

§11.10: Taylor and Maclaurin Series

Ch 11: Infinite Sequences and Series Math 5B: Calculus II

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Class #24 Notes

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Guiding Questions

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Proof of Taylor's Theorem

Outline



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Guiding Question(s)

- What are Taylor series?
- 2 How can we use Taylor series to evaluate transcendental functions via approximation?
- What are the Taylor series for important functions?

Introduction



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Hold up!

- Why on earth would we take a perfectly good function like $f(x) = \frac{1}{1+x^2}$ and write it as a power series, especially since it only works for a restricted domain!?!?
- A few reasons:
 - Multiplying and adding on a computer is really fast and easy
 - Dividing by hand and on a computer is harder
 - We can use partial sums to approximate complicated functions with only adding and multiplying.
 - We can find derivatives of f(x) easily using the power series
 - We can find anti-derivatives of f(x), $\int \frac{1}{1+x^2} dx$, easily using power series
 - These are the heart of the chapter.

Introduction



 We've mentioned a few times now that this chapter is built on the foundation of the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad \text{for} \quad x \in (-1,1)$$

• Other important functions we found a PSR for:

•
$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$
 for $x \in (-1,1)$
• $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $x \in (-1,1)$

•
$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$
 for $x \in (-1,1)$

• What about other important functions?

$$sin(x), e^x$$
, and others

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Intro

Introduction



 We will develop a method to find power series expansions/representations for a wider range of functions and devise a method to identify the values of x for which the function equals the power series expansion. (This is not always the entire interval of convergence of the power series.) §11.10

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• Assume that a function f(x) has a power series representation at x = a with radius of convergence R > 0.

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

- Recall that by the "Just like Poly" theorem, we can find all the derivatives of f(x) inside the interval of convergence, (a R, a + R).
- Then f'(x) =
- Then f''(x) =
- Then f'''(x) =
- Then $f^{(4)}(x) =$

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Activity 1:

- (a) Find a formula for the *n*th derivative of f(x), $f^{(n)}(x) = \frac{d^n f}{dx^n}$.
- (b) Find a formula for $f^{(n)}(a)$, for n = 0, 1, 2, 3, 4 and general n.
- (c) Find a formula for c_n in terms of $f^{(n)}(a)$.



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Definition 1: Taylor Series

Let f(x) be a function that is infinitely differentiable at x = a.

• The Taylor series of f(x) at x = a is

$$T(x) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(a)}{n!} \right] (x-a)^n$$

= $f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots$

- Recall: $f^{(0)}(x) = f(x)$, that is, the zeroth derivative is simply the function itself. Also, 0! = 1 by definition.
- The Maclaurin series of f(x) is a Taylor series at the origin, x = 0:

$$T(x) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(0)}{n!} \right] x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(1)}{2!} x^2 + \frac{f'''(2)}{3!} x^3 + \cdots$$

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Theorem 1: Taylor Series Part 1

- If f(x) has a power series representation at x = a, then the Taylor series of f at x = a is the same as the power series representation.
- That is, if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $x \in (a-R,a+R)$ and

$$T_a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$
, then

$$c_n = \frac{f^{(n)}(a)}{n!} \ .$$

- When a function f(x) is infinitely differentiable at x = a, then $T_a(x)$ exists and must be its PSR.
- We also have: $T'_a(a) = f'(a)$.

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Theorem 2: Taylor Series for e^x

The Taylor series for $f(x) = e^x$ at x = 0 is

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

with radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

Remark: By plugging-in x = 1 into the formula above, we prove Definition 3 in our "Eight Definitions of e" handout:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

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Activity 2:

Prove Theorem 2: That is, find the Taylor series for e^x at x=0 and it's interval of convergence.

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Theorem 3: Taylor Series for sin(x)

The Taylor series for $f(x) = \sin(x)$ at x = 0 is

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

with radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

Remark: By plugging-in x = 1 into the formula above, we arrive at an answer to one of the questions from our introduction:

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} + \dots \approx \frac{101}{120} = 0.841\overline{6}$$

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Activity 3:

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Theorem 4: Taylor Series for cos(x)

The Taylor series for f(x) = cos(x) at x = 0 is

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

with radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

Taylor Series for cos(x)



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Activity 4:

Prove Theorem 4: That is, find the Taylor series for cos(x) at x=0 and it's interval of convergence.

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Theorem 5: Taylor's Theorem

- Let f(x) be an infinitely differentiable function at x = a.
- Let $T_a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ be the Taylor series of f(x) at x=a defined for |x-a| < R for some R > 0.
- Let $T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n$ called the Nth Taylor Polynomial of f(x).
- Let $R_N(x) = f(x) T_N(x)$ called the remainder term of the Nth Taylor Polynomial of f(x). Note: $f(x) = T_N(x) + R_N(x)$.
- If $\lim_{N\to\infty} R_N(x) = 0$, for |x-a| < R, then f(x) equals the sum of its Taylor series on the interval |x-a| < R.
- Moreover,

$$f(x) = \lim_{N \to \infty} T_N(x) = T_a(x).$$

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The "hard part" of Taylor's Theorem is showing that $\lim_{N\to\infty} R_N(x) = 0$. This is done using:

Theorem 6: Taylor's Remainder Theorem (aka Taylor's Inequality)

• If $|f^{(N+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_N(x)$ of the Taylor series satisfies the inequality:

$$|R_N(x)| \le \frac{M}{(N+1)!} |x-a|^{N+1}$$
 for $|x-a| \le d$

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Example

• If $f(x) = \sin(x)$, then $f^{(N+1)}(x)$ is either $\pm \sin(x)$ or $\pm \cos(x)$. In either case, we have $|f^{(N+1)}(x)| \le 1$ for all values of x.

Therefore, a = 0, M = 1 and d (can be any positive number).

Taylor's Remainder theorem then says: $|R_N(x)| \le \frac{1}{(N+1)!} |x|^{N+1}$ for $|x| \le d = \infty$, i.e. any number x.

- Now, we can show directly that: $\lim_{N\to\infty} \frac{|x|^{N+1}}{(N+1)!} = 0$, for any fixed x.
- Thus, by Taylor's Theorem, we conclude: that sin(x) is equal to the sum of its the Taylor series at x = 0.

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Example

• If $f(x) = e^x$, then $f^{(N+1)}(x) = e^x$. Since e^x is increasing, we have $|f^{(N+1)}(x)| = |e^x| \le e^d$ for |x| < d, ie $x \in (-d, d)$.

Therefore, a = 0, $M = e^d$ and d (can be any positive number).

Taylor's Remainder theorem then says: $|R_N(x)| \le \frac{e^d}{(N+1)!} |x|^{N+1}$ for $|x| \le d$.

- Now, we can show directly that: $\lim_{N\to\infty}\frac{\mathrm{e}^d|x|^{N+1}}{(N+1)!}=0$, for any fixed x,d.
- Thus, by Taylor's Theorem, we conclude: that e^x is equal to the sum of its the Taylor series at x = 0.

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Activity 5:

Find the Taylor series for $f(x) = x^4 e^{-3x^2}$ at x = 0 and the radius of convergence.



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Activity 6:

Find the Taylor series for $f(x) = \sqrt{x}\sin(x^2)$ at x = 0 and the radius of convergence.



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Activity 7:

Find the Taylor series for $f(x) = e^x$ at x = -1 and the radius of convergence.



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Activity 8:

Find the Taylor series for $f(x) = \ln(x)$ at x = 2 and the radius of convergence.



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Activity 9:

Evaluate:

$$\lim_{x\to 0}\frac{\cos(x^5)-1}{x^{10}}$$

Note: use power series. You can use L'Hôpital's Rule, but it is a very long calculation.



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