Section 9.2 - The Dot Product

Objectives:

- The Dot Product of Vectors
- Orthogonal Vectors
- The Component of u along v
- The Projection of u along v
- Work

• The Dot Product

$\vec{v} = \langle v_1, v_2 \rangle$ & $\vec{w} = \langle w_1, w_2 \rangle$

Definition: The **Dot Product** between two vectors $\vec{v}$ and $\vec{w}$ is the real number (scalar)

$$\boxed{\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2}$$

Ex1: Find the dot product of the two vectors given:

(a) $\vec{v} = \langle 5, -2 \rangle$ and $\vec{w} = \langle -8, 9 \rangle$

$\vec{v} \bullet \vec{w} = \langle 5, -2 \rangle \bullet \langle -8, 9 \rangle$

$= (5)(-8) + (-2)(9)$

$= -40 - 18 = \boxed{-58}$

(b) $\vec{v} = 2\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = \hat{\imath} + \hat{\jmath}$

$\vec{v} \bullet \vec{w} = \langle 2, 7 \rangle \bullet \langle 1, 1 \rangle$

$= (2)(1) + (7)(1)$

$= \boxed{9}$

PROPERTIES OF THE DOT PRODUCT

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ The length of vector squared = dot product of vector w/ itself.

GTQ!

Proof of property 4:

LHS:

Let $\vec{u} = \langle u_1, u_2 \rangle$. Then $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$. So $|\vec{u}|^2 = u_1^2 + u_2^2$.

RHS:

$\vec{u} \bullet \vec{u} = \langle u_1, u_2 \rangle \bullet \langle u_1, u_2 \rangle = (u_1)(u_1) + (u_2)(u_2) = u_1^2 + u_2^2$.

Therefore, $|\vec{u}|^2 = u_1^2 + u_2^2 = \vec{u} \bullet \vec{u}$. □

Review: Law of Cosines from Section 5.6:

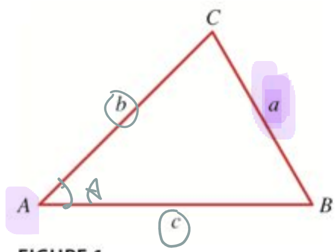


FIGURE 1

THE LAW OF COSINES

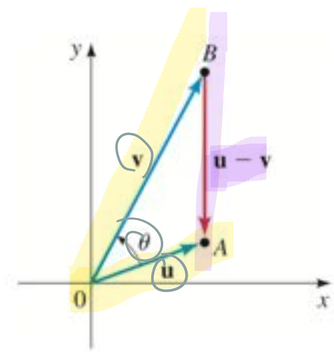
In any triangle ABC (see Figure 1) we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Theorem: The Dot Product Theorem

For any two vectors $\vec{u}$ and $\vec{v}$, let θ be the angle between them ($\theta \in [0,\pi]$), then we have

$$\boxed{\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta}$$



Proof: Apply the Law of Cosine to the figure and property 4 to

$$|\vec{u}-\vec{v}|^2 = |\vec{v}|^2 + |\vec{u}|^2 - 2|\vec{v}|\,|\vec{u}|\cos\theta$$

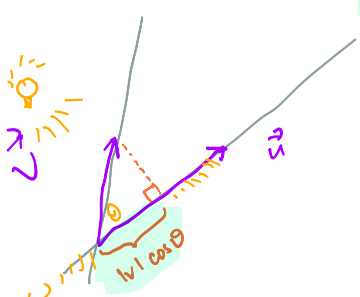
Use Theorem $|\vec{u}|^2 = \vec{u}\cdot\vec{u}$

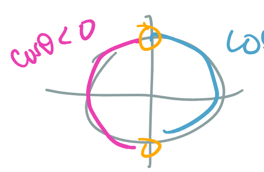
LHS: $|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\bullet(\vec{u}-\vec{v}) = u\cdot u \underbrace{- u\cdot v - v\cdot u} + v\cdot v$
$= |u|^2 - 2u\cdot v + |v|^2$

$$\cancel{|u|^2} - 2u\cdot v + \cancel{|v|^2} = \cancel{|u|^2} + \cancel{|v|^2} - 2|u||v|\cos\theta$$
$$-\cancel{2}(u\cdot v) = -\cancel{2}|u||v|\cos\theta. \quad \square$$

- **Geometry of Dot Product**

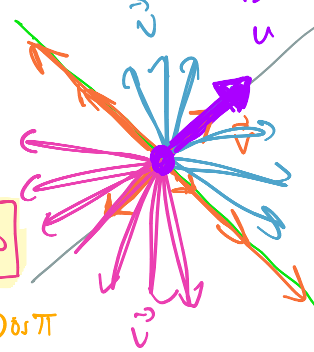
$\vec{u}\cdot\vec{v} = |u||v|\cos\theta$
$= |u|(\text{shadow of } \vec{v} \text{ onto } \vec{u})$





$\boxed{\vec{u}\cdot\vec{v} < 0}$

$\theta = 0$ or π



$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$
$\cos\theta < 0$

$\cos 0 = 1$
$\cos \pi/2 = 0$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\cos\theta > 0$
$\boxed{\vec{u}\cdot\vec{v} > 0}$

$\theta = \pi/2$
$\boxed{\vec{u}\cdot\vec{v} = 0}$

Theorem: Angle between Two Vectors

For any two vectors $\vec{u}$ and $\vec{v}$, let θ be the angle between them ($\theta \in [0,\pi]$), then we have

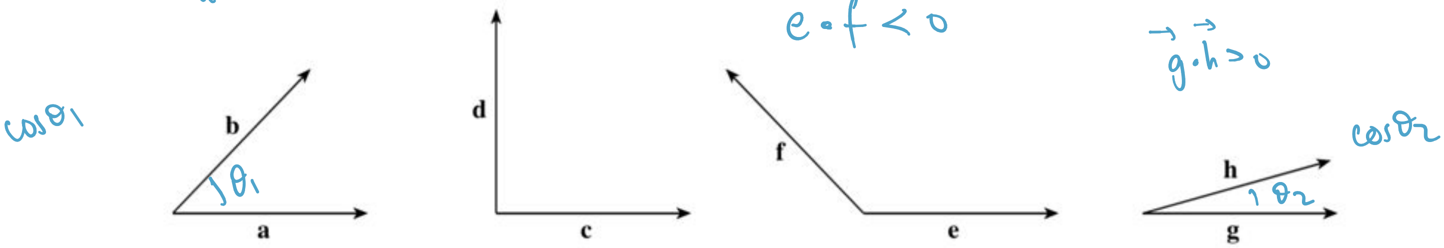
$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}\right)}$$

Pf $\quad \vec{u}\cdot\vec{v} = |u||v|\cos\theta$

$$\underbrace{\cos^{-1}(\cos\theta)}_{\theta} = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}\right). \quad \square$$

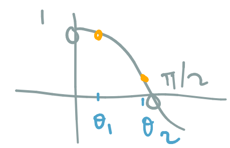
smallest: $\vec{e}\cdot\vec{f}$, $\vec{c}\cdot\vec{d}$, $\vec{a}\cdot\vec{b}$, $\vec{g}\cdot\vec{h}$

Ex2: $\vec{a}\cdot\vec{b}>0$ $\quad \vec{c}\cdot\vec{d}=0$ $\quad \vec{e}\cdot\vec{f}<0$ $\quad \vec{g}\cdot\vec{h}>0$



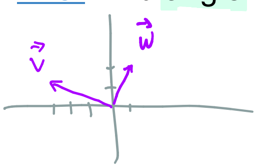
Put the following quantities in order, from smallest to largest:

$\mathbf{a}\cdot\mathbf{b}$ ③ $\qquad \mathbf{c}\cdot\mathbf{d}$ ② $\qquad \mathbf{e}\cdot\mathbf{f}$ ① $\qquad \mathbf{g}\cdot\mathbf{h}$ ④



$\cos\theta_1 > \cos\theta_2$

Ex3: Find angle between $\vec{v}=\langle -3,1\rangle$ and $\vec{w}=\hat{\imath}+2\hat{\jmath}$. $=\langle 1,2\rangle$



$\theta=\cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{|v|\,|w|}\right)=\cos^{-1}\left(\frac{(-3)(1)+(1)(2)}{\sqrt{10}\,\sqrt{5}}\right)=\cos^{-1}\left(\frac{-1}{5\sqrt{2}}\right)$

$\cdot\ |v|=\sqrt{(-3)^2+1^2}=\sqrt{10}$ $\qquad = \boxed{1.713 \text{ (radians)}}$

$\cdot\ |w|=\sqrt{(1)^2+(2)^2}=\sqrt{5}$ $\qquad \approx 98.1^\circ$

- **Orthogonal Vectors**

Definition: Two vectors are said to be **orthogonal** (or **perpendicular**) if the angle between them is $90^\circ=\pi/2$.

Theorem: Orthogonal Vectors
Two vectors are orthogonal if and only if their dot product equals zero!

$\vec{u}\perp\vec{v}$ iff $\vec{u}\cdot\vec{v}=0$

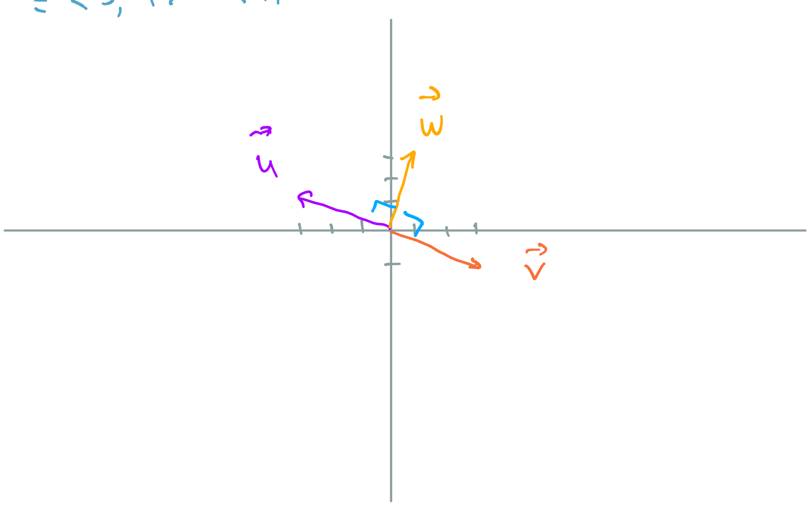
Ex4: Which pairs of vectors are orthogonal to each other?

$\vec{u}=\langle -3,1\rangle$, $\qquad \vec{v}=3\hat{\imath}-\hat{\jmath}$, $\qquad \vec{w}=\hat{\imath}+3\hat{\jmath}$

$\vec{u}\cdot\vec{v}=\langle -3,1\rangle\cdot\langle 3,-1\rangle=(-3)(3)+(1)(-1)=-10\neq 0 \longrightarrow$ not orthogonal

$\vec{u}\cdot\vec{w}=\langle -3,1\rangle\cdot\langle 1,3\rangle=(-3)(1)+(1)(3)=0 \longrightarrow$ orthogonal!

$\vec{v}\cdot\vec{w}=\langle 3,-1\rangle\cdot\langle 1,3\rangle=(3)(1)+(-1)(3)=0 \longrightarrow$ orthogonal!



- ## The Component of u along v

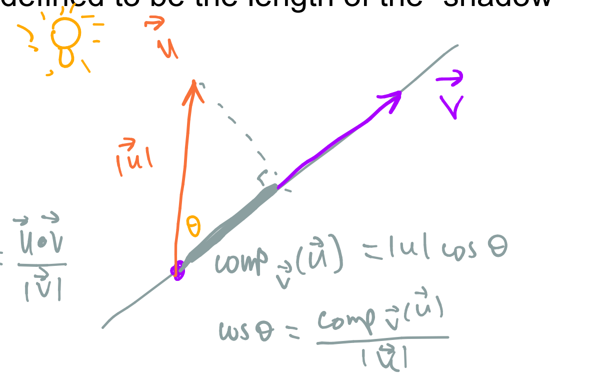
Definition: The **component of $\vec{u}$ along $\vec{v}$** is defined to be the length of the "shadow" that $\vec{u}$ casts onto $\vec{v}$. More precisely,

projection

This is a scalar!

$$\boxed{\text{comp}_{\vec{v}}(\vec{u}) = |\vec{u}| \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}$$

$$\text{comp}_{\vec{v}}(\vec{u}) = |u| \cos\theta \, \frac{|v|}{|v|} = \frac{|u|\,|v| \cos\theta}{|v|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

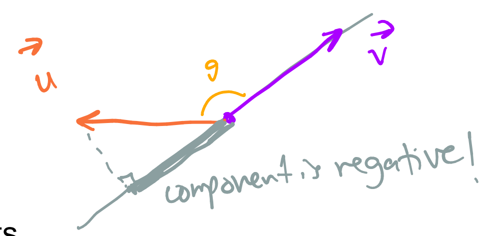


Because this is a length, it is also called the **scalar projection of $\vec{u}$ onto $\vec{v}$**.

Notation: $\text{comp}_{\vec{v}}(\vec{u}) = \text{s.proj}_{\vec{v}}(\vec{u})$

(Jorge's notation not in book)

What happens when $\pi/2 < \theta \le \pi$?

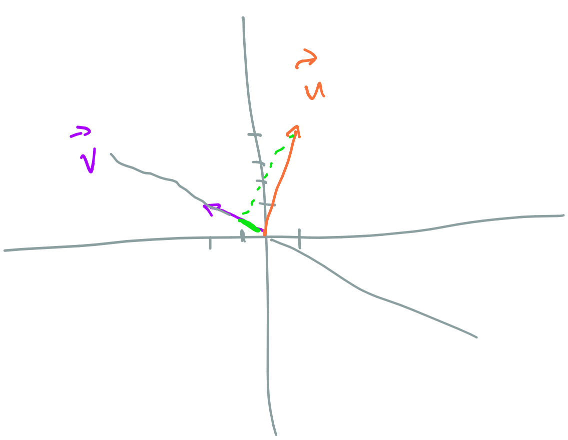


Theorem: Scalar Projection and Dot Products

$$\boxed{\text{comp}_{\vec{v}}(\vec{u}) = |\vec{u}| \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}$$

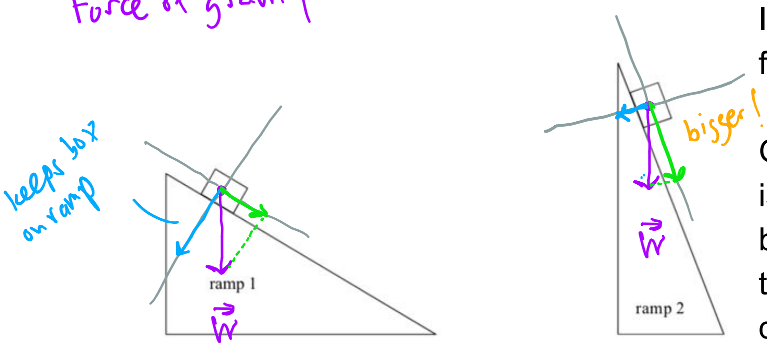
Ex5: Find the scalar projection of $\vec{u} = \hat{\imath} + 4\hat{\jmath}$ onto $\vec{v} = -2\hat{\imath} + \hat{\jmath}$.

$$\text{s.proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{(1)(-2) + (4)(1)}{\sqrt{(-2)^2 + (1)^2}} = \boxed{\frac{2}{\sqrt{5}}} \approx 0.9$$



- **Weight**

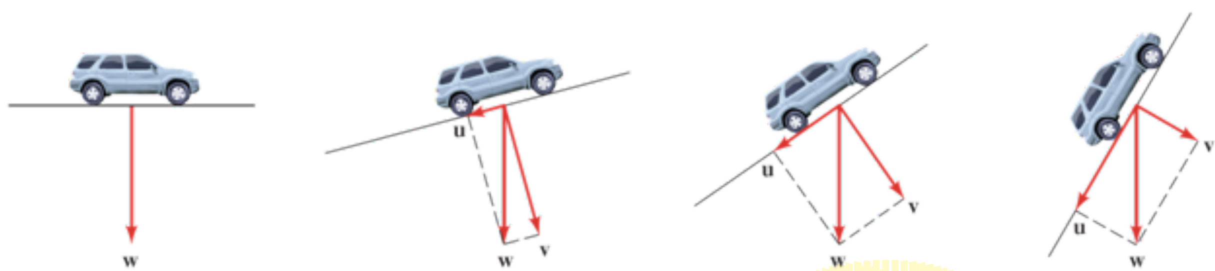
Force of gravity



It is clear that the block on ramp 2 will slide faster than the block on ramp 1. Why?

Gravity is the same in both cases, yet there is a definite difference in speed. The reason behind this is interesting. Gravity is doing two things at once: it is letting the box slide down, and it is also preventing the box from flying off the ramp and into outer space.

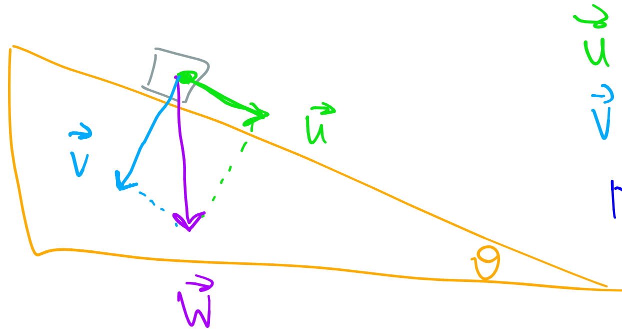
We can draw a “free body diagram” that shows how the component of gravity pulling the box down the ramp is affected by the angle of the ramp. The box on ramp 2 slides faster because the vector in the direction of the ramp is larger. That is, more of the force of gravity is pushing the box down ramp 2 and less of it is holding it down to the ramp.



The force of gravity times the mass of an object is called its **weight**. Therefore, weight is a force vector that always points towards the Earth: $\vec{W} = -mg\hat{\jmath}$ where m is the mass of the object and g is the gravitational acceleration due to the Earth which is constant and equals $g = 9.8m/s^2$ or $g = 32ft/s^2$. (For those who studied physics, Newton’s Second Law says “force = mass x acceleration”. The gravity constant g is the acceleration that the Earth produces on objects.)

Taking the road into consideration we want to express the weight, $\vec{W}$, as the vector sum of two forcers: one force parallel to the road (in picture $\vec{u}$) and the other force perpendicular to the road (in picture $\vec{v}$).

Thus, we want to “decompose” or “resolve” $\vec{W}$ as follows: $\vec{W} = \vec{u} + \vec{v}$.



$\vec{u}$ parallel to ramp

$\vec{v}$ perpendicular to ramp

Note

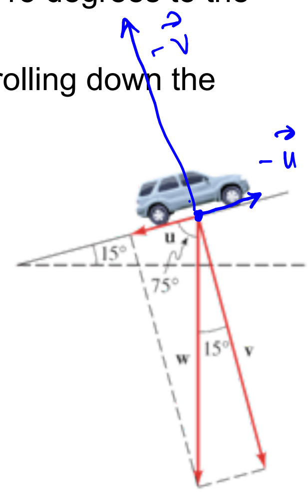
$\vec{u} + \vec{v} = \vec{W}$

We now have a good interpretation of components:

The component of $\vec{W}$ along a unit vector parallel to the road is the length of $\vec{u}$.
The component of $\vec{W}$ along a unit vector orthogonal to the road is the length of $\vec{v}$.

For applications, however, it is also important to find the vectors themselves.

Ex6: A car weighing 3000 lb is parked on a driveway that is inclined 15 degrees to the horizontal.
(a) Find the magnitude of the force required to prevent the car from rolling down the driveway.



$|\vec{W}| = 3000$ $\vec{W} = -3000\hat{j}$

- force to prevent sliding $= -\vec{u}$
- magnitude $|-\vec{u}| = |\vec{u}|$

$= \text{s.proj}_{\vec{u}}(\vec{w}) = |\vec{w}| \cos 75°$

$= 3000 \cos 75°$

$= \boxed{776.5 \text{ lbs}}$

(b) Find the magnitude of the force experienced by the driveway due to the weight of the car.

$\text{s.proj}_{\vec{v}}(\vec{w}) = |\vec{w}| \cos 15°$

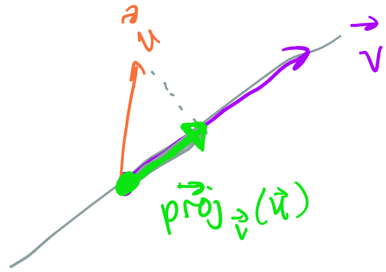
$= \boxed{2897.8 \text{ lbs}}$

- ## The Projection of u along v

Think back to the car on a ramp. The force of gravity can be "decomposed" into the sum of two forces: the component parallel to the ramp and the component perpendicular to the ramp. We can write the force as: $\vec{W} = \vec{u} + \vec{v}$. This motivates the following definition.

Definition: The projection of $\vec{u}$ along $\vec{v}$.
Given two vectors $\vec{u}$ and $\vec{v}$, we define the **projection of $\vec{u}$ along $\vec{v}$** to be the vector parallel to $\vec{v}$ and whose length is the scalar projection of $\vec{u}$ along $\vec{v}$.

Picture:



Key:
Scalar projection (or component) of u along v is a Scalar
Projection of u along v is a Vector

Theorem: The formula for projection vector

$$\boxed{\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}}$$

Pf

$$\text{proj}_{\vec{v}}(\vec{u}) = \text{length}\cdot\begin{pmatrix}\text{unit}\\\text{vector}\end{pmatrix} \text{in direction of } v\Big)$$
$$= \left(s.\text{proj}_{\vec{v}}(\vec{u})\right)\frac{\vec{v}}{|\vec{v}|}$$
$$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\left(\frac{\vec{v}}{|\vec{v}|}\right) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}.$$

Ex7: Compute the projection of $\vec{u} = \langle -2, 9\rangle$ along $\vec{v} = -\hat{\imath} + 2\hat{\jmath}$.

$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{u\cdot v}{|v|^2}\right)\vec{v} = \left(\frac{(-2)(-1)+(9)(2)}{(-1)^2+(2)^2}\right)\langle -1, 2\rangle$$

Recall: $|v|^2 = v\cdot v$

$$= \frac{2+18}{5}\langle -1, 2\rangle$$
$$= 4\langle -1, 2\rangle$$
$$= \boxed{\langle -4, 8\rangle}$$
$$= -4\hat{\imath} + 8\hat{\jmath}$$

- **Decomposing or Resolving Vectors**

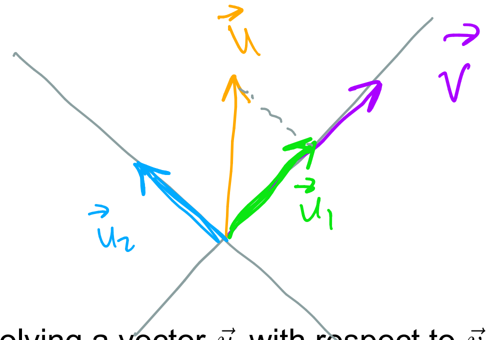
Definition: When asked to **resolve a vector $\vec{u}$ with respect to $\vec{v}$** we mean writing

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$

where

$\vec{u}_1$ is parallel vector to $\vec{v}$

and $\vec{u}_2$ is perpendicular to $\vec{v}$



$\vec{u}_1 = proj_{\vec{v}}(\vec{u})$

$\vec{u}_2 = \vec{u} - \vec{u}_1$

Theorem: Formulas for resolving a vector $\vec{u}$ with respect to $\vec{v}$

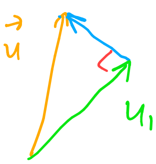
$\vec{u}_1 =$ and $\vec{u}_2 =$

Proof • Look at picture & it's clear by def of projection that $\vec{u}_1 = proj_{\vec{v}}(\vec{u})$

• Look at picture: $\vec{u} - \vec{u}_1$

recall pro tip

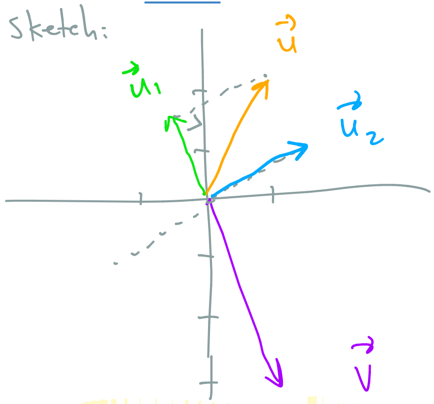
$\vec{u} \leftarrow \vec{u}_1$ points to $\vec{u}$



If you slide blue arrow then it clearly is $\vec{u}_2$. $\square$

Ex8: Resolve the vector $\vec{u} = \langle 1, 2 \rangle$ with respect to $\vec{v} = \langle 1, -3 \rangle$.

Sketch:



$\vec{u}_1 = proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{(1)(1) + (2)(-3)}{1^2 + (-3)^2} \langle 1, -3 \rangle$

$= \frac{-5}{10} \langle 1, -3 \rangle = -\frac{1}{2} \langle 1, -3 \rangle = \boxed{\langle -\frac{1}{2}, \frac{3}{2} \rangle = \vec{u}_1}$

$\vec{u}_2 = \vec{u} - \vec{u}_1 = \langle 1, 2 \rangle - \langle -\frac{1}{2}, \frac{3}{2} \rangle = \langle 1 + \frac{1}{2}, 2 - \frac{3}{2} \rangle$

$= \boxed{\langle \frac{3}{2}, \frac{1}{2} \rangle = \vec{u}_2}$

- **Work**

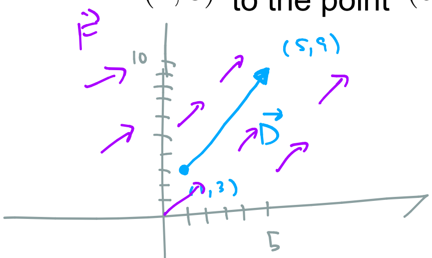
In everyday use, the term work means the amount of effort required to perform a task. However, in physics, it has a more technical meaning: force times direction.

Definition: The work done by a force $\vec{F}$ along the displacement vector $\vec{D}$ is defined by

$$W = \vec{F} \bullet \vec{D} \ .$$

The units of work are foot-pounds. The reason we take the dot product is because only the component of the force in the direction of $\vec{D}$ affects the object. Note that work is a scalar.

Ex9: A force is given by the vector $\vec{F} = 2\hat{\imath} + 2\hat{\jmath}$ and moves an object from the point $(1,3)$ to the point $(5,9)$. Find the work done.



$\vec{F} = \langle 2,2 \rangle$

draw $\vec{F}$ everywhere : a "force field" ☺

Displacement Vector:

$\vec{D}$ = end point − initial point

$= \langle 5-1, 9-3 \rangle$

$= \langle 4,6 \rangle$

Work:

$W = \vec{F} \cdot \vec{D}$

$= \langle 2,2 \rangle \cdot \langle 4,6 \rangle$

$= (2)(4) + (2)(6)$

$= 8 + 12$

$\boxed{W = 20 \text{ ft-lbs}}$