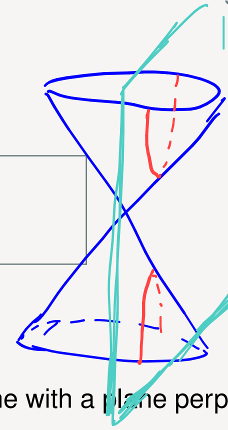
Section 12.3 Hyperbolas



Objectives
- Geometric Definition of a Hyperbola
- Equations and Graphs of Hyperbolas

• Geometric Definition of a Hyperbolas

The next conic section we study are called "hyperbolas" and they occur when we slice a double-cone with a plane perpendicular to the cone base.

Their shapes are very interesting. They look like a "double parabola" but they are not. They have asymptotes which means that the straighten out at infinity unlike parabolas.

Their definition is similar to the definition of an ellipse but instead of "sum" we replace it with "difference."

Defn 1 Geometric Definition of a Hyperbola

A **hyperbola** is the set of all points in the plane the **difference** of whose distances from two fixed points F_1 and F_2 is a constant. D

The two fixed points F_1 and F_2 are called the **foci** (plural of focus) of the ellipse.

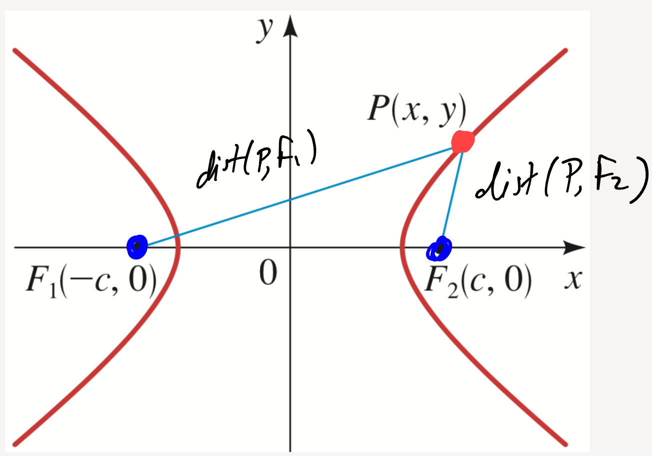
It's much harder to draw this by hand. Instead we'll look at the nice graphic:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

+ ↳ foci on x

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

+ ↳ foci on y



parabolas

hyperbolas

Using the notation as in the figure, the geometric definition says:

$$\boxed{\text{dist}(P, F_1) - \text{dist}(P, F_2) = D} \quad \text{OR} \quad \boxed{\text{dist}(P, F_2) - \text{dist}(P, F_1) = D} \tag{1}$$

12 − 11 = 1
4 − 3 = 1
−3 − (−4) = 1

where D is the constant in the definition. Notice we have two equations because of the asymmetry of subtraction.
As we did in the previous sections, we can derive the equation of the hyperbola with from equations given in (1). We will skip the details–read the textbook if you're interested.

If we set $D = 2a$ and $\boxed{b^2 = c^2 - a^2}$ (notice it's different than the ellipse which we set $\boxed{b^2 = a^2 - c^2}$), the equations in (1) simplify to:

HYP   Ellipse

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

Unsurprisingly, it's almost exactly the same as the ellipse–with one major difference: a minus sign!

DESMOS: https://www.desmos.com/calculator/cm6nomoy2q

• Equations and Graphs of Hyperbolas

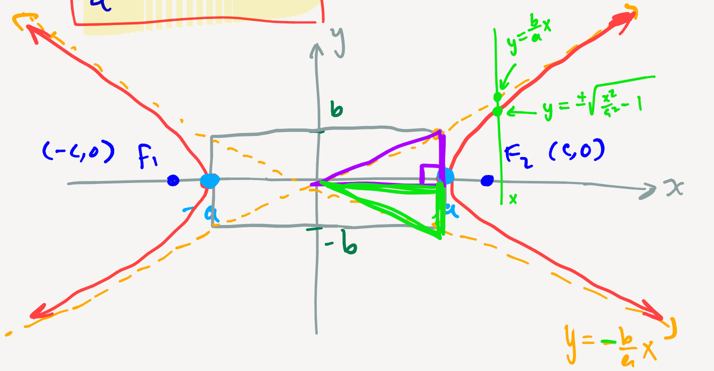
Theorem 1 Equation of a Hyperbola

Horizontal Foci

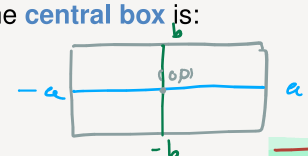
If a hyperbola has foci along the x-axis, at $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ with $c > 0$, and the difference of the distances is $\pm D = \pm 2a$, $a > 0$, then

- The equation of the hyperbola in **graphing form** is:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$



- We define $\boxed{b^2 = c^2 - a^2}$ and take $b > 0$. Note that $b < c$.
- The foci are: $(\pm c, 0)$
- The vertices are: $(\pm a, 0)$
- The central box is:



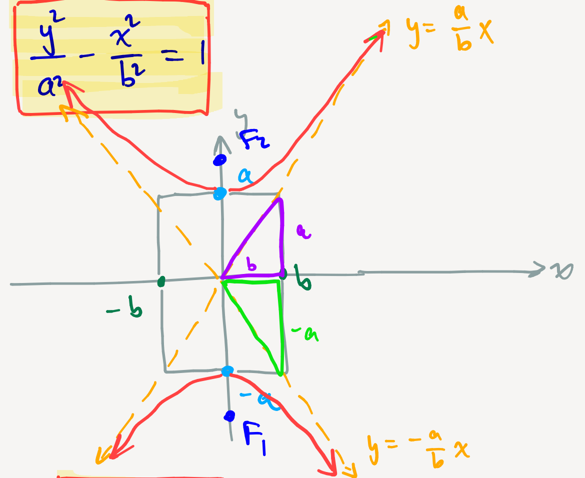
- The asymptotes are: $\boxed{y = \pm \frac{b}{a} x}$

Vertical Foci

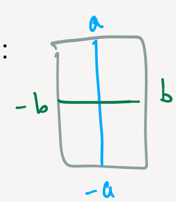
If a hyperbola has foci along the y-axis, at $F_1 = (0, -c)$ and $F_2 = (0, c)$ with $c > 0$, and the difference of the distances is $\pm D = \pm 2a$, $a > 0$, then

- The equation of the hyperbola in **graphing form** is:

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$



- We define $\boxed{b^2 = c^2 - a^2}$ and take $b > 0$. Note that $b < c$.
- The foci are: $(0, \pm c)$
- The vertices are: $(0, \pm a)$
- The central box is:

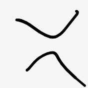


- The asymptotes are: $\boxed{y = \pm \frac{a}{b} x}$

Remarks Some helpful comments:

- Hyperbolas "eats" the foci: The vertices and foci occur on the axis that corresponds to the variable with the "positive sign" $(+)$.
- **Tips for graphing hyperbolas:** draw central box; draw asymptotes; determine vertices; draw hyperbola.
- A note on asymptotes: Let's rewrite the equation as a function: $y = y(x)$.

  Let's work with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Solving for y:



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{EQ HYP OPENING UP/DOWN}$$

$$\frac{y^2}{a^2} = 1 + \frac{x^2}{b^2} \quad \text{ADD } x^2/b^2 \text{ to both sides}$$

$$y^2 = a^2 + \frac{a^2}{b^2}x^2 \quad \text{Multiply both sides by } a^2$$

$$y = \pm\frac{a}{b}\sqrt{b^2 + x^2} \quad \text{Take square roots}$$

$$\boxed{y = \pm\frac{a}{b}x\sqrt{1 + \frac{b^2}{x^2}}}$$

$$a^2 + \frac{a^2}{b^2}x^2 = a^2\left(1 + \frac{1}{b^2}x^2\right) = \frac{a^2}{b^2}\left(b^2 + x^2\right) = \frac{a^2}{b^2}x^2\left(\frac{b^2}{x^2} + 1\right)$$

Thus for large values of x, the term $\sqrt{1 + \frac{b^2}{x^2}}$ is really close to 1. So, $y \approx \pm\frac{a}{b}x$ for large values of x.

Ex 1 What we call graphing form is also called **standard form**.
(a) Write the equation of the hyperbola in graphing form: $9x^2 - 16y^2 = 144$, and sketch its graph.
(b) Find the vertices, asymptotes, and foci.

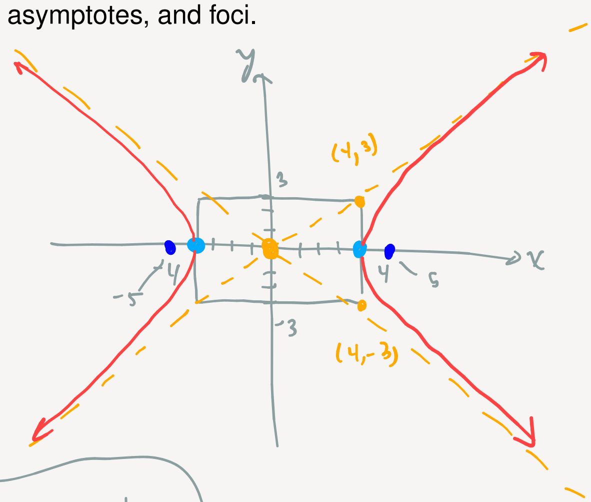
$$\frac{9x^2}{144} - \frac{16y^2}{144} = \frac{144}{144}$$

$$\boxed{\frac{x^2}{16} - \frac{y^2}{9} = 1}$$

$+$

$a = \sqrt{16} = 4$

$b = \sqrt{9} = 3$



$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \qquad a = 4 \quad b = 3$$

(b) Vertices: $\boxed{(\pm 4, 0)}$

Asymptotes $\boxed{y = \pm\frac{3}{4}x}$ $\quad y = \frac{3}{4}x \quad y = \frac{-3}{4}x$

Foci $b^2 = c^2 - a^2$ (cf Ellipse: $b^2 = a^2 - c^2$)

Recall when using these $a > b$.

$c^2 = a^2 + b^2$
$= 16 + 9$
$= 25$

$c = 5$

Foci $\boxed{(\pm 5, 0)}$

Ex 2 What we call graphing form is also called **standard form**.
(a) Write the equation of the hyperbola in graphing form: $x^2 - 9y^2 + 9 = 0$, and sketch its graph.
(b) Find the vertices, asymptotes, and foci.

a) $x^2 - 9y^2 + 9 = 0$

$$\frac{x^2}{-9} - \frac{9y^2}{-9} = \frac{-9}{-9}$$

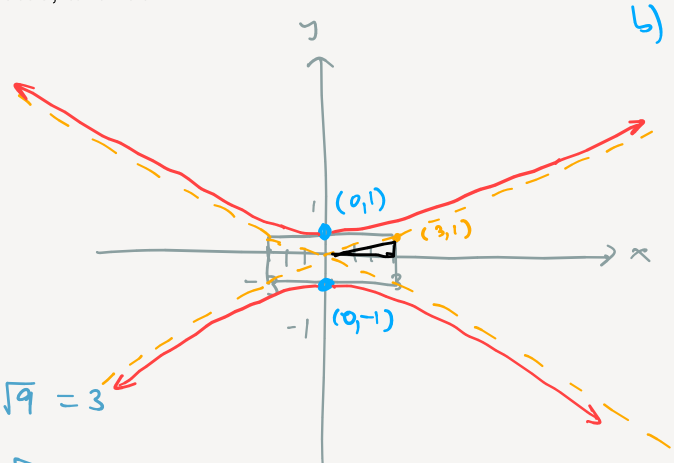
$$-\frac{x^2}{9} + y^2 = 1$$

$$\boxed{\frac{y^2}{1} - \frac{x^2}{9} = 1}$$

$+$

$a = \sqrt{9} = 3$

$b = \sqrt{1} = 1$



b) vertices $\boxed{(0, \pm 1)}$

Asymptotes $\boxed{y = \pm\frac{1}{3}x}$

Foci $\boxed{(0, \pm\sqrt{10})}$

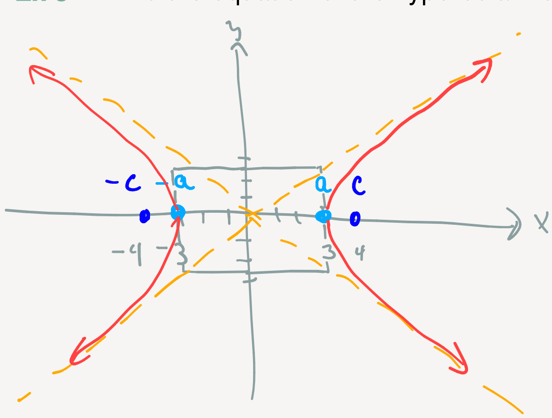
$b^2 = c^2 - a^2$ or $c^2 = a^2 + b^2$

$c^2 = 3^2 + 1^2 = 9 + 1 = 10$

$c = \sqrt{10}$

Ex 3 Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci at $(\pm 4, 0)$. Also, sketch its graph.

↳ x axis



• $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ $\quad$ ($a > b$ both)

• $a = 3$

• $c = 4$

• $b^2 = c^2 - a^2$

$b^2 = 16 - 9 = 7$

$b = \sqrt{7} \approx 2.6$

$$\boxed{\frac{x^2}{9} - \frac{y^2}{7} = 1}$$

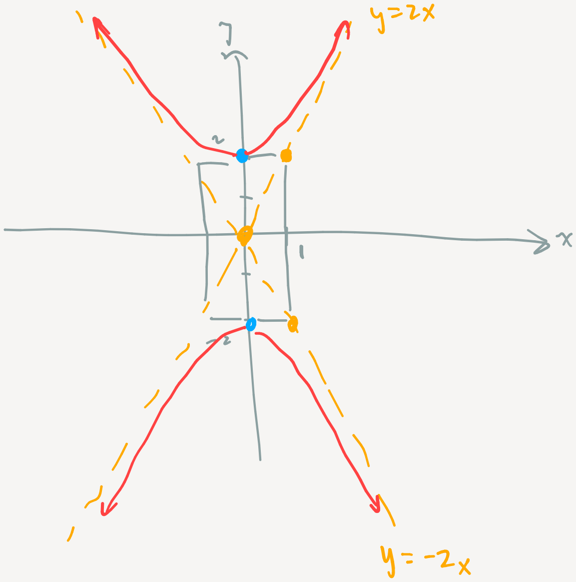
Ex 4 Find the equation of the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm 2x$. Also, sketch its graph.

↳ $a = 2$

↳ $b = 1$

↳ $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

$$\boxed{\frac{y^2}{4} - \frac{x^2}{1} = 1}$$



• Applications of Hyperbolas

Hyperbolic mirrors have the interesting property that light directed at one foci is reflected and aimed towards the other foci.

The famous Cassegrain-type telescopes use this property and the parabolic mirror property.

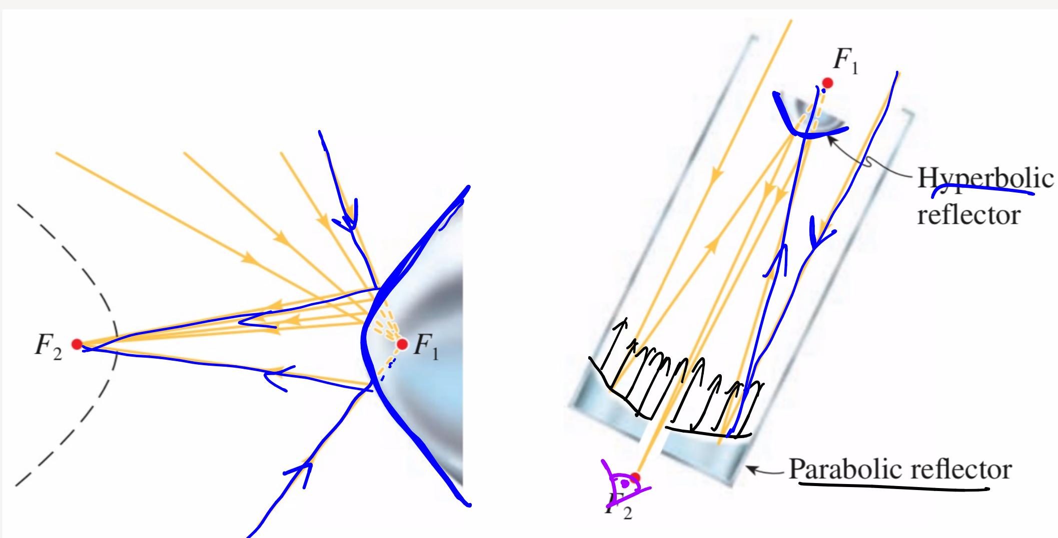


FIGURE 7 Reflection property of hyperbolas

FIGURE 8 Cassegrain-type telescope

The paths of comets can be ellipses, parabolas or hyperbolas. This can be proven by calculus (a triumph of Newton).