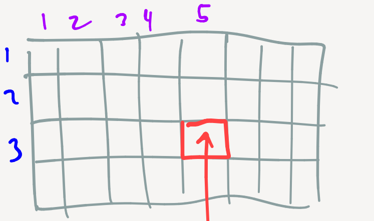
$$\begin{array}{c|ccccc} & Var1 & Var2 & & & Varn \\ \hline EQ1 & a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\ EQ2 & a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2,n} \\ & & & \vdots & & \\ EQm & a_{m,1} & a_{m,2} & \cdots & a_{m,n-1} & a_{m,n} \end{array} \quad \left|\begin{array}{c} b_1 \\ b_2 \\ \\ b_m \end{array}\right.$$

→ to study this in total generality take a course called "Linear Algebra"

We write this in concise notation as $\boxed{[\mathbf{A} \,|\, \mathbf{b}]}$ where

rows

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2,n} \\ & & \vdots & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n-1} & a_{m,n} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$




in row 3, column 5

We call $\mathbf{A}$ is itself a **matrix of coefficients** and $b$ the **matrix of b values**.
**Other key terms** **Rows** **Columns** **Dimension (or Size)**

"rows by columns"
"m × n"

**Act 1** Give an example of matrices of size: $2 \times 2$, $2 \times 3$, $3 \times 1$, and $1 \times 4$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix}_{2\times 3} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} \qquad \begin{bmatrix} 0 & 1 & -1 & 3 \end{bmatrix}_{1\times 4}$$

2 by 2 matrix

## • Elementary Row Operations

Let's consider a simple SOE:

$$\begin{cases} x + 4y & = 14 \\ 3x - 2y & = 0 \end{cases}$$

There's simple rules we know we are allowed to do when trying to solve the SOE. We can add the two equations together, subtract them, multiply one equation by 2, etc.

multiply EQ2 by 2

$$\begin{cases} x + 4y = 14 \\ 6x - 4y = 0 \end{cases}$$

$+$

$7x = 14$

$x = 2$

multiply by $(1/7)$ →

& then back-sub

Since the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right] \xrightarrow{\text{multiply EQ2 by 2}} \begin{bmatrix} 1 & 4 & 14 \\ 6 & -4 & 0 \end{bmatrix}$$

contains exactly the same information as the SOE, we can apply these same rules to the matrix itself.

$$\xrightarrow[\text{\& put into row 2}]{\text{add two EQ together}} \begin{bmatrix} 1 & 4 & 14 \\ 7 & 0 & 14 \end{bmatrix} \xrightarrow[\frac{1}{7}R_2 \to R_2]{\text{multiply by } 1/7 \text{ on Row 2}} \begin{bmatrix} 1 & 4 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

$R_2 + 1 \cdot R_1 \to R_2$

**Important** These rules do not change the solution set!

The upshot is that its way easier to keep track of!

**Elementary Row Operations (EROs)** are the following:

1. Add a multiple of one row to another $(R_i + cR_j \to R_i)$
2. Multiply a row by a nonzero constant $(cR_i \to R_i)$ replacing $R_i$ with $c \cdot R_i$
3. Interchange rows $(R_i \leftrightarrow R_j)$

**Important** Performing EROs to an augmented matrix do not change the solution set to a SOE.

Why do we use EROs? What's the purpose?

The answer lies in the Elimination Method we studied earlier in solving SOEs.

The goal is to eliminate all the variables except one in one of the rows (usually the last row). If we do this then we know what the value is and can use back-substitution to continue to solve the problem completely.

Before we actually use EROs, let's look at two trivial examples:

**Ex 3 Trivial Example 1** Let's look at a SOE where we already know the solutions are $x = 1$, $y = 2$, and $z = 3$. Write the augmented matrix of this "system".

$$\begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases} \qquad \begin{cases} x + 0y + 0z = 1 \\ 0x + y + 0z = 2 \\ 0x + 0y + z = 3 \end{cases}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

identity matrix $I_3$

Last Column is "solution matrix"

"one solution"

$$\boxed{X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

**Ex 4 Trivial Example 2**

Next, we look at a system in a special form, called **triangular** form:
$$\begin{cases} x - 2y + z = 0 \\ \quad\;\; y - 3z = -8 \\ \qquad\quad\;\; z = 3 \end{cases}$$

Back sub ($z = 3$ & $y = 1$)

Back Substitution ($z = 3$)

$y - 3(3) = -8$

$y = 1$

Write the augmented matrix of this "system".
Recall we can solve the above system easily using back-substitution.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

**Solving SOEs using EROs (part 1)** Perform EROs to the augmented matrix until

1. the matrix is in triangular form → zeros below diagonal & 1s on the diagonal
2. continue solving using back-substitution

**Ex 5** Solve the SOE using EROs:
$$\begin{cases} x - 2y + z = 0 \\ \qquad 2y - 8z = 8 \\ -4x + 5y + 9z = -9 \end{cases}$$

**Solution**

We start with the augmented matrix:

EROs

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \end{array} \xrightarrow{R_3 + 4R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{(1/2)R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 + 3R_2 \to R_3} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \end{array} \qquad \begin{cases} x - 2y + z = 0 \\ y - 4z = 4 \\ z = 3 \end{cases}$$

triangular form!

Back sub!

Back sub!

Then continue solving using back-substitution.

**Important Note** You must write the EROs with each as part of your work as shown in this example.

## • Gaussian Elimination and Gauss-Jordan Elimination

**GOAL of GJE** **START** $[A\,|\,b]$ $\xrightarrow[\text{Apply EROs}]{\text{GJE}}$ **END** $[I_n^*\,|\,c]$

RREF

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ solutions

$X = c$

$\begin{cases} x_1 = c_1 \\ x_2 = c_2 \\ \vdots \end{cases}$

**What is $I_n^*$?** It depends on the number of solutions. We'll discuss this in more detail later.

For now, we simply think of this as an identity matrix $I_n$ as in "Trivial example 1" (Example 3).

**Ex 6 Identity Matrices** Write $I_2$, $I_3$, and $I_n$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_n = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$

**What is $c$?** This column of numbers is the solution matrix!

Recall trivial example from earlier. This explains the **"end goal"** of GJE.

**Defn 3 Row Echelon Form (REF)**

A matrix is said to be in **row echelon form** if
1. The first nonzero number in each row is a 1 (from left to right).
   This is called a **leading 1**. And the variable it corresponds to is called a **leading variable**. (leaders)
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

**Defn 4 Reduced Row Echelon Form (RREF)**

A matrix is said to be in **reduced row echelon form** if
1. it is in REF (satisfies all conditions for REF above)
2. Every number **above and below leading 1s are zeros**.

**REF vs RREF** **Important Difference** REF is **NOT** unique whereas **R**REF IS UNIQUE

**Bonus: Etymology of "echelon"** What does the word "echelon" mean?

It comes from the French word relating to hierarchy of rank.
This is used in English too ("The are part of the upper echelon of society").

In French military terms, *echellon* refers to a "step-like arrangement of troops". Further etymology of the word suggests it comes from "ladder" or "steps."

Of course, because of the leading 1s moving to the right, as we go down the matrix, this idea of "steps" makes sense!

**Act 2** Determine if the following matrices are in REF, RREF, or neither.
Identify all leading 1s.

(a) $\begin{bmatrix} 0 & 1 & -3 & 0 & -3 \\ \cancel{1} & 0 & 4 & -4 & 8 \\ 0 & 0 & 0 & 1 & 0.2 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

neither

(b) $\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & 8 \\ 0 & 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REF ✓
not in RREF

(c) $\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REF ✓
RREF ✓

(d) $\begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REF ✓
RREF ✓
(both)

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cancel{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Neither

(f) $\begin{bmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REF ✓

**How to do Gaussian Elimination** The goal of Gaussian Elimination is to put an augmented matrix into reduced row echelon form (REF) using EROs.
The key is to follow a special order, or algorithm.

First Column

- **Step 1** Start by obtaining a 1 in the top-left corner.
Then obtain zeros below the 1 by adding/subtracting appropriate multiples of the first row to the rows below it. That is, use ERO1 to eliminate the numbers below the leading 1.

When you're done, your first column should look like this: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. ← start here

next

$* =$ ignore until later

- **Step 2** Next, obtain a leading 1 in the next column.
Then obtain zeros below the leading 1 by adding/subtracting appropriate multiples of the first row to the rows below it. Notice that we don't care or worry about the first number in the second column. That is why we use an astrix ($*$) there.

When you're done, your second column should look like this: $\begin{bmatrix} * \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. ← start here

- **Step k** Continue "downward and rightward"
At each step, you ignore the first few numbers until you create a leading 1, then work downwards creating zeros.
Keep in mind that you want to make sure that every leading 1 is to the right of the leading 1 in the row above it–rearrange the rows if necessary.
The end result should look like this:

$$\begin{bmatrix} 1 & * & * & \cdots & * & * & * \\ 0 & 1 & * & \cdots & * & * & * \\ 0 & 0 & 1 & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 1 & * & * \\ 0 & 0 & 0 & \cdots & 0 & 1 & * \end{bmatrix}$$

when done w/ these steps you're in REF

Gauss-JORDAN "upwards & backwards"

- **Back Substitution** Once you have put the matrix in REF, you can continue to solve using back substitution.

Solving SOEs using this technique is called **Gaussian Elimination** in honor of it's German inventor Carl F. Gauss.
This is also called **row reducing** because we are applying EROs (in a specific way).

**Ex 7** Steps of Gaussian Elimination for a $3 \times 3$ system. That is, a system with 3 variables.
Note that the augmented matrix has size $3 \times 4$.

**Gaussian Elimination "downwards and forwards"**

start / start

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right] \to \begin{array}{cccc} x & y & z & \\ \end{array} \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

**Summary of Gaussian Elimination** Do the following steps:
Step 1: Write the Augmented Matrix.
Step 2: Row-reduce until you are in REF. (Follow the steps in the "algorithm")
Step 3: Finish solving using Back-substitution.

**Act 3** Solve the system of equations using Gaussian Elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

EROs

① $R_i + cR_j \to R_i$

② $cR_i \to R_i$

③ $R_i \leftrightarrow R_j$

$$\left[\begin{array}{ccc|c} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right] \xrightarrow[R_1 - R_2 \to R_1]{\text{ERO①}} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 15 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

$$\xrightarrow[R_2 - 3R_1 \to R_2]{\text{ERO①}} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 15 \\ 0 & 8 & 32 & -56 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

$$\xrightarrow[R_3 + 2R_1 \to R_3]{\text{ERO①}} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 15 \\ 0 & 8 & 32 & -56 \\ 0 & 1 & -6 & 13 \end{array}\right] \quad \text{"old"}$$

step1 GEA

EROs down here use rows from "new" matrix

$$\xrightarrow[(\frac{1}{8})R_2 \to R_2,\ R_3 - R_2 \to R_3]{\text{ERO②}} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 15 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{array}\right] \quad \text{"new"}$$

step2 GEA

$$\xrightarrow{(-\frac{1}{10})R_3 \to R_3} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -9 & 15 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right] \end{array}$$

IS IN REF ?!

write SOE

$$\begin{cases} x - 9z = 15 \\ y + 4z = -7 \\ \boxed{z = -2} \end{cases}$$

Back sub

- $z = -2$
- $y + 4(-2) = -7 \longrightarrow y = 1$
- $x - 9(-2) = 15 \longrightarrow x = -3$

solution set: "one solution"

$$\boxed{X = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \right\}$$

• **Gaussian Elimination vs Gauss-Jordan Elimination**

When we do EROs and arrive at REF, then continue to solve using back-sub, this is called Gaussian Elimination.

However, we can solve the SOE completely using EROs.

So, if we continue to perform EROs **"upwards and backwards"** we can arrive at the **R**REF of the augmented matrix.
If we use row reduction to obtain the RREF then we do not need to use back-substitution to solve the system.

**How to do Gauss-Jordan Elimination** Use the following steps:
Step 1: Follow the steps in Gaussian Elimination to arrive at REF.
Step 2: Continue "upwards and backwards": use EROs to obtain zeros above the leading 1s starting from the right-most leading 1.
That is, try to make the augmented matrix into the identity matrix $I_n^*$.

**Ex 8** Steps of Gaussian Elimination for a $3 \times 3$ system. That is, a system with 3 variables.
Note that the augmented matrix has size $3 \times 4$.

**Gaussian Elimination "downwards and forwards"**

$$\begin{bmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \to \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{bmatrix} \to \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

**Gauss-Jordan Elimination "upward and backwards"**

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \to \begin{bmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

**Act 4** Solve the system of equations using Gauss-Jordan Elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Note: this is the same system as in Activity 3 so start from the REF.

$I_3$ idenity matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -9 & 15 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_1 + 9R_3 \to R_1]{R_2 - 4R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

start w/ REF

now in RREF & are done!

$$\begin{cases} x + 0y + 0z = -3 \\ 0x + y + 0z = 1 \\ 0x + 0y + z = -2 \end{cases} \qquad \begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

$$\boxed{\underline{X} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}}$$

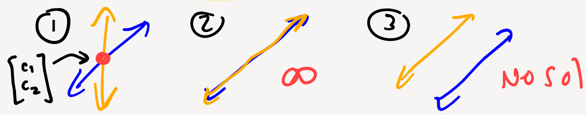
# • Inconsistent and Dependent Systems

**Intro** Why bother? Do we really need GJE?

Yes! Recall 3 types of solution sets:

**Number of Solutions**
1. One
2. Infinitely Many
3. No solutions

When we study **linear** SOEs in several variables, there are still only 3 possibilities for the solution set!




**Defn 5 Consisten vs Inconsistent SOEs**

A SOE with **at least one solution** (case 1 or 2) is called **consistent**.

A SOE with **no solutions** is called **inconsistent** (case 3).

In higher dimensions where we can't use lines, it can be difficult or impossible to visualize the solution set. However, the beauty of GJE and putting a matrix in RREF is that we can easily answer which case we are in.

**Number of Solutions and RREF** Let $M = [A \mid b]$ (Starting) denote the starting augmented matrix. Let $\widetilde{M} = [I^* \mid c]$ (end of GJE) denote the end result of GJE and be in RREF.

1. **One Solution** $I^*$ is the identity matrix and every variable has a leading 1.

   Explanation:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_n \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & & 0 & c_1 \\ 0 & 1 & 0 & & 0 & c_2 \\ 0 & 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \\ 0 & 0 & 0 & & 1 & c_n \end{array}\right] \\ I_n \end{array} \qquad \begin{cases} x_1 = c_1 \\ x_2 = c_2 \\ \vdots \\ x_n = c_n \end{cases} \qquad \boxed{X = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}$$

"one (matrix) solution"

"n-dimensional point"

2. **Infinitely Many** There is a row made entirely of zeros in $\widetilde{M}$.

   Explanation:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & x_n \end{array} \\ \left[\begin{array}{cccc|c} & & & & \\ 0 & 0 & \cdots & 0 & 0 \\ & & & & \end{array}\right] \end{array} \to 0 = 0$$

always true!

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & \boxed{0} & 0 \end{array}\right] \end{array} \qquad \boxed{0 \cdot z = 0}$$

↳ z can be anything!

always true!

Notation

Let $z = t$ (parameter) $t \in \mathbb{R}$

called a free variable

3. **No solutions** There is a row with zeros everywhere except the very last column. That is, we have a row like this: $[00 \cdots 0|c]$ where $c \neq 0$.

   Explanation:

$$\left[\begin{array}{cccc|c} & & & & \\ 0 & 0 & \cdots & 0 & c \end{array}\right]$$

NOT ZERO!

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 4 \end{array}\right] \end{array} \to \boxed{0 \cdot z = 4}$$

↳ This has no solutions!

Always False

$\boxed{\text{a contradiction}}$

consistent : at least one sol
↳ case 1 or 2
inconsistent : no sol (case 3)

**Act 5** The RREF of a given SOE is given below.
(i) State whether the SOE is consistent or inconsistent.
(ii) Determine how many solutions the SOE has.
(iii) Write the solution set.

(a) $\begin{array}{c} \begin{matrix} x & y & z & & \end{matrix} \\ \left[\begin{array}{ccc|c} \boxed{1} & -3 & 0 & -3 \\ 0 & 0 & \boxed{1} & 0.2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ (b) $\left[\begin{array}{ccc|c} 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 5 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & \pi \end{array}\right]$

(a) $\infty$ ly many sol case!

i) $\boxed{\text{consistent}}$

ii) infinitely many sol

iii) $\begin{cases} x - 3y + 0z = -3 \\ z = 0.2 \\ 0 \cdot y = 0 \rightarrow y \text{ free} \end{cases}$

$y = t \in \mathbb{R}$

$\begin{cases} x - 3t = -3 \longrightarrow x = -3 + 3t \\ z = 0.2 \longrightarrow z = 0.2 \\ y = t \longrightarrow y = t \end{cases}$

solution set: $\boxed{\left\{ \begin{bmatrix} -3+3t \\ t \\ 0.2 \end{bmatrix} : t \in \mathbb{R} \right\}}$

(b) i) $\boxed{\text{inconsistent}}$ ii) no sol iii) $\emptyset$

(c) i) $\boxed{\text{inconsistent}}$ ii) no sol iii) $\emptyset$

**How to write solutions sets for infinitely many solutions** If a SOE is in RREF and there is an entire row of zeros, then the variable which "should have a 1 but doesn't" is called a **free variable**. That is, a free variable is a variable that does not have a leading 1.

We denote free variables using a parameter such as $z = t$, where $t \in \mathbb{R}$. If there is more than one free variable use $r, s, t, \ldots$

Next, solve for the leading variables (variables with a leading 1) in terms of the free variables.

**Act 6** The RREF of a given SOE is given by:

$\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 9 \\ 0 & \boxed{1} & -2 & 5 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right] \end{array}$ → $\infty$ ly many sol.

x & y are leaders
z free

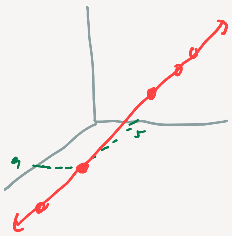
(i) State whether the SOE is consistent or inconsistent.
(ii) Determine how many solutions the SOE has.
(iii) Write the solution set.

$\begin{cases} x + 4z = 9 \longrightarrow x = 9 - 4z = 9 - 4t \\ y - 2z = 5 \longrightarrow y = 5 + 2z = 5 + 2t \\ 0 \cdot z = 0 \longrightarrow z = t \in \mathbb{R} \end{cases}$

z can be any real #

Parametric EQs → $\boxed{\begin{cases} x = 9 - 4t \\ y = 5 + 2t \\ z = t \end{cases} \quad (t \in \mathbb{R})}$

In this example, a line in 3 dimensional space



solution set: $\boxed{\left\{ \begin{bmatrix} 9-4t \\ 5+2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}}$ ✱

For example: $t = 0$: $\begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix}$ is a solution to original SOE

Set Notation
{ thing : conditions }

## Bonus Example

Solve SOE:

$$\begin{cases} -4x - y + 36z = 24 \\ x - 2y + 9z = 3 \\ -2x + y + 6z = 6 \end{cases}$$

$$\left[\begin{array}{ccc|c} -4 & -1 & 36 & 24 \\ 1 & -2 & 9 & 3 \\ -2 & 1 & 6 & 6 \end{array}\right] \xrightarrow[R_3 + 2R_2 \to R_3]{R_1 + 4R_2 \to R_1} \left[\begin{array}{ccc|c} 0 & -9 & 72 & 36 \\ 1 & -2 & 9 & 3 \\ 0 & -3 & 24 & 12 \end{array}\right]$$

$$\xrightarrow[(\frac{-1}{3})R_3 \to]{(\frac{-1}{9})R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & -8 & -4 \\ 1 & -2 & 9 & 3 \\ 0 & 1 & -8 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & -8 & -4 \\ 1 & -2 & 9 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & \boxed{-2} & 9 & 3 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here it is in REF

$$\xrightarrow{R_1 + 2R_2 \to R_1} \begin{array}{c} \begin{array}{cccc} x & y & & \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 0 & -7 & -5 \\ 0 & \boxed{1} & -8 & -4 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right] \end{array}$$

z free

x, y leaders

Now this is in RREF

$$\begin{array}{cccc} x & y & & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -7 & -5 \\ 0 & \boxed{1} & -8 & -4 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

z free

x, y leaders

Now this is in RREF

$$\begin{cases} x - 7z = -5 \\ y - 8z = -4 \\ 0 \cdot z = 0 \longrightarrow z \text{ free} \end{cases}$$

$z = t$

$$\begin{cases} x = -5 + 7t \\ y = -4 + 8t \\ z = t \end{cases}$$

Solution set :

$$\boxed{\left\{ \begin{bmatrix} -5 + 7t \\ -4 + 8t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}}$$

## • Bonus Problem: Special $k$

**Act 7 Special $k$** Consider the following system:

$$\begin{cases} x + 2y + 3z = 0 \\ 3x + 6y + kz = 0 \\ x + 4z = 0 \end{cases}$$

(a) Explain why regardless of the value of $k$, $X_0 = [0, 0, 0]$ is always a solution to the SOE. That is, $x = 0$, $y = 0$, $z = 0$. This is called a **trivial solution**.

(b) Find a value of $k$ to ensure that this system has a non-trivial solution. That is, at least one other solution that is not $X_0$.