§7.7: Approximate Integration

Ch 7: Techniques of Integration

Math 5B: Calculus II

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Class #12 Notes

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Outline

1. Guiding Questions
2. Review of definitions of Integration
3. Endpoint Approximations
4. Midpoint Approximations
5. Trapezoid Approximations

Guiding Questions for §7.7

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2. What are error bounds for the midpoint and trapezoid rules
3. How can we use them to anticipate the number of sums needed to achieve a certain accuracy?

Review Integration

- Definition of anti-derivative. Given a function $f(x)$, we say $F(x)$ is an anti-derivative of $f(x)$ if

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  This is way more complicated. Notation: $\int_a^b f(x)\,dx$.
- Important point: though the notation is similar, they are very different.

$$\int f(x)\,dx = \text{infinitely many functions} \quad \text{and} \quad \int_a^b f(x)\,dx = \text{a single number}$$

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  Pick a random sample of $c_i \in [x_{i-1}, x_i]$ and compute the area of the ith sub-rectangle: Area (one rectangle) $= f(c_i)\Delta x_i$ where $\Delta x_i = x_{i-1} - x_i$

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  To get the exact area we take a limit:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

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- At this point, you now have many tools in your integration toolbox to compute a lot of integrals. But...
- There's still many, many integrals we can't find anti-derivatives of!
- So we will go back to the definition of the definite integral and settle for approximation.

Approximate Integration Techniques

Essentially, we have many choices for how to pick $c_i \in [x_{i-1}, x_i]$
(Note: the book uses x_i^* instead of c_i).

- **Left-endpoint approximation (LEA)**: $c_i = x_{i-1}$.

  Ex: when f is increasing, this underestimates the exact area.
- **Right-endpoint approximation (REA)**: $c_i = x_i$.

  Ex: when f is increasing, this overestimates the exact area.
- **Midpoint approximation (MPA)**: $c_i = \frac{x_{i-1}+x_i}{2}$.

  this is much better than the LEA and REA, in general, since it averages the them!

In my opinion, it's a bit silly to write out he full formulas for these approximations. If you know the idea, then you can just find these values by hand with patience.

Endpoint Approximations

Set up for all approximations: n a positive integer. $a, b \in \mathbb{R}$. for $i = 0, 1, 2, \ldots, n$:

$\boxed{\Delta x = \frac{b-a}{b}}$ and $\boxed{x_i = a + i \cdot \Delta x}$ and subintervals $[x_{i-1}, x_i]$

Definition 1: Endpoint Approximations

- Left Endpoint approximation (LEA):

$$\int_a^b f(x)\,dx \approx \Delta x\,[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})] = S_n(LEA)$$

- Right Endpoint approximation (REA):

$$\int_a^b f(x)\,dx \approx \Delta x\,[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)] = S_n(REA)$$

Midpoint Approximations

Set up for all approximations: n a positive integer. $a, b \in \mathbb{R}$. for $i = 0, 1, 2, \ldots, n$:

$\boxed{\Delta x = \dfrac{b-a}{b}}$ and $\boxed{x_i = a + i \cdot \Delta x}$ and subintervals $[x_{i-1}, x_i]$

Definition 2: Midpoint Approximations

- Midpoint approximation (MPA): if we average the endpoints of the subintervals, we get the midpoints, given by $\overline{x_i} = \frac{x_{i-1}+x_i}{2} = a + i \cdot (\Delta x/2)$. Then

$$\int_a^b f(x)\,dx \approx \Delta x \left[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + \cdots + f(\overline{x_n}) \right] = S_n(MPA)$$

Trapezoid Approximations

Set up for all approximations: n a positive integer. $a, b \in \mathbb{R}$. for $i = 0, 1, 2, \ldots, n$:

$\boxed{\Delta x = \frac{b-a}{b}}$ and $\boxed{x_i = a + i \cdot \Delta x}$ and subintervals $[x_{i-1}, x_i]$

Definition 3: Trapezoid Approximations

- Midpoint approximation (MPA): if we average the height of the function at the endpoints of the subintervals, then we get trapezoids.

$$\int_a^b f(x)\,dx \approx \Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] = S_n(TrapA)$$

Note: Areas(Trap) $= \frac{f(x_{i-1})+f(x_i)}{2}\Delta x$