§6.8: Indeterminate Forms & L'Hôpital's Rule

Ch 6: Exponentials, Logs, & Inverse Trig Functions

Math 5B: Calculus II

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Class #9 Notes

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Guiding Questions for §6.8

Guiding Question(s)

1. What are indeterminate forms?
2. What is L'Hôpital's Rule?
3. What are applications of L'Hôpital's Rule?

Back to limits

- In the beginning of Calculus, we studied limits of form $\frac{0}{0}$ since this is what happens in the definition of the derivative.
- For example, if we want to find the slope of the tangent line of the function $f(x) = x^2$, we must compute:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{0}{0} \quad \text{when we plug-in } h = 0$$

- For many functions, we can use algebra techniques to evaluate them: for example,

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} 2x + h = 2x.$$

Introducing L'Hôpital's Rule

Back to limits

- A important example from Calc 1 was the derivative of $\sin(x)$:

$$\frac{d}{dx}[\sin(x)] = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

  which was solved by either a geometric argument or using some trigonometric identities.

- But there are many examples of limits of the form $\frac{0}{0}$ where we can't evaluate with previous tricks. For example,

$$\lim_{x\to 1}\frac{\ln(x)}{x-1} \quad \text{or} \quad \lim_{x\to 0}\frac{\tan(x)}{x^3}$$

- L'Hôpital's Rule will give us a new technique to help us evaluate such limits

Introducing L'Hôpital's Rule

Limits like $\frac{0}{0}$ come up often and are given a special name: indeterminate forms. There are other limits that can be found using similar tricks.

$$\text{Goal: Evaluate } \lim_{x \to a} \frac{f(x)}{g(x)}$$

Definition 1: Indeterminate Forms

- An indeterminate form is a limit of the above form that results with one of the following when plugging in $x = a$:
  - Fractions: $\dfrac{0}{0}, \quad \dfrac{\infty}{\infty}$
  - Products: $0 \cdot \infty$
  - Differences: $\infty - \infty$
  - Powers: $0^0, \quad \infty^0, \quad 1^\infty$

Theorem 1: L'Hôpital's Rule

(a) **Type $\frac{0}{0}$:** Assume that f and g are differentiable functions on an open interval I containing $a \in \mathbb{R}$. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ on I (except possibly at a), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided both limits exits or equals $\pm\infty$.

(b) **Type $\frac{\infty}{\infty}$:** Assume that f and g are differentiable functions on an open interval I containing $a \in \mathbb{R}$. If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$, and $g'(x) \neq 0$ on I (except possibly at a), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided both limits exits or equals $\pm\infty$.

Remarks

- It's important to check whether the conditions for using L'Hôpital's Rule are satisfied. You can run into trouble and get wrong answers otherwise.
$$\lim_{x\to 1} \frac{x^2+1}{2x+1} = \lim_{x\to 1} \frac{2x}{2} = 1 \quad \textit{FAIL!}$$
- L'Hôpital's Rule works also for one-sided limits ($x \to a^+$ and $x \to a^-$) and also with limits to infinity ($x \to \pm\infty$)
- When applying L'Hôpital's Rule, differentiate the numerator and denominator **separately** and do NOT use the quotient rule.
- You can apply L'Hôpital's Rule multiple times (as long as the assumptions are still satisfied)!

Activity 1:

State the type of indeterminate form and evaluate using L'Hôpital's Rule:

(a) $\displaystyle\lim_{x\to 2} \frac{x^3 - 8}{x^4 + 2x - 2}$

(b) $\displaystyle\lim_{x\to 1} \frac{\ln(x)}{x - 1}$

(c) $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{\cos(x) - 1}$

(d) $\displaystyle\lim_{x\to \infty} \frac{e^x}{x^2}$

L'Hôpital's Rule: Type $0/0$ or ∞/∞

Type $0 \cdot \infty$:

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ then try

$$\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)}$$

Activity 2:

State the type of indeterminate form and evaluate using L'Hôpital's Rule:

$$\lim_{x\to\infty} x^3 e^{-x^2}$$

L'Hôpital's Rule: Type $0 \cdot \infty$

Type $\infty - \infty$:

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then try to write as a single fraction (using common denominator or factoring).

Activity 3:

State the type of indeterminate form and evaluate using L'Hôpital's Rule:

$$\lim_{x\to 0} \left(\csc(x) - \frac{1}{x} \right)$$

L'Hôpital's Rule: Type $\infty - \infty$

Type 0^0, ∞^0, or 1^∞:
In this case, use the inverse properties trick: $x = e^{\ln(x)}$ and the fact that e^x is continuous:

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a}\ln(g(x))}$$

Activity 4:

State the type of indeterminate form and evaluate using L'Hôpital's Rule:

(a) $\displaystyle\lim_{x\to 0^+} x^{\sqrt{x}}$

(b) $\displaystyle\lim_{x\to 0^+} (1+4x)^{1/2x}$

Applications of L'Hôpital's Rule

Two applications of L'Hôpital's Rule:

- Computing limits at infinity for curve sketching
- Computing limits at infinity for comparing growth of functions

Example 1: Curve Sketching

Sketch the graph of $y = x^2 e^{-x}$ using the "CSI technique"

"CSI technique" stands for Curve Sketching Info:

1. Domain
2. Intercepts
3. Symmetry (Even, Odd, ...)
4. Asymptotes

1. Build CSI Line for f': locate CPs and local extrema
2. Build CSI Line for f'': locate CP2s (where $f''(c) = 0$), concavity, and points of inflection (if any)

Applications of L'Hôpital's Rule: Curve Sketching

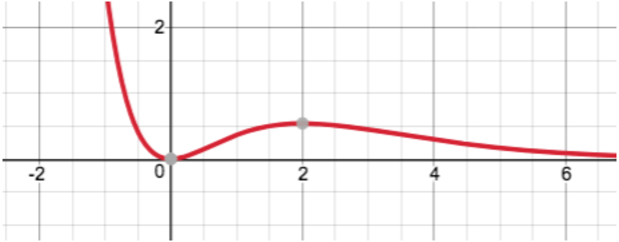
"CSI technique" stands for Curve Sketching Info: $y = x^2e^{-x}$

1. Domain: $\mathbb{R}$
2. Intercepts: $(0,0)$ only
3. Symmetry: not sure
4. Asymptotes: use L'Hop to find:
   $\lim_{x\to\infty} x^2e^{-x} = 0$
   $\lim_{x\to-\infty} x^2e^{-x} = +\infty$

1. Build CSI Line for f':
   $f'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x} = x(2-x)e^{-x} \implies$ CPs: $x = 0$, $x = 2$. Decreasing: $(-\infty, 0) \cup (2, \infty)$, Increasing: $(0, 2)$
2. Build CSI Line for f'':
   $f''(x) = (2-2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x}$. CP2s: $x = 2 \pm \sqrt{2}$. Concave up: $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$, Concave down: $(2-\sqrt{2}, 2+\sqrt{2})$.

Applications of L'Hôpital's Rule: Curve Sketching

The graph of $y = x^2 e^{-x}$:



Applications of L'Hôpital's Rule: Comparing Growth of Functions

- Consider two functions $f(x)$ and $g(x)$ that grow to infinity as x grows to infinity.
- There are many applications where we want to compare two functions that are going to infinity and we want to ask: which grows faster for large values of x? That is, who "wins" as $x \to \infty$, $f(x)$ or $g(x)$?
- Encoding this question using limits, we are interested in:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \quad \text{or} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

  where, in the first case, $g(x)$ wins and, in the second case, $f(x)$ wins.
- When $g(x)$ wins, we say $g(x)$ grows faster at infinity than $f(x)$, and write: $f(x) \ll g(x)$.

Applications of L'Hôpital's Rule: Comparing Growth of Functions

Quick Sort vs Bubble Sort

- In Computer Science, we study algorithms and consider the "cost" or "performance" of executing an algorithm. Two examples are sorting algorithms: Quick Sort and Bubble Sort.
  - Quick Sort: the average time it takes to sort a list of size n is $n\ln(n)$
  - Bubble Sort: the average time it takes to sort a list of size n is n^2.
- If n is small, then a computer will sort a list of size n pretty quickly and we don't care which algorithm we use. But if n is very large, which is better?

Example 2: Quick Sort vs Bubble Sort

We want to know whether $n^2 \ll n\ln(n)$ or $n\ln(n) \ll n^2$. We compute:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n\ln(n)}{n^2} &= \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{[1]\ln(n) + n[\frac{1}{n}]}{2n} \quad \text{(by L'Hop)} \\
&= \lim_{n\to\infty} \frac{\ln(n)+1}{2n} \; \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{2} \quad \text{(by L'Hop)} \\
&= \lim_{n\to\infty} \frac{1}{2n} = 0
\end{aligned}$$

So, we conclude $n\ln(n) \ll n^2$, or Quick Sort is a faster algorithm than Bubble Sort (for large lists).

Applications of L'Hôpital's Rule: Comparing Growth of Functions

Theorem 2: Rates of Growth

$$\ln(x) \ll x \ll x^2 \ll \cdots \ll x^k \ll e^x \ll x! \ll x^x$$

Roughly speaking the key point is: exponential functions grow faster than any polynomial whereas the logarithm grows slow than any polynomial.