

Solving the Many-Electron Schrödinger Equation With a Transformer Architecture

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The Schrödinger equation

On 1926 Schrodinger derived the Time Dependent Form.(TDSE)

$$i \hbar \partial_t \Psi = \hat{H} \Psi, \quad (1)$$

- $\Psi \in \mathcal{H}$ is a complex value function called **wave function**.
- \hat{H} is called the **Hamiltonian Operator** , encodes all the information of the energy of the system.
- Depends on the position \vec{r} of a particle and the temporal evolution (t).
- Ψ encodes all information about the system; $|\Psi|^2$ gives a probability density that integrates to 1.

Hamiltonian

In the position basis:

$$\hat{H} = \frac{\hat{\vec{P}}^2}{2m} + \hat{V} = -\frac{\hbar^2}{2m} \nabla^2 + \hat{V} \quad (2)$$

Time Dependent Form

When the wave function could be written as:

$$\psi(\vec{\mathbf{r}}, t) = R(\vec{\mathbf{r}})T(t) \quad (3)$$

TDSE returns you that:

$$T(t) = e^{-iEt/\hbar} \wedge \hat{H}R(\vec{\mathbf{r}}) = ER(\vec{\mathbf{r}}) \quad (4)$$

Where E is the total energy of the system. The eigen-problem becomes obtain R solving:

Find a Ψ , such that:

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) \right] \psi(\vec{\mathbf{r}}) = E\psi(\vec{\mathbf{r}}) \quad (5)$$

Find the potential V of the system.

Many-Body System

When considering a many-body system, we need to consider the position of each electron like also the spin of it. When considering n bodies, we have:

$$\hat{H}\psi(\mathbf{x}_1, \dots, \mathbf{x}_n) = E\psi(\mathbf{x}_1, \dots, \mathbf{x}_n) \quad (6)$$

With $\mathbf{x}_i = \{\mathbf{r}_i, \sigma\}$, where $\mathbf{r}_i \in \mathbb{R}^3$ is the position of each particle and $\sigma \in \{\uparrow, \downarrow\}$ is the so called spin.

Considerations

- Each particle interact with all the another particles in specific ways.
- For atoms, consider all the protons, electrons and neutrons.
- Solution obey physical laws.

Setting up the Hamiltonian

The first step is obtain a practical form of the **Hamiltonian**.

- Kinetic energy: $T = -\frac{1}{2} \sum_{i=1}^N \nabla_i^2$.
- Electron-nuclear attraction: $V_{en} = - \sum_{i,I} \frac{Z_I}{r_{iI}}$.
- Electron-electron repulsion: $V_{ee} = \sum_{i < j} \frac{1}{r_{ij}}$.

$$\begin{aligned} \hat{H} = & - \sum_{i=1}^N \frac{1}{2} \nabla_i^2 - \sum_{I=1}^M \frac{1}{2M_I} \nabla_I^2 - \sum_{i=1}^N \sum_{I=1}^M \frac{Z_I}{|\mathbf{r}_i - \mathbf{R}_I|} \\ & + \sum_{1 \leq i < j \leq N} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{1 \leq I < J \leq M} \frac{Z_I Z_J}{|\mathbf{R}_I - \mathbf{R}_J|} \end{aligned} \quad (7)$$

Borh Oppenheimer approximation helps with.

Fermi-Dirac statistics

All the fermions follow the Fermi-Dirac Statistics, this is.

- Electrons are indistinguishable fermions.
- Exchanging two electrons flips the wavefunction's sign:
 $\Psi(\dots i, j \dots) = -\Psi(\dots j, i \dots)$.
- Pauli exclusion: no two electrons can occupy the same

Slater Determinant

We can enforce this using a determinant to enforce an antisymmetric Ψ .

$$\psi = \begin{vmatrix} \phi_1^k(\mathbf{x}_1) & \dots & \phi_1^k(\mathbf{x}_n) \\ \vdots & & \vdots \\ \phi_n^k(\mathbf{x}_1) & \dots & \phi_n^k(\mathbf{x}_n) \end{vmatrix} \quad (8)$$

Where ϕ are called spin orbitals

Kato cusp conditions, Jastrow Factor

When two electrons

- Coulomb potentials cause a sharp cusp in Ψ when particles overlaps.
- Electron–nucleus cusp: $\frac{\partial \Psi}{\partial r_{iI}} \Big|_{r_{iI}=0} = -Z_I \Psi(0).$
- Electron–electron cusp: $\frac{\partial \Psi}{\partial r_{ij}} \Big|_{r_{ij}=0} = \frac{1}{2} \Psi(0).$

Jastrow Factor $\exp(\mathcal{J})$

In this work we are going to use this specific form:

$$\mathcal{J}_\theta(x) = \sum_{i < j; \sigma_i = \sigma_j} -\frac{1}{4} \frac{\alpha_{par}^2}{\alpha_{par} + |\mathbf{r}_i - \mathbf{r}_j|} + \sum_{i, j; \sigma_i \neq \sigma_j} -\frac{1}{2} \frac{\alpha_{anti}^2}{\alpha_{anti} + |\mathbf{r}_i - \mathbf{r}_j|} \quad (9)$$

Loss: Variational Principle

Try an ansatz and optimize it.

Variational principle states that the energy of any ansatz is greater than the true energy.

$$E[\Psi] = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \geq E_0$$

Minimizing $E[\Psi]$ drives the ansatz toward the ground state. E_0 is the true ground energy (minimum possible).

$$\mathcal{L}(\Psi_\theta) = \frac{\langle \Psi_\theta | \hat{H} | \Psi_\theta \rangle}{\langle \Psi_\theta | \Psi_\theta \rangle} = \frac{\int d\mathbf{r} \Psi^*(\mathbf{r}) \hat{H} \Psi(\mathbf{r})}{\int d\mathbf{r} \Psi^*(\mathbf{r}) \Psi(\mathbf{r})}$$

Define:

$$p_\theta(x) = \frac{\Psi_\theta^2(x)}{\int dx' \Psi_\theta^2(x')} \quad \text{and} \quad E_L(x) = \Psi_\theta^{-1}(x) \hat{H} \Psi_\theta(x)$$

Then:

$$\mathcal{L}_\theta = \mathbb{E}_{x \sim p_\theta}[E_L(x)] \tag{10}$$

Variational Monte Carlo

To evaluate that expectation we use Monte Carlo estimator.

Quantum Monte Carlo

With the samples $\mathbf{R}_1, \dots, \mathbf{R}_M \sim p_\theta(\mathbf{R})$ we can make:

$$\mathcal{L}_\theta = \mathbb{E}_{x \sim p_\theta}[E_L(x)] \approx \frac{1}{M} \sum_{i=1}^m E_L(\mathbf{R}_k) \quad (11)$$

With:

$$E_L(\mathbf{R}_k) = \frac{\hat{H}\psi(\mathbf{R}_k)}{\psi(\mathbf{R}_k)} = -\frac{1}{2} \frac{\nabla^2 \psi(\mathbf{R}_k)}{\psi(\mathbf{R}_k)} + V(\mathbf{R}_k)$$

To obtain those samples we use the **Metropolis-Hastings Algorithm** which handles well high dimensions.

Metropolis-Hastings Algorithm

1. Take a initial configuration $\mathbf{X}_0 \in E$ arbitrary:
2. Propose $\mathbf{X}' = \mathbf{X}_0 + \eta$, where $\eta \sim q(\eta)$, q is a probability density on E called proposal kernel. symmetric Gaussian
3. Compute the quantity:

$$A(\mathbf{X}_0, \mathbf{X}') = \min \left(1, \frac{\rho(\mathbf{X}')}{\rho(\mathbf{X}_0)} \frac{q(\mathbf{X}' - \mathbf{X}_0)}{q(\mathbf{X}_0 - \mathbf{X}')} \right)$$

Where ρ is the target distribution where we want sample, In the case where q is symmetric, this simplifies to:

$$A(\mathbf{X}_0, \mathbf{X}') = \min \left(1, \frac{\rho(\mathbf{X}')}{\rho(\mathbf{X}_0)} \right)$$

4. Generate a uniform number $U \in [0, 1]$
5. If: $U < A(\mathbf{X}_0 \rightarrow \mathbf{X}'_l)$ then $\mathbf{X}_1 = \mathbf{X}'$, otherwise try another \mathbf{X}' .
Accept or decline.

6. Repeat until obtain N_{eq} accepted, the change stabilizes (stationary distribution) this phase is called **burn in**.
7. From $\mathbf{X}_{N_{eq}}$ generate M samples until reach the sample $\mathbf{X}_{N_{eq}+M+1}$. In each sample generates $E_L(\mathbf{R}_k)$ then average to obtain $\mathbb{E}(E_L)$ and begin the back propagation step.

Gradients of the Loss

Using calculus you obtain:

$$\nabla_{\theta} \mathcal{L} = 2\mathbb{E}_{x \sim \Psi^2}[(E_L(x) \quad (12)$$

This expectation is calculated in the same way.

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Fisher Information Matrix

Gradient descent assumes that parameters lives on the Euclidian Space \mathbb{R}^n . ($\mathcal{L}(\theta_1, \dots, \mathbf{x})$)

But in the case where parameters lives on a probability distribution $p(x, \theta)$ that assumption don't work to well. A natural **metric** when looking for compare the distributions $p(x, \theta)$ and $p(x, \theta + \delta\theta)$ is the **Fisher information Matrix**.

Fisher Matrix $F(\theta)$

Let $x \sim p(x|\theta)$. Define the score:

$$s_{\theta}(x) = \nabla_{\theta} \log p(x|\theta)$$

$$F(\theta) = \mathbb{E}_{x \sim p(\cdot|\theta)} [s_{\theta}(x) s_{\theta}(x)^{\top}]$$

$s \in \mathbb{R}^d$ a column vector. d number of parameters.

Natural Gradient

Using the Fisher Metric, the steepest descent direction of the loss is:

$$-F(\theta)^{-1}\nabla_{\theta}\mathcal{L}$$

The updates becomes:

$$\theta_{k+1} = \theta_k - \eta F(\theta_k)^{-1} \delta_{\theta} \mathcal{L}$$

- Compute F , expensive.
- KFAC ameliorate this with two approximations.

Optimizer: KFAC (natural gradient)

The first approximation is: F_{ij} is assumed zero when θ_i and θ_j are on different layers (neural network). So we just care about the same layer ℓ . Calculus say.

$$\mathbb{E}_{p(\mathbf{X})} \left[\frac{\partial \log p(X)}{\partial \text{vec}(\mathbf{W}_\ell)} \frac{\partial \log p(X)}{\partial \text{vec}(\mathbf{W}_\ell)}^\top \right] = \mathbb{E}_{p(\mathbf{X})} [(\mathbf{a}_\ell \otimes \mathbf{e}_\ell)(\mathbf{a}_\ell \otimes \mathbf{e}_\ell)^\top]$$

Where \mathbf{a}_ℓ are the forward activation and \mathbf{e}_ℓ are the backward sensitivities for that layer.

$$\mathbb{E}_{p(\mathbf{X})} [(\mathbf{a}_\ell \otimes \mathbf{e}_\ell)(\mathbf{a}_\ell \otimes \mathbf{e}_\ell)^\top]^{-1} \approx \mathbb{E}_{p(\mathbf{X})} [\mathbf{a}_\ell \mathbf{a}_\ell^\top]^{-1} \otimes \mathbb{E}_{p(\mathbf{X})} [\mathbf{e}_\ell \mathbf{e}_\ell^\top]^{-1} \quad (13)$$

Attention on the room

Multi Head Attention Let d be the embedding dimension, n_h be the number of attention heads, d_h be the dimension per head, and $\mathbf{h}_t \in \mathbb{R}^d$ be hidden dimension. The learn matrices are:

$$W^Q, W^K, W^V \in \mathbb{R}^{d_h n_h \times d}$$

We obtain the key, queries, and values vectors with:

$$\mathbf{k}_i = \mathbf{W}^k \mathbf{h}_i, \mathbf{q}_i = \mathbf{W}^q \mathbf{h}_i, \mathbf{v}_i = \mathbf{W}^v \mathbf{h}_i$$

We slice this vectors:

$$[\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{n_h}] = \mathbf{q}$$

$$[\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{n_h}] = \mathbf{k}$$

$$[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n_h}] = \mathbf{v}$$

In the $i - th$ head:

$$\mathbf{o}_{t,i} = \sum_{j=1}^t \text{Softmax} \left(\frac{\mathbf{q}_{t,i}^T \mathbf{k}_{j,i}}{\sqrt{d_h}} \right) \mathbf{v}_{j,i} \quad (14)$$

Attention!

Let $W^O \in \mathbb{R}^{d \times d_h n_h}$ be the output projection matrix.

$$\mathbf{u}_t = W^O[\mathbf{o}_{t,1}; \mathbf{o}_{t,2}; \dots; \mathbf{o}_{t,n_h}]$$

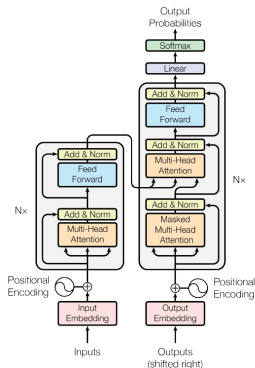


Figure 1: Original Transformer

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Orbital

To compute the orbitals we need to compute.

$$\phi_i^{k\alpha}(\mathbf{r}_j^\alpha; \{\mathbf{r}_{/j}^\alpha\}; \{\mathbf{r}^{\bar{\alpha}}\}) = \left(\mathbf{w}_i^{k\alpha} \cdot \mathbf{h}_j^{L\alpha} + g_i^{k\alpha} \right) \sum_m \pi_{im}^{k\alpha} \exp \left(-|\boldsymbol{\Sigma}_{im}^{k\alpha}(\mathbf{r}_j^\alpha - \mathbf{R}_m)| \right) \quad (15)$$

Where the hidden states are:

$$\begin{aligned} \mathbf{h}_i^{\ell+1\alpha} &= \tanh \left(\mathbf{V}^\ell \mathbf{f}_i^{\ell\alpha} + \mathbf{b}^\ell \right) + \mathbf{h}_i^{\ell\alpha} \\ \mathbf{h}_{ij}^{\ell+1\alpha\beta} &= \tanh \left(\mathbf{W}^\ell \mathbf{h}_{ij}^{\ell\alpha\beta} + \mathbf{c}^\ell \right) + \mathbf{h}_{ij}^{\ell\alpha\beta} \end{aligned}$$

Finally:

$$\psi(\mathbf{r}_1^\uparrow, \dots, \mathbf{r}_{n\downarrow}^\downarrow) = \sum_k \omega_k \left(\det \left[\phi_i^{k\uparrow}(\mathbf{r}_j^\uparrow; \{\mathbf{r}_{/j}^\uparrow\}; \{\mathbf{r}^\downarrow\}) \right] \right. \\ \left. \det \left[\phi_i^{k\downarrow}(\mathbf{r}_j^\downarrow; \{\mathbf{r}_{/j}^\downarrow\}); \{\mathbf{r}^\uparrow\}; \right] \right). \quad (16)$$

Fermi Net Architecture

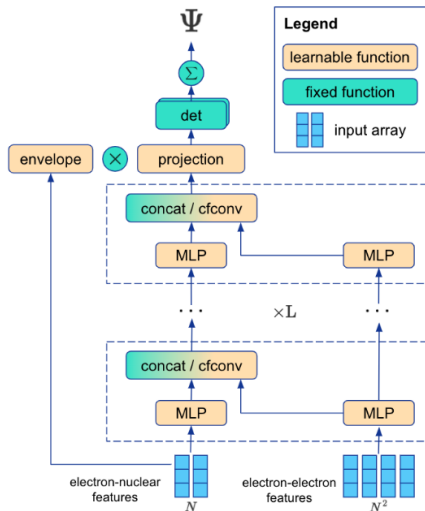


Figure 2: Fermi Net Architecture

The Psiformer Ansatz (overview)

The output of **Psiformer** is:

$$\Psi_{\theta}(\mathbf{x}) = \exp(\mathcal{J}_{\theta}(\mathbf{x})) \sum_{k=1}^{N_{\text{det}}} \det \left[\boldsymbol{\Phi}_{\theta}^k(x) \right] \quad (17)$$

We obtain the hidden states:

$$A_h^l = [\text{SelfAttn}(\mathbf{h}_1^l, \dots, \mathbf{h}_N^l; \mathbf{W}_q^{\ell h}, \mathbf{W}_k^{\ell h}, \mathbf{W}_v^{\ell h})]$$

$$A^{\ell} = \text{concat}_h[A_h^{\ell}]$$

And then used it apply:

$$\mathbf{f}_i^{\ell+1} = \mathbf{h}_i^{\ell} + W_o^{\ell} A^{\ell}$$

And generate a new hidden state:

$$\mathbf{h}_i^{\ell+1} = \mathbf{f}_i^{\ell+1} + \tanh\left(\mathbf{W}^{\ell+1} \mathbf{f}_i^{\ell+1} + \mathbf{b}^{\ell+1}\right)$$

Psi Former Architecture

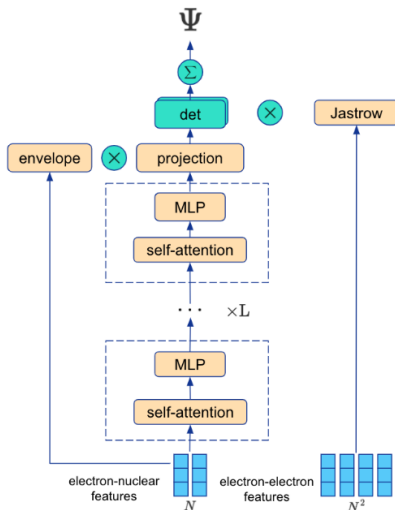


Figure 3: Psi Former Architecture

Thanks

Thanks.