Math 5B - Calculus II | Spring 2019

Exam 3

§7.7, 7.8, 11.1–11.5 | **Dr. Jorge Basilio**

May_9 | **gbasilio@pasadena.edu**

Honesty Pledge

On my honor, by printing and signing my name below, I vow to neither receive nor given any unauthorized assistance on this examination:

NAME (PRINT): Solution SIGNATURE: ______________________

Directions

- YOU ARE ALLOWED TO USE ONLY A SCIENTIFIC CALCULATOR ON THIS EXAM.
- You have 90 minutes to complete this exam.
- The exam totals **110 points**. There are two extra-credit problems also, each worth 5 points (so 120 points are possible out of 110).
- There are 11 problems, many of them with multiple parts.
- Place all of your belongings in the front of the classroom and I will assign you a seat. Bring with you your writing utensils.
- Cell phones must be turned off and put away in with your items in the front of the classroom.
- Handwriting should be neat and legible. If I cannot read your writing, zero points will be given.
- Some questions contain multiple-parts which you must do individually and the parts are denoted by (a), (b), (c), etc. Some questions are multiple-choice and the choices are denoted with (A), (B), (C), (D), and (E).
- Make sure to ALWAYS SHOW YOUR WORK; you will not receive any partial credits unless work is clearly shown. *If in doubt, ask for clarification.*
- Leave answers in exact form (as simplified as possible), unless told otherwise.
- Put a $\boxed{\text{box around your final answer}}$ where applicable.
- Give $\boxed{\text{UNITS}}$ where applicable.
- **PLEASE CHECK YOUR WORK!!!**
- If you need extra space, there is extra space on the back of the cover page and clearly indicate that you are continuing your work there in the original location.
- If you finish early, you may take a break but you must come back to class by 5:30 and we will have class.
- I will take attendance at the end of class

Score	Grade

This page is intentionally blank. It may be used for scratch paper. If you wish for me to grade your work on this page, please (i) label the problem you are working on, (ii) box your answer, (iii) indicate in the original problem's location that you will continue your work on this page.

Problem 1: 16 pts (2 pts each)

Fill-in the blank:

(a) Express the series $1 - \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} - \frac{1}{\sqrt[4]{4}} + \cdots$ using Sigma notation: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[4]{n}}$ (other ans possible)

$+ \quad - \quad + \quad -$

(b) The fourth partial sum S_4 of $\sum_{j=0}^{\infty} \frac{1}{j!}$ is $\frac{65}{24}$ or $2.708\overline{3}$

$S_4 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24} = 2.708\overline{3}$
$= 65/24$

(c) The general term of the sequence $2, -\frac{4}{3}, \frac{6}{9}, -\frac{8}{27}, \ldots$ is $(-1)^n \frac{(2+2n)}{3^n}$, $n = 0, 1, 2, 3, \ldots$

$\frac{2}{3^0}$

$(-1)^{\#} \frac{\text{even}}{3^{\#}}$

(d) List all the requirements needed to apply the integral test:
1. $f(x)$ continuous
2. $f(x)$ positive
3. $f(x)$ decreasing

on $[1, \infty)$

4. $f(n) = a_n$

(e) List all the requirements needed to apply the alternating series test: (1) $b_n > 0$ (2) $\{b_n\}$ decreasing (3) $\lim_{n\to\infty} b_n = 0$

$\sum a_n = \sum (-1)^n b_n$

TRUE or **FALSE** (please spell out/write the entire word for credit). *(No work needed)*

(a) FALSE If $0 \leq a_n \leq b_n$, for all n, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ ~~diverges~~. Cannot be determined based on assumptions.

(b) FALSE If $\lim_{n\to\infty} a_k = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. This isn't true, but the converse is.
Consider: $\sum \frac{1}{k}$ harmonic! $a_k = \frac{1}{k} \to 0$ but series diverges

(c) FALSE The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ ~~diverges~~. converges

Problem 2: 6 pts

Multiple-choice: Please circle the entire correct answer.
Consider:

(I) $\int_{-1}^{1} \frac{1}{x^2} dx$ (II) $\int_{0}^{1} \frac{1}{\sqrt{x}} dx$ (III) $\int_{-1}^{1} \frac{1}{\sqrt[3]{x^3}} dx$

One of the following statements is true. Which one? *Show some work.*

~~(A) I is convergent; II and III are divergent~~

(B) II and III are convergent; I is divergent

$\boxed{\text{(C) II is convergent; I and III are divergent}}$

~~(D) All are divergent~~

~~(E) All are convergent~~

(I) $\underbrace{\int_{-1}^{1} \frac{1}{x^2} dx}_{\text{type II improper int}} = \int_{-1}^{0} \frac{1}{x^2} dx + \underbrace{\int_{0}^{1} \frac{1}{x^2} dx}$

$\hookrightarrow = \lim_{r\to 0^+} \int_r^1 x^{-2} dx = \lim_{r\to 0^+} \overbrace{\frac{x^{-1}}{-1}}^{-\frac{1}{x}} \Big|_r^1 = \lim_{r\to 0^+} \frac{-1}{x} \Big|_r^1 = \lim_{r\to 0^+} \left(-1 + \frac{1}{r}\right) = -1 + \frac{1}{+0} = +\infty$ $\boxed{D}$

eliminate (E) & (A)

$\hookrightarrow$ or use p-test type II: $p = 2 > 1 \to \boxed{D}$

(II) $\int_0^1 \frac{1}{\sqrt{x}} dx = \int_0^1 \frac{1}{x^{1/2}} dx \to$ p-test type II
$p = \frac{1}{2} < 1 \to \boxed{C} \to$ eliminate (D)

(III) $\underbrace{\int_{-1}^{1} \frac{1}{\sqrt[3]{x^3}} dx}_{\text{type II improper}} = \int_{-1}^{0} \frac{1}{x} dx + \underbrace{\int_{0}^{1} \frac{1}{x} dx}$

$\hookrightarrow p = 1 \geq 1 \to \boxed{D} \to$ answer is (C)
p-test type II

Problem 3: 6 pts

Multiple-choice: Please circle the entire correct answer.
Consider:

$$\text{(I)}\quad \left\{(-1)^{n+1}\frac{n^2-1}{2n^2+1}\right\}_{n=1}^{\infty} \qquad \text{(II)}\quad \left\{\frac{\ln(n)}{n^2}\right\}_{n=1}^{\infty} \qquad \text{(III)}\quad \left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$$

One of the following statements is true. Which one? *Show some work.*

(A) All are divergent

(B) All are convergent

(C) I is convergent; II and III are divergent

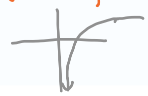
(D) II and III are convergent; I is divergent

(E) $\boxed{\text{(E) II is convergent; I and III are divergent}}$

(I) $\lim_{n\to\infty}\left((-1)^{n+1}\frac{n^2-1}{2n^2+1}\right) \underset{\text{DTT}}{=} \lim_{n\to\infty}\left((-1)^{n+1}\frac{n^2}{2n^2}\right) = \lim_{n\to\infty}\left(\frac{(-1)^{n+1}}{2}\right) =$ DNE (oscillates b/w $-\frac{1}{2}$ & $\frac{1}{2}$) $\boxed{D}$

(II) $\lim_{n\to\infty}\left(\frac{\ln(n)}{n^2}\right) = \underbrace{\frac{\infty}{\infty}}_{\text{indeterminate form}} \overset{\text{L'Hôp}}{=} \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{2n}\right) = \lim_{n\to\infty}\frac{1}{2n^2} = 0. \to \boxed{C}$ & limit is 0

(III) $\lim_{n\to\infty}\left(\ln\left(\frac{1}{n}\right)\right) = \ln(0) = -\infty$ $\boxed{D}$



Problem 4: 15 pts

Consider

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

(3 pts) (a) Show the conditions for the integral test are satisfied.

let $f(x) = \frac{1}{x^4}$. let $a_n = \frac{1}{n^4}$.

Then ① $f(n) = \frac{1}{n^4} = a_n$ ✓

② $f(x)$ continuous on $[1,\infty)$ ✓

③ $f(x)$ positive on $[1,\infty)$ ✓

④ $f(x)$ decreasing on $[1,\infty)$

Method 1: x^4 is increasing on $[1,\infty)$, so for $x,y \in [1,\infty)$ $x<y$, we have $x^4<y^4$. Thus, dividing (x mult) we get $\frac{1}{y^4} < \frac{1}{x^4}$. So $f(y)<f(x)$ which shows f is decreasing.

Method 2: $f'(x) = -4x^{-5} = \frac{-4}{x^5}$. For $x\in[1,\infty)$, $f'(x)<0$. By IDTest $\to$ f is decreasing on $[1,\infty)$.

(6 pts) (b) Use the integral test to determine whether the series converges or diverges.

$$\int_1^{\infty}\frac{1}{x^4}dx = \lim_{R\to\infty}\int_1^R x^{-4}dx = \lim_{R\to\infty}\frac{x^{-3}}{-3}\Big|_1^R = \lim_{R\to\infty}\frac{-1}{3x^3}\Big|_1^R = \lim_{R\to\infty}\left[\frac{-1}{3R^3}+\frac{1}{3}\right] = \frac{1}{\infty}+\frac{1}{3} = \boxed{\frac{1}{3}} \to \int_1^{\infty}\frac{1}{x^4}dx\ \boxed{C}$$

Integral test says: both C or both D.

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^4}$ $\boxed{\text{converges.}}$

(6 pts) (c) How many terms are needed for the partial sum S_N to be accurate to within 0.005?

The error is R_N (since $S = S_N + R_N$). Remainder estimate says: $R_N \le \int_N^{\infty} f(x)dx$. So if $\int_N^{\infty} f(x)\,dx \le 0.005$ then $R_N \le 0.005$ as well. Find N:

- $\int_N^{\infty}\frac{1}{x^4}dx = \lim_{R\to\infty}\frac{-1}{3x^3}\Big|_N^R = \lim_{R\to\infty}\left[\frac{-1}{3R^3}+\frac{1}{3N^3}\right] = \frac{1}{3N^3}.$

- $\int_N^{\infty}\frac{1}{x^4}dx \le 0.005 \to \frac{1}{3N^3} \le 0.005$

$$\frac{1}{3*0.005} \le N^3$$

$$4.054\ldots \approx \sqrt[3]{\frac{1}{3*0.005}} \le N \longrightarrow \text{so take } \boxed{N=5}.$$

So need 5 terms in order for partial sum to be accurate within 0.005.

Problem 5: 6 pts

Evaluate the improper integral: $\displaystyle\int_{\frac{2}{\pi}}^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx.$

$$\int_{2/\pi}^{\infty} \frac{\sin(\frac{1}{x})}{x^2}dx = \lim_{R\to\infty}\int_{2/\pi}^{R} \frac{\sin(\frac{1}{x})}{x^2}dx = \lim_{R\to\infty}\int_{\pi/2}^{1/R} \sin(u)\cdot -du = \lim_{R\to\infty} \cos(u)\Big|_{\pi/2}^{1/R} = \lim_{R\to\infty}\left[\cos(\tfrac{1}{R}) - \underbrace{\cos(\tfrac{\pi}{2})}_{=0}\right] = \lim_{R\to\infty}\cos(\tfrac{1}{R}) = \cos(0) = \boxed{1}$$

since $\cos(x)$ is continuous.

$u = \frac{1}{x} = x^{-1}$

$du = -x^{-2}dx = -\frac{1}{x^2}dx \to \frac{1}{x^2}dx = -du$

$x = R \to u = \frac{1}{R}$

$x = \frac{2}{\pi} \to u = \frac{\pi}{2}$

Problem 6: 12 pts

(a) Use the (integral) **comparison test** to determine whether the integral converges or diverges: $\displaystyle\int_1^{\infty} \frac{\cos\left(x^3\right)+2}{x}\, dx.$

- $-1 \le \cos(x) \le 1$ for all x

  $-1 \le \cos(x^3) \le 1$ for all x

  $+2 \quad +2 \quad +2$

  $\boxed{1 \le \cos(x^3)+2} \le 3$ for all x

- so for $x \ge 1$,

  $\int_1^R \frac{1}{x} \le \int_1^R \frac{\cos(x^3)+2}{x}$ for any $R \ge 1$

$\to \lim_{R\to\infty}\int_1^R \frac{1}{x}dx \le \lim_{R\to\infty}\int_1^R \frac{\cos(x^3)+2}{x}dx$

- Since $\int_1^\infty \frac{1}{x}dx$ diverges by p-test (type I) ($p=1$)

and $\underbrace{\int_1^\infty \frac{1}{x}dx}_{=\infty} \le \int_1^\infty \frac{\cos(x^3)+2}{x}dx$ we conclude by the integral comparison test that the integral $\boxed{\text{diverges}}$.

(b) Use the (integral) **comparison test** to determine whether the integral converges or diverges: $\displaystyle\int_1^{\infty} \frac{\cos\left(x\right)}{x^3}\, dx.$

- $-1 \le \cos(x) \le 1$ for all x
- For $x \ge 1$,

  $\frac{-1}{x^3} \le \frac{\cos(x)}{x^3} \le \frac{1}{x^3}$

- $\int_1^\infty \frac{1}{x^3}dx \longrightarrow$ converges by p-test w/ $p = 3 > 1$

- Since $\frac{\cos(x)}{x^3} \le \frac{1}{x^3}$ for $x \ge 1$

  & $\int_1^\infty \frac{1}{x^3}dx$ converges

  by the integral comparison test we conclude that $\int_1^\infty \frac{\cos(x)}{x^3}dx$ $\boxed{\text{converges}}$.

Problem 7: 10 pts

Pick ONE (and only one) of the following statements to prove. Do not do more!
Circle ONE: "I will prove (a)" or "I will prove (b)"

(a) State and Prove: the theorem on Geometric Series.

(4 pts) **Statement of theorem:**

Geometric Series Fix $a \in \mathbb{R}$, $r \in \mathbb{R}$.

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \text{Converges } \& = \frac{a}{1-r}, & \text{when } |r|<1 \\ \text{diverges}, & \text{when } |r| \geq 1 \end{cases}$$

(6 pts) **Proof of** (a) & (b) :

Proof of Geometric Series

- $S_N = \sum_{n=0}^{N} ar^n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^N}$

$- r \cdot S_N = \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^N} + ar^{N+1}$

$S_N - rS_N = a - ar^{N+1}$

$S_N(1-r) = a(1-r^{N+1})$

$\to S_N = \frac{a}{1-r}(1-r^{N+1})$.

- $\sum_{n=0}^{\infty} ar^n = S = \lim_{N\to\infty}(S_N)$

$= \lim_{N\to\infty}\left(\underbrace{\frac{a}{1-r}}_{\text{constant in } N}(1-r^{N+1})\right)$

$= \frac{a}{1-r} \underbrace{\lim_{N\to\infty}(1-r^{N+1})}$

now use theorem on $\lim_{n\to\infty}(r^n) = \begin{cases} c=0 & \text{if } |r|<1 \\ D & \text{if } |r|\geq 1 \end{cases}$

This final limit converges when $|r|<1$ and diverges when $|r|\geq 1$.

When it converges, we have

$S = \frac{a}{1-r}\lim_{N\to\infty}(1-r^{N+1}) = \frac{a}{1-r}\cdot 1 - 0$

$= \frac{a}{1-r}$. ☐

(b) State and prove: the p-test type I

P-Test Type I $\int_1^{\infty} \frac{1}{x^p}dx = \begin{cases} \text{converges } \& = \frac{1}{p-1}, & \text{when } p>1 \\ \text{diverges}, & \text{when } p \leq 1 \end{cases}$

Proof of P-Test Type I For $p \neq 1$:

- $\int_1^{\infty} \frac{1}{x^p}dx = \lim_{R\to\infty}\int_1^R x^{-p}dx = \lim_{R\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^R = \lim_{R\to\infty}\left[\frac{R^{1-p}}{1-p} - \frac{1}{1-p}\right]$

$= \lim_{R\to\infty}\frac{R^{1-p}}{1-p} - \frac{1}{1-p} = \lim_{R\to\infty}\frac{R^{-(p-1)}}{-(p-1)} - \left(\frac{1}{-(p-1)}\right)$

$= \lim_{R\to\infty}\underbrace{\left(\frac{-1}{(p-1)R^{p-1}}\right)} + \frac{1}{p-1}$

- When $p>1$, $p-1>0$ so $R^{p-1} \to \infty$ when $R\to\infty$.

So $\lim_{R\to\infty}\left(\frac{-1}{(p-1)R^{p-1}}\right) = \frac{-1}{\infty} = 0$. So, the integral converges to $1/p-1$.

- When $p<1$, $p-1<0$ so $R^{p-1} \to 0$ when $R\to\infty$.

So $\lim_{R\to\infty}\left(\frac{-1}{(p-1)R^{p-1}}\right) = \frac{1}{0} = \infty$, so diverges. So the integral diverges.

- Case $p=1$ $\int_1^{\infty}\frac{1}{x}dx = \lim_{R\to\infty}\int_1^R \frac{1}{x}dx$

$= \lim_{R\to\infty} \ln(x)\Big|_1^R$

$= \lim_{R\to\infty}[\ln(R) - \ln(1)]$

$= \infty$ diverges! ☐

Problem 8: 12 pts

Find the $\boxed{\text{exact sum}}$ of the following series:

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n-1}}{4^{n+1}}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n \cdot 3^{-1}}{4^n \cdot 4^1} = \underbrace{\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n \cdot \frac{1}{3\cdot 4}}_{\text{geometric series with } a = 1/12,\ r = -3/4}$ constant

$= \frac{1/12}{1-[-\frac{3}{4}]} = \frac{1/12}{1+\frac{3}{4}} = \frac{1/12}{\frac{4+3}{4}} = \frac{1}{12}\cdot\frac{4}{7} = \boxed{\frac{1}{21}}$.

(b) $\sum_{n=0}^{\infty}\left(\frac{3+2^n}{5^n} - 2\frac{1}{8^n}\right)$

$= \sum_{0}^{\infty}\left[\frac{3}{5^n} + \frac{2^n}{5^n} - 2\frac{1}{8^n}\right]$

$= \underbrace{\sum_{0}^{\infty} 3\cdot\left(\frac{1}{5}\right)^n}_{\text{Geometric } a=3,\ r=1/5} + \underbrace{\sum\left(\frac{2}{5}\right)^n}_{\text{Geometric } a=1,\ r=2/5} - \underbrace{\sum 2\left(\frac{1}{8}\right)^n}_{\text{Geometric } a=2,\ r=1/8}$

$= \frac{3}{1-[1/5]} + \frac{1}{1-[\frac{2}{5}]} - \frac{2}{1-[1/8]}$

$= \frac{3}{\frac{4}{5}} + \frac{1}{\frac{3}{5}} - \frac{2}{\frac{7}{8}}$

$= \frac{15}{4}\frac{21}{21} + \frac{5}{3}\frac{28}{28} - \frac{16}{7}\frac{12}{12}$

$= \frac{315 + 140 - 192}{84} = \boxed{\frac{263}{84}}$

Problem 9: 15 pts (5 pts each)

(a) Use the **comparison test** to show that $\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{2}{3}\right)^n$ converges.

• Since $n \le n+1$ for all $n \ge 1$

$\frac{n}{n+1} \le 1$ for all $n \ge 1$

• So $\frac{n}{n+1}\left(\frac{2}{3}\right)^n \le \left(\frac{2}{3}\right)^n$.

• $\sum \left(\frac{2}{3}\right)^n \rightsquigarrow$ Geometric series w/ $r = 2/3$ converges!

• let $a_n = \frac{n}{n+1}\left(\frac{2}{3}\right)^n$ & $b_n = \left(\frac{2}{3}\right)^n$. Since $\sum b_n$ converges and $a_n \le b_n$ for all $n \ge 1$, by comparison test, our series $\boxed{\text{converges}}$.

(b) Use the **limit comparison test** to show that $\sum_{j=1}^{\infty} \frac{2j^2 - 5j - 15}{j^3 + 10j^2 + 100}$ diverges. $\rightsquigarrow$ Squint Test: $b_j = \frac{2j^2}{j^3} = \frac{2}{j} \rightsquigarrow$ like harmonic $\to$ D

• let $b_j = \frac{1}{j}$. We know $\sum b_j = \sum_1^{\infty} \frac{1}{j}$ diverges since it is a harmonic series.

• $L = \lim_{j\to\infty}\left(\frac{a_j}{b_j}\right) = \lim_{j\to\infty}\left(\frac{\frac{2j^2-5j-15}{j^3+10j^2+100}}{\frac{1}{j}}\right) = \lim_{j\to\infty}\left(\frac{2j^3-5j^2-15j}{j^3+10j^2+100}\right) = \lim_{j\to\infty}\left(\frac{2j^3}{j^3}\right) = 2.$

• Since $0 < L = 2 < \infty$, by limit comparison test we conclude our series $\boxed{\text{diverges}}$.

(c) Use any test to determine whether $\sum_{k=1}^{\infty} \frac{e^{2k}}{e^{2k} + e^k + 1}$ converges or diverges. $\to$ SQUINT: $\frac{e^{2k}}{e^{2k}} = 1$. so suspect diverges.

Use Test for Divergence: $\lim_{k\to\infty} a_k \ne 0$ then $\boxed{D}$

• $\lim_{k\to\infty}\left(\frac{e^{2k}}{e^{2k}+e^k+1}\right) \overset{\frac{\infty}{\infty}}{=} \lim_{k\to\infty}\left(\frac{e^{2k}}{e^{2k}+e^k+1}\cdot\frac{\frac{1}{e^{2k}}}{\frac{1}{e^{2k}}}\right) = \lim_{k\to\infty}\left(\frac{1}{1+\frac{e^k}{e^{2k}}+\frac{1}{e^{2k}}}\right) = \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{e^k}+\frac{1}{e^{2k}}}\right) = \frac{1}{1+\frac{1}{\infty}+\frac{1}{\infty}} = \frac{1}{1+0+0} = 1.$

DT is e^{2k}

• $\boxed{\text{Diverges}}$ by the Test for Divergence.

Problem 10: 10 pts

(4 pts) (a) The **trapezoid approximation (TrapA)** to

$$\int_1^5 \frac{1}{x+1}\,dx$$

using $\boxed{n = 4}$ is

$f(x) = \frac{1}{x+1}$

(A) $\cancel{\frac{1}{4}}\left[\frac{1}{2}+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{6}\right]$

(B) $\boxed{\frac{1}{2}\left[\frac{1}{2}+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{6}\right]}$

(C) $\cancel{\frac{1}{4}}\left[\frac{1}{2}+\frac{4}{3}+\frac{1}{2}+\frac{4}{5}+\frac{1}{6}\right]$

(D) $\frac{1}{2}\left[\frac{1}{2}+\frac{4}{3}+\frac{1}{2}+\frac{4}{5}+\frac{1}{6}\right]$

(E) Cannot be determined

- $n = 4$
- $\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$
- $x_i = a + i\Delta x = 1 + i,\ i = 0,1,2,3,4$

$x_0 = 1$
$x_1 = 2$
$x_2 = 3$
$x_3 = 4$
$x_4 = 5$

$\underline{\text{TrapA}} = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$

$= \frac{1}{2}\left[f(1)+2f(2)+2f(3)+2f(4)+f(5)\right]$

$= \frac{1}{2}\left[\frac{1}{2}+2\left(\frac{1}{3}\right)+2\left(\frac{1}{4}\right)+2\left(\frac{1}{5}\right)+\frac{1}{6}\right]$

$= \frac{1}{2}\left[\frac{1}{2}+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\frac{1}{6}\right]$

(6 pts) (a) Recall that the **error estimate** for TrapA of an integral $\int_a^b f(x)dx$ is given by

$$|Err(\text{TrapA}(n))| \le \frac{K(b-a)^3}{12n^2}$$

where K is a bound on the second derivative: $|f''(x)| \le K$ for $x \in [a,b]$.

How many trapezoids are needed to approximate the integral in part (a) with an error less than $\frac{1}{3}10^{-4}$?

(Hint: to approximate f'' show it is decreasing)

$f(x) = \frac{1}{x+1} = (x+1)^{-1}$

$f'(x) = -(x+1)^{-2}\cdot[1] = \frac{-1}{(x+1)^2}$

$f''(x) = 2(x+1)^{-3}[1] = \frac{2}{(x+1)^3}$

if $f'''(x) < 0$ then f'' is decreasing.

$f'''(x) = 2(-3)(x+1)^{-4} = \frac{-6}{(x+1)^4} < 0$

so f'' is decreasing.

- Estimate: $|f''(x)| \le K$ for $x \in [1,5]$

Since $f''(x) = \frac{2}{(x+1)^3}$ is decreasing on $[1,5]$

it is largest @ $x = 1$. so: $f''(x) \le f''(1)$.

$f''(1) = \frac{2}{2^3} = \frac{1}{4}$.

So let $\boxed{K = 1/4}$

- Need $\frac{K(b-a)^3}{12n^2} \le \frac{1}{3}10^{-4}$:

$\frac{(1/4)(5-1)^3}{12\ n^2} \le \frac{1}{3}10^{-4}$

$\frac{\cancel{(1/4)}\cdot 4^{\cancel{3}\,2}}{12\ n^2} \le \frac{1}{3}10^{-4}$

$3\cdot 10^4\left(\frac{4^2}{12\,n^2}\right) \le \left(\frac{1}{3}10^{-4}\right)3\cdot 10^4$

$\frac{\cancel{3}\cdot 10^4\cdot\cancel{4}\cdot 4}{\cancel{3}\cdot\cancel{4}\cdot n^2} \le 1$

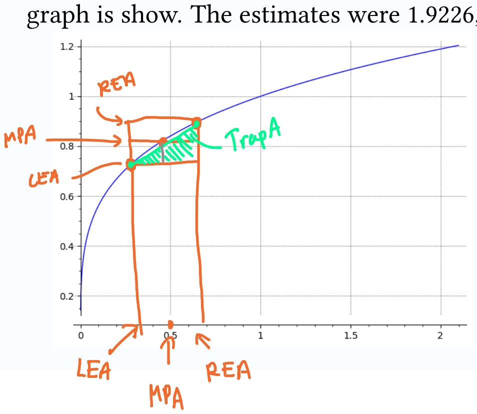
$\sqrt{4\cdot 10^4} \le \sqrt{n^2}$

$2\cdot 10^2 \le n \rightsquigarrow \boxed{n \ge 200}$

- So, we need 200 trapezoids to have an error less than $1/3\ 10^{-4}$.

Problem 11: 2 pts

The LEA (left), REA (right), and MPA (midpoint) approximations were uses to estimate $\int_0^2 f(x)dx$, where f is the function whose graph is show. The estimates were 1.9226, 1.4738, and 2.0684. Which approximation produced which estimate?



LEA < TrapA < MPA < REA   since f is increasing.

LEA: 1.4738   MPA: 1.9226   REA: 2.0684

Problem 12: Extra-credit (5 pts each)

You may attempt these only if every questions on the exam has an attempted solution. Otherwise they will not be graded.

(a) Does $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ equal any know real number? If so, find it. Prove your result for full credit. *(Hint: try to find a recursive sequence. Then use the Monotone Convergence Theorem)*

- Consider: $a_1 = \sqrt{2}$

$$a_2 = \sqrt{2+\sqrt{2}} = \sqrt{2+a_1}$$
$$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} = \sqrt{2+a_2}$$
$$\vdots$$
$$a_n = \sqrt{2+a_{n-1}}$$
$$\vdots$$

- $\{a_n\}_{n=1}^{\infty}$: So can define recursively: $a_1 = \sqrt{2}$, $a_n = \sqrt{2+a_{n-1}}$.
- We're interested in $\lim_{n\to\infty} a_n$.
- If limit exist (we'll verify this later) then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{2+a_{n-1}}\right)$$
$$L = \sqrt{2+\lim_{n\to\infty} a_{n-1}}$$
$$L = \sqrt{2+L}$$
$$L^2 = 2+L$$
$$L^2-L-2=0$$
$$(L-2)(L+1)=0$$
$$\cancel{L=-1},\ L=2$$

so $L = 2 = \sqrt{2+\sqrt{2+\sqrt{2}}}\ldots$. So it must be 2!

- We now prove it is equal to 2. We need to prove the limit exist. We use Monotone Convergence Theorem: we need to show 1) $\{a_n\}$ monotonic 2) $\{a_n\}$ bounded.
  - $\{a_n\}$ bounded:
    $a_1 = \sqrt{2} \approx 1.4$, $a_2 = \sqrt{2+\sqrt{2}} \approx 1.85$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \approx 1.96$
    so suspect $a_n \le 2$.

To prove this we use: Mathematical Induction.

- Claim $a_n \le 2$ for all $n \ge 1$.

Pf Base case: $n=1$.
$a_1 = \sqrt{2} \approx 1.41 \le 2$ ✓ true!

Induction Hypothesis Assume true for n, show it then true for $n+1$:
Assume: $a_n \le 2$.
want to show: $a_{n+1} \le 2$

$\Leftrightarrow \sqrt{2+a_n} \le 2$

$\Leftrightarrow 2+a_n \le 4$

$\Leftrightarrow a_n \le 2$ ✓

which is true by induction hypothesis! ◻
Done. We've proven the claim.

- $\{a_n\}$ is monotonic increasing.

claim $a_{n+1} > a_n$ for all $n \ge 1$.

Pf We use Mathematical induction again.

Base case $n=1$: $a_2 > a_1 \Leftrightarrow \sqrt{2+\sqrt{2}} > \sqrt{2}$

$\Leftrightarrow 2+\sqrt{2} > 2$

$\Leftrightarrow \sqrt{2} > 0$ ✓ true!

Induction Hypothesis Assume true for n.

Assume: $a_{n+1} > a_n$.

Want to show: $a_{(n+1)+1} > a_{n+1}$.

$\Leftrightarrow a_{n+2} > a_{n+1}$

$\Leftrightarrow \sqrt{2+a_{n+1}} > \sqrt{2+a_n}$

$\Leftrightarrow 2+a_{n+1} > 2+a_n$

$\Leftrightarrow a_{n+1} > a_n$ ✓ which is true by induction hypothesis!

Done. We've proven the claim that $\{a_n\}$ is increasing. ◻

(b) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$.

(Hint: find the partial sum S_N)

- $$S_N = \sum_{n=2}^{N} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{N} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{N}\left[\ln(\overset{(n+1)(n-1)}{n^2-1}) - \ln(n^2)\right]$$

$$= \sum_{n=2}^{N}\Big[\ln(n+1)+\ln(n-1) - [\ln(n)+\ln(n)]\Big] = \sum_{n=2}^{N}\Big\{[\ln(n+1)-\ln(n)] - [\ln(n)-\ln(n-1)]\Big\}$$

$$= \overset{n=2}{\Big\{[\cancel{\ln(3)}\ \boxed{-\ln(2)}] - [\cancel{\ln(2)} - \overset{=0}{\cancel{\ln(1)}}]\Big\}} + \overset{n=3}{\Big\{[\cancel{\ln(4)}-\cancel{\ln(3)}] - [\cancel{\ln(3)}-\cancel{\ln(2)}]\Big\}}$$
$$+ \overset{n=4}{\Big\{[\cancel{\ln(5)}-\cancel{\ln(4)}] - [\cancel{\ln(4)}-\cancel{\ln(3)}]\Big\}} + \overset{n=5}{\Big\{[\cancel{\ln(6)}-\cancel{\ln(5)}] - [\cancel{\ln(5)}-\cancel{\ln(4)}]\Big\}}$$

cancels w/ $n=6$ $-[\ln(n-1)]$

$$+ \overset{n=N-1}{\Big\{[\cancel{\ln(N)}-\cancel{\ln(N-1)}] - [\cancel{\ln(N-1)}-\cancel{\ln(N-2)}]\Big\}} + \overset{n=N}{\Big\{[\ln(N+1)-\cancel{\ln(N)}] - [\ln(N)-\cancel{\ln(N-1)}]\Big\}}$$

$$= -\ln(2) + \ln(N+1) - \ln(N) = -\ln(2) + \ln\left(\frac{N+1}{N}\right).$$

- Thus, exact sum is

since $\ln(x)$ is continuous

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left[-\ln(2) + \ln\left(\frac{N+1}{N}\right)\right] = -\ln(2) + \ln\Big(\underbrace{\lim_{N\to\infty}\left(\frac{N+1}{N}\right)}_{=1 \text{ by DTT}}\Big)$$

$$= -\ln(2) + \overset{=0}{\cancel{\ln(1)}}$$

$$= \boxed{-\ln(2)}.$$

Post Exam Survey

Now that you have finished the exam, please take a few minutes to reflect on how you prepared for the exam and how you think you did. Then answer these questions.

1. When taking the exam I felt
   (a) Rushed. I wanted more time.
   (b) Relaxed. I had enough time.
   (c) Amazed. I had tons of extra time.
2. The week before the test I did all my homework on time: YES NO
3. The week before the test, in addition to the homework I followed a study plan. YES NO
   (a) I think this helped: YES NO
4. The day before the test I spend ________ hours studying and reviewing.
   (a) I think that was enough time: YES NO
5. The night before the test:
   (a) I stayed up very late cramming for the test
   (b) I stayed up very late, but I wasn't doing math
   (c) I didn't need to cram because I was prepared
   (d) I got a good night's sleep so my brain would function well.
6. I think I got the following grade on this test: ________
7. Strategies that worked well for me were (please elaborate):
8. Next time I will do an even better job preparing for the test by: