§6.4: Derivatives of Logarithmic Functions

Ch 6: Exponentials, Logs, & Inverse Trig Functions

Math 5B: Calculus II

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Outline

Guiding Questions for §6.4

Guiding Question(s)

1. What is the derivative rule for the natural logarithmic function?
2. What are the derivative rules for logarithmic and exponential functions for other bases?
3. Can we exploit the nice properties of logarithms to compute derivatives of complicated functions more efficiently?
4. How can we prove a more convenient limit definition for e?

Derivative of Natural Logarithm

- How can we determine the derivative of $g(x) = \ln(x)$?
- So far we know that $g(x) = f^{-1}(x)$ is the inverse function of $f(x) = e^x$.
- The derivative of $f(x)$ is itself: $f'(x) = e^x$.
- If we write $y = \ln(x)$ then the corresponding exponential equation is $e^y = x$. Using implicit differentiation we can take the derivative of both sides and derive the following:

Derivative of Natural Logarithm

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- So far we know that $g(x) = f^{-1}(x)$ is the inverse function of $f(x) = e^x$.
- The derivative of $f(x)$ we've calculated previously to be: $f'(x) = e^x$.
- If we write $y = \ln(x)$ then the corresponding exponential equation is $e^y = x$. Using implicit differentiation we can take the derivative of both sides and derive the following:

Theorem 1: Derivative of Natural Logarithm

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}, \qquad x > 0 \tag{1}$$

Derivative of Natural Logarithm

Proof:

You try it! Give two proofs: (1) by supplying the remaining details of the argument sketched above; and (2) using the derivative rule for inverse functions from §6.1.

Derivative of Natural Logarithm

Keep in mind:

- The derivative rule of $\ln(x)$ only works for positive values of x.
- The derivative of $\ln(x)$ is neither an exponential function nor a logarithm! This is surprising. It's actually a tame rational function.

Derivative of Natural Logarithm

The derivative rules for $\ln(x)$ combined with the chain rule ma be stated as:

$$\frac{d}{dx}[\ln(u)] = \frac{1}{u}\frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)}f'(x)$$

Activity 1: Derivative of Natural Logarithm

Find the derivatives of the following functions:

(a) $h(x) = x^2 \ln(x)$

(b) $p(t) = \frac{\ln(t)}{e^t+1}$

(c) $s(y) = \ln(\cos(y) + 2)$

(d) $z(x) = \tan(\ln(x))$

(e) $m(z) = \ln(\ln(z))$

Derivative of Natural Logarithm

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Activity 2: Derivative of Natural Logarithm

Find the derivative of $f(x) = \ln\left(\frac{2x+1}{\sqrt{x+6}}\right)$ in two ways:

(a) using derivative rules directly

(b) by using the properties of log to simplify before you apply derivative rules

(c) Which method do you prefer?

(d) Find as many pros/cons of each method.

Derivative of Natural Logarithm

True or False or Maybe

$$\frac{d}{dx}\left[\,\ln|x|\,\right] = \frac{1}{|x|} \qquad \text{for all } x \neq 0$$

Hint: draw a picture!

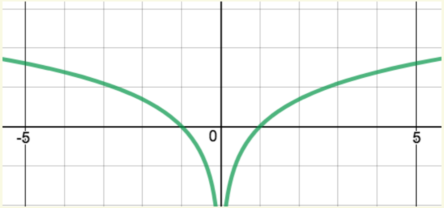
Derivative of Natural Logarithm

True or False or Maybe

$$\frac{d}{dx}[\ln|x|] = \frac{1}{|x|} \qquad \text{for all } x \neq 0 \qquad \textit{FALSE!}$$

Hint: draw a picture!

- Notice that for $x < 0$, the slope of the tangent line is negative
- So for $x > 0$, $1/x$ is negative
- The correct derivative is: $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$, for all $x \neq 0$.



Anti-Derivative of $\frac{1}{x}$

This helps with the anti-derivative rule for the function $1/x$. By taking anti-derivatives on both sides of $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$, for all $x \neq 0$, we get:

$$\int \frac{d}{dx}[\ln|x|]dx = \int \frac{1}{x}dx.$$

This simplifies because $\int$ and d/dx undo each other by definition! We've just proven:

Theorem 2: Anti-Derivative of Natural Logarithm

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{2}$$

Anti-Derivative of $\frac{1}{x}$

- Recall the Power Rule (for anti-derivatives) says:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

- We can fill in the gap now! The gap is at $n = -1$
- Since for $n = -1$, $x^n = x^{-1} = \frac{1}{x}$ and we now know the anti-derivative rule for $\frac{1}{x}$, we can now extend the power rule: to all integers!

Power Rule for ADs: $$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & n \neq 1 \\ \ln|x| + C, & n = -1 \end{cases}$$

Anti-Derivative of $\frac{1}{x}$

Activity 3: Anti-derivatives of $1/x$

(a) Evaluate: $\int \frac{2}{x}\,dx$

(b) Find the area under the hyperbola $xy = 2$ from $x = 1$ to $x = 2$. Round your answer to three decimal places.

(c) Compute: $\int \frac{2x}{x^2+4}\,dx$

(d) Find: $\int_1^e \frac{\ln(x)}{x}\,dx$

(e) What is $\int \tan(x)\,dx$?

DRs and ADRs for General Exp. and Log. Functions

Theorem 3: DRs and ADRs for Exp and Logs

We state all the Derivative and Anti-Derivative Rules for Exponential and Logarithmic Functions for general bases $b > 0$, $b \neq 1$:

DR 1 $\frac{d}{dx} b^x = b^x \cdot \ln(b)$

DR 2 $\frac{d}{dx} \log_b(x) = \frac{1}{x} \cdot \frac{1}{\ln(b)}$

ADR 1 $\int b^x \, dx = \frac{b^x}{\ln(b)} + C$

DRs and ADRs for General Exp. and Log. Functions

We start by proving DR2: $\frac{d}{dx}\log_b(x) = \frac{1}{x} \cdot \frac{1}{\ln(b)}$

Proof: Proof of DR for $\log_b(x)$

- By the change of base formula: $\log_b(x) = \frac{\ln(x)}{\ln(b)}$
- Since we know the derivative of $\ln(x)$ and $\ln(b)$ is a constant, we can take the derivative on both sides to get:
- $$\frac{d}{dx}\log_b(x) = \frac{d}{dx}\frac{\ln(x)}{\ln(b)} = \frac{1}{x} \cdot \frac{1}{\ln(b)}$$
- And we're done :-) $\square$

DRs and ADRs for General Exp. and Log. Functions

Next, we show DR1: $\frac{d}{dx} b^x = b^x \cdot \ln(b)$

Proof: Proof of DR for $\log_b(x)$

- Recall in §6.2 we started to study the derivative of b^x. In fact, we proved that: setting $f(x) = b^x$

$$f'(x) = f'(0)f(x) = C \cdot f(x) \qquad \text{where } C = f'(0) = \lim_{h \to 0} \frac{b^h - 1}{h}$$

- What is this constant? It turns out to be $C = \ln(b)$.
- We need to use a trick. We use the inverse properties of exponential and logarithmic functions. Why do we use this? Because it allows us to use the natural exponential and natural logarithm functions, and we know the DRs for these already

DRs and ADRs for General Exp. and Log. Functions

Next, we show DR1: $\frac{d}{dx} b^x = b^x \cdot \ln(b)$

Proof: Proof of DR for $\log_b(x)$

Continued:

- We use the inverse property: $x = e^{\ln(x)}$ and apply it to b^x: $b^x = e^{\ln(b^x)}$. Then take the derivative:

$$\begin{aligned}
\frac{d}{dx} b^x &= \frac{d}{dx} e^{\ln(b^x)} && \text{(inv prop)} \\
&= \frac{d}{dx} e^{x \ln(b)} && \text{(log prop)} \\
&= e^{x \ln(b)} \frac{d}{dx}[x \ln(b)] && \text{(chain rule)} \\
&= e^{\ln(b^x)} \ln(b) && \text{(note } \ln(b) \text{ is const.)} \\
&= b^x \ln(b).
\end{aligned}$$

- Done! :-)

$\square$

Next, we show ADR1: $\int b^x \, dx = \frac{b^x}{\ln(b)} + C$

Proof: Proof of ADR for b^x

- This follows immediately from DR 1, $[b^x]' = b^x \ln(b)$. $\square$

Activity 4:

(a) If $y = \log_{10}(1 + x + \tan(x))$, find y'

(b) Compute: $\frac{d}{dx}[10^{x^2}]$

(c) Evaluate: $\int_0^4 3^x \, dx$

Logarithmic Differentiation

The technique used in Part (b) of Activity 2 is so useful that it can be generalized and used whether or not we are taking derivatives involving a logarithm. Is given a special name, logarithmic differentiation.

Definition 1: Logarithmic Differentiation

Logarithmic Differentiation is a technique for computing derivatives that follows the following basic steps:

Step (1) Take the natural log, ln(), on both sides of an equation $y = f(x)$

Step (2) Expand using log properties

Step (3) Use Implicit Differentiation to differentiate the equation with respect to x

Step (4) Solve for $\frac{dy}{dx}$ (or y')

Logarithmic Differentiation

Activity 5:

Use Log Diff to find the derivatives of

(a) $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+5)^5}$

(b) $y = x^x$

A Limit formula for e

- Recall in my hand-out, I gave eight definitions of e. The "official definition" is that the constant $C = 1$ in the derivative rule $f'(x) = Cf(x)$ mentioned previously in these slides (namely, $C = f'(0) = \lim_{h\to 0}(e^h - 1)/h$). The official definition corresponds to definition (6) in the hand-out
- Our goal is to prove the following limit formula for e:

$$e = \lim_{x\to 0^+}(1 + x)^{1/x}. \tag{3}$$

- This means we will prove that "definition (6)" implies "definition (2)" from the hand-out

A Limit formula for e

Proof: Proof of a limit formula for e

- From the official definition of e we derived that $\frac{d}{dx}e^x = e^x$.
- Then we showed the inverse of e^x, $\ln(x)$, has derivative rule: $\frac{d}{dx}\ln(x) = 1/x$.
- Applying the derivative of $\ln(x)$ at $x = 1$ gives: $\frac{d}{dx}\ln(x)\big|_{x=1} = 1$.
- Now, we work some magic:

$$\begin{aligned}
\frac{d}{dx}\ln(x)\Big|_{x=1} &= \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} \\
&= \lim_{h\to 0}\frac{\ln(1+h)}{h} \qquad (\text{since } \ln(1)=0) \\
&= \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \qquad (\text{why?})
\end{aligned}$$

A Limit formula for e

Proof: Proof of a limit formula for e

- Magic continued:

$$\begin{aligned}
\frac{d}{dx}\ln(x)\Big|_{x=1} &= \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \qquad \text{(why?)}\\
&= \lim_{x\to 0^+}\frac{1}{x}\ln(1+x)\\
&= \lim_{x\to 0^+}\ln(1+x)^{1/x} \qquad \text{(log prop)}\\
&= \ln\left(\lim_{x\to 0^+}(1+x)^{1/x}\right) \qquad \text{(continuity)}
\end{aligned}$$

- Since $\frac{d}{dx}\ln(x)\big|_{x=1} = 1$, the above calculation gives:

$$\ln\left(\lim_{x\to 0^+}(1+x)^{1/x}\right) = 1$$

A Limit formula for e

Proof: Proof of a limit formula for e

- Since $\frac{d}{dx}\ln(x)\big|_{x=1} = 1$, the above calculation gives:

$$\ln\left(\lim_{x\to 0^+}(1+x)^{1/x}\right) = 1$$

- But, we know that $\ln(e) = 1$ by the inverse properties. So because $\ln(x)$ is one-to-one,the limit inside the parentheses must be e! $\square$