§6.3: Logarithmic Functions

Ch 6: Exponentials, Logs, & Inverse Trig Functions
Math 5B: Calculus II

Dr. Jorge Eduardo Basilio

Department of Mathematics & Computer Science
Pasadena City College

Class Notes #3

February 26, 2019
Spring 2019

Outline

1. Guiding Questions
2. Basics of Logarithmic Functions
3. Special Bases of Logarithms

Guiding Questions for §6.2

Guiding Question(s)

1. What are logarithmic functions?
2. What do they look like and what are their important properties?

Guiding Questions for §6.2

In these section, we cover just the basics of logarithms. Many of these properties will be familiar to you from your precalculus class. In the next section, we'll study the calculus of logarithms.

Basics of Logarithmic Functions

Definition 1:

A logarithmic function base b, denoted by $\log_b(x)$, is defined as the inverse function of the corresponding exponential function $f(x) = b^x$.

$$f^{-1}(x) = \log_b(x) \tag{1}$$

where b is a real number (briefly, $b \in \mathbb{R}$) satisfying

$$b > 0 \qquad \text{and} \qquad b \neq 1 \tag{2}$$

Graph on Desmos:

Example 1:

- $f(x) = \log_2(x)$
- $f(x) = \log_{5.89}(x)$

Example 2:

- $g(x) = \log_{1/2}(x)$
- $g(x) = \log_{0.15}(x)$

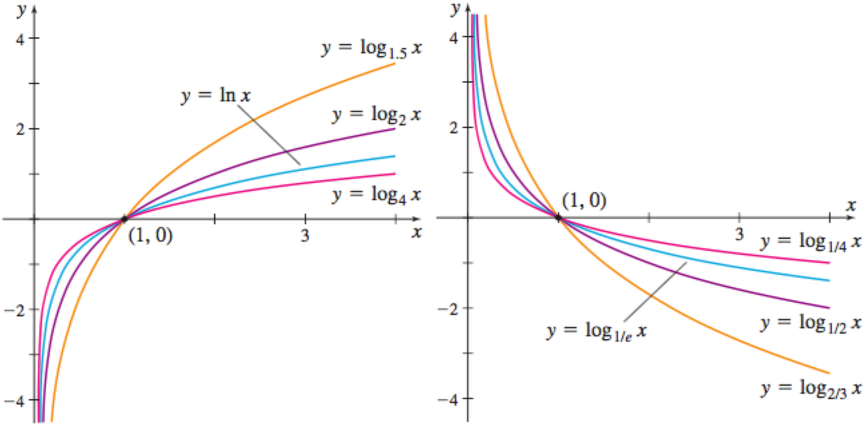
Basics of Logarithmic Functions

Activity 1:

(a) Sketch $f(x) = 2^x$ and $f^{-1}(x)$ on the same coordinate plane. Use: $x = -2, -1, 0, 1, 2$.

(b) Sketch $f(x) = \left(\frac{1}{2}\right)^x$ and $f^{-1}(x)$ on the same coordinate plane. Use: $x = -2, -1, 0, 1, 2$.

Basics of Logarithmic Functions



Case $b > 1$ Case $0 < b < 1$

Basics of Logarithmic Functions

We summarize some important properties of exponential functions:

Theorem 1: Properties of Logarithmic Functions

Let $f(x) = b^x$, with $b > 0$, $b \neq 1$. Let $f^{-1}(x) = \log_b(x)$.

1. f^{-1} is continuous & one-to-one
2. $D(f^{-1}) = (0, +\infty)$ (Note: $\log_b(x)$ defined only for $x > 0$)
3. $R(f^{-1}) = (-\infty, +\infty)$
4. the line $x = 0$ is a vertical asymptote as $x \to 0^+$
5. Graphing Properties:

Case: $b > 1$

- increasing
- $\lim_{x\to 0^+} \log_b(x) = -\infty$
- $\lim_{x\to +\infty} \log_b(x) = +\infty$

Case: $0 < b < 1$

- decreasing
- $\lim_{x\to 0^+} \log_b(x) = +\infty$
- $\lim_{x\to +\infty} \log_b(x) = -\infty$

Basics of Logarithmic Functions

Definition 2: Alternate

Let $b > 0$, $b \neq 1$.
The logarithm base b of x, denoted by $\log_b(x)$, is defined as the power to which you raise b in order to produce x. That is,

$$y = \log_b(x) \qquad \Leftrightarrow \qquad b^y = x \tag{3}$$

This definition comes from trying to find the inverse function for $f(x) = b^x$ using the algebraic steps outlines in §6.1:

- Find the inverse: $f(x) = b^x$
- Replace $f(x)$ with y: $y = b^x$
- Interchange roles of x & y: $x = b^y$
- Solve for y: ?????????

Notice (3) basically solves the ???? step.

Basics of Logarithmic Functions

Example 3:

Switching between the "log equation" and "exponential equation" is extremely helpful:

- $\log_3(81) = 4 \quad \Leftrightarrow \quad 3^4 = 81$ ✓
- $\log_{25}(5) = \frac{1}{2} \quad \Leftrightarrow \quad 25^{1/2} = 5$ ✓
- $\log_{10}(0.001) = -3 \quad \Leftrightarrow \quad 10^{-3} = 0.001$ ✓

Basics of Logarithmic Functions

The exponential rules can be converted into logarithm rules:

Theorem 2: Logarithm Properties (or Logarithm Rules)

Let $b > 0$, $b \neq 1$. Let $x, y > 0$. Let $r \in \mathbb{R}$

1. $\log_b(xy) = \log_b(x) + \log_b(y)$
2. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
3. $\log_b(x^r) = r \cdot \log_b(x)$

These can be rigorously proved using the Exponent Rules.

Theorem 3: Inverse Properties

Let $b > 0$, $b \neq 1$. Let $x, y > 0$. Let $r \in \mathbb{R}$

1. $\log_b(b^x) = x$ for all $x \in \mathbb{R}$
2. $b^{\log_b(x)} = x$ for all $x > 0$

Recall: (1) $\log_b(xy) = \log_b(x) + \log_b(y)$.

Proof:

We prove (1).

Let $A = \log_b(x)$ and $B = \log_b(y)$. Then we convert these into exponential equations: $b^A = x$ and $b^B = y$. Using the exponent rules, we get: $xy = b^A b^B = b^{A+B}$. Thus, converting $xy = b^{A+B}$ back into a log equation: $\log_b(xy) = A + B$. And we're done!

The proof of (3) is more interesting. Recall: (3) $\log_b(x^r) = r \cdot \log_b(x)$.

Proof:

The proof of (3) uses a lot of calculus tools! Namely, continuity. Let $A = \log_b(x^r)$. Then we convert it into an exp equation: $b^A = x^r$. We would like to take r roots but this is only defined for positive integers. So to prove this for $r \in \mathbb{R}$ we need to do it in steps.

- Step 1: If r is a positive integer: we can take r roots of $b^A = x^r$! So, $(b^A)^{1/r} = x$. Thus, converting $b^{A/r} = x$ back into log equation, we get: $\log_b(x) = A/r$ so multiplying by r yields: $r \log_b(x) = A$. Done!
- Step 2: If $r = 0$ then there's really nothing to prove. (why?)

Basics of Logarithmic Functions

Proof:

- Step 3: If r is a negative integer: this follows from Step 3 and Property 1: write $r = -n$ with n a positive integer. Then:

$$\log_b(x^r) = \log_b(x^{-n}) = \log_b\left(\frac{1}{x^n}\right) = \log_b(1) - \log_b(x^n) = -n \cdot \log_b(x)$$

- Step 4: r real number, $r \neq 0$: We pick a sequence of rational numbers $q \to r$. Then: the functions x^y and $\log_b(x)$ are continuous:

$$\begin{aligned}
\log_b(x^r) &= \log_b(x^{\lim_{q\to r} q}) = \log_b(\lim_{q\to r} x^q)(\text{by cont of } x^r) \\
&= \lim_{q\to r} \log_b(x^q)(\text{by cont of } \log) \\
&= \lim_{q\to r} q \log_b(x)(\text{by Step } 3) \\
&= r \log_b(x).
\end{aligned}$$

Activity 2:

Use the log properties to evaluate:

(a) $\log_4(2) + \log_4(32)$

(b) $\log_2(80) - \log_2(5)$

Basics of Logarithmic Functions

Activity 3:

Use the log properties to evaluate: $\lim_{x\to 0} \log_5\left(\sin^2(x)\right)$

Special Bases of Logs

Two special bases are important:

Definition 3:

- Special Base: $b = 10$. $\log(x) = \log_{10}(x)$ called common log
- Special Base: $b = e \approx 2.718\ldots$. $\ln(x) = \log_e(x)$ called natural log

Example 4:

- $\log(1000) = 3$ because $\log(1000) = \log_{10}(10^3) = 3$ by the inverse properties.
- $\ln(\sqrt{e}) = \frac{1}{2}$ because $\ln(\sqrt{e}) = \log_e(e^{1/2}) = \frac{1}{2}$ by the inverse properties.

Special Bases of Logs

Inverse Properties in the special base 10:

- $\log(10^x) = x$ for all $x \in \mathbb{R}$ $\leftarrow$ useful! & basically it's the definition of log!
- $10^{\log(x)} = x$ for all $x > 0$

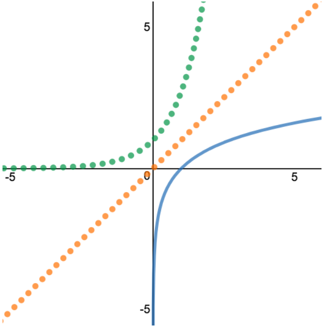
Inverse Properties in the special base e:

- $\ln(e^x) = x$ for all $x \in \mathbb{R}$ $\leftarrow$ useful! & basically it's the definition of ln!
- $e^{\ln(x)} = x$ for all $x > 0$

Also all previous properties proved hold for these special bases.

Natural Logarithm

- Since $e \approx 2.718 > 1$ this means the natural logarithm function, $\ln(x)$, is increasing.
- $\ln(e) = 1$
- $y = \ln(x) \qquad \Leftrightarrow \qquad e^y = x$



Graph of the natural logarithm $\ln(x)$ is in blue

Activity 4:

Find x if $\ln(x^2) = 2$.

Solve Activity 4 in two ways: (1) By re-writing the log eq into an exp eq; (2) By using the inverse properties and raising both sides by e

Activity 5:

Find x if $e^{5-3x} = 10$.

Solve Activity 5 in two ways: (1) By re-writing the exp eq into a log eq; (2) By using the inverse properties and applying ln() on both sides

Activity 6:

Sketch the graph of $y = \ln(x + 3) + 1$

Basics of Logarithmic Functions

Theorem 4: Change of Base Formula

For any two bases of logarithms, $b > 0$, $b \neq 1$ and $c > 0$, $c \neq 1$, we have:

$$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}. \tag{4}$$

A scientific calculator only has one or two bases programmed: $b = 10$ or $b = e$. So, usually one uses the Change of Base formula in one of the two forms:

$$\log_b(x) = \frac{\log(x)}{\log(b)} \qquad \text{or} \qquad \log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Basics of Logarithmic Functions

Activity 7:

Use your calculator to approximate $\log_8(5)$ to the nearest *thousandths*.