MATH 5B - Single Variable Calculus II — Spring 2019

Exam 1

§6.1-6.7 — Dr. Jorge Basilio

March_12 — gbasilio@pasadena.edu

Honesty Pledge

On my honor, by printing and signing my name below, I vow to neither receive nor given any unauthorized assistance on this examination:

NAME (PRINT): Solutions SIGNATURE: ____________________

Directions

- YOU ARE ALLOWED TO USE ONLY A SCIENTIFIC CALCULATOR ON THIS EXAM.
- You have 80 minutes to complete this exam.
- The exam totals **100 points** (+5 EC possible)
- There are 10 problems, many of them with multiple parts.
- Place all of your belongings in the front of the classroom and I will assign you a seat. Bring with you your writing utensils.
- Cell phones must be turned off and put away in with your items in the front of the classroom.
- Handwriting should be neat and legible. If I cannot read your writing, zero points will be given.
- Some questions contain multiple-parts which you must do individually and the parts are denoted by (a), (b), (c), etc. Some questions are multiple-choice and the choices are denoted with (A), (B), (C), (D), and (E).
- Make sure to ALWAYS SHOW YOUR WORK; you will not receive any partial credits unless work is clearly shown. *If in doubt, ask for clarification.*
- Leave answers in exact form (as simplified as possible), unless told otherwise.
- Put a $\boxed{\text{box around your final answer}}$ where applicable.
- **PLEASE CHECK YOUR WORK!!!**
- If you need extra space, there is extra space on the back of the cover page and clearly indicate that you are continuing your work there in the original location.
- If you finish early, you may take a break but you must come back to class by 5:30 and we will have class.
- I will take attendance at the end of class

Score	Grade

This page is intentionally blank. It may be used for scratch paper. If you wish for me to grade your work on this page, please (i) label the problem you are working on, (ii) box your answer, (iii) indicate in the original problem's location that you will continue your work on this page.

Problem 1: ~~10~~ 15 pts

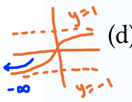
Fill in the blanks. (2 pts each)

(a) $\frac{d}{dx}[e + e^x + x^e] =$ $e^x + ex^{e-1}$

(b) $\int e^{x^2} 2x\,dx =$ $e^{x^2} + C$

(c) $\tan^{-1}(-1/2) =$ $\tan^{-1}\left(-\frac{1}{2}\right)$ (exact)

[Tricky Q! there's only a few angles where we know the exact angle. This isn't one of them.]



(d) $\lim_{x \to -2^-} \tanh^{-1}\left(\frac{2}{2+x}\right) =$ -1

$\frac{2}{-0} = -\infty$

(e) $\int_1^e \frac{1}{t}\,dt =$ 1

$= \ln|t|\Big|_1^e = \ln e - \ln 1 = 1$

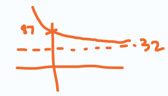
True or False. (5 pts - 1pt each)
Circle T for True, F for False.

T F (f) $\ln\left(\frac{x}{y}\right) = \frac{\ln(x)}{\cancel{\ln(y)}}$ $- \ln(y)$ — F circled

T F (g) $\int x^{-1}\,dx = \cancel{-\frac{1}{2}x^{-2}} + C$ $\ln|x|$ — F circled

T F (h) If $8^x = 0.1$ then $x = \log_8(0.1)$ — T circled

T F (i) $\lim_{t \to \infty} 32 + 55e^{-0.1t} = 32$ — T circled



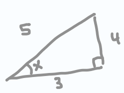
T F (j) $\frac{d}{dx}\ln|x^4 - x^3| = \frac{1}{|x^4 - x^3|}$, $x \neq 0$ (no abs val) — F circled

Problem 2: 6 pts

Simplify each of the following (exactly...i.e. no calculator):

(a) $\csc(\arccos(3/5)) = \boxed{\frac{5}{4}}$

$x = \cos^{-1}\left(\frac{3}{5}\right) \longleftrightarrow \cos(x) = \frac{3}{5}$

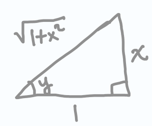


(3-4-5 triangle)
or use Pythagorean id:
$3^2 + ?^2 = 5^2 \rightarrow ?^2 = 25 - 9 = 16$
$? = \sqrt{16} = 4$

$\csc(x) = \frac{hyp}{opp} = \frac{5}{4}$.

(b) $\sin(\tan^{-1}(x)) = \boxed{\frac{x}{\sqrt{1+x^2}}}$

$y = \tan^{-1}(x) \longleftrightarrow \tan(y) = \frac{x}{1}$



$1^2 + x^2 = hyp^2$
$\sqrt{1+x^2} = hyp$

$\sin(y) = \frac{opp}{hyp} = \frac{x}{\sqrt{1+x^2}}$

Problem 3: 9 pts

Let $f(x) = x^5 + x^3 + 2x - 2$.

(a) Show that f is one-to-one on $\mathbb{R}$.

(b) Find $f^{-1}(-2)$

(c) Compute $(f^{-1})'(-2)$

(a) $f'(x) = \underbrace{5x^4 + 3x^2 + 2}_{\text{even } \geq 2} > 0$

$f'(x) > 0$ for all x

By ID test of Calc I, $f'(x) > 0$ implies f is increasing for all x

So, f is 1-1.

(b) $x = f^{-1}(-2) \longleftrightarrow f(x) = -2$

$x^5 + x^3 + 2x \cancel{-2} = \cancel{-2}$

$x(\underbrace{x^4 + x^2 + 2}_{\geq 2}) = 0$

So $x = 0$ is our solution

Thus, $\boxed{f^{-1}(-2) = 0}$.

(c) $(f^{-1})'(-2) = \frac{1}{f'(f^{-1}(-2))} = \frac{1}{f'(0)} = \frac{1}{5(0)^4 + 3(0)^2 + 2} = \frac{1}{2}$. So, $\boxed{(f^{-1})'(-2) = \frac{1}{2}}$.

formula from 6.1 by (b)

Problem 4: 24 pts

Find the derivative of each of the following functions and simplify:

(a) $E(z) = 2^{z\ln(z)}$

$$\frac{dE}{dz} = \underbrace{2^{z\cdot\ln(z)}\cdot\ln 2}_{(a^x)' = a^x\ln a}\cdot\underbrace{\frac{d}{dz}\left[z\cdot\ln(z)\right]}_{\text{chain rule}}$$

$$= 2^{z\ln(z)}\cdot\ln 2\cdot\left[[1]\ln(z) + z\left[\frac{1}{z}\right]\right]$$

$$= \boxed{2^{z\cdot\ln(z)}\cdot\ln 2\cdot\left(\ln(z)+1\right)}$$

(b) $L(w) = \overbrace{w\sec^{-1}}^{\text{product rule!}}\left(\frac{1}{w}\right)$, assume $w>0$

$$\frac{dL}{dw} = [1]\sec^{-1}\left(\frac{1}{w}\right) + w\left[\frac{1}{\left(\frac{1}{w}\right)\sqrt{\left(\frac{1}{w}\right)^2-1}}\cdot\underbrace{\frac{d}{dw}\left(\overset{w^{-1}}{\frac{1}{w}}\right)}_{\text{chain rule}}\right]$$

$$= \sec^{-1}\left(\frac{1}{w}\right) + w^2\cdot\frac{1}{\sqrt{\frac{1-w^2}{w^2}}}\cdot\left[-w^{-2}\right]$$

$$= \sec^{-1}\left(\frac{1}{w}\right) + \cancel{w^2}\cdot\frac{\sqrt{w^2}}{\sqrt{1-w^2}}\cdot\frac{-1}{\cancel{w^2}}$$

$$= \boxed{\sec^{-1}\left(\frac{1}{w}\right) - \frac{w}{\sqrt{1-w^2}}}$$

also accepted unsimplified:

$$= \sec^{-1}\left(\frac{1}{w}\right) - \frac{1}{\sqrt{\left(\frac{1}{w}\right)^2-1}}$$

(c) $y = \sinh^{-1}\left(\frac{1}{x+1}\right)$

$(x+1)^{-1}$

$$y' = \frac{1}{\sqrt{1+\left(\frac{1}{x+1}\right)^2}}\cdot\frac{d}{dx}\left(\frac{1}{x+1}\right)$$

$$= \frac{1}{\sqrt{\frac{(x+1)^2+1}{(x+1)^2}}}\cdot(-1)(x+1)^{-2}$$

$$= \frac{-1}{\frac{\sqrt{(x+1)^2+1}}{|x+1|}}\cdot\frac{1}{(x+1)^{\cancel{2}\,1}}$$

$$= \boxed{\frac{-1}{|x+1|\sqrt{(x+1)^2+1}}}.$$

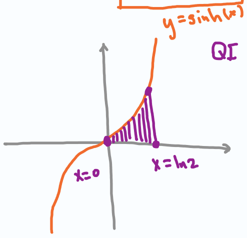
(d) $s = \ln\left(e^t+e^{-t}\right) + \tanh(e^{3t}) + e^{\tan(3t)}$

$$\dot{s} = \frac{1}{e^t+e^{-t}}\cdot\left[e^t-e^{-t}\right] + \operatorname{sech}^2(e^{3t})\cdot\left[e^{3t}\cdot 3\right] + e^{\tan(3t)}\cdot\left[\sec^2(3t)\cdot 3\right]$$

$$= \boxed{\frac{e^t-e^{-t}}{e^t+e^{-t}} + 3e^{3t}\operatorname{sech}^2(e^{3t}) + 3\sec^2(3t)\,e^{\tan(3t)}}$$

Problem 5: 6 pts

Find the $\boxed{\text{exact area}}$ of the region in $\boxed{\text{Quadrant I}}$ bounded by the curve $y = \sinh(x)$, the x-axis, and the line $x = \ln(2)$.



$\cdot$ Area $= \int_0^{\ln 2} \sinh(x)\,dx = \cosh(x)\Big|_0^{\ln 2} = \cosh(\ln 2) - \cosh(0) = \frac{5}{4} - 1 = \boxed{\frac{1}{4} \text{ units}^2}$

$\cdot \cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + e^{\ln 2^{-1}}}{2} = \frac{2 + 2^{-1}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$

$\cdot \cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$

Problem 6: 5 pts

At what $\boxed{\textbf{point}}$ on the curve $y = (2 + x)e^{-x}$ is the tangent line horizontal?

slope=0

$y' = 0$

$y' = [1]e^{-x} + (2+x)[e^{-x} \cdot -1]$

$= e^{-x} - e^{-x}(2+x)$

$= e^{-x}(1-(2+x))$

$= e^{-x}(-1-x)$

Thus, $y' = 0$:

$\underbrace{e^{-x}}_{\text{never } 0!}(-1-x) = 0$

$-1-x=0$

$x=-1$

But need point, so find y too:

$y = (2+(-1))e^{-(-1)}$

$= 1 \cdot e^1$

$y = e$

$\boxed{@ (-1, e) \text{ the tangent line is horizontal}}$.

Problem 7: 2 pts

Which of the statements below could be used as the definition of the irrational number "e":

✓ (A) e is the base of the exponential function $f(x) = e^x$ whose slope of the tangent line at $P = (0, 1)$ is one.

✓ (B) $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

✓ (C) $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

✓ (D) $e = \lim_{h \to 0} (1 + h)^{1/h}$.

$\boxed{\text{(E) All of the above.}}$

Problem 8: 24 pts

Find the value of each of the following integrals (simplify answers):

(a) $\int \left(x^7 + 7^x\right)\, dx$

$= \boxed{\frac{1}{8}x^8 + \frac{7^x}{\ln 7} + C}$

(b) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, d\cancel{x} =$

$\begin{cases} u = \sqrt{x} = x^{1/2} \\ du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}dx \longrightarrow 2du = \frac{1}{\sqrt{x}}dx \end{cases}$

$= \int e^u\, 2du = 2e^u + C = \boxed{2e^{\sqrt{x}} + C}$

(c) $\int_0^{1/2} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\, dx$

$\underbrace{\frac{dx}{\sqrt{1-x^2}}}_{du}$

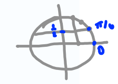
$\begin{cases} u = \sin^{-1}(x) \\ du = \frac{1}{\sqrt{1-x^2}}dx \end{cases}$

$= \int u\,du = \frac{1}{2}u^2 \Big| = \frac{1}{2}\left(\sin^{-1}(x)\right)^2\Big|_0^{1/2} = \frac{1}{2}\left(\sin^{-1}\left(\frac{1}{2}\right)\right)^2 - \frac{1}{2}\left(\sin^{-1}(0)\right)^2$

$= \frac{1}{2}\left(\frac{\pi}{6}\right)^2 - \frac{1}{2}(0)^2 = \boxed{\frac{\pi^2}{72}}$

$=$

- $x = \sin^{-1}\left(\frac{1}{2}\right) \longleftrightarrow \sin(x) = \frac{1}{2} \rightarrow x = \frac{\pi}{6}$
- $x = \sin^{-1}(0) \longleftrightarrow \sin(x) = 0$
  $\rightarrow x = 0$



(d) $\int_0^1 \frac{1+x}{1+x^2}\, dx = \int_0^1 \frac{1}{1+x^2}dx + \int_0^1 \frac{x}{1+x^2}dx =$

(hint: break up into two integrals)

$= \boxed{\frac{\pi}{4} + \frac{1}{2}\ln(2)}$ or $= \boxed{\frac{\pi}{4} + \ln\sqrt{2}}$

- $\int_0^1 \frac{1}{1+x^2}dx = \tan^{-1}(x)\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$
- $\int_0^1 \frac{x}{1+x^2}dx = \int_\bullet^\bullet \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\ln|u|\Big|_\bullet^\bullet = \frac{1}{2}\ln|1+x^2|\Big|_0^1 = \frac{1}{2}\ln|2| - \frac{1}{2}\underbrace{\ln 1}_{=0} = \frac{1}{2}\ln 2$

$\begin{cases} u = 1+x^2 \\ du = 2x\,dx \longrightarrow \frac{1}{2}du = x\,dx \end{cases}$

Problem 9: 8 pts

means: $\frac{dP}{dt} = -kP$

(a) A company decides to dump toxic waste into nearby Lake Muerto in 1975. Local scientists determine that the population of a species of fish living in Lake Muerto a dying off at a rate proportional to its size when the time is measured in years. If the constant of proportionality is $\boxed{k = -1.2}$ and the initial population of fish is $\boxed{300}$, measured in thousands fish, $\boxed{\text{when}}$ will the fish population decreases to one fish? Give the $\boxed{\text{exact answer}}$ ~~($M \to E$)~~

↳ find t

Given

$\frac{dP}{dt} = -1.2P$

solutions are:

$P(t) = P_0 e^{-1.2t}$

w/ initial pop $P_0 = 300$ (thousand)

$P(t) = 300e^{-1.2t}$

one fish: $P = \frac{1}{1000}$ (in thousands)

$P = 0.001$

solve for t: $P(t) = 0.001$

$300e^{-1.2t} = 0.001$

$\ln\left(e^{-1.2t}\right) = \ln\left(\frac{0.001}{300}\right)$

$-1.2t = \ln\left(\frac{0.001}{300}\right)$

$\boxed{t = \frac{\ln\left(\frac{0.001}{300}\right)}{-1.2} \text{ years}}$ (exact ans)

$t \approx 10.5096\ldots$ years

$t \approx 10.5$ years

(note: also accepted $P(t) = 1$

↳ $t = \frac{\ln\left(\frac{1}{300}\right)}{-1.2}$ or $t = \frac{\ln(300)}{1.2}$ years)

(b) How long will it take an investment of \$10,000 to double in value if the $\boxed{\text{interest rate is } 2.5\%}$ $\boxed{\text{when}}$ interest is compounded continuously? Give the $\boxed{\text{exact answer}}$ and an approximation rounded to the nearest thousandths. ~~($M \to E$)~~

find t

↳ use $A = Pe^{rt}$

solve for t: $\cancel{20{,}000}^{\,2} = \cancel{10{,}000}\, e^{0.025t}$

$\ln(2) = \ln\left(e^{0.025t}\right)$

$\frac{\ln 2}{0.025} = \frac{\cancel{0.025}\, t}{\cancel{0.025}}$

$\boxed{\frac{\ln 2}{0.025} \text{ years} = t}$ (exact)

$\boxed{t \approx 27.726 \text{ years}}$ (approx)

$A = Pe^{rt}$

- invest $= P = 10{,}000$
- rate $= 0.025$
- amount $= A = 2P = 20{,}000$

Problem 10: 6 pts

Pick ONE (and only one) of the following statements to prove. Do not do more!
Circle ONE: "I will prove (a)" or "I will prove (b)" or "I will prove (c)" or "I will prove (d)"

(a) Prove: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

(b) Prove: $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$.

(c) Prove: $\dfrac{d}{dx}[\tan^{-1}(x)] = \dfrac{1}{1+x^2}$.

(d) Prove: $\dfrac{d}{dx}[\sinh(x)] = \cosh(x)$.

Proof of ______ :

Proof of (a)

$y = f^{-1}(x) \longleftrightarrow f(y) = x$

$\frac{d}{dx}\left[f(y)\right] = \frac{d}{dx}\left[x\right]$ (use implicit diff)

$\underbrace{f'(y) \cdot \frac{dy}{dx}}_{\text{chain rule}} = 1$

$\frac{dy}{dx} = \frac{1}{f'(y)}$

use orig. eq

$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$. $\square$

Proof of (b):

$y = \ln(x) \longleftrightarrow e^y = x$ (re-write as exp eq)

$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right]$ (use implicit diff)

$\underbrace{e^y \cdot \frac{dy}{dx}}_{\text{chain rule}} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$ use orig. eq

$= \frac{1}{x}$. $\square$

Proof of (c):

$y = \tan^{-1}(x) \longleftrightarrow \tan(y) = x$

$\frac{d}{dx}\left[\tan(y)\right] = \frac{d}{dx}\left[x\right]$

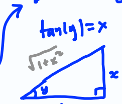
$\underbrace{\sec^2(y) \cdot \frac{dy}{dx}}_{\text{chain rule}} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

$= \frac{1}{1+\tan^2(y)}$ (use Pythag. ID: $1+\tan^2(x) = \sec^2(x)$)

alternatively: use triangle method

$\tan(y) = x$



so: $\sec(y) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{1+x^2}}{1}$

so: $\sec^2(y) = \left(\sqrt{1+x^2}\right)^2 = 1+x^2$

Same!

$\frac{dy}{dx} = \frac{1}{1+x^2}$. $\square$

Proof of (d):

$\frac{d}{dx}\left[\sinh(x)\right] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{d}{dx}\left[\frac{1}{2}e^x\right] + \frac{d}{dx}\left[-\frac{1}{2}e^{-x}\right] = \frac{1}{2}e^x - \frac{1}{2}e^{-x}(-1) = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2} = \cosh(x)$. $\square$

use def of sinh

def of $\cosh(x)$

Problem 11: Extra-credit (4 pts each)

You may attempt these only if every questions on the exam has an attempted solution. Otherwise they will not be graded.

(a) Einstein's Special Theory of Relativity states that the mass of an object traveling with velocity v satisfies

$$m(v) = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where m_0 is a constant (inertial mass) and c is a constant (speed of light). Find the inverse function of m, that is, find $v(m)$

$$(m(v))^2 = \left(\frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right)^2$$ (solve for v in terms of m)

$$m^2 = \frac{m_0^2}{1-\left(\frac{v}{c}\right)^2}$$

$$1-\left(\frac{v}{c}\right)^2 = \frac{m_0^2}{m^2}$$

$$\sqrt{1-\left(\frac{m_0}{m}\right)^2} = \sqrt{\left(\frac{v}{c}\right)^2}$$

$$\sqrt{1-\left(\frac{m_0}{m}\right)^2} = \frac{v}{c}$$

$$\boxed{v(m) = c\cdot\sqrt{1-\left(\frac{m_0}{m}\right)^2}}$$

(b) Given $f(x) = \dfrac{(x^2+5)\sqrt{x}}{(2x+1)^3}$, find $f'(x)$ using logarithmic differentitation

$$\ln f(x) = \ln\left(\frac{(x^2+5)\sqrt{x}}{(2x+1)^3}\right)$$

$$\ln f(x) = \ln\left((x^2+5)\cdot x^{1/2}\right) - \ln\left((2x+1)^3\right)$$

$$\ln f(x) = \ln(x^2+5) + \frac{1}{2}\ln(x) - 3\ln(2x+1)$$

now: implicitly differentiate both sides!

$$\frac{1}{f(x)}\cdot f'(x) = \frac{1}{x^2+5}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x} - 3\cdot\frac{1}{2x+1}\cdot 2$$

$$f'(x) = f(x)\left[\frac{2x}{x^2+5} + \frac{1}{2x} - \frac{6}{2x+1}\right]$$

$$\boxed{f'(x) = \frac{(x^2+5)\sqrt{x}}{(2x+1)^3}\left(\frac{2x}{x^2+5} + \frac{1}{2x} - \frac{6}{2x+1}\right)}$$

Post Exam Survey

Now that you have finished the exam, please take a few minutes to reflect on how you prepared for the exam and how you think you did. Then answer these questions.

1. When taking the exam I felt
   (a) Rushed. I wanted more time.
   (b) Relaxed. I had enough time.
   (c) Amazed. I had tons of extra time.
2. The week before the test I did all my homework on time: YES NO
3. The week before the test, in addition to the homework I followed a study plan. YES NO
   (a) I think this helped: YES NO
4. The day before the test I spend ________ hours studying and reviewing.
   (a) I think that was enough time: YES NO
5. The night before the test:
   (a) I stayed up very late cramming for the test
   (b) I stayed up very late, but I wasn't doing math
   (c) I didn't need to cram because I was prepared
   (d) I got a good night's sleep so my brain would function well.
6. I think I got the following grade on this test: ________
7. Strategies that worked well for me were (please elaborate):
8. Next time I will do an even better job preparing for the test by: