Section 10.3 Partial Fraction Decomposition

Objectives
- Method of Partial Fractions
- Case 1: Distinct Linear Factors
- Case 2: Repeated Linear Factors
- Case 3: Irreducible Quadratic Factors
- Case 4: Repeated Irreducible Quadratic Factors

• Method of Partial Fractions

Intro Let's start with two simple rational expressions and add: $\frac{3}{x+4} + \frac{2}{x-3}$ start w/ common denominator!

$$\frac{3}{x+4} \cdot \frac{x-3}{x-3} + \frac{2}{x-3} \cdot \frac{x+4}{x+4} = \frac{3(x-3)+2(x+4)}{(x+4)(x-3)} = \frac{3x-9+2x+8}{(x+4)(x-3)} = \boxed{\frac{5x-1}{(x+4)(x-3)}}$$

Reverse this!

What if we started instead with one rational expression and wanted to write it as a sum of two simpler fractions: $\frac{5x-1}{x^2+x-12}$?
This is much harder! However, this method is useful in certain situations used in calculus.

Defn 1
The **method of partial fractions** is an algebraic technique where we start with one rational expression and we expand it into many simpler separate terms, called **partial fractions**. In some sense, we are "undoing" adding the simple separate terms.
We call $\frac{3}{x+4} + \frac{2}{x-3}$ the **partial fraction decomposition** of $\frac{5x-1}{x^2+x-12}$.

Defn 2
Recall that a rational expression is an expression $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
It is called **proper** if $deg(P(x)) < deg(Q(x))$, that is the denominator has a larger degree than the numerator, otherwise $R(x)$ is called **improper**.

Theorem 1 Partial Fraction Decomposition

P & Q must be polynomials ~~sin(x)~~ ~~e^x~~ ~~$\sqrt{x}$~~

Start with a proper rational expression $R(x) = \frac{P(x)}{Q(x)}$. We break up into cases depending on the denominator $Q(x)$.

Case 1 **Distinct Linear Factors**
Assume that we can factor $Q(x)$ into distinct linear factors: $\boxed{Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}$.
Then the partial fraction decomposition of $R(x)$ takes the form

$$R(x) = \boxed{\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n},}$$

where the A_i's are real numbers.

Case 2 **Repeated Linear Factors**
Assume the factorization of $Q(x)$ contains the factor $\boxed{(ax+b)^k}$–that is, the linear factor $ax+b$ is repeated k times.
Then corresponding to each such repeated factor, the the partial fraction decomposition of $R(x)$ contains the form

$$\boxed{\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}}$$

where the A_i's are real numbers.

Case 3 **Irreducible Quadratic Factors** (power 2)
Assume the factorization of $Q(x)$ contains the irreducible quadratic factor $\boxed{ax^2+bx+c}$, which can't be factored further.
Then corresponding to each such repeated factor, the the partial fraction decomposition of $R(x)$ contains the form

doesn't factor

$$\boxed{\frac{Ax+B}{ax^2+bx+c},}$$

Main Ex x^2+1 x^2+4

where the A and B are real numbers.

Case 4 **Repeated Irreducible Quadratic Factors**

Assume the factorization of $Q(x)$ contains the factor $(ax^2+bx+c)^k$ where the quadratic factor ax^2+bx+c is irreducible and is repeated k times.
Then corresponding to each such repeated factor, the the partial fraction decomposition of $R(x)$ contains the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

where the A_i's are real numbers.

Notes:

- To use the method of partial fractions, keep in mind that the denominator must have a degree greater than the numerator (or be the same degree).
- Don't assume that a quadratic term is irreducible. Check if it factors first! irreducible x^2+1 vs $x^2-1=(x+1)(x-1)$
- Practice setting-up a complicated partial fractions problem without trying to solve it completely.
- It is not always possible to perform the method of partial fractions since you can't always factor expressions of high degree (starting at degree 5 due to a deep theorem in advanced mathematics).
- Recall that an improper fraction can be reduced via long division to a polynomial and a proper fraction. For example, you can check using long division that

$$\frac{2x^3+3x^2+7x+4}{2x+1}=x^2+x+3+\frac{1}{2x+1}.$$

$2x+1 \overline{)\,2x^3+3x^2+7x+4}$

- **Pro Tip:** The JB Method (aka Short-cut). Instead of setting up a system of equations

Ex 1 Here are some examples of each case in the above theorem:

(a) Case 1: $\dfrac{x-7}{(x-2)(x+3)}=\dfrac{-1}{x-2}+\dfrac{2}{x+3}$ (b) Case 2: $\dfrac{3x^2-x-3}{x^2(x+1)}=\dfrac{1}{x+1}+\dfrac{2}{x}+\dfrac{-3}{x^2}$

repeated linear
$x \cdot x$

(c) Case 3: $\dfrac{3}{(x-1)(x^2+2)}=\dfrac{1}{x-1}-\dfrac{x+1}{x^2+2}$ (d) Case 4: $\dfrac{1}{x(x^2+1)^2}=\dfrac{1}{x}-\dfrac{x}{x^2+1}-\dfrac{x}{(x^2+1)^2}$

irreducible

repeated irreducible

GTQ

Ex 2 Write **the form** of the partial fraction decomposition for each of the following. **Do not solve!**

(a) $\dfrac{x^2+7x-2}{x(x+1)(x-1)}$ (b) $\dfrac{x^3-x+5}{(x+1)(x-2)^2(x^2+1)}$ (c) $\dfrac{x^3-x+5}{(x+1)(x-2)^2(x^2+x+1)^3}$ $= \dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}+\dfrac{Dx+E}{x^2+x+1}+\dfrac{Fx+G}{(x^2+x+1)^2}+\dfrac{Hx+I}{(x^2+x+1)^3}.$

check if factors

$x^2+x+1 \overset{?}{=} (x+1)(x+1)$
$\overset{?}{=} (x-1)(x-1)$

neither work! irreducible!

a) $\dfrac{x^2+7x-2}{x(x+1)(x-1)} = \boxed{\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}}$

distinct linear

b) $\dfrac{x^3-x+5}{(x+1)(x-2)^2(x^2+1)} = \boxed{\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}+\dfrac{Dx+E}{x^2+1}}$

irreducible

Procedure for finding the partial fraction decomposition Follow the steps:

1. If the $R(x)$ is proper, go to step 2. Otherwise, use long division to divide and continue with the rational term.
2. Factor the denominator into a product of linear, $(ax+b)^k$, and irreducible quadratic factors, $(ax^2+bx+c)^k$.
3. Set $R(x)$ equal to the sum of the partial fractions with unknown coefficients according to the above theorem. (Ex 2)
   Pro Tip Avoid using subscripts and use letters in the alphabet: A, B, C, . . . as needed
4. Clear the denominator to obtain the **basic equation**.
5. Solve the basic equation:
   a) **The JB Method (aka Short-cut)** Plug-in zeros of the denominator, $Q(x)$, into the basic equation to solve for the coefficients.
   b) **The Long Way** Expand both sides, collect like-terms, and solve the corresponding system of equations

Ex 3 Find the partial fraction decomposition for each of the following.

(a) $\dfrac{x^2+7x-2}{x^3-x}$

(b) $\dfrac{x^2+11x+4}{x^3+4x^2+4x}$

(c) $\dfrac{3x-5}{x^3-1}$

(d) $\dfrac{x^3+x^2}{(x^2+4)^2}$

Ex 4 Find the partial fraction decomposition for each of the following.

(a) $\dfrac{x^3-3x^2+1}{x^2+5x+6}$

(b) $\dfrac{x^3-8}{x^2(x-1)^3}$

Ex3 a) $\dfrac{x^2+7x-2}{x^3-x} = \dfrac{x^2+7x-2}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

Q(x) Q(x)

Case 1 (3) (4) clear Denom

(2) factor denom?

$x^3-x = x(x^2-1) = x(x+1)(x-1)$

BASIC EQ

Clearing Denom: $x^2+7x-2 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

JB Method plug-in $x=0,\ x=1,\ x=-1$

$\boxed{x=0}$ $-2 = A\cdot 1\cdot(-1) + B\cdot 0 + C\cdot 0$

$-2 = -A \longrightarrow \boxed{A=2}$

$\boxed{x=1}$ $1+7-2 = 0 + 0 + C(1)(2)$

$6 = 2C \longrightarrow \boxed{C=3}$

$\boxed{x=-1}$ $1-7-2 = 0 + B(-1)(-2) + 0$

$-8 = 2B \longrightarrow \boxed{B=-4}$

$$\boxed{\dfrac{x^2+7x-2}{x^3-x} = \dfrac{2}{x} - \dfrac{4}{x+1} + \dfrac{3}{x-1}}$$

position for each of the foll

(b) $\dfrac{x^2 + 11x + 4}{x^3 + 4x^2 + 4x} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ $Q(x)$

$Q(x)$

① Factor Denom: $x^3 + 4x^2 + 4x = x(x^2 + 4x + 4) = x(x+2)^2$

$= x(x+2)(x+2)$

② Clear Denom: $1 \cdot x^2 + 11x + 4 = A(x+2)^2 + Bx(x+2) + Cx$

expand & group like terms

$= A(x^2 + 4x + 4) + Bx^2 + 2Bx + Cx$

$= x^2[A+B] + x[4A + 2B + C] + 1 \cdot [4A]$

③ Set-up SOE:

$$\begin{cases} 1 = A + B \\ 11 = 4A + 2B + C \\ 4 = 4A \end{cases}$$

$4 = 4A \rightarrow \boxed{A = 1}$

back sub

$1 = 1 + B \rightarrow \boxed{B = 0}$

plug in $A = 1$ & $B = 0$: $11 = 4(1) + 2(0) + C$

$11 = 4 + C$

$\boxed{C = 7}$

$$\frac{x^2 + 11x + 4}{x^3 + 4x^2 + 4x} = \frac{1}{x} + \frac{0}{x+2} + \frac{7}{(x+2)^2} = \boxed{\frac{1}{x} + \frac{7}{(x+2)^2}}$$

Check

$$\frac{1}{x} \cdot \frac{(x+2)^2}{(x+2)^2} + \frac{7}{(x+2)^2} \cdot \frac{x}{x} = \frac{(x+2)^2 + 7x}{x(x+2)^2} = \frac{(x^2 + 4x + 4) + 7x}{x(x+2)^2}$$

$$= \frac{x^2 + 11x + 4}{x(x+2)^2} \checkmark$$

$x^3 - a^3 = (x-a) \ldots\ldots$

(c) $\dfrac{3x-5}{x^3-1} = \dfrac{3x-5}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$

① Factor Denom $x^3 - 1 = ?$

- clearly $x=1$ is a root since $1^3 - 1 = 0$, so $(x-1)$ factors out!
- Don't remember this? Use long division!

$$
\begin{array}{r l}
 & x^2 + x + 1 \\
x-1 \,\big) & \cancel{x^3} - 1 \\
 & -(\cancel{x^3} - x^2) \\
\hline
 & \cancel{x^2} - 1 \\
 & -(\cancel{x^2} - x) \\
\hline
 & x - 1 \\
 & -(x-1) \\
\hline
 & R\ 0
\end{array}
$$

$\dfrac{x^3-1}{x-1} = x^2+x+1$

multiply by $x-1$

$$\underline{\underline{x^3 - 1 = (x-1)(x^2+x+1)}}$$

factor!

$x^2+x+1 =$ factor? No!

-5
-1

$Q(x)\left(\dfrac{3x-5}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}\right)$ $(x-1)(x^2+x+1) = Q(x)$

Clear Denom $\quad 3x-5 = A(x^2+x+1) + (Bx+C)(x-1)$

- JBM $x=1$: $\quad 3-5 = A(3) + (\cancel{B+C})\cdot 0 \Rightarrow -2 = 3A$ $\quad \boxed{A = -2/3}$
- Plug-in: $A = -2/3$

$3\left(3x-5 = -\frac{2}{3}(x^2+x+1) + Bx^2 - Bx + Cx - C\right)$

$9x - 15 = -2x^2 - 2x - 2 + 3Bx^2 - 3Bx + 3Cx - 3C$

$$0 \cdot x^2 + 9x - 15 = \underline{-2x^2} \underline{-2x} \underline{-2} + \underline{3Bx^2} \underline{-3Bx + 3Cx} \underline{-3C}$$

$$= x^2\left[3B - 2\right] + x\left[-2 - 3B + 3C\right] + 1 \cdot \left[-2 - 3C\right]$$

Set-up SOE

$$\begin{cases} ① \quad 0 = \cancel{3}B - 2 \\ ② \quad 9 = -\cancel{3}B + \cancel{3}C - 2 \\ ③ \quad -15 = \quad -\cancel{3}C - 2 \end{cases}$$

Use Elimination method

① + ② $\quad 9 = 3C - 4$

$13 = 3C$

$\boxed{C = 13/3}$

② + ③ $\quad -6 = -3B - 4$

$-2 = -3B$

$\boxed{B = 2/3}$

$A = -2/3 \quad B = 2/3 \quad C = 13/3$

$$\frac{3x-5}{x^3-1} = \frac{-2/3}{x-1} + \frac{2/3\,x + 13/3}{x^2+x+1}$$

$$= \boxed{\frac{-2}{3(x-1)} + \frac{2x+13}{3(x^2+x+1)}}$$

Ex

$$\frac{5x-6}{x^3+2x^2-3x} = \frac{5x-6}{x(x^2+2x-3)} = \frac{5x-6}{x(x+3)(x-1)}$$

Case 1 only! 3 distinct linear use JBM

$$= \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$$

$$0x^2 + 5x - 6 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3)$$

$$= A(x^2 - x + 3x - 3) + B(x^2 - x) + C(x^2 + 3x)$$

$2x$

$$= x^2 \cdot [A+B+C] + x \cdot [2A - B + 3C] + 1 \cdot [-3A]$$

$$\begin{cases} 0 = A + B + C \\ 5 = 2A - B + 3C \\ -6 = -3A \end{cases}$$

$-6 = -3A \longrightarrow \frac{-6}{-3} = A \Rightarrow \boxed{A = 2}$

$$\begin{cases} 0 = 2 + \cancel{B} + C \\ 5 = 4 - \cancel{B} + 3C \end{cases}$$

$+$

$5 = 6 + 4C$

$-1 = 4C$

$\boxed{C = -1/4}$

plug in $C = -1/4$ & solve for B

$(0 = 2 + B - \frac{1}{4})\,4$

$0 = 8 + 4B - 1$

$-7 = 4B$

$\boxed{B = -7/4}$

$$\frac{5x-6}{x^3+2x^2-3x} = \frac{2}{x} - \frac{7/4}{x+3} - \frac{1/4}{x-1} = \frac{2}{x} - \frac{7}{4(x+3)} - \frac{1}{4(x-1)}$$