Discrete Response: The Logit Model

Advanced Microeconometrics

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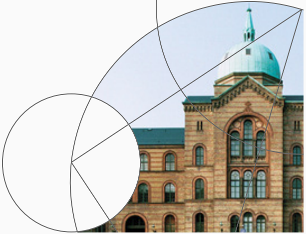


Plan for lectures: Helicopter

Part I: Linear methods. ✓

Part II: High-dimensional methods. ✓

Part III: M-estimation, theory ✓

Part IV: M-estimation, examples ←

Where are we in the course?

Part	Topic	Parameterization non-linear	Estimation non-linear	Dimension dim(x)	Numerical optimization	M-estimation (Part III)	Outcome (y_i)	Panel (c_i)
I	OLS	÷	÷	low	÷	✓	$\mathbb{R}$	✓
II	LASSO	÷	✓	high	✓	÷	$\mathbb{R}$	÷
IV	Probit	✓	✓	low	✓	✓	$\{0, 1\}$	÷
	Logit	✓	✓	low	✓	✓	$\{1, 2, \ldots, J\}$	÷
	Tobit	✓	✓	low	✓	✓	$[0; \infty)$	÷
	Sample selection	✓	✓	low	✓	✓	$\mathbb{R}$ and $\{0, 1\}$	÷
	Simulated Likelihood	✓	✓	low	✓	✓	Any	✓
	Quantile Regression	÷	✓	(low)	✓	✓	$\mathbb{R}$	÷
	Non-parametric	✓	(✓)	∞	÷	÷	$\mathbb{R}$	÷

Outline

1. Overview

2. Random Utility Model

3. Criterion Function

  3.1. Identification

4. Features of Interest

5. Specific Issues

  5.1. The IIA Restriction

  5.2. Market Share Data

Big Big Picture: Modeling Approaches

- **Simplest:** with $q_{jt} = \#$ sold cars and $\mathbf{x}_{jt}$ = car attributes (price, horsepower, km/l, etc.)

$$q_{jt} = \mathbf{x}_{jt}\beta + \text{error}_{jt}$$

  - **Con:** Competitor prices / selection does not affect sales. E.g. zero cross-price elasticity. NB! Of course it matters what my competitor is doing with their prices to demand of my car

- **Big OLS:** Include prices of *all* cars in explaining j's sales

$$q_{jt} = \mathbf{x}_{jt}\beta + \sum_{k \neq j} \mathbf{x}_{kt}\gamma_{jk} + \text{error}_{jt}.$$

'Almost ideal demand system' (AIDS)

  - **Cons:**
    - *many* parameters (e.g. J^2 cross-price coefficients!!), J-cars?
    - **Too flexible?** Might violate basic logic (Slutsky symmetry, adding up, etc.)

if you don't have much variation in x's, then it might be that you impose some basic logical restrictions on how demand works -> slutsky symmetry

- **Logit:**

$$\frac{q_{jt}}{\sum_{k=1}^{J} q_{kt}} = \frac{\exp(\mathbf{x}_{jt}\beta)}{\sum_{k=1}^{J} \exp(\mathbf{x}_{kt}\beta)}.$$

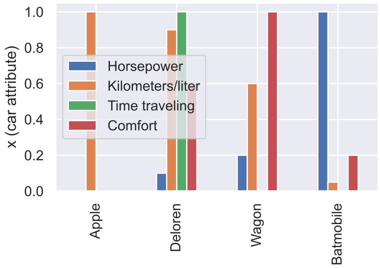
Essentially, it maps a market share of a j car

  - **Pros:** Few parameters ($\dim \beta = K$), cross-product "market stealing" (substitution)
  - **Cons:** Restricted substitution patterns (the "IIA property")

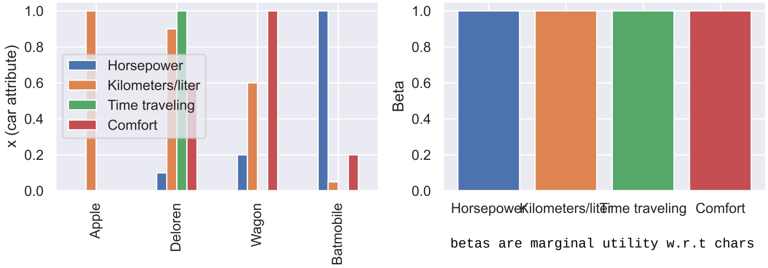
Logit Intuition

- **Choices:** four different cars.
- **Preferences:** $u^{\text{bat}} \succ u^{\text{Delorean}} \succ u^{\text{wagon}} \succ u^{\text{apple}}$.
- **Data:**
  - Characteristics of cars, x_j (e.g. price, horsepower, flux-capacitor dummy, ...)
  - Characteristics of decision maker, x_i (e.g. income, work distance).
  - The final decision (which e.g. says $u^{\text{batmobile}} \succ u^j$ for all other j)
- **Model:** A *utility function*, $u : (\mathbf{x}_{ij}, \theta) \mapsto \mathbb{R}$.
  - **Goal:** Choose θ st. $u(\mathbf{x}_{i\text{bat}}, \theta) > u(\mathbf{x}_{ij}, \theta)$ for $j \neq \text{bat}$.

Logit intuition

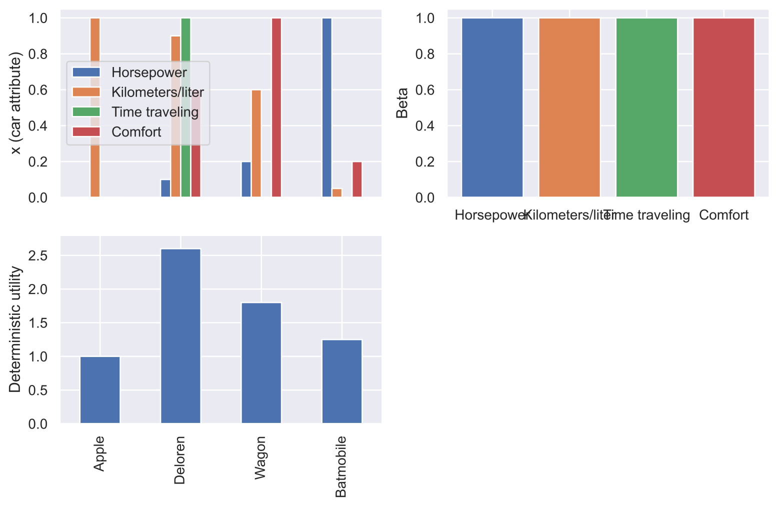


Logit intuition

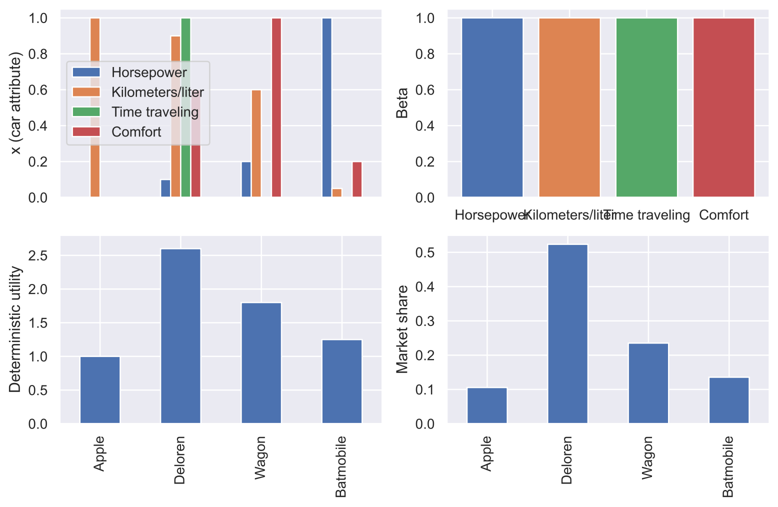


betas are marginal utility w.r.t chars

Logit intuition

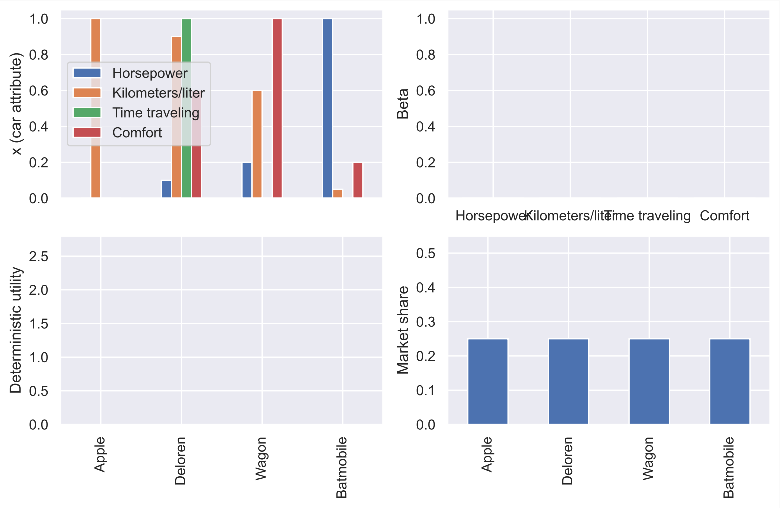


Logit intuition



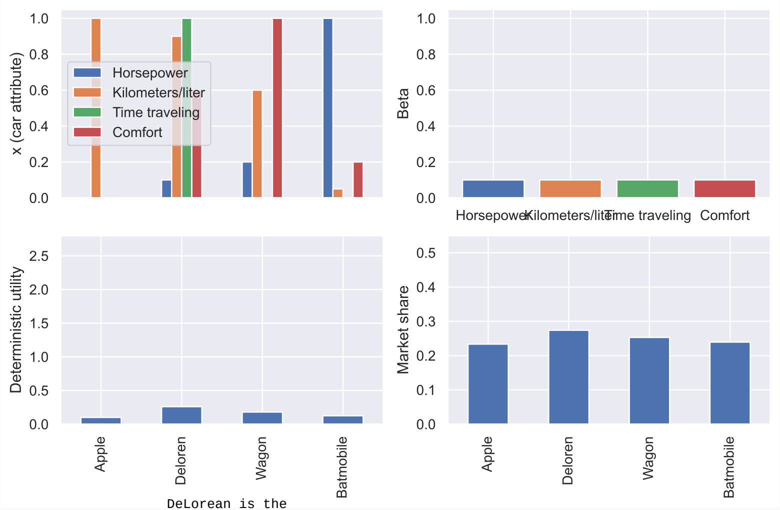
Logit intuition: $\beta = 0$

what if the decision makers do not care about any of the characteristics?



No utility from any characteristics ⇒ identical market shares.

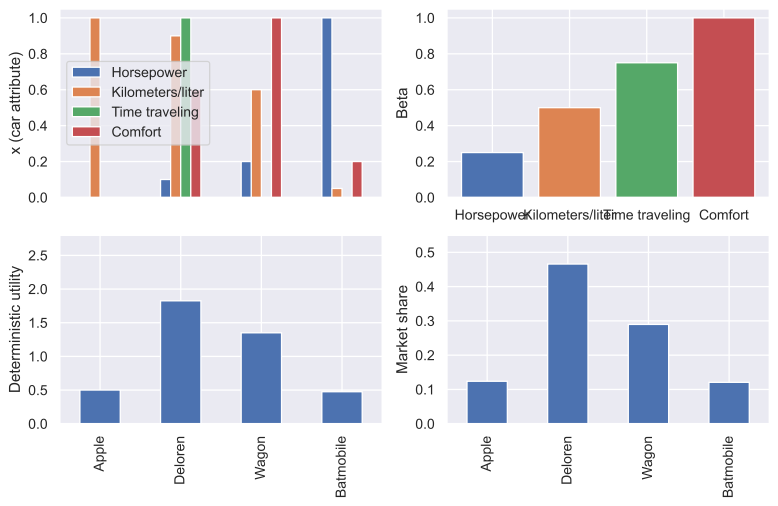
Logit intuition: β "small"




```
DeLorean is the
'nicest'
```

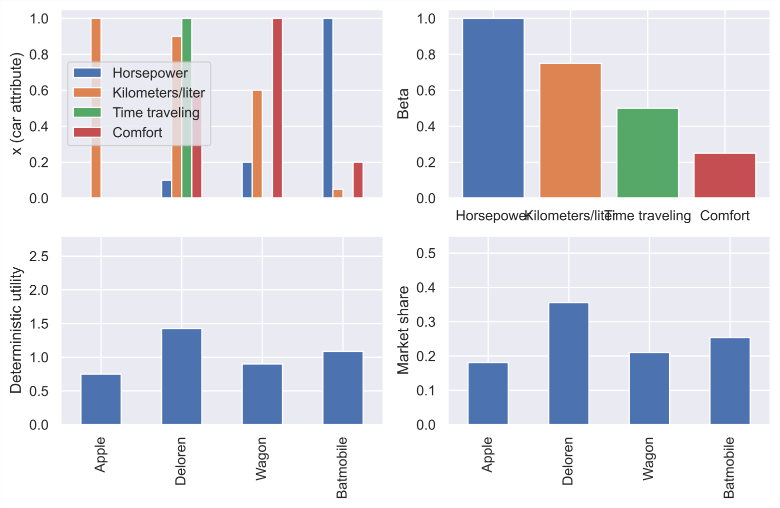
Here, $\beta = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$

Logit intuition



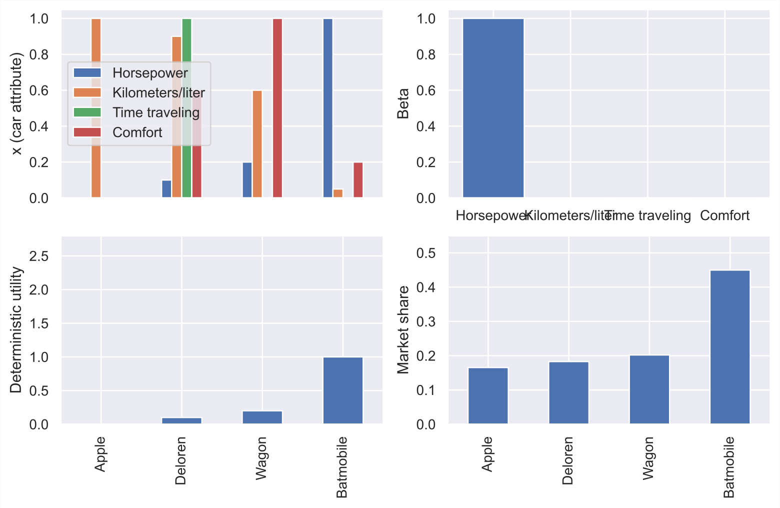
Consumers prefer comfort and time traveling capabilities

Logit intuition



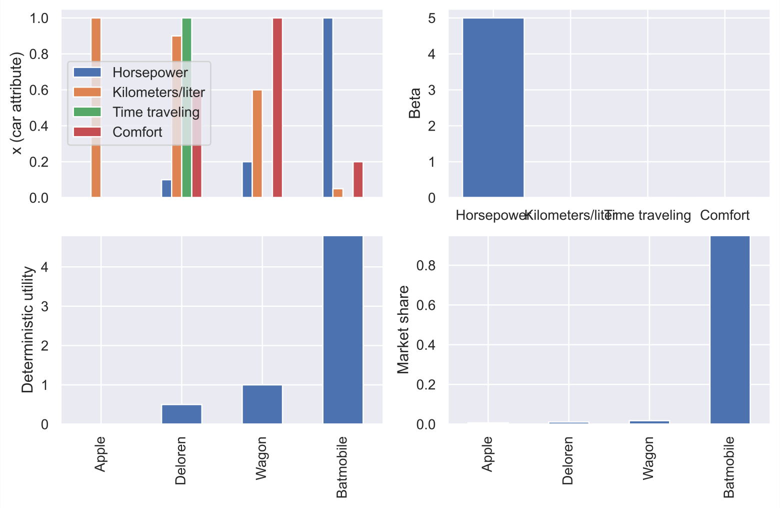
Consumers prefer horsepower and fuel economy.

Logit intuition: $\beta = (1, 0, 0, 0)$



Consumers only care about horsepower (and idiosyncraticies)

Logit intuition: $\beta = (5, 0, 0, 0)$



Consumers only care about horsepower, and by a lot!

Overall Intuition: Identification?

Model, 1st idea

For cars $j = 1, ..., J$,

$$u_{ij} = \beta_0 + \beta_1 \text{horsepower} + \beta_2 \text{income} + \text{error}, \quad \text{error} \sim \text{IID}(\mu, \sigma^2).$$

Discuss

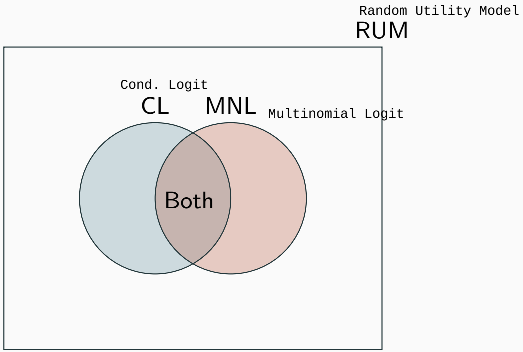
What can we hope to identify here?

No way we can identify \mu and \beta_0 -> we can't have two constants

Have to normalize by sigma^2

\beta_0 = doesn't affect choice behavior, 'baseline' utility -> utility you gain regardless of \beta_K's -> just pushes coef's up -> ordering will not change

Models



Random Utility Model
RUM
Cond. Logit
CL
MNL
Multinomial Logit
Both

Overview of Logit Models

Random Utility Model (RUM)

latent, unobservable
utility

$$\begin{aligned} u_{ij} &= v_{ij} + \varepsilon_{ij}, \\ y_i &= \underset{j \in \{1,\ldots,J\}}{\operatorname{argmax}}\, u_{ij}. \end{aligned}$$

i**:** Individual, $i = 1, ..., N$,

j**:** Alternative, $j = 1, ..., J$,

y_i**:** Chosen alternative by i,

u_{ij}**:** Utility from choosing alternative j,

ε_{ij}**:** Unobserved error term,

v_{ij}**:** Observed part (regressors and parameters).

- **Primary** two types of RUM:
  1. **Conditional logit:** $v_{ij} = \mathbf{x}_j\beta$, varies over cars J
  2. **Multinomial logit:** $v_{ij} = \mathbf{x}_i\beta_j$, varies over decision makers i

Overview of models

Random Utility Model

$$
\begin{aligned}
u_{ij} &= v_{ij} + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{IID Extreme Value Type I}, \\
y_i &= \operatorname{argmax}_{j \in \{1,\dots,J\}} u_{ij}.
\end{aligned}
$$

- **Conditional choice probabilities (CCPs):** to be shown...

$$
\Pr(d|v_{ij}) = \frac{\exp(v_{id})}{\sum_{j=1}^{J} \exp(v_{ij})}.
$$

  - due to Dan McFadden.

CL vs. MNL

Conditional logit (CL)

- $\mathbf{x}_j$: characteristics of the *alternatives*.
- **Example**: car choice, $\mathbf{x}_j = (\text{price, horsepower, ...})$.
- **Normalization**: No intercept in $\mathbf{x}$.
- β_k **interpretation**: marginal utility of characteristic k.

Multinomial logit (MNL)

- $\mathbf{x}_i$: characteristics of the *individual*.
- **Example**: car choice, $\mathbf{x}_i = (\text{income, work distance, ...})$.
- **Normalization**: $\beta_1 = \mathbf{0}_{K\times 1}$.
- β_{jk} **interpretation**: change in utility of car j (relative to car 1) when (e.g.) income increases (if $\mathbf{x}_{ik}$ is income).

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Random Utility Model

Random Utility Model

$$
\begin{aligned}
u_{ij} &= v(\mathbf{x}_{ij}, \theta) + \varepsilon_{ij}, \quad \text{IID Extreme Value Type I,} \\
y_i &= \operatorname{argmax}_{j \in \{1,\ldots,J\}} u_{ij}.
\end{aligned}
$$

$v(\cdot,\cdot)$**:** Known functional form,

$\mathbf{x}_{ij}$**:** Data (characteristics of individuals and/or alternatives),

- **Functional form:** CL and MNL are two *linear* forms of v
  - **CL:** $v(\mathbf{x}_{ij}, \beta) = \mathbf{x}_{ij}'\beta$
    - Note: no intercept in $\mathbf{x}_{ij}$ (intercept is normalized)
  - **MNL:** $v(\mathbf{x}_i, \beta_j) = \mathbf{x}_i'\beta_j$.
    - Note: $\theta = (\beta_2, \ldots, \beta_J)$ ($\beta_1 := 0$ is normalized)

Code for MNL

Multinomial Logit

$$
\begin{aligned}
u_{ij} &= \mathbf{x}_i \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{IID Extreme Value Type I}, \ \beta_1 = \mathbf{0}_{K\times 1}, \\
y_i &= \operatorname{argmax}_{j\in\{1,\dots,J\}} u_{ij}.
\end{aligned}
$$

```python
from scipy.stats import genextreme # generalized EV distr.
def sim_data(theta, N):
    K, J_1 = theta.shape # theta must be (K, J-1)
    J = J_1 + 1
    xx = np.random.normal(size=(N,K-1))
    oo = np.ones((N,1)) # constant term
    x  = np.hstack([oo,xx]) # full x matrix
    zz = np.zeros((K,1)) # normalized coefficients
    beta = np.hstack([zz, theta]) # full (K,J) matrix
    v = x @ beta # (N,J): observable utility
    uni = np.random.uniform(size=(N,J)) # uniform draws
    e = genextreme.ppf(uni, c=0) # generalized extreme value errors
    u = v + e # full (unobserved) utility
    y = u.argmax(axis=1) # observed, chosen alternative
    return y, x
```

Conditional Logit

Conditional Logit

$$
\begin{aligned}
u_{ij} &= \mathbf{x}_{ij}'\beta + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{IID Extreme Value Type I}, \\
y_i &= \operatorname{argmax}_{j\in\{1,\ldots,J\}}\, u_{ij}.
\end{aligned}
$$

```
def sim_data(theta, N, J):
        K = theta.size
        assert theta.ndim == 1 # theta should be (K,)
        x = np.random.normal(size=(N,J,K)) # Explanatory variables
        uni = np.random.uniform(size=(N,J))
        e = genextreme.ppf(uni, c=0) # generalized extreme value
        v = x @ theta # (N,J) matrix of "observable utilities"
        u = v + e # full utility (unobserved)
        y = u.argmax(axis=1) # observed chosen alternative
        return y,x
```

\theta = how many characteristics
N = no obs
J = alternatives

NO CONSTANT TERM

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Overview

- **Goal:** Derive the likelihood function.
- **How** do we begin?
- **Normally:** we always try to isolate the error terms..
- **This time:** there are J of them and only one y_i is observed...

Deriving the likelihood

Conditional Logit Model

$$
\begin{aligned}
u_{ij} &= \mathbf{x}_j \boldsymbol{\beta}_o + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{Extreme Value Type I.} \\
y_i &= \text{argmax}_{j \in \{1,\ldots,J\}} \; u_{ij}.
\end{aligned}
$$

- **Choice probability:**

$$
\begin{aligned}
\Pr(y = j|\mathbf{X}) &= \Pr(u_{ij} > u_{ik} \, \forall k \neq j|\mathbf{X}) \\
&= \Pr(\mathbf{x}_j \boldsymbol{\beta}_o + \varepsilon_{ij} > \mathbf{x}_k \boldsymbol{\beta}_o + \varepsilon_{ik} \; \forall k \neq j|\mathbf{X}) \\
&= \Pr\left(\varepsilon_{ik} < \mathbf{x}_j \boldsymbol{\beta}_o - \mathbf{x}_k \boldsymbol{\beta}_o + \varepsilon_{ij} \; \forall k \neq j|\mathbf{X}\right) \\
&= \ldots \\
&= \frac{\exp(\mathbf{x}_j \boldsymbol{\beta}_o)}{\sum_{k=1}^{J} \exp(\mathbf{x}_k \boldsymbol{\beta}_o)}.
\end{aligned}
$$

▸ Details

Random Utility Model

Note: Nothing relied on the precise form of v_{ij}.

Random Utility Model

$$\begin{aligned} u_{ij} &= v_{ij} + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{Extreme Value Type I.} \\ y_i &= \text{argmax}_{j \in \{1,\dots,J\}} \; u_{ij}. \end{aligned}$$

- **Choice probability:**

$$\begin{aligned} \Pr(y = j|\mathbf{X}) &= \Pr(u_{ij} > u_{ik} \; \forall k \neq j|\mathbf{X}) \\ &= \Pr(v_{ij} + \varepsilon_{ij} > v_{ik} + \varepsilon_{ik} \; \forall k \neq j|\mathbf{X}) \\ &= \Pr\left[\varepsilon_{ik} < v_{ij} - v_{ik} + \varepsilon_{ij} \; \forall k \neq j|\mathbf{X}\right] \\ &= \cdots \\ &= \frac{\exp(v_{ij})}{\sum_{k=1}^{J} \exp(v_{ik})}. \end{aligned}$$

Likelihood function

We showed

$$\Pr(y_i = j|\mathbf{X}) = \frac{\exp(v_{ij})}{\sum_{k=1}^{J} \exp(v_{ik})}.$$

Log Likelihood function

$$\ell_i(\boldsymbol{\theta}) = v_{iy_i} - \log\left[\sum_{k=1}^{J} \exp(v_{ik})\right].$$

```python
def util(theta, x):
        v = x @ theta # (N,J,K) * (K,) = (N,J)
        return v

def loglikelihood(theta, y, x):
        N,J,K = x.shape
        v = util(theta, x)
        v_i = v[np.arange(N), y] # (N,) vector
        denom = np.exp(v).sum(axis=1) # (N,) vector
        ll_i = v_i - np.log(denom)
        return ll_i
```

Max-rescaling

- **Crucial trick:** *Analytically*, the following holds

$$\frac{\exp(v_{ij})}{\sum_{k=1}^{J}\exp(v_{ik})} = \frac{\exp(v_{ij} - K_i)}{\sum_{k=1}^{J}\exp(v_{ik} - K_i)}, \quad \forall K_i \in \mathbb{R}.$$

- **Numerically** however, `exp()` quickly becomes highly imprecise...
  - `np.exp(800) = inf`: overflow
  - `np.exp(-800) = 0.0`: underflow
- **Both** types of error are bad...
  - ... but overflow is much worse.
- **Solution:** ***Max-rescaling*** our utilities

```python
def util(theta,x):
        v = x @ theta
        vmax = v.max(axis=1, keepdims=True) # (N,1)
        return v - vmax # subtract columns-wise max
```

Normalization in CL

Conditional Logit

$$\begin{aligned} u_{ij} &= \mathbf{x}_j \beta + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{IID Extreme Value Type I}, \\ y_i &= \underset{j \in \{1,\ldots,J\}}{\operatorname{argmax}} u_{ij}. \end{aligned}$$

Discuss

- Why can't we allow for an intercept (constant) in x_j?
- Why can't we estimate $\mathrm{Var}(\varepsilon_{ij})$ here?
- How does this relate to standard micro with *ordinal* vs. *cardinal* utility?

Normalization in MNL

Multinomial Logit

$$
\begin{aligned}
u_{ij} &= \mathbf{x}_i \boldsymbol{\beta}_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{IID Extreme Value Type I}, \\
y_i &= \underset{j \in \{1,\dots,J\}}{\operatorname{argmax}} \, u_{ij},
\end{aligned}
$$

where we normalize $\boldsymbol{\beta}_1 := \mathbf{0}_{K\times 1}$.

Discuss

- Why do we *have* to normalize β for one alternative?
- Why can we allow for an intercept in x_i and estimate β_{j0} for all j except the normalized alternative?
- What is the utility of the normalized alternative?

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Features of Interest

- **Generally:** We are rarely interested in βs directly.
- **Note:** The average choice probability is the *market share*.
  - so average partial effects say how market shares change.
- **Car example:** Interested in how car choice is affected by x'es.
  - I.e. $\frac{\partial}{\partial x_{ijk}} \Pr(j|x)$.
  - Standard errors: via the delta method.
- **Conditional logit:** E.g. the effect raising the price of Volvo V70, holding fixed all other car prices

$$\frac{\partial}{\partial x_{kl}} \Pr(j) = \Pr(j) \left[\mathbf{1}_{\{k=j\}} \beta_l - \Pr(l) \beta_l \right].$$

- **Multinomial logit:** E.g. the effect on Pr(V70) of raising household income,

$$\frac{\partial}{\partial x_k} \Pr(j) = \Pr(j) \left[\beta_{jk} - \sum_{l=1}^{J} \Pr(l) \beta_{lk} \right].$$

Multinomial Logit: Parameter Interpretation

- **Interpreting β_{jp}, warning:** Partial effect is complicated.
  - The sign is not determined by β_{jp} alone!!!
- **Instead:** Consider the log Odds-ratio (and use $\boldsymbol{\beta}_1 = \mathbf{0}$)

$$\log \frac{\Pr(y = j|\mathbf{x}_i)}{\Pr(y = 1|\mathbf{x}_i)} = \log \frac{\frac{\exp(\mathbf{x}_j \boldsymbol{\beta}_j)}{\sum_{k=1}^{J} \exp(\mathbf{x}_k \boldsymbol{\beta})}}{\frac{\exp(\mathbf{x}_1 \boldsymbol{\beta}_1)}{\sum_{k=1}^{J} \exp(\mathbf{x}_k \boldsymbol{\beta})}} = \mathbf{x}_i \boldsymbol{\beta}_j.$$

- **Linear!** This is why some software reports (log) odds ratios.

More generally

- **Larger choicesets:** If $J > 1000$, partial effects are not interesting.
- **Instead:** Predicted characteristics,

$$\mathbb{E}(\text{horsepower}) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{J}\Pr(j) \times \text{horsepower}_j.$$

- **Elasticities?** Computed numerically;
  1. Compute baseline probabilities,
  2. increase the horsepower of car$j$ by 1%,
  3. Compute new probabilities,
  4. Elasticity is $\cong$ change in probabilities.

Welfare

Discuss

How does measures like *compensating variation* relate to our model?

Definition: The compensating variation is the monetary transfer that would make an individual indifferent between two regimes.

Welfare Analysis

- **Result:** With $u_{ij} = v_{ij} + \varepsilon_{ij}$, it can be shown that

$$\mathbb{E}[\max_j u_{ij}] = \log \sum_{j=1}^{J} \exp(v_{ij}) + \gamma,$$

where $\gamma \cong .5772$ is Euler's constant.

- **Suppose** one regressor is price, e.g. x_{j1}.
- **Then** $\frac{1}{\beta_1}$ converts from utils to money.
- **Hence** we can compare "welfare" under $\{v_{ij}\}$ and $\{\tilde{v}_{ij}\}$.

$$\mathrm{CV} = \frac{1}{\beta_1} \log \sum_{j=1}^{J} \exp(v_{ij}) - \frac{1}{\beta_1} \log \sum_{j=1}^{J} \exp(\tilde{v}_{ij}).$$

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Independence of Irrelevant Alternatives (IIA)

- **Model:** Consider the Conditional Logit model,

$$u_{ij}^* = \mathbf{x}_j \boldsymbol{\beta}_o + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \text{Extreme Value Type I.}$$

- **Property:** Let $\Lambda(\mathbf{X}) \equiv \sum_{j=1}^{J} \exp(\mathbf{x}_j \beta)$.

$$\begin{aligned} \frac{\Pr(y=j)}{\Pr(y=k)} &= \frac{\exp(\mathbf{x}_j \beta)/\Lambda(\mathbf{X})}{\exp(\mathbf{x}_j \beta)/\Lambda(\mathbf{X})} \\ &= \exp\left[(\mathbf{x}_j - \mathbf{x}_k)\beta\right]. \end{aligned}$$

  This is called the IIA assumption.

- **I.e.** The "odds-ratio" (relative prob.) *does not depend on*
  - x for any $h \neq j, k$,
  - Size of the choiceset.

- **Implication:** Restricts substitution patterns...

  ... two examples clarify.

IIA: Bus Example

- **Assume:** $j = 0$ for (red) bus and $j = 1$ for car.
  - With $\mathbf{x} = (\text{price}, \text{time})$, assume $\mathbf{x}_0 = (1, 2)$, $\mathbf{x}_1 = (2, 1)$.
- **Assume further:** That $\beta_o = (1, 1) \Rightarrow$ then $\Pr(y = 1) = .5$.
- **Since** prob.s sum to 1, $\Pr(y = 0) = .5$.

  $(*)$ Therefore $\Pr(y = 1)/\Pr(y = 0) =$.
- **Now:** Add a new (blue) bus; $j = 2$ with $\mathbf{x}_2 = (1, 2)$.
  [identical to red bus]
  - $\Rightarrow \Pr(y = 2) =$
- **The problem:** But by IIA and unchanged β_o, $(*)$ must still hold,
  $\Rightarrow \Pr(y = 1) =$.
- **Implication:** The share choosing car would fall to $\frac{1}{3}$...
- **... counterintuitive:** We added an alternative *identical* to an existing.
  - How can that lure people away from cars?

IIA: Car Example

- **Another example:** Suppose $y \in \{1, ..., J\}$ denotes car type, $\mathbf{x}_j = (\text{price}_j, \text{size}_j)$.
- **Suppose** the price of car J increases.
  - i.e. $x_{J1} := x_{J1} + \Delta$
- **Then** the change in probability (i.e. market share) for *all other* cars, k, are changed **similarly**,

$$\forall k \neq J : \Pr(y = k|\mathbf{X}) = \frac{\exp(\mathbf{x}_k\beta)}{\exp(\beta_1 \cdot \Delta) \exp(\mathbf{x}_J\beta) + \sum_{j \neq J} \exp(\mathbf{x}_j\beta)}.$$

- **Implication:**
  - Car j of course loses some of the market...
  - ... but that loss is divided *similarly* across the remaining cars.
  - ... i.e. not just to the closest substitute.
- **Interpretation:** IIA is a restriction on the possible *substitution patterns*.

Without micro data

- **Model:** Conditional logit.
- **Data:** Aggregate market shares and characteristics of choices.
- **Note:** Expected market share = avg. choice probability (!)

$$
\begin{aligned}
s_j &\equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{y = j\}, \\
s_j &\overset{N\to\infty}{\to} \Pr(y = j).
\end{aligned}
$$

- **Data:** Time-series information on
  - Characteristics of the available cars at time t, x_{jt},
  - Realized market shares during that year, s_{jt}.

Model

Conditional logit in market shares

$$s_{jt} = \Pr(y = j | \mathbf{X}_t) = \frac{\exp(\mathbf{x}_{jt}\beta)}{\sum_{k=1}^{J} \exp(\mathbf{x}_{kt}\beta)}.$$

- **Relative** market share of j to l

$$\frac{s_{jt}}{s_{lt}} = \frac{\frac{\exp(\mathbf{x}_{jt}\beta)}{\sum_{k=1}^{J} \exp(\mathbf{x}_{kt}\beta)}}{\frac{\exp(\mathbf{x}_{lt}\beta)}{\sum_{k=1}^{J} \exp(\mathbf{x}_{kt}\beta)}} = \frac{\exp(\mathbf{x}_{jt}\beta)}{\exp(\mathbf{x}_{lt}\beta)}.$$

- **In logs**

$$\log s_{jt} - \log s_{lt} = (x_{jt} - x_{lt})\beta.$$

Regression idea

- **We showed:**
$$\log s_{jt} - \log s_{lt} = (\mathbf{x}_{jt} - \mathbf{x}_{lt})\beta.$$
- **Idea:**
  - $\tilde{y}_{jt} := \log s_{jt} - \log s_{1t}$,
  - $\tilde{\mathbf{x}}_{jt} := \mathbf{x}_{jt} - \mathbf{x}_{1t}$.
- **Regression:** Obtain $\hat{\beta}$ by regressing $\tilde{y}_{jt}$ on $\tilde{\mathbf{x}}_{jt}$ for all $j \neq 1$ and all t.
- **Applications:** Requires many markets with much exogenous variation in $\mathbf{x}$...
  - but a good first approximation.

Outline

6. Appendix

Deriving the likelihood: A few more steps

Choice probability:

$$
\begin{aligned}
\Pr(y=j|z) &= \Pr(u_{ij} > u_{ik}\,\forall k \neq j|z) \\
&= \Pr(x_j\beta_o + \varepsilon_{ij} > x_k\beta_o + \varepsilon_{ik}\ \forall k \neq j|z) \\
&= \Pr\left[\varepsilon_{ik} < (x_j - x_k)\beta_o + \varepsilon_{ij}\ \forall k \neq j|z\right] \\
&= \mathbb{E}_{\varepsilon_{ij}}\left\{\Pr\left[\varepsilon_{ik} < (x_j - x_k)\beta_o + \varepsilon_{ij}\ \forall k \neq j|z, \varepsilon_{ij}\right]\right\} \\
&= \mathbb{E}_{\varepsilon_{ij}}\left\{\prod_{k\neq j} G\left[(x_j - x_k)\beta_o + \varepsilon_{ij}\right]\right\} \\
&= \int \left\{\prod_{k\neq j} G\left[(x_j - x_k)\beta_o + \epsilon\right]\right\} e^{-\epsilon} e^{-e^{-\epsilon}}\, \mathrm{d}\epsilon \\
&= \cdots \\
&= \frac{\exp(x_j\beta_o)}{\sum_{k=0}^{J}\exp(x_k\beta_o)}.
\end{aligned}
$$

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