§7.4: Integration of Rational Functions by Partial Fractions

Ch 7: Techniques of Integration
Math 5B: Calculus II

Dr. Jorge Eduardo Basilio

Department of Mathematics & Computer Science
Pasadena City College

Class #11 Notes

March 28, 2019
Spring 2019

Outline

1. Guiding Questions
2. Introduction
3. Method of Partial Fractions
4. Case I: distinct linear factors of $Q(x)$
5. Case II: repeated linear factors of $Q(x)$
6. Case III: repeated irreducilbe factors of $Q(x)$
7. Case IV: non-proper rational functions

Guiding Questions for §7.4

Guiding Question(s)

1. How do we integrate rational functions using the method of partial fractions?
2. What are some blockbuster applications illustrating the technique?

- Recall the common denominator trick from algebra:

$$\frac{1}{x-3} - \frac{1}{x-2} = ???$$
$$= \frac{(x-2)-(x-3)}{(x-3)(x-2)} = \frac{1}{x^2-5x+6}$$

- So, we can put this to work to evaluate an integral:

$$\int \frac{1}{x^2-5x+6}\,dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$
$$= \ln|x-3| - \ln|x-2| + C$$
$$= \ln\left|\frac{x-3}{x-2}\right| + C$$

- So, we'd like to generalize this idea:

$$\text{Start: } \frac{1}{x^2-5x+6} \longrightarrow \text{End (with lower order): } \frac{1}{x-3} - \frac{1}{x-2}$$

Introduction

- Our goal is in this section is easy to state. Evaluate integrals of the form:
$$\int \frac{P(x)}{Q(x)}\,dx, \text{ where } P(x) \text{ and } Q(x) \text{ are polynomials.}$$
- Functions of the form $\dfrac{P(x)}{Q(x)}$ are called rational functions.
- Rational functions are proper when $\deg(Q) > \deg(P)$–that is, the denominator has a higher degree than the numerator.
$$\text{Proper: } \frac{x^2+6x+1}{x^5-7x+1} \qquad \text{NOT Proper: } \frac{x^2+6x+1}{x^2-7x+1} \text{ or } \frac{x^5-1}{x^2-9}$$
- We'll study techniques for integrating proper rational functions first.
- If a rational function is NOT proper, then we use long division first, then the remainder is proper and we can use the previous techniques.

Method of Partial Fractions

Definition 1: Method of Partial Fractions

Assume $\dfrac{P(x)}{Q(x)}$ is proper and $P(x)$ and $Q(x)$ are polynomials. Let $k = \deg(Q)$.

To integrate: $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$, try:

- **Case I:** $Q(x) = (a_1x - b_1)(a_2x - b_2)\cdots(a_kx - b_k)$ is product of distinct linear factors in denominator.
  - Then $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{a_1x + b_1} + \dfrac{A_2}{a_2x + b_2} + \cdots + \dfrac{A_k}{a_kx + b_k}$ (with $A_i \in \mathbb{R}$)
- **Case II:** $Q(x) = (ax + b)^k$ repeated roots of Q.
  - Then $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{ax + b} + \dfrac{A_2}{(ax + b)^2} + \cdots + \dfrac{A_k}{(ax + b)^k}$ (with $A_i \in \mathbb{R}$)
- **Case III:** $Q(x) = (ax^2 + bx + c)^k$ repeated irreducible.
  - Then $\dfrac{P(x)}{Q(x)} = \dfrac{A_1x + B_1}{ax^2 + bx + c} + \dfrac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \dfrac{A_kx + B_k}{(ax^2 + bx + c)^k}$

Method of Partial Fractions

Definition 2: Method of Partial Fractions

Assume $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

To integrate: $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$, try:

- **Case IV:** Assume $\deg(P) \geq \deg(Q)$.
  - Use long division first to write $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$ where $S(x)$ is a polynomial and $R(x)$ is the remainder with $\deg(R) < \deg(Q)$.
  - Then $\dfrac{R(x)}{Q(x)}$ is proper and use Case I, II, or III.

Case I: distinct linear factors of $Q(x)$

Activity 1: Case I: distinct linear factors of $Q(x)$

(a) $\displaystyle\int \frac{x+1}{x^2-4x+3}\,dx$

(b) $\displaystyle\int \frac{3x+2}{x^3-x^2-2x}\,dx$

Case I: distinct linear factors of $Q(x)$

Case II: repeated linear factors of $Q(x)$

Activity 2: Case II: repeated linear factors of $Q(x)$

Evaluate: $\displaystyle\int \frac{x-2}{(x-1)(2x-1)^2}\,dx$

Case II: repeated linear factors of $Q(x)$

Case III: repeated irreducilbe factors of $Q(x)$

Activity 3: Case III: repeated irreducilbe factors of $Q(x)$

Evaluate:

(a) $\displaystyle\int \frac{2x-3}{x^3+x}\,dx$

(b) $\displaystyle\int \frac{1}{x^3-8}\,dx$

Case III: repeated irreducilbe factors of $Q(x)$

Case IV: non-proper rational functions

Activity 4: Case IV: non-proper rational functions

Evaluate: $\displaystyle\int \frac{x^2+4}{x^2-4}\,dx$