Lecture 4:
Predetermined Regressors

Jesper Riis–Vestergaard Sørensen

University of Copenhagen, Department of Economics

Plan for Panel Data Lectures

Lecture 1: Linear model + OLS in cross section (W.4)

Lecture 2: Fixed effects + First differences (W.10)

Lecture 3: Random effects + Hausman test (W.10)

Lecture 4: Predetermined regressors (W.11)

Lecture 5: First-Differencing IV Methods and GMM (W.11)

Exogeneity Assumptions for POLS, FE/D, RE

Starting point still

$$y_{it} = \mathbf{x}_{it}\beta + c_i + u_{it}, \quad t = 1, 2, \ldots, T.$$

Assumptions used for identification/consistency:

POLS:

- $E\left(u_{it} \middle| \mathbf{x}_{it}, c_i\right) = 0$ (contemporaneous exogeneity).
- $E\left(c_i \middle| \mathbf{x}_i\right) = E\left(c_i\right) = 0$. unobserved heterogeneity is uncorrelated w/ regressors in all time periods

  x-vector stacked over all time periods

FD/E:

- $E\left(u_{it} \middle| \mathbf{x}_i, c_i\right) = 0$ (strict exogeneity).
- $E\left(c_i \middle| \mathbf{x}_i\right) \neq 0$ allowed.

RE:

- $E\left(u_{it} \middle| \mathbf{x}_i, c_i\right) = 0$ (strict exogeneity).
- $E\left(c_i \middle| \mathbf{x}_i\right) = E\left(c_i\right) = 0$.

Strict Exogeneity too Strong?

Strict exogeneity restrictive.

- u_{it}s uncorrelated with *past, current and future* $\mathbf{x}_{it}$s.
- Need not be plausible—or even possible.
- Model may imply current u_{it} affects *future* $\mathbf{x}_{it}$.

$\Rightarrow$ Need less restrictive notion of exogeneity.

Outline

Sequential Exogeneity

Sequential Exogeneity

$\{\mathbf{x}_{it}\}_{t=1}^{T}$ sequentially exogenous conditional on unobserved effect if

PAST

$$E\left(u_{it} \middle| \mathbf{x}_{it}, \mathbf{x}_{it-1}, \ldots, \mathbf{x}_{i1}, c_i\right) = 0, \quad t = 1, 2, \ldots, T.$$

Will also call $\{\mathbf{x}_{it}\}_{t=1}^{T}$ predetermined.

Implications:

$$E\left(y_{it} \middle| \mathbf{x}_{it}, \mathbf{x}_{it-1}, \ldots, \mathbf{x}_{i1}, c_i\right) = E\left(y_{it} \middle| \mathbf{x}_{it}, c_i\right) = \mathbf{x}_{it}\boldsymbol{\beta} + c_i.$$

Controlling for $(\mathbf{x}_{it}, c_i)$, no *past* $\mathbf{x}_{it}$s predict outcome.

Dynamics allowed? Feedback?

A Dynamic Model

Example: A Dynamic Model

Consider **first-order autoregressive [AR(1)] model**

$$y_{it} = \rho y_{it-1} + c_i + u_{it},$$
$$E(u_{it}|y_{it-1}, y_{it-2}, \ldots, y_{i0}, c_i) = 0, \quad t = 1, 2, \ldots, T.$$

- In previous notation, $x_{it} = y_{it-1}$.

Does $\{y_{it}\}_{t=1}^{T}$ exhibit state dependence?

- Controlling for c_i, does last period's outcome help predict next period's outcome?
- Yes, provided $\rho \neq 0$.

Strict Exogeneity?

Q: Is $\{y_{it}\}_{t=1}^{T}$ <u>strictly</u> exogenous? (Would justify FE/D.)

- Does $E\left(u_{it} | y_{i0}, y_{i1}, \ldots, y_{iT}, c_i\right) = 0$? $\Rightarrow E(y_{iT} u_{it}) = 0$

$$= E\left[y_{iT}\, \underbrace{E(u_{it} | y_{iT})}_{=0}\right]$$

under strict exog.

Consider $x_{it+1} = y_{it}$. Then

$$
\begin{aligned}
\boxed{E\left(x_{it+1} u_{it}\right)} &= E\left[\left(\overbrace{\rho y_{it-1} + c_i + u_{it}}^{= y_{it}}\right) u_{it}\right] \\
&= \rho E\left(y_{it-1} u_{it}\right) + E\left(c_i u_{it}\right) + E\left(u_{it}^2\right) \\
&= E\left(u_{it}^2\right) \qquad \text{(sequential exogeneity)} \\
&> 0 \qquad \text{(in general)}
\end{aligned}
$$

$= 0$ (LIE)

law of iterated expectation

Conflicts with strict exogeneity (LIE).

Conclude: Lagged dependent variables (LDVs) rule out strict exogeneity.

Spurious State Dependence

Q: Is $E\left(y_{it-1}c_i\right) = 0$? (Would justify POLS.)

A: No. At $t-1$, y_{it-1} on LHS—necessarily depends on c_i.

- If c_i not controlled for, persistence in $\{y_{it}\}_{t=1}^{T}$ due to c_i may be incorrectly attributed to LDV.
- Creates spurious state dependence.

Static Model with Feedback

Example: Static Model with Feedback

$$y_{it} = \mathbf{z}_{it}\boldsymbol{\beta} + \delta h_{it} + c_i + u_{it}$$
$$E(u_{it}|\mathbf{z}_{i}, h_{it}, \ldots, h_{i1}, c_i) = 0, \quad t = 1, 2, \ldots, T.$$

- $\mathbf{z}_{it}$s strictly exogenous,
- h_{it}s sequentially exogenous.

Specifically, h_{it} influenced by past outcome

$$h_{it} = \mathbf{z}_{it}\boldsymbol{\xi} + \eta y_{it-1} + \psi c_i + r_{it}.$$

Examples:

More studying (h), higher grades (y)

FDA opioid approval (h), overdose deaths (y)

- HIV infections (h) and condom usage (y).
- R&D expenditures (h) and patents awarded (y).
- Fertility (h) and female labor supply (y).

Strict Exogeneity?

Q: Is $\{h_{it}\}_{t=1}^{T}$ strictly exogenous?

Consider h_{it+1} which is in $\mathbf{x}_{it+1} := (\mathbf{z}_{it+1}, h_{it+1})$.

$$
\begin{aligned}
E\left(h_{it+1}u_{it}\right) &= E\left[\left(\mathbf{z}_{it+1}\boldsymbol{\xi} + \eta y_{it} + \psi c_i + r_{it+1}\right)u_{it}\right] \\
&= E\left(u_{it}\mathbf{z}_{it+1}\right)\boldsymbol{\xi} + \eta E\left(y_{it}u_{it}\right) + \psi E\left(c_i u_{it}\right) + E\left(r_{it+1}u_{it}\right) \\
&= \underline{\eta E\left(y_{it}u_{it}\right)} + E\left(r_{it+1}u_{it}\right). \qquad \text{(strict exogeneity)}
\end{aligned}
$$

Even if $E\left(r_{it+1}u_{it}\right) = 0$, requires $E\left(y_{it}u_{it}\right) = 0$, in general.

$= 0$

strict exog of z_{it}s (LIE)

Strict Exogeneity?

But

$$
\begin{aligned}
E\left(y_{it}u_{it}\right) &= E\left[\overbrace{\left(\mathbf{z}_{it}\boldsymbol{\beta} + \delta h_{it} + c_i + u_{it}\right)}^{=y_{it}} u_{it}\right] \\
&= E\left(u_{it}\mathbf{z}_{it}\right)\boldsymbol{\beta} + \delta E\left(h_{it}u_{it}\right) + E\left(c_i u_{it}\right) + E\left(u_{it}^2\right) \\
&= E\left(u_{it}^2\right) && \text{(strict/sequential exogeneity)} \\
&> 0. && \text{(in general)}
\end{aligned}
$$

=

So cannot expect $E\left(h_{it+1}u_{it}\right) = 0$.

Conclude: Feedback effects rule out strict exogeneity.

FE and FD with Sequential Exogeneity

FE with Sequential Exogeneity

Probability Limit of FE

Under Sequential Exogeneity

$T \times K$ time-demeaned/within transformed regressors

May show

$$\widehat{\beta}_{FE} - \beta = \left(\frac{1}{N} \sum_{i=1}^{N} \ddot{\mathbf{X}}_i' \ddot{\mathbf{X}}_i \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^{N} \ddot{\mathbf{X}}_i' \mathbf{u}_i \right)$$

$T \times 1$

$$\xrightarrow{p} [E(\ddot{\mathbf{X}}_i' \ddot{\mathbf{X}}_i)]^{-1} E(\ddot{\mathbf{X}}_i' \mathbf{u}_i)$$

using FE.2 + LLN + product rule for plims.

FE rank condition

$$E(\ddot{\mathbf{X}}_i' \mathbf{u}_i) = \sum_{t=1}^{T} E\left(\ddot{\mathbf{x}}_{it}' u_{it}\right).$$

For consistency, suffices that $E\left(\ddot{\mathbf{x}}_{it}' u_{it}\right) = \mathbf{0}$, all t.

FE Inconsistency under Sequential Exogeneity

Now $\ddot{x} = x_{it} - \bar{x}_i$, $\bar{x}_i = \frac{1}{T}\sum_{s=1}^{T} x_{is}$

$$E\left(\ddot{\mathbf{x}}_{it}' u_{it}\right) = E\left(\mathbf{x}_{it}' u_{it}\right) - E\left(\bar{\mathbf{x}}_i' u_{it}\right).$$

$$= -E\left(\bar{\mathbf{x}}_i' u_{it}\right). \qquad \text{(contemporaneous exogeneity)}$$

However, $\bar{\mathbf{x}}_i$ averages over *all* time periods,

$$E\left(\bar{\mathbf{x}}_i' u_{it}\right) = \frac{1}{T}\sum_{s=1}^{T} E\left(\mathbf{x}_{is}' u_{it}\right)$$

$E(u_{it} | x_{is}) = 0$ provided $s \leq t$

$$= \frac{1}{T}\sum_{s=t+1}^{T} \underbrace{E\left(\mathbf{x}_{is}' u_{it}\right)}_{\overset{?}{=} \underline{\underline{0}}} \qquad \text{(sequential exogeneity)}$$

$$\neq \mathbf{0}. \qquad \text{(in general)}$$

Conclude: Under sequential exogeneity, FE inconsistent.

FE Inconsistency under Sequential Exogeneity

$E(\ddot{X}_i'\ddot{X}_i)$ $\quad\quad$ $= E(\ddot{X}_i' u_i)$

$$\underset{N\to\infty}{\text{plim}}\left(\widehat{\beta}_{FE}\right) = \beta + \left[\frac{1}{T}\sum_{t=1}^{T} E\left(\ddot{\mathbf{x}}_{it}'\ddot{\mathbf{x}}_{it}\right)\right]^{-1}\left[-\frac{1}{T}\sum_{t=1}^{T} E\left(\overline{\mathbf{x}}_i' u_{it}\right)\right].$$

- ***IF*** we assume process $\{(\mathbf{x}_{it}, u_{it})\}_{t=1}^{\infty}$ weakly dependent...

$$\frac{1}{T}\sum_{t=1}^{T} E\left(\overline{\mathbf{x}}_i' u_{it}\right) = E\left(\overline{\mathbf{x}}_i'\overline{u}_i\right) = O\left(T^{-1}\right) \text{ as } T\to\infty.$$

- Inconsistency of FE of order $O\left(T^{-1}\right)$ as $T\to\infty$.
- Weak dependence $\approx$ dependence vanishing with time gap.
- But we work with T small, so FE inconsistent.

FD with Sequential Exogeneity

Probability Limit of FD

Under Sequential Exogeneity

(T-1) x K matrix of Δx_{it} s

(T-1) x 1 vector of Δu_{it} s

May show

$$\widehat{\boldsymbol{\beta}}_{FD} - \boldsymbol{\beta} = \left(\frac{1}{N} \sum_{i=1}^{N} \Delta \mathbf{X}_i' \Delta \mathbf{X}_i \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^{N} \Delta \mathbf{X}_i' \Delta \mathbf{u}_i \right)$$

$$\xrightarrow{p} \left[E \left(\Delta \mathbf{X}_i' \Delta \mathbf{X}_i \right) \right]^{-1} E \left(\Delta \mathbf{X}_i' \Delta \mathbf{u}_i \right)$$

using FD.2 + LLN + product rule for plims.

FD rank condition

$$E \left(\Delta \mathbf{X}_i' \Delta \mathbf{u}_i \right) = \sum_{t=2}^{T} E \left(\Delta \mathbf{x}_{it}' \Delta u_{it} \right).$$

Again, for consistency, suffices that $E \left(\Delta \mathbf{x}_{it}' \Delta u_{it} \right) = \mathbf{0}$, all t.

FD Inconsistency under Sequential Exogeneity

Now

$$
\begin{aligned}
E\left(\overset{x_{it}-x_{it-1}}{\Delta \mathbf{x}'_{it}}\,\overset{u_{it}-u_{it-1}}{\Delta u_{it}}\right) &= \overset{=0}{E\left(\mathbf{x}'_{it}u_{it}\right)} - \overset{=0}{E\left(\mathbf{x}'_{it-1}u_{it}\right)} \\
&\quad - E\left(\mathbf{x}'_{it}u_{it-1}\right) + \overset{=0}{E\left(\mathbf{x}'_{it-1}u_{it-1}\right)} \\
&= -E\left(\mathbf{x}'_{it}u_{it-1}\right) && \text{(sequential exogeneity)} \\
&\neq \mathbf{0}. && \text{(in general)}
\end{aligned}
$$

under

Conclude: Under sequential exogeneity, FD inconsistent.

FD Inconsistency under Sequential Exogeneity

$= E(\Delta X_i' \Delta X_i$

$$\operatorname*{plim}_{N\to\infty}(\widehat{\boldsymbol{\beta}}_{FD}) = \boldsymbol{\beta} + \left[\frac{1}{T-1}\sum_{t=2}^{T} E\left(\Delta \mathbf{x}_{it}'\Delta \mathbf{x}_{it}\right)\right]^{-1}$$

$\Delta X_i' \Delta u_i$

$$\times \left[\frac{1}{T-1}\sum_{t=2}^{T} E\left(\Delta \mathbf{x}_{it}'\Delta u_{it}\right)\right].$$

- Latter $[\cdot]$ need not vanish as $T \to \infty$ even with weak dependence...
- As opposed to FE, FD inconsistency not alleviated by long panel.

Empirical Strategy

Orthogonality Conditions

With sequential exogeneity, can't rely on $E\left(\Delta \mathbf{x}'_{it} \Delta u_{it}\right) = 0$.

Instead we will...

- Search for other orthogonality conditions suggesting instrumental variables (IVs).
- Use IVs for estimation à la two-stage least squares (2SLS)...
- ... or possibly generalized method of moments (GMM).

Advantage of FD: Only creates correlation within one lag,

$$E\left(\Delta \mathbf{x}'_{it} \Delta u_{it}\right) = -E\left(\mathbf{x}'_{it} u_{it-1}\right). \qquad \text{(sequential exogeneity)}$$

- FE more problematic since $\ddot{\mathbf{x}}_{it}$ involves all periods.

Orthogonality Conditions

Q: Valid instruments for $\Delta \mathbf{x}_{it}$?

Sequential exogeneity $\Rightarrow \{\mathbf{x}_{is}\}_{s=1}^{t-1}$ orthogonal to Δu_{it},

$$\begin{aligned} E\left(\mathbf{x}_{is}' \Delta u_{it}\right) &= E\left(\mathbf{x}_{is}' u_{it}\right) - E\left(\mathbf{x}_{is}' u_{it-1}\right) \\ &= 0, \quad s = 1, 2, \ldots, t-1. \end{aligned}$$

At t, available instruments $\boxed{\mathbf{x}_{it-1}^{o}}$ where

$$\mathbf{x}_{it}^{o} := \left(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{it}\right). \qquad (1 \times tK)$$

- Any function of $\mathbf{x}_{it-1}^{o}$ valid instrument too.
- Potential issue: $\Delta \mathbf{x}_{it}$ may have little correlation with $\mathbf{x}_{it-1}^{o}$.
  - A problem of weak instruments.

Orthogonality Conditions in AR(1) Model

Model:

$$y_{it} = \rho y_{it-1} + c_i + u_{it},$$
$$E(u_{it} | y_{it-1}, y_{it-2}, \ldots, y_{i0}, c_i) = 0, \quad t = 1, 2, \ldots, T.$$

In first differences:

$$\Delta y_{it} = \rho \Delta y_{it-1} + \Delta u_{it}, \quad t = 2, 3, \ldots, T.$$

May show (using sequential exogeneity)

$$E\left(\Delta y_{it-1} \Delta u_{it}\right) = -E\left(u_{it-1}^2\right) < 0.$$

$\Rightarrow \Delta y_{it-1}$ endogenous $\Rightarrow$ need instrument.

Instruments in AR(1) Model

Anderson and Hsiao (1982)

- Pooled IV estimation of FD equation with single instrument y_{it-2} (or Δy_{it-2}).

Arellano and Bond (1991)

- Full GMM estimation using all of the available instruments at time t.

  // x^o_{it-1}

- Here: $\mathbf{y}^o_{it-2} = (y_{i0}, y_{i1}, \ldots, y_{it-2})$ is available as IVs for Δy_{it-1}.
- Implies $\Delta\mathbf{y}^o_{it-2} = (\Delta y_{i1}, \ldots, \Delta y_{it-2})$ valid IVs.
- Next: Instrumentation and estimation of FD'd system.