§11.3: The Integral Test and Estimating Sums

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Class #19 Notes

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Outline

1. Guiding Questions
2. Introduction
3. Integral Test
4. *p*-series
5. Estimating Sums of Series

Guiding Questions for §11.3

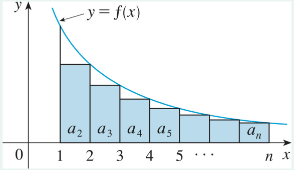
Guiding Question(s)

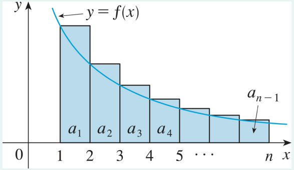
1. What is the integral test?
2. What are p-series?
3. What is the remainder of a series?
4. How can we estimate the remainder of a series using the integral test?

Introduction

- Finding the exact value (sum) of a general series is very difficult. We typically can only estimate the series with a partial sum, S_n.
- Still, we must know whether or not the series converges before we try to use the estimate (say, in solving an applied problem).
- Our goal in this section is to develop a tool that helps us determine whether a series converges or diverges.
- The tool we learn in this section is called the integral test.
- This question is so important that we will learn additional tools in future sections.

Integral Test

- We focus on series with positive terms only. I.e. $a_n > 0$ (or non-negative $a_n \geq 0$)
- The basic idea of the integral test is to use integrals to test whether a series converges (duh! :p)
- 



- By comparing the series with the integral (again, since both are positive!), if the integral is bigger than the series and the integral converges (first picture), then the series converges as well.
- Similarly, if the integral is smaller than the series and the integral diverges (second picture), then the series diverges as well.

Theorem 1: Integral Test

Assume that f satisfies the following on $[1, \infty)$:

1. continuous
2. positive
3. decreasing
4. $f(n) = a_n$

The series converges

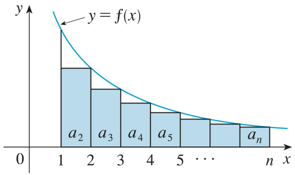
if and only if

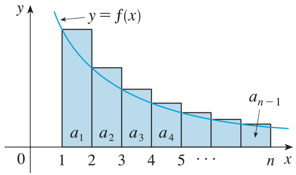
the integral converges.

More explicitly,

(a) IF $\int_1^\infty f(x)dx < \infty$ THEN $\sum_{n=1}^\infty a_n < \infty$

(b) IF $\int_1^\infty f(x)dx = \infty$ THEN $\sum_{n=1}^\infty a_n = \infty$





Integral Test

Integral Test

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Activity 1:

Use the integral test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges or diverges.

Integral Test

Activity 2:

Use the integral test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or diverges.

Integral Test

Integral Test

Activity 3:

Use the integral test to determine whether $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ converges or diverges.

Integral Test

Integral Test

Activity 4:

Use the integral test to determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

Integral Test

Definition 1: p-series

- Series of the form: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ are called p-series.

Theorem 2: p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \leq 1 \end{cases}$$

WARNING: When a p-series converges, we do not have a formula for it's sum (except for $p = 2$ which was due to Euler, others are very hard to find!). Do not use the formula for the p-test for the integral! i.e. $\sum \frac{1}{n^p} \neq \frac{1}{p-1}$.

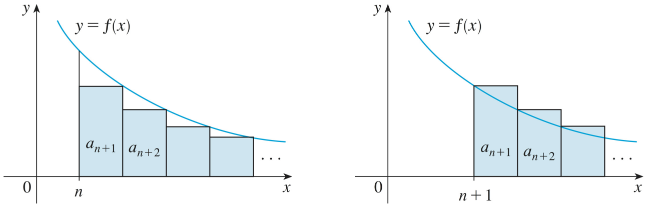
Definition 2: Partial Sums and Remainders of a series

- $S = \displaystyle\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{n} a_k + \sum_{k=n+1}^{\infty} a_k$
- nth Partial sum: $S_n = \sum_{k=0}^{n} a_k$
- nth Remainder: $R_n = \sum_{k=n+1}^{\infty} a_k$
- $\boxed{S = S_n + R_n}$
- $R_n = S - S_n$.
- R_n is also called the "error" (recall: exact - approx)

Theorem 3: Remainder Estimate using the Integral Test

With $f(x)$ and a_n exactly the same as in the statement of the integral test:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$



Activity 5:

(a) Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ by finding the sum of the first 10 terms (use Sage). Estimate the error involved in this.

(b) How many terms are needed to ensure the partial sum is accurate to within 0.000005.

Estimating Sums of Series

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Estimating Sums of Series

Recall: S, S_n, R_n, $S = S_n + R_n$

Theorem 4: Estimating the series using the Integral Test

With $f(x)$ and a_n exactly the same as in the statement of the integral test:

- $$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq S_n + \int_{n}^{\infty} f(x)\,dx$$
- $S_n + \displaystyle\int_{n+1}^{\infty} f(x)\,dx$ is a lower bound for S.
- $S_n + \displaystyle\int_{n}^{\infty} f(x)\,dx$ is an upper bound for S.

Activity 6:

Give an upper and lower bound for Euler's series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 100$.

Estimating Sums of Series