§7.8: Improper Integrals

Ch 7: Techniques of Integration
Math 5B: Calculus II

Dr. Jorge Eduardo Basilio

Department of Mathematics & Computer Science
Pasadena City College

Class #14 Notes

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Outline

Guiding Questions for §7.8

Guiding Question(s)

1. Can we extend definite integrals, $\int_a^b f(x)\,dx$, to more general intervals?
2. How can we deal with integrals with an infinite interval of integration?
3. How can we deal with integrals where $f(x)$ has an infinite discontinuity inside the interval of integration?

Introduction

- What's wrong with the following argument?

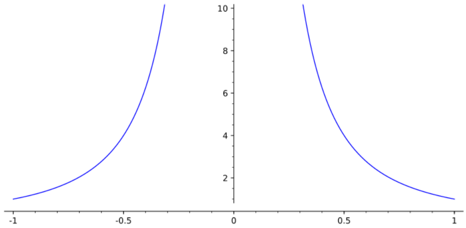
$$\int_{-1}^{1} \frac{1}{x^2}\, dx = [\frac{x^{-1}}{-1}]_{-1}^{1}$$
$$= -1 + -1$$
$$= -2$$

  But, we know that $f(x) = \frac{1}{x^2}$ is always positive, and the integral is the area under the curve. What went wrong?

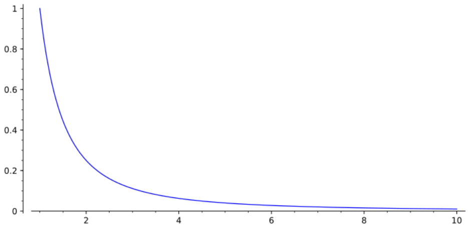
- What about:

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx = \left[\frac{x^{-1}}{-1}\right]_{1}^{\infty}$$
$$= \frac{-1}{\infty} + 1$$
$$= 1.$$

Introduction



Graph of $f(x) = \frac{1}{x^2}$ over $[-1, 1]$



Graph of $f(x) = \frac{1}{x^2}$ over $[1, 10]$

Introduction

- The integral value $\int_1^\infty \frac{1}{x^2}\,dx = 1$ seems reasonable.
- We can make it rigorous by using limits and setting:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$$

- Since we can evaluate as before:

$$\begin{aligned} &= \lim_{R\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^R \\ &= \lim_{R\to\infty} \left[\frac{-1}{R} + 1\right] \\ &= 1 \end{aligned}$$

Definition 1: Improper Integrals: Type I

- Assume $\displaystyle\int_a^b f(x)\,dx$ exists for all $b \geq a$. We define $\int_a^\infty f(x)\,dx$ to be the limit (when it exists):
$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$
- Similarly, assume $\displaystyle\int_a^b f(x)\,dx$ exists for all $a \leq b$. We define $\int_{-\infty}^b f(x)\,dx$ to be the limit (when it exists):
$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx$$
- These are called improper integrals of type I.

Improper Integrals: Type I

Definition 2: Improper Integrals: Type I

Given improper integrals: $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$

- Converge: if the limit exists & is a finite number. We say that the improper integral converges.
- Diverge: if the limit does no exist OR is not a finite number. We say that the improper integral diverges.

When both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ converge,

- We define $\int_{-\infty}^\infty f(x)\,dx$ to be:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Where c can be any real number. Usually, we pick $c = 0$ out of convenience.

Activity 1:

C or D? That is, do the following improper integrals converge or diverge? If they converge, find their value.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

(b) $\displaystyle\int_{-\infty}^1 xe^x\,dx$

(c) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Improper Integrals: Type I

Improper Integrals: Type I

Improper Integrals: Type I

Activity 2:

Investigate numerically using Sage whether $\int_1^\infty e^{-x^2}\,dx$ converges or diverges. If it appears to converge, estimate it's value.

Improper Integrals: p-Test Type I

An important test for convergence and divergence is the following:

Theorem 1: p-test Type I

- $$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{converges,} & \text{if } p > 1 \\ \text{diverges,} & \text{if } p \le 1 \end{cases}$$
- In fact, when $p > 1$, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$

Proof:

Proof breaks into several steps:

- $p = 1$ we showed earlier that it diverges
- Compute $\int_1^R x^{-p}\,dx$ for any $R \geq 1$.
- Case: $p > 1$
- Case $p < 1$

Improper Integrals: p-Test

Improper Integrals: Type II

Definition 3: Improper Integrals: Type II

Assume f has a vertical asymptote at $x = c$. We sometimes call this a "infinite discontinuity" at $x = c$ but otherwise is continuous on the interval (a, b) with $a < c < b$. We define:

- $$\int_a^c f(x)\,dx = \lim_{b \to c^-} \int_a^b f(x)\,dx$$
- $$\int_c^b f(x)\,dx = \lim_{a \to c^+} \int_a^b f(x)\,dx$$
- When both integrals above converge, we define
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
- These are called improper integrals of type II.

Improper Integrals: Type II

Activity 3:

C or D? That is, do the following improper integrals converge or diverge? If they converge, find their value.

(a) $\displaystyle\int_0^9 \frac{1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^1 \frac{1}{x}\,dx$

(c) $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{(x-1)^2}}\,dx$

Improper Integrals: Type II

Activity 4:

C or D? That is, do the following improper integrals converge or diverge? If they converge, find their value.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int_0^{\pi/2} \sec(x)\,dx$

Improper Integrals: Type II

Improper Integrals: p-Test Type II

An important test for convergence and divergence is the following:

Theorem 2: p-test Type II

- $$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{diverges,} & \text{if } p \geq 1 \\ \text{converges,} & \text{if } p < 1 \end{cases}$$
- In fact, when $p < 1$, $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{p-1}$

NOTICE: that this is similar to the p-test for type I integrals but the roles switch!

Improper Integrals: Comparison Test

Many times the integral is either impossible to evaluate directly or too difficult. The following test gives us an easy way to determine convergence or divergence. Note it does not give use a value of the integral if it converges (but it does give us an estimate, however).

Theorem 3: Comparison Test

Assume that f and g are continuous on $[a, \infty)$.
Assume that $f(x) \geq g(x) \geq 0$. Then

(a) If $\int_a^\infty f(x)\,dx$ converges, THEN $\int_a^\infty g(x)\,dx$ also converges.

(b) If $\int_a^\infty g(x)\,dx$ diverges, THEN $\int_a^\infty f(x)\,dx$ also diverges.

Improper Integrals: Comparison Test

Activity 5:

Use the comparison test to show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

(Hint: split into three integrals: $\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty}$. Use the comparison test with $e^{-x^2} \leq e^{-x}$.)

Activity 6:

Use the comparison test to determine whether the integral converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$

(b) $\displaystyle\int_1^\infty \frac{1}{e^{3x}+\sqrt{x}}\,dx$

(c) $\displaystyle\int_1^\infty \frac{1+\sin^2(x)}{\sqrt{x}}\,dx$

Improper Integrals: Comparison Test