7.4 Orthogonal Complements and Decompositions

Orthogonal Complements

Definition/Theorem: Let W be any subspace of an inner product space V. We define the ***orthogonal complement*** of W, another subspace of V, by:

$$W^{\perp} = \left\{ \vec{v} \in V \mid \left\langle \vec{v} \mid \vec{w} \right\rangle = 0 \text{ for all } \vec{w} \in W \right\}$$

Theorem: Let $B = \left\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\right\}$ be a ***basis*** for a subspace W of a finite dimensional inner product space V. Then $\vec{v} \in V$ is a member of $W^{\perp}$ ***if and only if***:

$$\left\langle \vec{v} \mid \vec{w}_i \right\rangle = 0 \text{ for all } i = 1, 2, ..k$$

Thus, it is both necessary and sufficient that we check that an arbitrary vector $\vec{v} \in V$ is orthogonal to every member of a basis B for W.

Strategy: Set-up and solve a homogeneous system of equations (nullspace!).

Finding Orthonormal Bases for W and $W^{\perp}$

Theorem (The Dimension Theorem for Orthogonal Complements):

Let $W = Span(B)$ be a subspace of a ***finite dimensional*** inner product space V, where $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for W.

If we enlarge B to:

$$B' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n\},$$

a basis for V, and

$$S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{u}_{k+1}, \ldots, \vec{u}_n\}$$

is the orthonormal basis for V obtained after applying the Gram-Schmidt Algorithm to B', then

$$S_1 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$$

is an orthonormal basis for W, and

$$S_2 = \{\vec{u}_{k+1}, \ldots, \vec{u}_n\}$$

is an orthonormal basis for $W^{\perp}$.

Consequently:

$$dim(W) + dim(W^{\perp}) = n.$$

More generally, if $S = S_1 \cup S_2$ is an orthonormal basis for V, where S_1 and S_2 are non-empty subsets of S with no member in common, then $W_1 = Span(S_1)$ and $W_2 = Span(S_2)$ are orthogonal complements of ***each other***. Thus:

$$(W^{\perp})^{\perp} = W.$$

Note: It is an interesting fluke in Linear Algebra that the last equation above is not necessarily true in an ***infinite dimensional*** inner product space.

However, it is ***always true*** that $W \subset (W^{\perp})^{\perp}$.

Orthogonal Decompositions

Theorem: Let W be a ***finite-dimensional subspace*** of an inner product space V. Then, any vector $\vec{v} \in V$ can be expressed ***uniquely*** as a sum:

$$\vec{v} = \vec{w}_1 + \vec{w}_2,$$

where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$. We refer to this as the ***orthogonal decomposition*** of $\vec{v}$ with respect to W and $W^\perp$.

Moreover, we can explicitly construct $\vec{w}_1$ and $\vec{w}_2$ as follows:

If $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\right\}$ is any orthonormal basis for W, then:

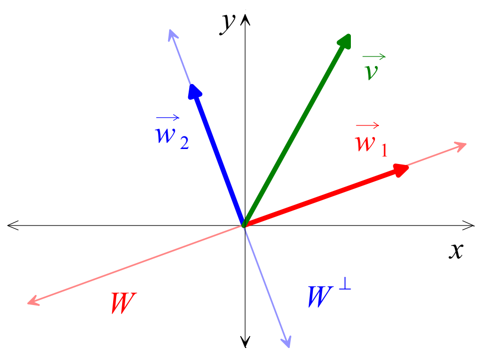
$$\vec{w}_1 = \left\langle \vec{v} \mid \vec{u}_1 \right\rangle \vec{u}_1 + \left\langle \vec{v} \mid \vec{u}_2 \right\rangle \vec{u}_2 + \cdots + \left\langle \vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k, \text{ and}$$

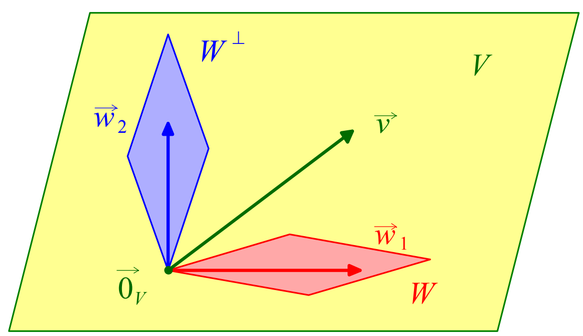
$$\vec{w}_2 = \vec{v} - \vec{w}_1$$

We call $\vec{w}_1$ the ***orthogonal projection*** of $\vec{v}$ onto W, and $\vec{w}_2$ the orthogonal projection of $\vec{v}$ onto $W^{\perp}$.

We write this as:

$$\vec{w}_1 = proj_W(\vec{v}) \quad \text{and}$$
$$\vec{w}_2 = proj_{W^{\perp}}(\vec{v})$$





The Orthogonal Decomposition $\vec{v} = \vec{w}_1 + \vec{w}_2$,

where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^{\perp}$