Quantile Regression

Advanced Microeconometrics

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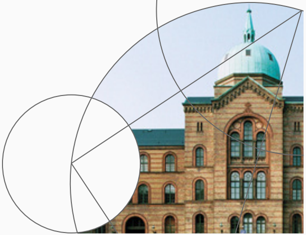
2022



Plan for lectures: Helicopter

Part I: Linear methods. ✓

Part II: High-dimensional methods. ✓

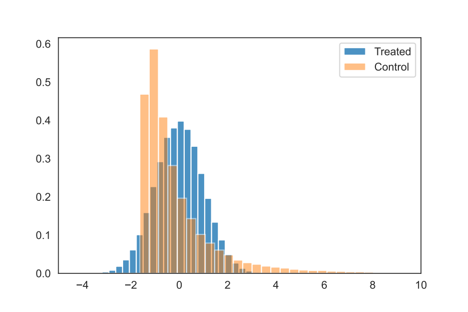
Part III: M-estimation, theory ✓

Part IV: M-estimation, examples $\leftarrow$

Where are we in the course?

Part	Topic	Parameterization non-linear	Estimation non-linear	Dimension dim(x)	Numerical optimization	M-estimation (Part III)	Outcome (y_i)	Panel (c_i)
I	OLS	$\div$	$\div$	low	$\div$	✓	$\mathbb{R}$	✓
II	LASSO	$\div$	✓	high	✓	$\div$	$\mathbb{R}$	$\div$
IV	Probit	✓	✓	low	✓	✓	$\{0, 1\}$	$\div$
	Logit	✓	✓	low	✓	✓	$\{1, 2, ..., J\}$	$\div$
	Tobit	✓	✓	low	✓	✓	$[0; \infty)$	$\div$
	Simulated Likelihood	✓	✓	low	✓	✓	Any	✓
	Quantile Regression	$\div$	✓	(low)	✓	✓	$\mathbb{R}$	$\div$
	~~Non-parametric~~	~~✓~~	~~(✓)~~	~~∞~~	~~$\div$~~	~~$\div$~~	~~$\mathbb{R}$~~	~~$\div$~~

Discussion time!



In the example in the graph, the *average* treatment effect is zero.

Question

What else can we say about the impact of treatment based on this graph?

Topic for today

- **Most** empirical work focuses on the conditional **mean**...
- **Today:** explore conditional **quantiles**.

Outline

Earnings Example

- **Bitler, Gelbach, & Hoynes (2006; AER):** The effect on earnings of Connecticut's Job First waiver program.
- **Finding:** Behind the mean response lies a large response by a small group.

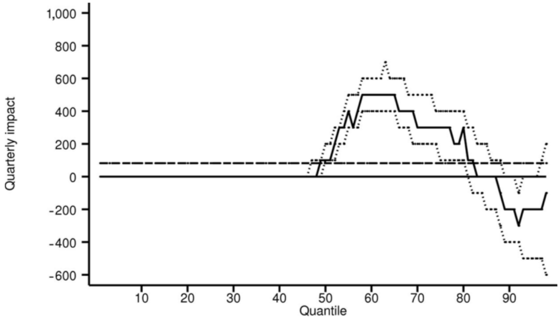
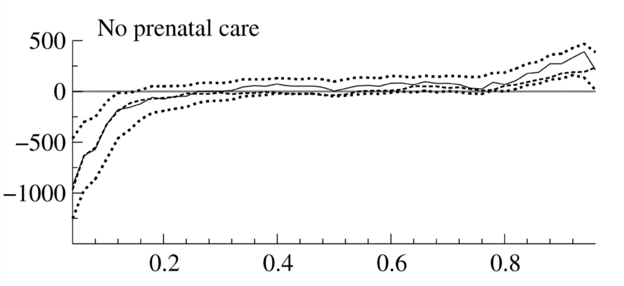


FIGURE 3. QUANTILE TREATMENT EFFECTS ON THE DISTRIBUTION OF EARNINGS, QUARTERS 1–7

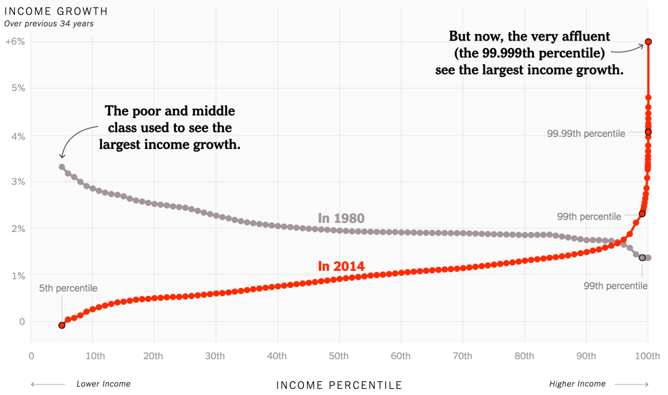
Notes: Solid line is QTE; dotted lines provide bootstrapped 90-percent confidence intervals; dashed line is mean impact; all statistics computed using inverse propensity-score weighting. See text for more details.

Birth weight example

- **Abrevaya & Dahl (2008):** Birth weight of newborns.
- **Example:** No prenatal visits makes a big difference at bottom quantiles (the "at risk" babies).



Income Inequality



Note: Inflation-adjusted annual average growth using income after taxes, transfers and non-cash benefits.

Source: David Leonhardt, NYTimes, `nytimes.com`

Outline

Introduction

- **Normally:** we are interested in the *conditional mean*,

$$\mathbb{E}(y|\mathbf{x}) = \mathbf{x}\boldsymbol{\beta}_m$$

- **Instead:** we might consider

$$\mathrm{Median}(y|\mathbf{x}) = \mathbf{x}\boldsymbol{\beta}_{0.5}$$

- **More generally:** conditional τ-percentiles, $\tau \in (0;1)$

$$\mathrm{Percentile}_\tau(y|\mathbf{x}) = \mathbf{x}\boldsymbol{\beta}_\tau$$

Quantiles

Definition: quantile

The τ'th quantile ($\tau \in (0;1)$) of y is μ_τ such that

$$\tau = \Pr(y \leq \mu_\tau) \equiv F_y(\mu_\tau).$$

Hence

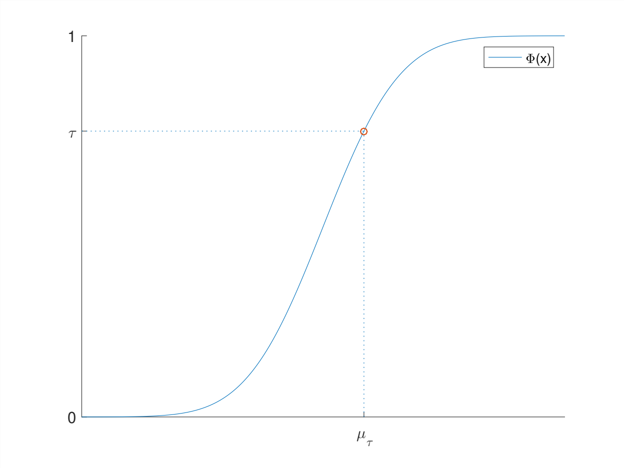
$$\mu_\tau = F_y^{-1}(\tau).$$

Definition: percentile

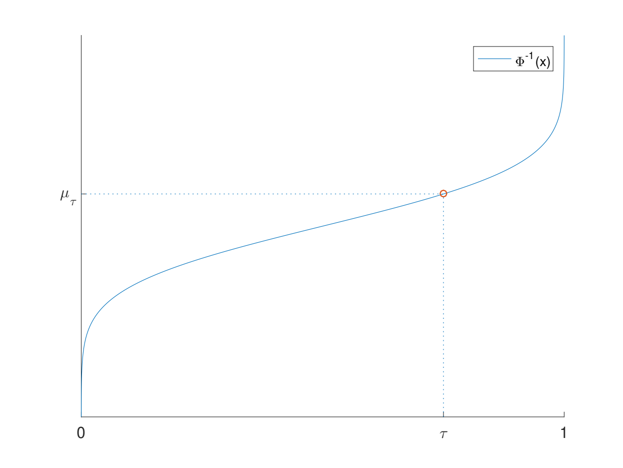
The τ'th quantile is the 100τ'th percentile.

E.g., the 0.23 quantile and the 23rd percentile are the same.

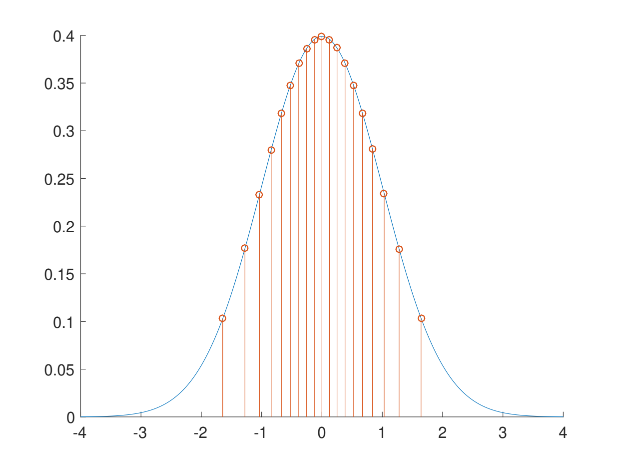
Graphically, $\tau \equiv F_y(\mu_\tau)$: The CDF



Graphically, $\mu_\tau = F^{-1}(\tau)$: The Quantile Function



Quantiles {.05, .1, .15, ..., .95}



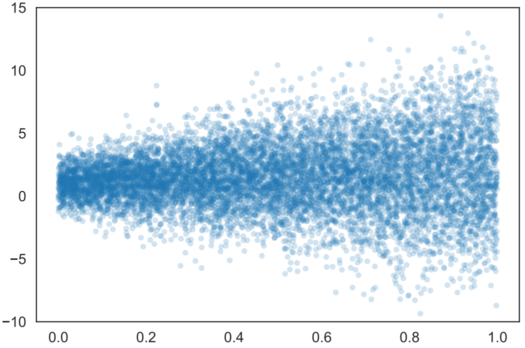
Conditional Quantiles

Definition: conditional quantile

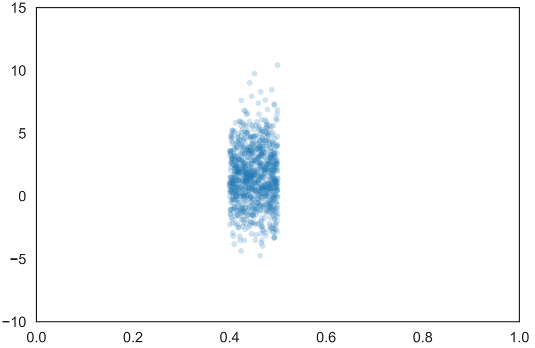
The conditional τ quantile of y is

$$\mu_\tau(\mathbf{x}) = F^{-1}_{y|\mathbf{x}}(\tau|\mathbf{x}).$$

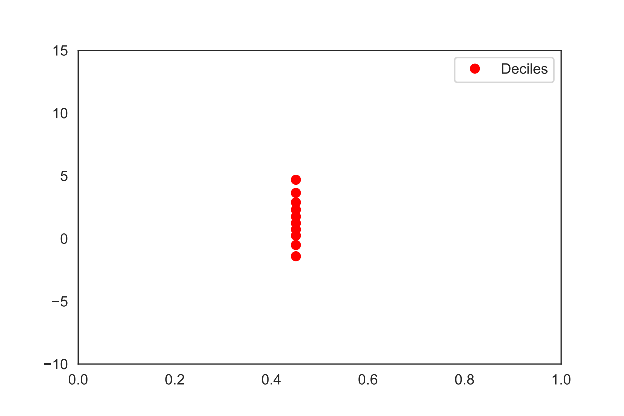
Conditional Quantiles: Dataset



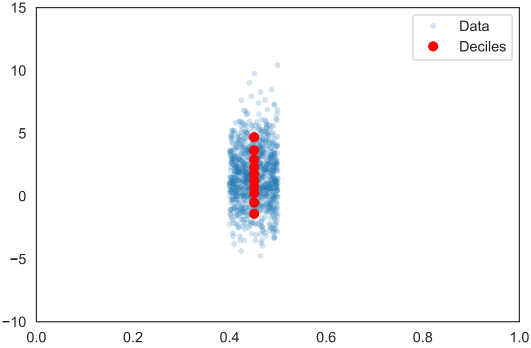
Conditional Quantiles: Focus on x_i



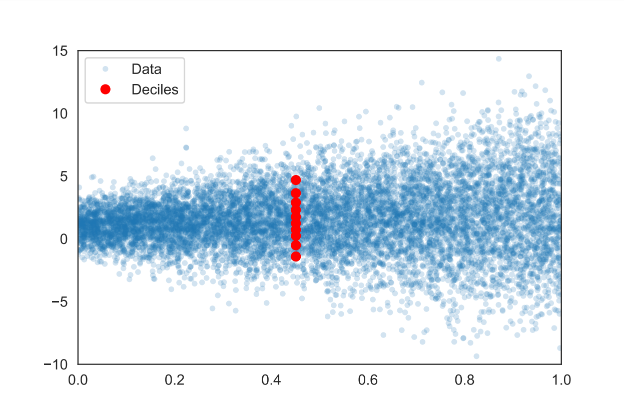
Conditional Quantiles, 0, 5, 10, ..., 100



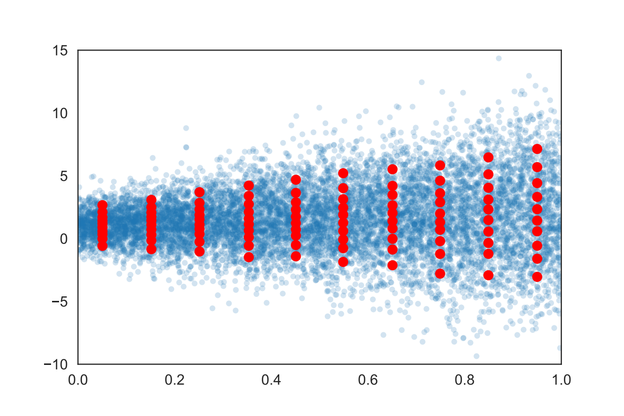
Conditional Quantiles over x



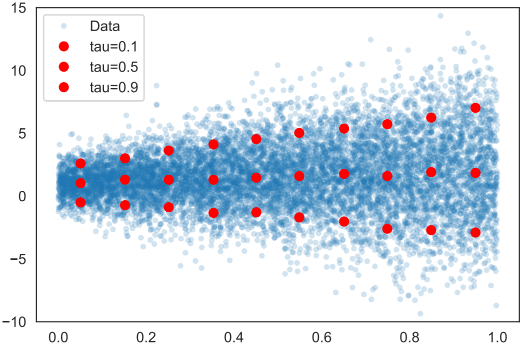
Conditional Quantiles over x



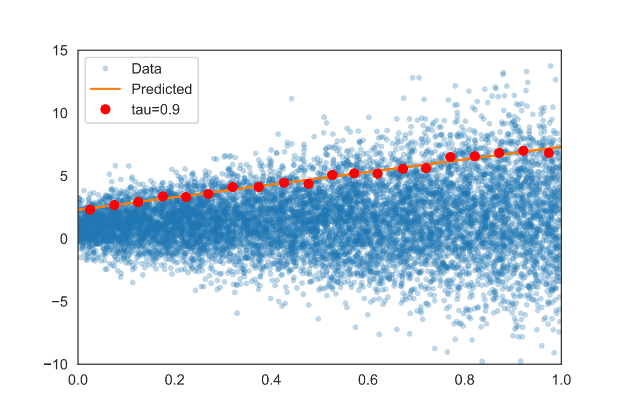
Conditional Quantiles over x



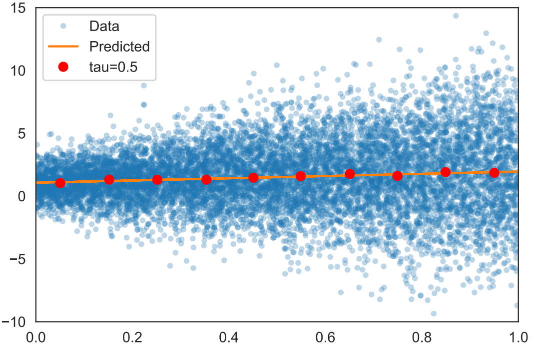
Conditional Quantiles over x



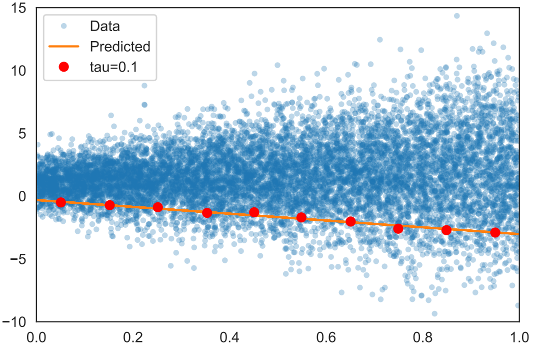
Quantile Regression, $\tau = 0.90$



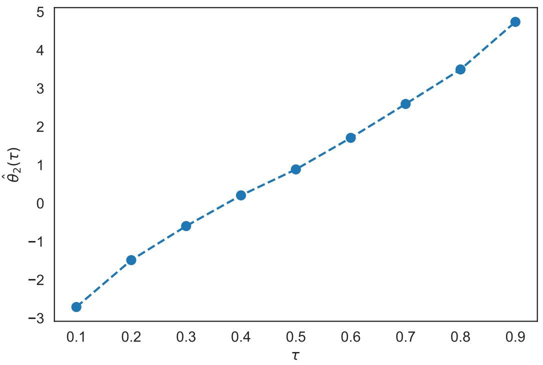
Quantile Regression, $\tau = 0.50$



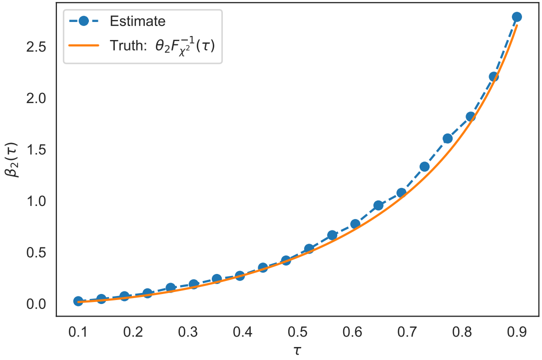
Quantile Regression, $\tau = 0.10$



Quantile Regression, $\tau = 0.10$



Quantile Regression, $\tau = 0.10$



Outline

Model

- Will cover **two specific models** as examples;
  1. Heteroscedasticity model,
     Pro: error-term form,
  2. Heterogenous parameter model.
     Pro: simpler to grasp what we estimate.
- **Finally:** Will cover the more general model.

Model

$$
\begin{aligned}
y &= \mathbf{x}\beta + u \\
u &= (\mathbf{x}\boldsymbol{\alpha})\,\varepsilon, \quad \varepsilon \sim F_\varepsilon,
\end{aligned}
$$

where we assume $\mathbf{x}\boldsymbol{\alpha} > 0$.

- **Goal:** Derive an expression for $\mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha})$.

$$
\begin{aligned}
\tau &= \Pr\left[y \leq \mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha})\right] \\
&= \Pr\left[\mathbf{x}\boldsymbol{\beta} + (\mathbf{x}\boldsymbol{\alpha})\,\varepsilon \leq \mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha})\right] \\
&= \Pr\left\{\varepsilon \leq \frac{1}{\mathbf{x}\boldsymbol{\alpha}}\left[\mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha}) - \mathbf{x}\beta\right]\right\} \\
&\equiv F_\varepsilon\left\{\frac{1}{\mathbf{x}\boldsymbol{\alpha}}\left[\mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha}) - \mathbf{x}\beta\right]\right\}.
\end{aligned}
$$

- **Qs:** Where did we need $\mathbf{x}\boldsymbol{\alpha} > 0$?

- We have

$$\tau = F_\varepsilon \left\{ \frac{1}{\mathbf{x}\boldsymbol{\alpha}} \left[\mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha}) - \mathbf{x}\boldsymbol{\beta} \right] \right\}.$$

- It follows,

$$\begin{aligned} F_\varepsilon^{-1}(\tau) &= \frac{1}{\mathbf{x}\boldsymbol{\alpha}} \left[\mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha}) - \mathbf{x}\boldsymbol{\beta} \right] \\ \Leftrightarrow \mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha}) &= \mathbf{x}\boldsymbol{\beta} + (\mathbf{x}\boldsymbol{\alpha}) F_\varepsilon^{-1}(\tau). \end{aligned}$$

$$\mu_\tau(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\alpha}) = \mathbf{x}\boldsymbol{\beta} + (\mathbf{x}\boldsymbol{\alpha})F_\varepsilon^{-1}(\tau).$$

Suppose $\mathbf{x}_i\alpha = 1$ for all i.

Discuss

What is the difference between two quantile regression lines in this case?

Heterogenous parameter model

Model

$$y_i \quad = \quad \mathbf{x}_i\beta(\tau_i), \quad \tau_i \sim \mathrm{Uniform}(0,1),$$

where $\beta(\cdot)$ maps $[0;1]$ into $\mathbb{R}^K$ subject to the *no quantile crossing* restriction [next slide].

- τ_i is interpreted as the conditional **quantile draw** of i.
- **Alternatively,** we can think of the model as

$$y_i = \mathbf{x}_i\boldsymbol{\beta} + e_i, \quad e_i \equiv \mathbf{x}_i\beta(\tau_i) - \mathbf{x}_i\boldsymbol{\beta}.$$

  - we then require $\Pr(e_i \leq \tau_i|\mathbf{x}_i) = 0$.

No quantile crossing

The mapping, $\tau_i \mapsto \mathbf{x}_i\beta(\tau_i)$ must be monotonic for all $\tau_i \in (0;1)$ where $\mathbf{x}_i$ has support (almost surely).

Discuss

Why do we need the no quantile crossing property?

[hint, consider the "constant only," example where $\mathbf{x}_i = 1$, so $\mathbf{x}_i\beta(\tau_i) = \beta_0(\tau_i)$]

Heteroscedasticity DGP in Python

```
def sim_data(theta, N, alpha):
    K = theta.size
    assert alpha.size == K, f'alpha must be same size as beta'

    # 1. regressors
    oo = np.ones((N,1))
    xx = np.random.uniform(size=(N,K-1)) # uniform => ensures posit
    x = np.hstack([oo, xx])

    # 2. error term
    epsilon = np.random.normal(size=(N,))
    sigma_i = x@alpha # individual variance (function of x)
    u = sigma_i * epsilon # heteroskedastic error

    # 3. outcome
    y = x@theta + u
    return y,x
```

- **Note**: we estimate $\hat{\beta}(\tau)$, but the DGP requires θ_o (K-vector) and α_o (K-vector)

```
def sim_data(theta, N, icdf):
    K = theta.size
    assert K == 2 # for simplicity
    assert callable(icdf)

    oo = np.ones((N,1))
    xx = np.random.uniform(size=(N,1)) + theta[0]
    x = np.hstack([oo, xx]) # (N,K)

    tau = np.random.uniform(size=(N,1))
    beta2 = icdf(tau) * theta[1]
    beta1 = np.ones((N,1)) # first coefficient constant across quar
    beta = np.hstack([beta1, beta2]) # (N,K)

    y = np.sum(x * beta, axis=1)
    return y,x
```

- **Note**: we estimate $\hat{\beta}(\tau)$, but the DGP requires θ_o (K-vector) and $F^{-1}(\cdot)$ (function)

Outline

Overview

- **Recall:** We minimize $\mathbb{E}[(y-\mu)^2]$ by setting $\mu = \mathbb{E}(y)$.
- **Similarly:** We minimize $\mathbb{E}|y-\mu|$ by setting $\mu = \mathrm{Median}(y)$.
- **Regression:** We minimize $\mathbb{E}[(y-\mu(x))^2|x]$ by setting $\mu(x) = \mathbb{E}(y|x)$.
  - Typically, we then further assume $\mathbb{E}(y|x) = \mathbf{x}\beta$.
- **Median regression:** minimize $\mathbb{E}\left(|y-\mu(x)|x\right)$ by setting $\mu(x) = \mathrm{Median}(y|x)$.
  - Similarly, we might assume $\mathrm{Median}(y|x) = \mathbf{x}\beta$.

Criterion function: rewriting the C&T form

- **In C&T:** They write

$$Q(\boldsymbol{\beta}_\tau) = \sum_{i:y_i \geq \mathbf{x}_i\boldsymbol{\beta}_\tau} \tau\,|y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau| + \sum_{i:y_i < \mathbf{x}_i\boldsymbol{\beta}_\tau} (1-\tau)|y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau|.$$

- **Equivalently**, we can write this as

$$Q(\boldsymbol{\beta}_\tau) = \textstyle\sum_{i=1}^N \left[\mathbf{1}_{\{y_i \geq \mathbf{x}_i\boldsymbol{\beta}_\tau\}}\tau|y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau| + \mathbf{1}_{\{y_i < \mathbf{x}_i\boldsymbol{\beta}_\tau\}}(1-\tau)|y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau|\right]$$

- **Getting** rid of the "$|\cdot|$",

$$Q(\boldsymbol{\beta}_\tau) = \textstyle\sum_{i=1}^N \left[\mathbf{1}_{\{y_i \geq \mathbf{x}_i\boldsymbol{\beta}_\tau\}}\tau(y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau) + \mathbf{1}_{\{y_i < \mathbf{x}_i\boldsymbol{\beta}_\tau\}}(\tau-1)(y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau)\right]$$

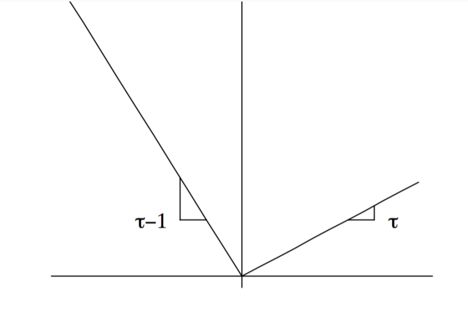
- **In my slides** and everywhere else,

$$\begin{aligned} Q(\boldsymbol{\beta}_\tau) &= \textstyle\sum_{i=1}^N \left(\tau - \mathbf{1}_{\{y_i < \mathbf{x}_i\boldsymbol{\beta}_\tau\}}\right)(y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau) \\ &\equiv \textstyle\sum_{i=1}^N \rho_\tau(y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau). \end{aligned}$$

More generally

Definition: Check function

$$\rho_\tau(u) \equiv \left(\tau - \mathbf{1}_{\{u<0\}}\right) u.$$



From Koenker (2000).

Criterion function

Criterion function

$$q(y_i, \mathbf{x}_i, \boldsymbol{\beta}_\tau) = \rho_\tau(y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau),$$

where ρ_τ is the *check function*.

- **Claim:** $\boldsymbol{\beta}_\tau = \arg\min_\beta \mathbb{E}[q(y_i, \mathbf{x}_i, \beta)|\mathbf{x}_i]$.
- ... we will show this for the intercept-only model.
  - [not the full proof]

Minimizer

- **To show:** $\beta_{0\tau} = \arg\min_{\beta_0} \mathbb{E}[\rho_\tau(y - \beta_0)|x]$.
- **Idea:** Split up $(-\infty; \infty)$ into $(-\infty; \beta_0] \cup [\beta_0; \infty)$ to get rid of $\mathbf{1}\{\cdot\}$.
- Re-write

$$
\begin{aligned}
\mathbb{E}[\rho_\tau(y - \beta_0)|x] \quad &\equiv \quad \int_{-\infty}^{\infty} [\tau - \mathbf{1}\{y - \beta_0 < 0\}]\,(y - \beta_0)\,\mathrm{d}F(y) \\
&= \quad \int_{-\infty}^{\beta_0} [\tau - \mathbf{1}\{y < \beta_0\}]\,(y - \beta_0)\,\mathrm{d}F(y) \\
&\quad\quad + \int_{\beta_0}^{\infty} [\tau - \mathbf{1}\{y < \beta_0\}]\,(y - \beta_0)\,\mathrm{d}F(y) \\
&= \quad (\tau - 1) \int_{-\infty}^{\beta_0} (y - \beta_0)\,\mathrm{d}F(y) \\
&\quad\quad + \tau \int_{\beta_0}^{\infty} (y - \beta_0)\,\mathrm{d}F(y).
\end{aligned}
$$

Step 2: FOC

- **We have**

$$\mathbb{E}[\rho_\tau(y-\beta_0)|x] = (\tau-1)\int_{-\infty}^{\beta_0}(y-\beta_0)\,\mathrm{d}F(y) + \tau\int_{\beta_0}^{\infty}(y-\beta_0)\,\mathrm{d}F(y).$$

- **FOCs** imply

$$\begin{aligned}\frac{\partial\mathbb{E}[\rho_\tau(y-\beta_0)|x]}{\partial\beta_0} &= 0 \\ &[...] \\ \Leftrightarrow F(\beta_0) &= \tau.\end{aligned}$$

- **Intuition:** Choose β_0 so that τ of the mass is to the left.
  - [cdf(x) $\equiv$ % of mass to the left of x]

- **Intuitively:** similar to OLS.
- **Further** details get much more complicated.

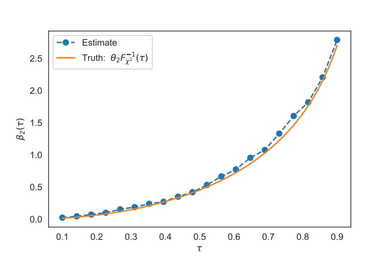
Question

Why can I not identify a model with two intercepts?

Model

$$y_i = \mathbf{x}_i \beta(\tau_i), \quad \tau_i \sim \mathrm{Uniform}(0,1),$$

where $\beta_0(\tau) = 1 \forall \tau$ and $\beta_1(\tau) = F^{-1}(\tau)$, and F is the χ^2 cdf.



Discuss

What are the parameter estimates? A vector? A function?

Outline

Features of Interest

- **If** we are interested in how x_{ik} affects conditional quantiles...
  - ... it is linear!
- **That is,**

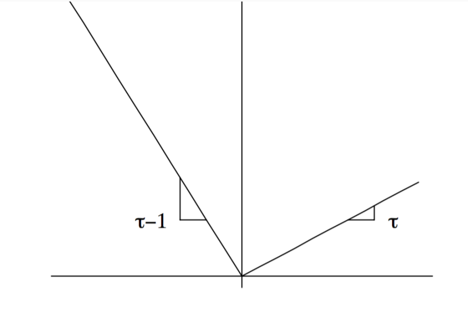
$$\frac{\partial \mu_\tau(\mathbf{x}_i, \boldsymbol{\beta}_\tau)}{\partial x_{ik}} = \beta_{\tau k}.$$

- **Often** we will plot all the estimates together, comparing with OLS.
  - [see examples earlier]

Outline

Definition: Check function

$$\rho_\tau(u) \equiv [\tau - \mathbf{1}\{u < 0\}]\, u.$$



From Koenker (2000).

Criterion function

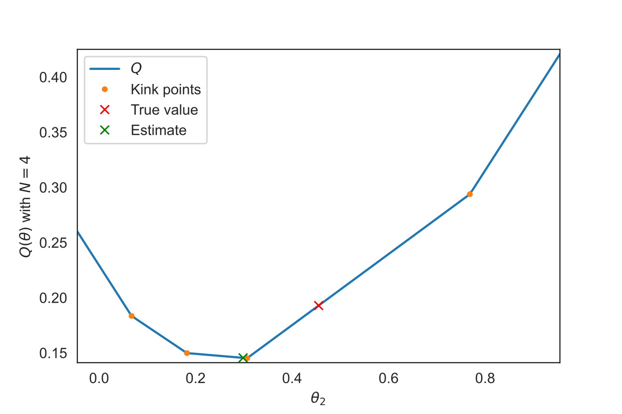
$$q(y_i, \mathbf{x}_i, \boldsymbol{\beta}_\tau) = \rho_\tau(y_i - \mathbf{x}_i\boldsymbol{\beta}_\tau),$$

where $\rho_\tau(u) \equiv [\tau - \mathbf{1}\{u < 0\}]\, u$.

- Two problems;
  1. Non-differentiable at $y_i = \mathbf{x}_i\boldsymbol{\beta}_\tau$,
  2. Hessian is zero everywhere.
- **Implications:**
  1. Optimization may be hard.
  2. Theorem 12.3 doesn't guarantee asymptotic normality,

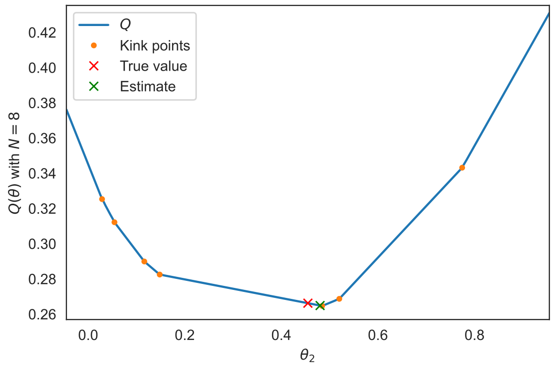
Criterion for various N

Figure 1: $N = 4$



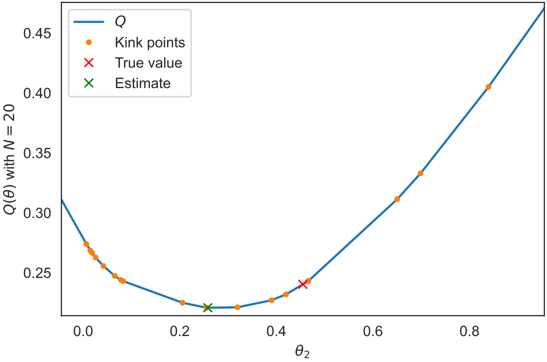
Criterion for various N

Figure 2: $N = 8$



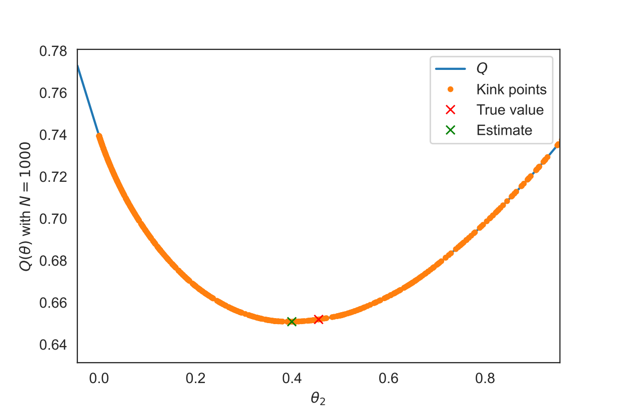
Criterion for various N

Figure 3: $N = 20$



Criterion for various N

Figure 4: $N = 1000$



Discuss

Discuss

Which problems do we face here:

1. The gradients may point the wrong way,
2. We cannot divide by the Hessian,
3. There are local optima.

Also: Does the primary problem get better or worse as $N \to \infty$? Why?

Standard errors

- **Note:** Criterion is not smooth,
  - Non-differentiable at $y_i = \mathbf{x}_i \beta_\tau$,
  - Hessian is zero everywhere else.
- **Fortunately:** Normality still comes about...

$$\sqrt{N}(\hat{\beta}_\tau - \beta_\tau^o) \xrightarrow{d} \mathcal{N}(0, \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}).$$

  - $\mathbf{A}_o = \operatorname{plim} \frac{1}{N} \sum_{i=1}^N f_{u_\tau|x}(0|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'$,
  - $\mathbf{B}_o = \operatorname{plim} \frac{1}{N} \sum_{i=1}^N \tau(1-\tau)\mathbf{x}_i\mathbf{x}_i'$,
  - $u_\tau \equiv y - \mathbf{x}\beta_\tau$, $f_{u_\tau|x}$ is the conditional pdf of u_τ given $\mathbf{x}_i$.

Variance estimation in practice

- **Bootstrap:** Use when in doubt.
- **Many variants** exist (non-parametric, parametric, block, ...).

Bootstrap

1. Draw a new dataset *with replacement* from the original dataset,
2. Estimate parameters, $\hat{\beta}_\tau^b$,
3. Repeat 1–2 for $b = 1, ..., B$.
4. The distribution of $\{\hat{\beta}_\tau^b\}_{b=1}^B$ is the approximation of the asymptotic distribution of $\hat{\beta}_\tau$.

Intuition

- **Original** dataset was an IID draw from the population.
- **Thus,** IID draws from this sub-population will match the original distribution closely.
- **Panel data:** Do the bootstrap sampling over i (not i, t);
  - i.e. take entire T-paths for individual i when he is drawn.