§11.1: Sequences

Ch 11: Infinite Sequences and Series
Math 5B: Calculus II

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Class #16 Notes

April 23, 2019
Spring 2019

Outline

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2. Introduction
3. Sequences
4. Limits of Sequences
5. Reccurence Relations
6. Monotonic Sequences
7. Bounded Sequences

Guiding Questions for §11.1

Guiding Question(s)

1. What are sequences?
2. What are limits of sequences?
3. What are some theorems on limits of sequences?
4. What are recurrence relations?
5. What are monotonic sequences?
6. What are bounded sequences?

Introduction

- In this chapter, we will encounter some very fundamental properties about numbers and functions. It is a very deep and challenging chapter.
- Some questions about the nature of real numbers and functions are:
  - What are irrational numbers like $\pi, e, \sqrt{2}$? How do we compute them?
  - How do we actually compute transcendental functions like $\sin(x)$. We only have a few values where we can compute the "exactly" but what about $\sin(1)$?
- Questions like the above are not easy to answer and are intimately tied to the "foundations of calculus."
- Many of the inventors of calculus didn't think about functions like we do today (ie. like a black box that only our calculators actually know how to compute)

Introduction

- The founders of calculus were very cavalier with infinite processes like "infinite sums:"
- Some interesting discoveries:
  - Leibniz: $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$
  - Euler: $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots$
  - $e = 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \cdots$
- Where do these come from and how can we understand them?

Sequences

Looking back at the examples of the “interesting discoveries” we see that they have simple patterns:

- Leibniz: $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$
- Euler: $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots$
- Euler: $e = 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \cdots$

Definition 1: Sequences

- A sequence is a lis of numbers arranged in definite order.
- Notation: $a_1, a_2, a_3, \ldots, a_{105}, a_{106}, \ldots, a_n, \ldots$ or $\{a_n\}_{n=1}^{\infty}$

Sequences

Definition 2: Sequences

- A sequence is a lis of numbers arranged in definite order.
- Notation: $a_1, a_2, a_3, \ldots, a_{105}, a_{106}, \ldots, a_n, \ldots$ or $\{a_n\}_{n=1}^{\infty}$
- General Term, or n^{th} term, of a sequence: a_n
- Formally: a sequence is a function: $f : \mathbb{N} \to \mathbb{R}$ that has inputs in $\mathbb{N} = \{0, 1, 2, \ldots\}$ (or $\mathbb{N} = \{1, 2, \ldots\}$) has has outputs in $\mathbb{R}$. The inputs just keeps track of the location of the number, or the term in the sequence.

Sequences

Activity 1:

List the first 5 terms of the sequence:

(a) $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{1}{n}$.

(b) $$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

(c) $$\left\{(-1)^n \frac{n}{2^n}\right\}_{n=1}^{\infty}$$

Sequences

Sequences

Activity 2:

Find the general term of the sequence determined by the terms of the sums:

(a) Leibniz: $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

(b) Euler: $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots$

(c) Euler: $e = 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \cdots$

Use Sage to visualize the graph of the sequences by plotting the points: (n, a_n)

Sequences

Limits of sequences

Notice all of the examples from Activity 2, after plotting the points we saw they approached the x-axis. In other words, the sequences approached 0.
Knowing the 'trend' of a sequence is exactly like a limit of a function.

Definition 3: Limit of a sequence

- The limit of a sequence, L, is the value such that a_n approaches as n grows arbitrarily large.
- Notation: $L = \lim_{n\to\infty}(a_n)$.
- We can do a rigorous $\epsilon - \delta$ definition as well: $L = \lim_{n\to\infty}(a_n)$ means: Given any $\epsilon > 0$, if there exists a positive integer N (depending on ϵ) so that for all $n \geq N$, we have $|a_n - L| < \epsilon$.
- When the limit exists, we say the sequence converges, otherwise it diverges.

Limits of sequences

Theorem 1: Limit Rules

Given two convergent sequences $\{a_n\}$ and $\{b_n\}$, and a fixed real number c:

(a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty}(a_n) \pm \lim\limits_{n\to\infty}(b_n)$

(b) $\lim\limits_{n\to\infty}(ca_n) = c \cdot \lim\limits_{n\to\infty}(a_n)$

(c) $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty}(a_n) \cdot \lim\limits_{n\to\infty}(b_n)$

(d) $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$, when $\lim_{n\to\infty}(b_n) \neq 0$

(e) $\lim\limits_{n\to\infty}(a_n^p) = \left[\lim\limits_{n\to\infty}(a_n)\right]^p$

Squeeze theorem: if $a_n \leq c_n \leq b_n$ for all $n \geq N$, and $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(b_n) = L$, then

$$\lim_{n\to\infty}(c_n) = L$$

Useful special case of squeeze theorem:
if $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty}(a_n) = 0$.
Continuous functions:
if f is continuous at $x = L$ and $L = \lim_{n\to\infty}(a_n)$, then $\lim_{n\to\infty}(f(a_n)) = f(L)$.

Limits of sequences

Activity 3:

Determine whether the sequences converge or diverge. If they converge, determine their limit.

(a) $\left\{ \dfrac{(-1)^n}{2n} \right\}_{n=1}^{\infty}$

(b) $\{(-1)^n\}_{n=0}^{\infty}$

(c) $\{\cos(\pi n)\}_{n=0}^{\infty}$

(d) $\left\{ \cos\left(\dfrac{\pi}{2} + \pi n \right) \right\}_{n=0}^{\infty}$

Limits of sequences

Limits of sequences

Activity 4:

Evaluate the limits of sequences:

(a) $\displaystyle\lim_{n\to\infty}\left(\frac{2n^2+n+1}{n^2+1}\right)$

(b) $\displaystyle\lim_{n\to\infty}\left(\frac{2n+1}{e^n-11}\right)$

Limits of sequences

It will be important for later study to understand the simple sequence: r^n where r is a fixed number. The following theorem tells us when it converges and diverges.

Theorem 2: Power of r

Let $r \in \mathbb{R}$ be a fixed number. Then

$$\lim_{n\to\infty} (r^n) = \begin{cases} \text{converges} = 0, & \text{if } |r| < 1 \quad (\text{i.e.} -1 < r < 1) \\ \text{coverges} = 1, & \text{if } r = 1 \\ \text{diverges}, & \text{if } r = -1 \\ \text{diverges}, & \text{if } |r| > 1 \end{cases}$$

Limits of sequences

Example 1:

(a) $\lim_{n\to\infty} 3^n$

(b) $\lim_{n\to\infty}(-3)^n$

(c) $\lim_{n\to\infty} \frac{1}{3^n}$

(d) $\lim_{n\to\infty} \left(\frac{-1}{3}\right)^n$

Activity 5:

Let $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n \geq 2$. Sequences defined in this way are called reccurrence relations.

(a) Compute the first 8 terms of the sequence

(b) Based on part (a) value do you predict that $\{a_n\}_{n=1}^{\infty}$ converges to?

(c) How can you prove your prediction correct?

Reccurence Relations

Monotonic sequences

Definition 4: Monotonic sequences

- We say a sequence is increasing if each successive term is greater than the previous term:

$$a_1 < a_2 < a_3 < a_4 < \cdots \quad \text{or} \quad a_n < a_{n+1} \quad \text{for all } n$$

- We say a sequence is decreasing if each successive term is less than the previous term:

$$a_1 > a_2 > a_3 > a_4 > \cdots \quad \text{or} \quad a_n > a_{n+1} \quad \text{for all } n$$

- When a sequence is either increasing or decreasing, we call it monotonic

Monotonic sequences

Activity 6:

(a) Show $\left\{ \dfrac{2}{n+3} \right\}_{n=1}^{\infty}$ is decreasing.

(b) Use the ID test to show that $\left\{ \dfrac{2n}{n^2+1} \right\}_{n=1}^{\infty}$ is decreasing.

Bounded sequences

Definition 5: Bounded sequences

- We say a sequence is bounded above if there is a number M so that for all n, we have
$$a_n \leq M$$
- We say a sequence is bounded below if there is a number m so that for all n, we have
$$m \leq a_n$$
- We say a sequence is bounded if it is both bounded above and below. In this case, we can find a number K so that $|a_n| \leq K$, or $-K \leq a_n \leq K$.

Bounded sequences

Example 2:

(a) The sequence $\{(-1)^n\}_{n=0}^{\infty}$ is bounded.

(b) The sequence $\{\cos(\pi n)\}_{n=0}^{\infty}$ is bounded.

(c) The sequence $\{1 - e^{-n}\}_{n=0}^{\infty}$ is monotic and bounded above by $M = 1$.

Bounded sequences

All convergent sequences give examples of bounded sequences:

Theorem 3: Convergent implies bounded

A convergent sequence is bounded.

Sketch of argument: Assume $a_n \to L$. Then by going far enough out in the sequence, we can assume that a_n are bounded between $L - \epsilon$ (from below) and $L + \epsilon$) (from above) for infinitely many values. Since there's only a remaining number of terms left to consider, the result follows.

Monotonic + Bounded = Converges

An extremely important theorem for this chapter (and in the theory of calculus) is that any monotonic sequence that is also bounded must converge to a limit L.

Theorem 4: Monotonic+Bounded Converges

Every bounded, monotonic sequence converges.

Even though this is important the proof is beyond the scope of this class. It requires an axiom about real number (so called "completeness axiom"). If you're interested you can read the proof in our textbook or ask me about it during office hours.

Monotonic + Bounded = Converges

Activity 7:

Verify that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded below. Does $\lim_{n\to\infty} a_n$ exist?

Monotonic + Bounded = Converges