

§11.4: The Comparison Tests

Ch 11: Infinite Sequences and Series Math 5B: Calculus II

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Class #19 Notes

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Guiding Questions for §11.4

Guiding Question(s)

- ① What is the **comparison test**?
- ② What is the **limit comparison test**?

- Finding the exact value (sum) of a general series is very difficult. We typically can only estimate the series with a partial sum, S_n .
- Still, we must know whether or not the series converges before we try to use the estimate (say, in solving an applied problem).
- Our goal in this section is to develop two more tools that helps us determine whether a series **converges** or **diverges**.
- The tools we learn in this section are called the **comparison test** and the **limit comparison test**.
- This question is so important that we will learn additional tools in future sections.

Comparison Test

- We focus on series with **positive terms** only. I.e. $a_n > 0$ (or non-negative $a_n \geq 0$)
- The basic idea of the comparison test is to use basic inequalities to compare with series we already know converge or diverge (so far: geometric series, harmonic series, p -series)
- We use the informal “squint test.”
- If the squint test suggest convergence, estimate with a series that’s larger
- If the squint test suggest divergence, estimate with a series that’s smaller

Comparison Test

- $\sum_{j=1}^{\infty} \frac{1}{2^j + 2^{j+1}}$ “squint test” should compare with $\sum_{j=1}^{\infty} \frac{1}{2^j}$ (which converges by geometric series with $r = 1/2$)

$$\text{each term: } \frac{1}{2^j + 2^{j+1}} \leq \frac{1}{2^j}, \text{ for each } j = 1, 2, 3, \dots$$

- $\sum_{m=1}^{\infty} \frac{1}{4m-3}$ “squint test” should compare with $\sum_{m=1}^{\infty} \frac{1}{m}$ (which diverges since harmonic)

$$\text{each term: } \frac{1}{4m-3} \geq \frac{1}{4m}, \text{ for each } j = 1, 2, 3, \dots$$

Theorem 1: Comparison Test

Assume that $a_n > 0$ and $b_n > 0$, i.e. the series $\sum a_n$ and $\sum b_n$ have **positive** terms.

- (a) If $a_n \leq b_n$ for all $n \geq 1$ and $\sum b_n < \infty$ [C] THEN $\sum a_n < \infty$ [C]
- (b) If $b_n \leq a_n$ for all $n \geq 1$ and $\sum b_n = \infty$ [D] THEN $\sum a_n = \infty$ [D]

Remark: the comparison test still works even if we ignore a finite number of terms. That is, if we know the comparison work *after ignoring a finite number of terms*, $n \geq N$ (for some large N).

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Activity 1:

Use the comparison test to determine whether $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

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Activity 2:

Use the comparison test to determine whether $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or diverges.

(Hint: ignore the first few terms then compare.)

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Limit Comparison Test

- Hard to make comparisons with inequalities with series like:

$$\sum_{n=0}^{\infty} \frac{1}{3^n - 4} \quad \text{or} \quad \sum_{k=0}^{\infty} \frac{2k^3 - 5k - 3}{\sqrt{7 + k^8}}$$

- “Squint Test” says the first is like $\sum \frac{1}{3^n}$ so should converge (geometric series with $r = 1/3$) and the second is like $\sum \frac{k^3}{k^{8/4}} = \sum \frac{1}{k}$ so should diverge (harmonic).
- The limit comparison test turns the squint test into a rigorous test!

Theorem 2: Limit Comparison Test

Assume that $a_n > 0$ and $b_n > 0$, i.e. the series $\sum a_n$ and $\sum b_n$ have **positive** terms. Define

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

If $0 < L < \infty$, THEN either both series converge, or both series diverge.

If $L = 0$ or $L = \infty$ then the test is inconclusive.

Remark: the limit comparison test still works even if we ignore a finite number of terms. That is, if the first few terms are negative, but are positive after then we can still use this test.

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Activity 3:

Use the limit comparison test to determine whether the series converges or diverges.

(a)
$$\sum_{n=0}^{\infty} \frac{1}{3^n - 4}$$

(b)
$$\sum_{k=0}^{\infty} \frac{2k^3 - 5k - 3}{\sqrt{7 + k^8}}$$

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