What is Calculus?

A Brief History of Mathematics and Calculus

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Outline

What is Calc?

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Intro
The APs
Cultural Context
AP1: Area
AP2: Inst. Velocity
AP3: Numbers
AP4: TLP
AP solved
A bit more history
Does $pi = 4$?

Calculus: Latin word for "pebble"

- In Roman times: used to count
- Now: a subject of math that includes tools to solve hard math problems
  - Common answer: "study of change"
    - Change is encoded by functions
    - Change is short for Rates of Change, which is short for Instantaneous Rates of Change
  - My answer: "a PROCESS developed to solve hard problems in the following steps:"
    - Step 1: find an approximate solution to the hard problem
    - Step $1\frac{1}{2}$: find a better approximate solution, & then a better one, & better ...
    - Step 2: the exact (ideal) answer = LIMIT of approximate solutions"

Ancient Problems

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The strategy outlined in the three steps which I call the "process of calculus" solved Ancient Problems

- Ancient Problem 1: Area under a curve (Egypt, Mesopotamia, Greek)
- Ancient Problem 2: Problem of Motion, i.e. Instantaneous Velocity (Greek)
- Ancient Problem 3: What is a (real) Number? (Egypt, Mesopotamia, Greek)
- Ancient Problem 4: Tangent Line Problem (Greek, Europe Middle Ages)

Cultural Context

Timeline for Ancient Cultures

- Mesopotamia: $\approx$ 10,000 BCE (earliest human) to 500 BCE
  - Notable inventions: humanity/culture?, earliest writing and number notation (especially decimal expansions!), math, astronomy
- Egypt: $\approx$ 5500 BCE to 30 BCE
  - Notable inventions: humanity/culture?, paper, writing system, engineering feats (Pyramids built $\approx$ 2600 BCE), math, astronomy
- Greek: $\approx$ 800 BCE to 600 AD
  - Notable inventions: alphabet, politics, philosophy, logic/reasoning, science, deductive method, idea of proof (in geometry)
- Also: India and China had ancient cultures which parallel much of the above[1]

[1] Apology: I don't know much of their Ancient Cultures in reference to math but also their historical record isn't as easily accessible

Want to know more?

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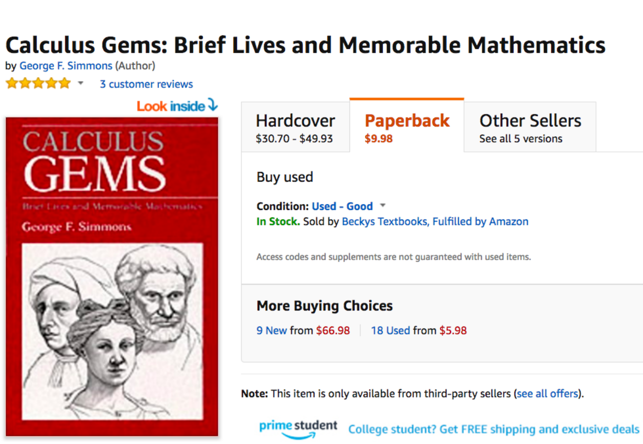
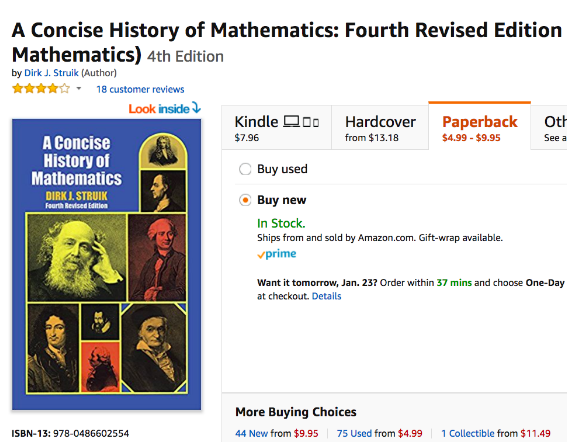
1. Calculus Gems: Brief Lives and Memorable Mathematics
2. A Concise History of Mathematics
3. & So much more! Curious? Just ask me :-)





Ancient Problem 1: Area under a curve

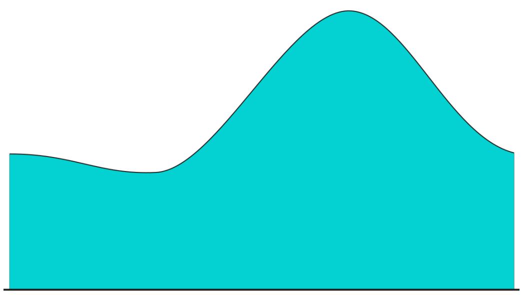


Figure: In Egypt, region determined by lines (road, neighbors) and the river Nile

In Egypt, taxes were collected according to how much land you had.

Because the Nile river floods, the amount of land you had would change through the seasons. Thus, it was necessary to figure out the area of your land according to a boundary shaped by a complex curve.

Notice: you can't use simple geometric shapes (rectangles, triangles, circles) to calculate such an area exactly.

AP1: Find the area under the curve? Asked by: Egypt, Mesopotamia, Greek

Ancient Problem 2: Instantaneous Velocity

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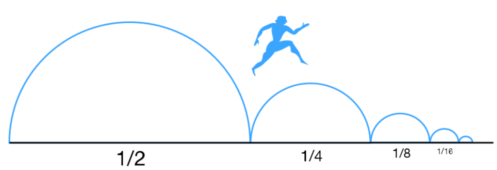


Zeno's Paradox:
Motion is impossible

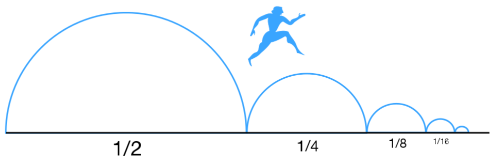
How does one define motion?

Velocity = Distance / time

Velocity at an instant?? Divide by 0?

AP2: What is motion, or instantaneous velocity? Asked by: Greek

Ancient Problem 2: Instantaneous Velocity



Zeno of Elea:
(c. 490 -430 CBE)

There are three main paradoxes attributed to Zeno extant from Aristotle's *Physics*:

Achilles and the Tortoise, The Dichotomy, The Arrow

The Dichotomy: *"That which is in locomotion must arrive at the half-way stage before it arrives at the goal."* –as recounted by Aristotle, Physics

Ancient Problem 3: (Real) Numbers

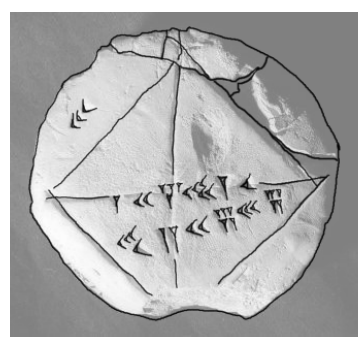
- Babylonians: invented very complex number system that included decimals called the positional notation

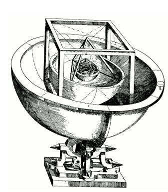
  Ex: $12.65 = 10 \cdot 1 + 1 \cdot 2 + 10^{-1} \cdot 6 + 10^{-2} \cdot 5$

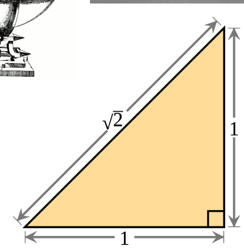
  Understood numbers like $\sqrt{2}$
- Greeks: "All is Number" (Pythagoas)

  π, $\sqrt{2}$ were NOT numbers

  Because of this: geometry and numbers developed separately







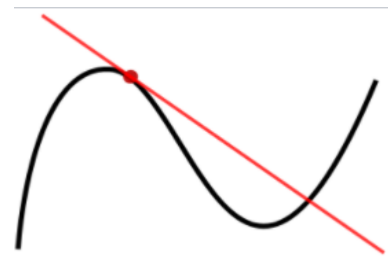
AP3: What is a (real) number? Asked by: Egypt, Mesopotamia, Greek

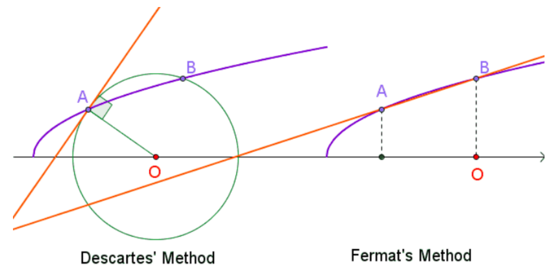
Ancient Problem 4: Tangent Line Problem

This one is clearly more modern:

> **AP4:** Given a curve and a point P on the curve, find the line that also passes through P that is "infinitely close to the curve" (Liebniz)

Tangent lines were important in Greek geometry. Archimedes ($\approx$ 300 BCE) found the tangent line to a spiral (first to find a tangent to a non-circular curve)
Many Renaissance mathematicians: Descartes, Fermat, & more solved it for specific functions like polynomials of low degrees





In both cases, O slides along horizontal axis until A=B.

How were these problems solved?

- Calculus: "a PROCESS developed to solve hard problems in the following steps:"
  - Step 1: find an approximate solution to the hard problem
  - Step $1\frac{1}{2}$: find a better approximate solution, & then a better one, & a better, ETC
  - Step 2: the exact (ideal) answer = LIMIT of approximate solutions"
- Open Stewart's "A Preview of Calculus"
  - AP1: Area of a circle via Archimedes' "Method of Exhaustion"
  - AP4: Tangent Line Problem
  - AP2: Instantaneous Velocity
  - Limit of a sequence (related to Zeno?s Paradox/AP2 and also irrational numbers/AP3)
  - Sum of a series (related to irrational numbers/AP3)

> Many different questions, the same approach towards a solution
> BUT MUST DEAL WITH **LIMITS/INFINITY**

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A bit more history...

Ingredients for Calculus?

The Greeks developed planar geometry to an impressive degree of sophistication and certainly were struggling with APs 1-4 mainly from a philosophical point of view and not necessarily concerned with applications to the real world.

A bit more history...

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What kept them from developing the calculus?

In short, notation. The modern notation of numbers was not yet invented. Babylonian system of numbers was thousands of years older than the Greek but it was not pursued by them in favor of the whole numbers due to mystic reasons (thanks Pythagoras!). It was hundreds of years after the Greeks when the Indian/Islamic mathematicians adopted the decimal notation from the Babylonians with a nicer way of writing numbers using the symbols 0,1,2,...,9, which you of course recognize. But "number" itself is a tool and not the impetus for applications. These came from algebra equations. The Indian/Islamic mathematicians developed the theory of algebra (briefly, solving polynomial eqs) with the notation of today. This helped pave the way for the quintessential invention needed for calculus– the function.

A bit more history. . .

What about Functions?

Essentially they encode "variation". Variation is code for change. Compare the "static" equation $2x + y = 3$ with the dynamic function $f(x) = 3 - 2x$. They encode the same information but the function point of view implies variation, dynamics, input/output. By the way, the function concept was a tricky thing. It was developed hundreds of years AFTER Newton/Leibniz invented calculus.

A bit more history. . .

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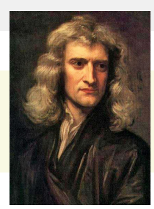
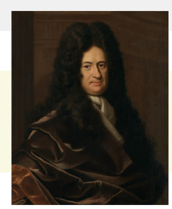
Wait! I thought Newton/Liebniz invented Calculus!





Sort of. The main reason for their fame: they are the first to observe and prove that AP1 and AP4 are related problems–they are "inverse to each other" or:

AP1 & AP4 are TWO SIDES of the SAME COIN

Besides solving the APs, the subject of calculus is unparalleled at solving hard problems. Two main branches of calculus:

1. **AP1** (Area Problem) became known as integral calculus
2. **AP4** (Tangent Line Problem) because known as the differential calculus

A Fear of Infinfty

Infinity has captivated mathematicians, philosophers, and poets like no other concept. It is as alluring as it is wicked.

The rules for how to correctly work with the concept are not obvious (unlike the rules for Euclidean geometry).

It took a long time to understand what is allowed and what is not allowed in infinite processes. The result of these studies is calculus, which you will now learn.

Here's two examples showing what can go wrong:

- **Zeno's Paradoxes** stumped the best minds for thousands of years
- $\pi = 4$

Does $\pi = 4$?

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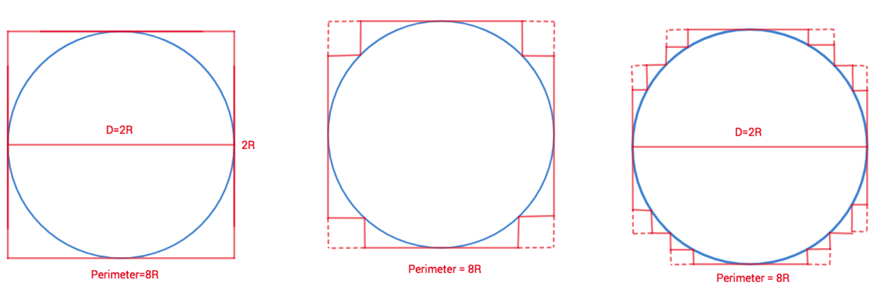
Theorem 1:

$\pi = 4$

Start with a circle of radius R. We first estimate the perimeter by $8R$ using a circumscribed square. Cut out four corners and use the new shorter sides to approximate the perimeter. Notice that the perimeter is still $8R$. Approximate the circle with more and more with sides parallel to axes. The perimeter is still $8R$. Limit of these jagged curves approximates the circle so the limit of these perimeter is the circumference of the circle. Thus, perimeter of full circle is $8R$. Because $C = 2\pi R$ we have $8R = 2\pi R$. We conclude: $4 = \pi$! $\square$



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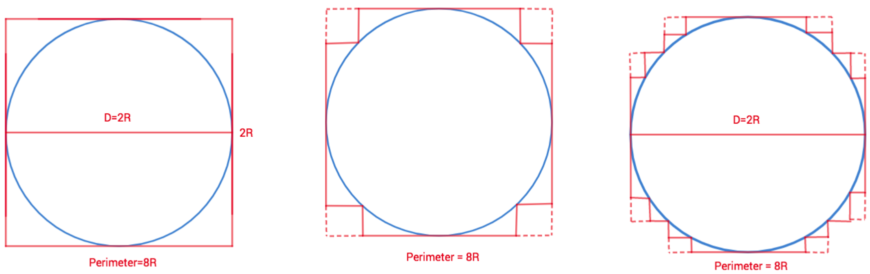
Of course this is wrong!

There's a flaw in this argument that's *very* difficult to catch.
Can you find it?



Does $\pi = 4$?

THANK YOU FOR YOUR ATTENTION