§9.3: Separable Equations

Ch 9: Differential Equations
Math 5B: Calculus II

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Class #8 Notes

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Outline

1. Guiding Questions
2. Separable Equations

Guiding Questions for §9.3

Guiding Question(s)

1. What are separable equations and how can solve them?
2. What are the important examples of separable equations?

Separable Equations - Introduction

Differential equations can be very hard to solve. So far, we've looked at:

- Solutions to a few DE:
    - Natural Growth/Decay: $\frac{dy}{dx} = ky \to y = Ce^{kx}$
    - Newton's Law of Cooling: $\frac{dT}{dt} = k(T - T_S) \to T(t) = T_S + Ce^{kt}$
    - Using anti-differentiation: $\frac{dy}{dx} = f(x) \to y(x) = \int f(x)\, dx$
- The qualitative behavior of a DE. That is, how the DE can tell us about the shape of the solutions.
- Slope Fields to help us visualize solutions
- Euler's method to approximate solutions (numerical algorthim)

Separable Equations - Introduction

- Let's look again at how we solved DEs of the form: $\dfrac{dy}{dx} = f(x)$.

$$\frac{dy}{dx}dx = f(x)$$
$$\int \frac{dy}{dx}dx = \int f(x)dx$$
$$y(x) = \int f(x)dx$$

- So can we generalize this simple trick to solve more complicated DEs?
- Yes, and we've given them a name: separable equations.

Separable Equations

Recall we looked at DE of the form, $\frac{dy}{dx} = F(x, y)$, but didn't try to solve them and instead looked at their slope fields. This gave us a general idea about the solution curves.

Definition 1: Separable Equations

- The separable equation is a DE of the form where $F(x, y) = f(x)g(y)$, or

$$\frac{dy}{dx} = f(x)g(y)$$

- That is, the right-hand side is a product of a function in x only and a function in y only.

Separable Equations

Examples

The following DEs are all separable equations:

(a) $\dfrac{dy}{dx} = x^2y^2$

(b) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$

(c) $y' + xe^y = 0$

(d) $\dfrac{dy}{dx} = y$

(e) $y' = \dfrac{6x^2}{2y + \cos(y)}$

(f) $\theta e^{t^2} \dfrac{d\theta}{dt} = t\sec(\theta)$

The following are NOT separable equations:

(g) $\dfrac{dy}{dx} = x^2 + y^2$

(h) $\dfrac{dy}{dx} = (x+y)^2$

Separable Equations

How to solve Separable EQs

- The name comes from the fact that we can "separate the derivative" $\frac{dy}{dx}$ into two pieces dy and dx and because the right-hand side is a product $f(x)g(y)$ we can re-arrange the DE by "separating each variable on one side of the equal sign."
- Here's the general strategy to solving these DEs:
  - $\dfrac{dy}{dx} = f(x)g(y)$
  - $\dfrac{dy}{g(y)} = f(x)dx$ ("separate the x and y")
  - $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\, dx$
  - After integrating both sides, solve for y

Separable Equations

Activity 1:

Find the general solutions of the following separable equations:

(a) $y\dfrac{dy}{dx} - x = 0$

(b) $y' = -ty$

Separable Equations

Separable Equations

We finish this section by proving the following theorem from §6.5:

Theorem 1: Exponential Growth & Decay Equation

The only solutions to the "natural growth/decay equation," $\frac{df}{dt} = kf(t)$, for a constant $k \neq 0$, are of the form:

$$f(t) = Ce^{kt}. \tag{1}$$

We already proved that the solutions $f(t) = Ce^{kt}$ do indeed solve the DE but we didn't prove these were the only solutions.
We can show these are the only solutions now.

Activity 2:

Prove that the only solutions to the natural growth/decay equation, $\frac{df}{dt} = kf$, are of the form $f(t) = Ce^{kt}$.

Separable Equations

Activity 3:

(a) Solve the IVP:

$$y' = \frac{xy^3}{1+x^2}; \quad y(0) = -1$$

(b) Discuss the maximal interval where the solution exits (called an interval of validity).

Separable Equations