AME

Week 3: Random Effects

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Today's Plan

- Random Effects
- Hausman Test
- Your time to shine!

Random Effects Model

- Linear Panel Data Model

$$y_{it} = c_i + \boldsymbol{x_{it}}\boldsymbol{\beta} + u_{it} \tag{1}$$

- Let's assume no confounding time-invariant heterogeneity $E(c_i x_{it}) = 0$ and strict exogeneity $E(u_{it} x_{is}) = 0 \forall s, t \in T$ holds. Add to that assumptions FE.3 $E(\boldsymbol{u_i}\boldsymbol{u_i}') = \sigma_u^2 I_T \quad \forall \quad i \in N$ and (new!) $E(\boldsymbol{c_i}\boldsymbol{c_i}') = \sigma_c^2 \mathbf{j}_T \mathbf{j}_T' \quad \forall \quad i \in N$ where $\mathbf{j}_T = [1 \dots 1]'$ is a $T \times 1$ vector of ones.
- Is POLS consistent? Yes
- Is POLS efficient? Error terms need to be homoskedastic (IID)
  only need contemp. effects to be uncorr.

RE Covariance Matrix

- The composite error covariance matrix has non-zero off-diagonals due to the time-invariant heterogeneity, c_i :

$$
\begin{aligned}
E(\mathbf{v}_i\mathbf{v}_i') &= \sigma_c^2 \mathbf{j}_T \mathbf{j}_T' + \sigma_u^2 \mathbf{I}_T \\
&= \begin{bmatrix} \sigma_c^2 + \sigma_u^2 & \sigma_c^2 & \dots & \sigma_c^2 \\ \sigma_c^2 & \sigma_c^2 + \sigma_u^2 & \dots & \sigma_c^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_c^2 & \sigma_c^2 & \dots & \sigma_c^2 + \sigma_u^2 \end{bmatrix}_{T \times T}
\end{aligned} \quad (2)
$$

- POLS is inefficient as the composite error term is not homoskedastic
- RE accounts for this feature of the panel data and efficiently weights the data to minimise the variance

RE Quasi Demeaning I

- RE is equivalent to POLS on "quasi-demeaned" data:

$$\check{y}_{it} = \check{\mathbf{x}}_{it}\beta + \check{v}_{it} \tag{3}$$

where $\check{y}_{it} = y_{it} - \hat{\lambda}\bar{y}_{it}$, $\check{\mathbf{x}}_{it} = \mathbf{x}_{it} - \hat{\lambda}\bar{\mathbf{x}}_{it}$ and $\check{v}_{it} = v_{it} - \hat{\lambda}\bar{v}_{it}$

- Quasi-demeaning subtracts means weighted by

$$\hat{\lambda} = 1 - \sqrt{\frac{\widehat{\sigma}_u^2}{(\widehat{\sigma}_u^2 + T\widehat{\sigma}_c^2)}} \tag{4}$$

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For \lambda = 0
   -> POLS

For \lambda -> 1:
   -> FE world,
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- To estimate λ we need estimates of σ_u^2 and σ_c^2.
  - "Between estimator" returns $\hat{\sigma}_v^2 = \hat{\sigma}_c^2 + \frac{1}{T}\hat{\sigma}_u^2$
  - Fixed Effects yields $\hat{\sigma}_u^2$
  - Combine these to obtain $\hat{\sigma}_c^2 = \hat{\sigma}_v^2 - \frac{1}{T}\hat{\sigma}_u^2$

RE Quasi Demeaning II

- Use perm function to quasi-demean the data. For this you'll need the matrix $\mathbf{C}_T = \mathbf{I}_T - \hat{\lambda}\mathbf{P}_T$ where $\mathbf{P}_T = [1/T \dots 1/T]'$
- Rearranging (4)

$$\hat{\lambda} = 1 - \sqrt{\frac{1}{(1 + T\widehat{\sigma}_c^2/\widehat{\sigma}_u^2)}}$$

we see that

$$\begin{aligned}
\hat{\lambda} \to 1 \quad &\textit{for} \quad T\widehat{\sigma}_c^2/\widehat{\sigma}_u^2 \to \infty \quad \Rightarrow \quad \hat{\boldsymbol{\beta}}_{RE} \to \hat{\boldsymbol{\beta}}_{FE} \\
\hat{\lambda} \to 0 \quad &\textit{for} \quad T\widehat{\sigma}_c^2/\widehat{\sigma}_u^2 \to 0 \quad \Rightarrow \quad \hat{\boldsymbol{\beta}}_{RE} \to \hat{\boldsymbol{\beta}}_{POLS}
\end{aligned}$$

- RE lies between POLS and FE estimators and puts more weight on observations for which there is more variation (e.g. if lots of within variation, σ_c^2 is large, efficient to be closer to FE)

RE Assumptions

- Random Effects Assumptions

$$
\begin{aligned}
RE.1a: &\quad E(\mathbf{x_{it}} u_{is}) = 0 \quad \forall \quad i \in N \quad \text{and} \quad s, t \in T \\
RE.1b: &\quad E(\mathbf{x_{it}} c_i) = 0 \quad \forall \quad i \in N \quad \text{and} \quad t \in T \\
RE.2: &\quad \text{rank}(E(\check{\mathbf{X}}' \check{\mathbf{X}})) = K \quad \forall \quad i \in N \\
RE.3: &\quad E(\mathbf{v_i} \mathbf{v_i}') = \sigma_u^2 I_T + \sigma_c^2 \mathbf{j}_T \mathbf{j}_T' \quad \forall \quad i \in N
\end{aligned}
$$

- We need both no confounding time-invariant heterogeneity (RE.1b) and strict exogeneity (RE.1a)
- The fact that RE tends to FE in one limit and POLS in another limit explains why we need "the worst of both worlds"
- RE in linear settings might seem silly but comes to shine in nonlinear settings :)

Hausman Test: RE vs FE

- Under assumptions RE.1a, RE.2 and RE.3 (and FE.2), we can test whether there are time invariant omitted variables i.e. whether $E(\mathbf{x}_{it} c_i) = 0$
- Hausman Test Statistic

$$H = (\hat{\boldsymbol{\beta}}_{RE} - \hat{\boldsymbol{\beta}}_{FE})'(\widehat{Avar}(\hat{\boldsymbol{\beta}}_{FE}) - \widehat{Avar}(\hat{\boldsymbol{\beta}}_{FE}))(\hat{\boldsymbol{\beta}}_{RE} - \hat{\boldsymbol{\beta}}_{FE}) \sim \chi^2_M$$

  where $M \leq K$ is the number of coefficients
- Only include estimated coefficients $\hat{\boldsymbol{\beta}}_{RE}$ on time varying regressors (since FE doesn't have the ones on time-variant regressors)

Your time to shine!

- Have a look at the toolbox LinearModelsWeek3.py
- Solve the problem set and use functions from the toolbox where necessary