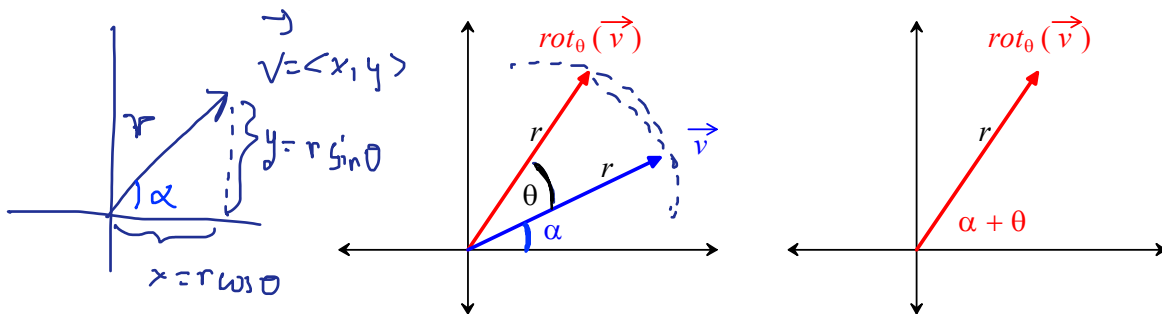# *2.2 Rotations, Projections and Reflections*

Rotations in $\mathbb{R}^2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$$

trig
ID

$$= \begin{bmatrix} r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ r(\sin\alpha\cos\theta + \cos\alpha\sin\theta) \end{bmatrix}_{2\times 1}$$

$\vec{v} = \langle x, y \rangle$

$y = r\sin\theta$

$x = r\cos\theta$

$rot_\theta(\vec{v})$

$rot_\theta(\vec{v})$

$\alpha + \theta$

A Vector $\vec{v}$ and $\boxed{rot_\theta(\vec{v}),}$

its Counterclockwise Rotation by θ

$$= \begin{bmatrix} (\cos\theta)(r\cos\alpha) + (-\sin\theta)(r\sin\alpha) \\ (\cos\theta)(r\sin\alpha) + (\sin\theta)(r\cos\alpha) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$
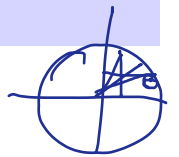
$$= \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}$$

$$Rot_\theta(\vec{v}) = \langle x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta \rangle$$

Theorem: The function $rot_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that takes a vector $\vec{v}$ and rotates $\vec{v}$ counterclockwise by an angle of θ about the origin is a **linear transformation**, with:

$$[rot_\theta] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Note
θ is fixed!
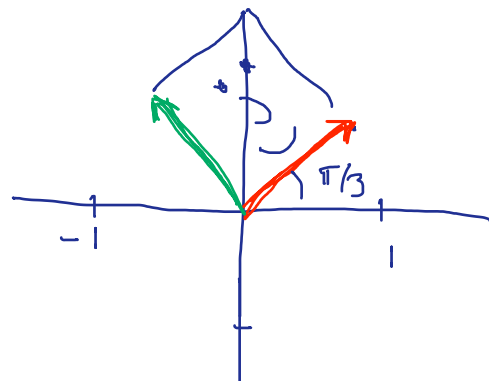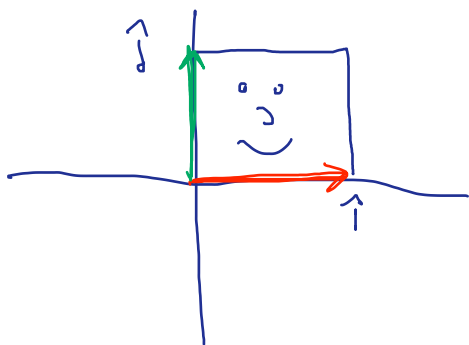so LT!

$\theta = \pi/6 = 30°$

$$\left[rot_{\pi/6}\right] = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \quad \& \quad \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$
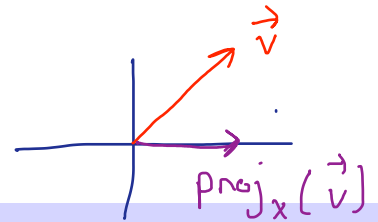
$\sqrt{3}/2 \approx 0.866$

$\approx \begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix}$

$\approx \begin{bmatrix} -0.5 \\ 0.9 \end{bmatrix}$

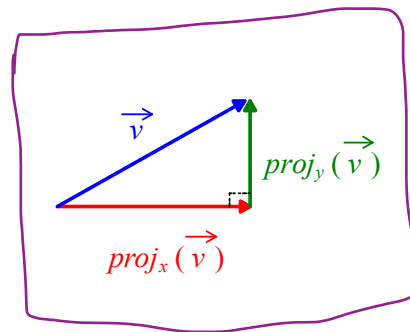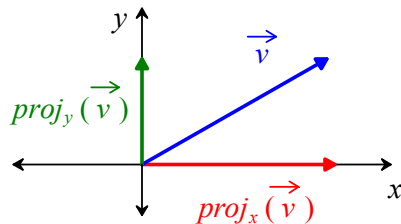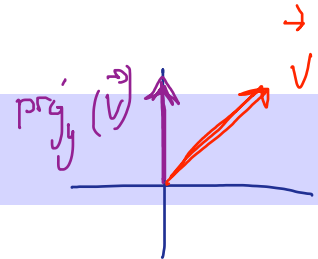# *Basic Projections in* $\mathbb{R}^2$

$$proj_x(\langle x, y \rangle) = \langle x, 0 \rangle.$$

$proj_x(\vec{v})$

$$proj_y(\langle x, y \rangle) = \langle 0, y \rangle.$$

$proj_y(\vec{v})$ $\vec{v}$

Key Relationship:

$$\vec{v} = \langle x, y \rangle$$
$$= \langle x, 0 \rangle + \langle 0, y \rangle$$
$$= proj_x(\vec{v}) + proj_y(\vec{v})$$

$$[Proj_x]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$
$$\|$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$[Proj_y]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$
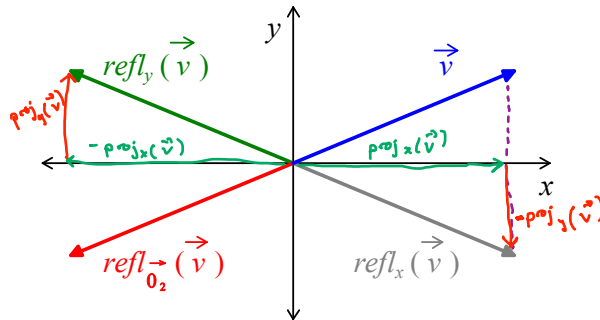
$$proj_x(\vec{v}) = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and}$$

$$proj_y(\vec{v}) = \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[proj_x] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[proj_y] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basic Reflections in $\mathbb{R}^2$

A Vector $\vec{v}$ and its Three Basic Reflections in $\mathbb{R}^2$.

across x-axis
across y-axis

- Reflection Across x-axis

$$\boxed{Refl_x(\vec{v}) = proj_x(\vec{v}) - proj_y(\vec{v})}$$

$$\boxed{[Refl_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Check

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x-0 \\ 0-y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

- Reflection Across y-axis

$$Refl_y(\vec{v}) = \begin{bmatrix} -x \\ y \end{bmatrix} = -proj_x(\vec{v}) + proj_y(\vec{v})$$

$$\boxed{[Refl_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$-\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$
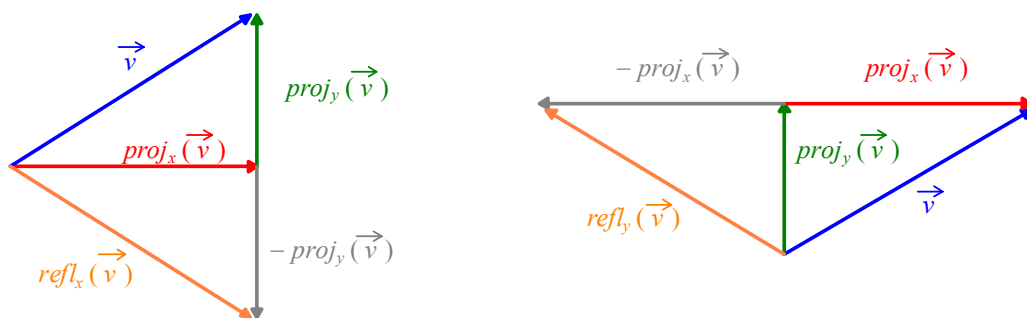
$$refl_x\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

$$refl_y\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \text{ and}$$

Across the origin

$$refl_{\vec{0}_2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$
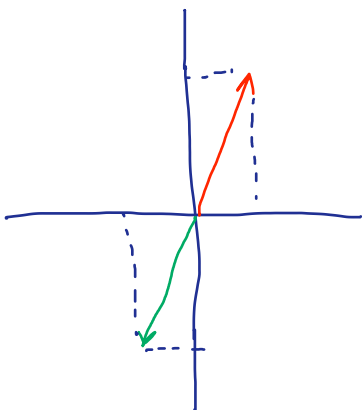
The Geometric Relationships Among

$\vec{v}$, $proj_x(\vec{v})$, $proj_y(\vec{v})$, $refl_x(\vec{v})$ and $refl_y(\vec{v})$

✓ $refl_x(\vec{v}) = proj_x(\vec{v}) - proj_y(\vec{v}),$ and
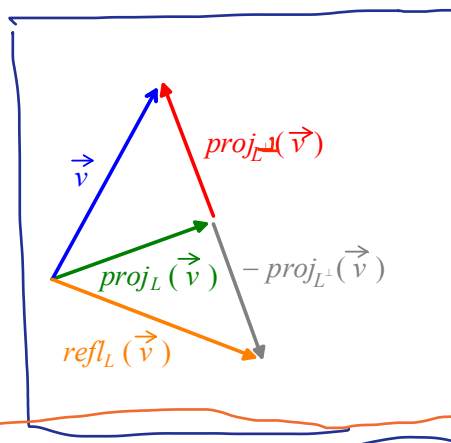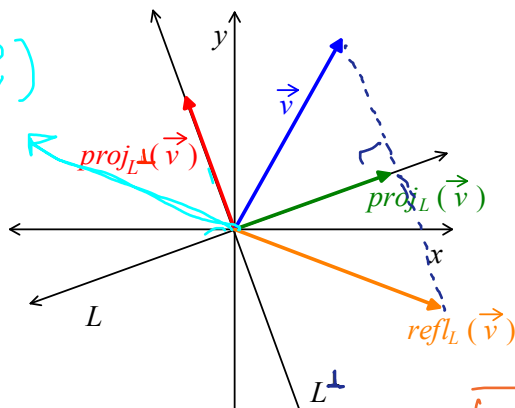
✓ $refl_y(\vec{v}) = proj_y(\vec{v}) - proj_x(\vec{v}).$

L: line through origin

$L^{\perp}$: line through origin perpendicular to L

$$\vec{v} = proj_L(\vec{v}) + proj_{L\perp}(\vec{v})$$

$refl_{L^{\perp}}(\vec{v})$

$proj_{L^{\perp}}(\vec{v})$

$proj_L(\vec{v})$

$\vec{v}$

$refl_L(\vec{v})$

L

$L^{\perp}$

$proj_{L^{\perp}}(\vec{v})$

$\vec{v}$

$proj_L(\vec{v})$

$-proj_{L^{\perp}}(\vec{v})$

$refl_L(\vec{v})$

$$Refl_{L^{\perp}}(\vec{v}) = -proj_L(\vec{v}) + proj_{L\perp}(\vec{v})$$

$$Refl_L(\vec{v}) = proj_L(\vec{v}) - proj_{L\perp}(\vec{v})$$

The Projections of $\vec{v}$ Onto a Line L and its Orthogonal Complement $L^{\perp}$, and the Reflection of $\vec{v}$ Across L.

Basic Projections in $\mathbb{R}^3$

$$proj_x(\langle x,y,z \rangle) = \langle x,0,0 \rangle$$

$$proj_y(\langle x,y,z \rangle) = \langle 0,y,0 \rangle$$

$$proj_z(\langle x,y,z \rangle) = \langle 0,0,z \rangle$$

$$proj_{xy}(\langle x,y,z \rangle) = \langle x,y,0 \rangle$$

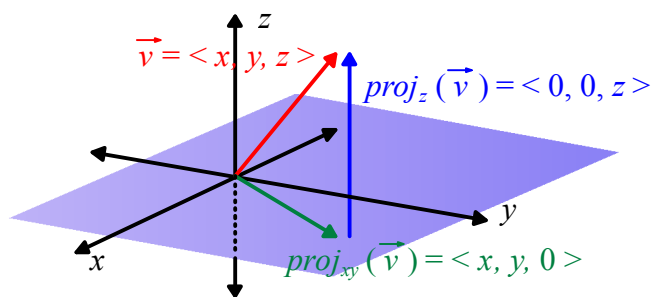$$proj_{xz}(\langle x,y,z \rangle) = \langle x,0,z \rangle$$

$$proj_{yz}(\langle x,y,z \rangle) = \langle 0,y,z \rangle$$

$$\vec{v} = \langle x, y, z \rangle$$
$$= \langle x, 0, 0 \rangle + \langle 0, y, z \rangle$$
$$= proj_x(\langle x, y, z \rangle) + proj_{yz}(\langle x, y, z \rangle),$$

$$\vec{v} = \langle x, y, z \rangle$$
$$= \langle 0, y, 0 \rangle + \langle x, 0, z \rangle$$
$$= proj_y(\langle x, y, z \rangle) + proj_{xz}(\langle x, y, z \rangle), \text{ and}$$

$$\vec{v} = \langle x, y, z \rangle$$
$$= \langle 0, 0, z \rangle + \langle x, y, 0 \rangle$$
$$= proj_z(\langle x, y, z \rangle) + proj_{xy}(\langle x, y, z \rangle)$$

The Relationships Among $\vec{v}$, $proj_{xy}(\vec{v})$ and $proj_z(\vec{v})$.

$$refl_{xy}(\langle x,y,z \rangle) = \langle x,y,-z \rangle$$
$$= \langle x,y,0 \rangle - \langle 0,0,z \rangle$$
$$= proj_{xy}(\langle x,y,z \rangle) - proj_z(\langle x,y,z \rangle).$$
$$refl_z(\langle x,y,z \rangle) = proj_z(\langle x,y,z \rangle) - proj_{xy}(\langle x,y,z \rangle)$$
$$= \langle 0,0,z \rangle - \langle x,y,0 \rangle$$
$$= \langle -x,-y,z \rangle.$$

The Basic *Reflection Operators*:

$$refl_x(\langle x, y, z \rangle) = \langle x, -y, -z \rangle$$

$$refl_y(\langle x, y, z \rangle) = \langle -x, y, -z \rangle$$

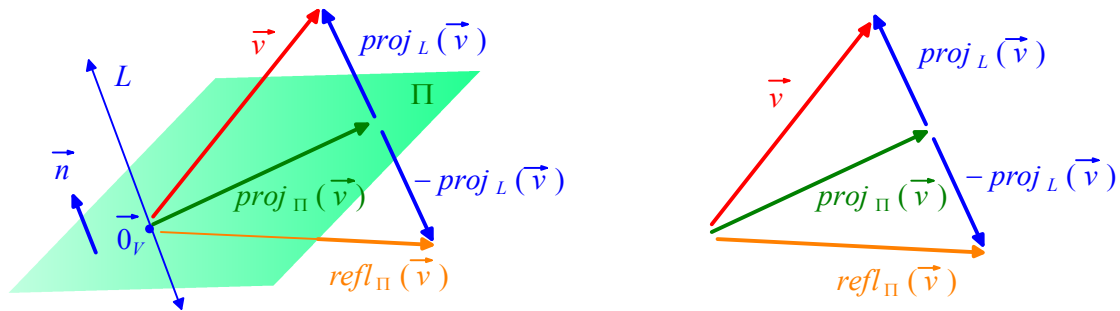$$refl_z(\langle x, y, z \rangle) = \langle -x, -y, z \rangle$$

$$refl_{xy}(\langle x, y, z \rangle) = \langle x, y, -z \rangle$$

$$refl_{xz}(\langle x, y, z \rangle) = \langle x, -y, z \rangle$$

$$refl_{yz}(\langle x, y, z \rangle) = \langle -x, y, z \rangle$$

$$refl_{\vec{0}_3}(\langle x, y, z \rangle) = \langle -x, -y, -z \rangle$$

General Projections and Reflections in $\mathbb{R}^3$

We need the decomposition:

$$\vec{v} = proj_\Pi(\vec{v}) + proj_L(\vec{v}),$$

$$\text{where } proj_\Pi(\vec{v}) \in \Pi \text{ and } proj_L(\vec{v}) \in L.$$

General Principle:

$$\text{If } \vec{v} = proj_W(\vec{v}) + proj_{W^\perp}(\vec{v}), \text{ then:}$$

$$refl_W(\vec{v}) = proj_W(\vec{v}) - proj_{W^\perp}(\vec{v})$$

For planes Π and lines L through the origin:

$$refl_\Pi(\vec{v}) = proj_\Pi(\vec{v}) - proj_L(\vec{v}), \text{ and}$$

$$refl_L(\vec{v}) = proj_L(\vec{v}) - proj_\Pi(\vec{v}).$$