Linear Models for Panel Data

Advanced Microeconometrics

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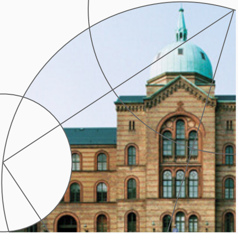


Introduction

Example: Wages and schooling

Example: Wages and schooling

$$w_{it} = \beta_0 + \beta_1 e_{it} + c_i + u_{it},$$

where w_{it} is wage and e_{it} is years of schooling.

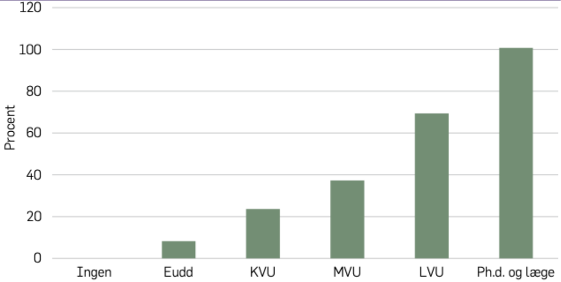
- we can **compute** the average wage among, say, people with $e_{it} = 10$ years of schooling, $\hat{\mathrm{E}}(w_{it}|e_{it} = 10)$
- **Intuitively**, you can think of the POLS estimator as finding $\hat{\beta}_1$ by seeing how $\hat{\mathrm{E}}(w_{it}|e_{it})$ varies with *educ*.
- **The problem:** what if high-IQ people (high c_i, and high $wage_i$ regardless of $educ_i$) take more education ($educ$)
  - Mathematically: $\mathrm{Cov}(e_{it}, c_i) > 0$
- **Decomposing** how $\hat{\mathrm{E}}(w_{it}|e_{it})$ increases with e_{it}:
  - one part is the value of education: $\beta_1 e_{it}$
  - one part is the *type composition* changing: higher IQ among higher educated.

Example: Wages and schooling

Estimates from regression of log wage on dummies for education types (and sector dummies and experience)

FIGUR 2.1

Lønafkast pr. uddannelseskategori på hele arbejdsmarkedet set i forhold til lønnen for ansatte uden uddannelse, 2011. Opgjort for standardberegnet timefortjeneste (ekskl. genebetaling). Procent.



Discussion

Education: Signaling or Skill Building

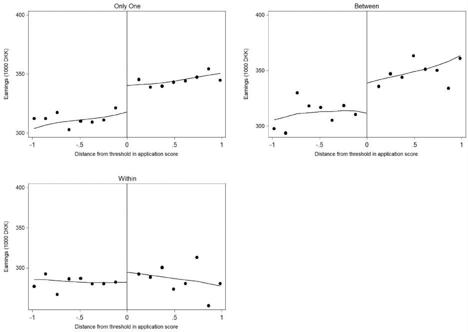
Clearly, highly educated earn more. Pick a side randomly and argue which is more important for economists:

1. **Skills:** education teaches analytical and abstract thinking which heightens productivity. (E.g. Scandinavian school reforms)
2. **Signaling:** smart people take a tough education to show firms how productive they will be. (E.g. almost nobody remembers trigonometry)

Relate the discussion to the linear model below.

Check out the discussion here: youtu.be/MvWnyUT7vPk

Admission Cutoffs



Cool empirical strategy: strict cutoff in admission for GPA

- $\Rightarrow$ *Regression Discontinuity* design.

Source: Daly, Jensen & le Maire (2022; Labour Economics).

Where are we in the course?

Part	Topic	Parameterization non-linear	Estimation non-linear	Dimension $\dim(x)$	Numerical optimization	M-estimation (Part III)	Outcome (y_i)	Panel (c_i)
I	OLS	$\div$	$\div$	low	$\div$	✓	$\mathbb{R}$	✓
II	LASSO	$\div$	✓	high	✓	$\div$	$\mathbb{R}$	$\div$
IV	Probit	✓	✓	low	✓	✓	$\{0, 1\}$	$\div$
	Tobit	✓	✓	low	✓	✓	$[0; \infty)$	$\div$
	Logit	✓	✓	low	✓	✓	$\{1, 2, ..., J\}$	$\div$
	Sample selection	✓	✓	low	✓	✓	$\mathbb{R}$ and $\{0, 1\}$	$\div$
	Simulated Likelihood	✓	✓	low	✓	✓	Any	✓
	Quantile Regression	$\div$	✓	(low)	✓	✓	$\mathbb{R}$	$\div$
	Non-parametric	✓	(✓)	∞	$\div$	$\div$	$\mathbb{R}$	$\div$

Outline

Panel model

$$y_{it} = \mathbf{x}_{it}\beta + c_i + u_{it}.$$

			Estimator				
Assumptions	Rank	Model name	POLS	BE	RE	FE	FD
$\mathrm{E}(u_{it}\|\mathbf{x}_{it}, c_i) = 0,\ c_i = c,$	$\mathbf{X}'\mathbf{X}$	No individual effects	$\checkmark + \star$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$\mathrm{E}(u_{it}\|\mathbf{X}_i, c_i) = 0,\ c_i \sim \mathrm{IID}(0, \sigma_c^2),$	$\mathbf{X}'\mathbf{X}$	Random effects	$\checkmark$	$\checkmark$	$\checkmark + \star$	$\checkmark$	$\checkmark$
$\mathrm{E}(u_{it}\|\mathbf{X}_i, c_i) = 0,\ \mathrm{E}(c_i\mathbf{x}_{it}) \neq 0,$	$\ddot{\mathbf{X}}'\ddot{\mathbf{X}}$	Fixed effects	$\div$	$\div$	$\div$	$\checkmark$	$\checkmark$
$\mathrm{E}(u_{it}\|\mathbf{X}_i, c_i) = 0,\ \mathrm{E}(c_i\mathbf{x}_{it}) \neq 0,$ and u_{it} IID	$\ddot{\mathbf{X}}'\ddot{\mathbf{X}}$	Fixed effects	$\div$	$\div$	$\div$	$\checkmark + \star$	$\checkmark$
$\mathrm{E}(u_{it}\|\mathbf{X}_i, c_i) = 0,\ \mathrm{E}(c_i\mathbf{x}_i) \neq 0$ and $u_{it} - u_{it-1}$ IID	$\Delta\mathbf{X}'\Delta\mathbf{X}$	Fixed effects	$\div$	$\div$	$\div$	$\checkmark$	$\checkmark + \star$
$\mathrm{E}(u_{it}\|\mathbf{x}_{it}, c_i) \neq 0$	–	Endogeneity	$\div$	$\div$	$\div$	$\div$	$\div$

($\div$ = inconsistent, $\checkmark$ = consistent, $\star$ = efficient)

First Differences

First Differences

- **Now,** consider the first-differences transformed model

$$\begin{aligned} y_{it} - y_{it-1} &= (\mathbf{x}_{it} - \mathbf{x}_{it-1})\beta + \overbrace{c_i - c_i}^{=0} + u_{it} - u_{it-1} \\ \Delta y_{it} &= \Delta \mathbf{x}_{it}\beta + \Delta u_{it} \end{aligned}$$

- **Voila!** Big beautiful success.
- **Implementation:** transform data and run POLS of $\Delta\mathbf{Y}$ on $\Delta\mathbf{X}$
- **Assumptions:** Translate between assumptions on raw vs. transformed dataset.
  - Analogously to what we did for FE.

First Differences (FD) Estimator

$$\hat{\beta}_{FD} = (\Delta \mathbf{X}' \Delta \mathbf{X})^{-1} \Delta \mathbf{X}' \Delta \mathbf{y}.$$

FD Assumptions for Consistency

1. $E(\Delta \mathbf{x}_{it} \Delta u_{it}) = \mathbf{0}$,
2. $E(\Delta \mathbf{X}' \Delta \mathbf{X})$ must have full rank

(Simply the POLS assumptions written for the transformed data.)

Assumptions

FD Assumptions for Consistency

1. $\mathrm{E}(\Delta \mathbf{x}_{it} \Delta u_{it}) = \mathbf{0}$,
2. $\mathrm{E}(\Delta \mathbf{X}' \Delta \mathbf{X})$ must have full rank

Translating assumptions on transformed to raw variables

- **Exogeneity**:

$$\mathrm{E}(\Delta \mathbf{x}_{it} \Delta u_{it}) = \mathrm{E}(\mathbf{x}_{it} u_{it}) - \mathrm{E}(\mathbf{x}_{it-1} u_{it}) + \mathrm{E}(\mathbf{x}_{it} u_{it-1}) + \mathrm{E}(\mathbf{x}_{it-1} u_{it-1})$$

  - Again, **contemporaneous** exog. is **insufficient**
  - Strict exogeneity is, since $\mathrm{E}(u_{it}|\mathbf{x}_i) = 0 \Rightarrow \mathrm{E}(\mathbf{x}_{it-1} u_{it}) = \mathbf{0}$.
  - **Sufficient:** $\mathrm{E}(u_{it}|\mathbf{x}_{it+1}, \mathbf{x}_{it}, \mathbf{x}_{it-1}) = 0$.
- **Rank condition:** Requires time-series variation.
  - Same as for FE.

Inference with FD

- **Similar** argumentation to the FE case yields:

Panel-robust variance estimator

$$\widehat{\text{Avar}}(\hat{\beta}_{FD}) = \left(\sum_i \Delta\mathbf{X}_i'\Delta\mathbf{X}_i\right)^{-1}\left(\sum_i \Delta\mathbf{X}_i'\widehat{\Delta\mathbf{u}}_i\widehat{\Delta\mathbf{u}}_i'\Delta\mathbf{X}_i\right)\left(\sum_i \Delta\mathbf{X}_i'\Delta\mathbf{X}_i\right)^{-1}$$

where $\widehat{\Delta\mathbf{u}}_i \equiv \Delta\mathbf{y}_i - \Delta\mathbf{X}_i\hat{\beta}_{FD}$ ($\Delta\mathbf{X}_i$ is $T-1\times K$)

Variance estimator for IID errors

Assuming that Δu_{it} are IID (implying that u_{it} is a random walk!),

$$\widehat{\text{Avar}}(\hat{\beta}_{FD}) = \hat{\sigma}_u^2\left(\sum_i \Delta\mathbf{X}_i'\Delta\mathbf{X}_i\right)^{-1},$$

where

$$\hat{\sigma}_u^2 = \frac{1}{NT-N-K}\sum_i\sum_t \widehat{\Delta u}_{it}^2.$$

FE vs. FD

How do we choose between FE and FD?

- **Identical** when $T = 2$, both numerically and algebraically:
  - E.g.: $\ddot{y}_{i2} = y_{i2} - \frac{y_{i1}+y_{i2}}{2} = \frac{1}{2}\Delta y_{i2}$, and $\ddot{y}_{i1} = -\frac{1}{2}\Delta y_{i1}$.
- **Consistency:**
  - FE: requires *strict* exogeneity, $\mathrm{E}(u_{it}|\mathbf{x}_{i1}, ..., \mathbf{x}_{iT}) = 0$
  - FD: only requires $\mathrm{E}(u_{it}|\mathbf{x}_{it+1}, \mathbf{x}_{it}, \mathbf{x}_{it-1}) = 0$, which is *implied* by strict exogeneity.
- **Efficiency:** in POLS $\Rightarrow$ occurs when all errors are IID.
  - FD: efficient if Δu_{it} is IID i.e. u_{it} is a unit root.
  - FE: efficient if u_{it} is IID i.e. u_{it} serially uncorrelated,
    - Why u_{it} and not $\ddot{u}_{it}$? ... next slide...

Efficiency

- **Question:** Why does FE.3 assume u_{it} is IID, and not $\ddot{u}_{it}$?
- **First:** note that $\ddot{u}_{it}$ can never be IID
  - $\ddot{u}_{it} \equiv u_{it} - \bar{u}_i$: so $\ddot{u}_{it}$ and $\ddot{u}_{is}$ share the component $\bar{u}_i$.
- **Second**: Recall the "demeaning" matrix, $\mathbf{Q}_T$, s.t. $\ddot{\mathbf{u}}_i = \mathbf{Q}_T\mathbf{u}_i$.
  - Note: $\mathbf{Q}_T\mathbf{Q}_T = \mathbf{Q}_T$ (idempotent, symmetric)
  - Next: $\ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_i = (\mathbf{Q}_T\mathbf{X}_i)'\mathbf{Q}_T\mathbf{u}_i = \mathbf{X}_i'\mathbf{Q}_T\mathbf{u}_i = \ddot{\mathbf{X}}_i'\mathbf{u}_i$
- **FE** takes the asymptotic form

$$\sqrt{N}(\hat{\beta}_{FE} - \beta) = \left(N^{-1}\textstyle\sum_{i=1}^{N}\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i\right)^{-1} N^{-1}\textstyle\sum_{i=1}^{N}\ddot{\mathbf{X}}_i'\mathbf{u}_i.$$

- **Conclusion:** Unweighted estimation on demeaned data (i.e. FE) is efficient when $\mathrm{E}(\mathbf{u}_i\mathbf{u}_i|\ddot{\mathbf{X}}_i)$ is diagonal, i.e. when u_{it} are IID.

Random Effects Model

Between Estimator

- **Stepping stone:** the Between Estimator (BE).
- **New transformation:** take the time-average,

$$\overline{y}_i = \overline{\mathbf{x}}_i\beta + c_i + \overline{u}_i,$$

where $\overline{\mathbf{x}}_i \equiv T^{-1}\sum_t \mathbf{x}_{it}$ is $K \times 1$.

- **Failure:** does *not* remove c_i.
- **Between (BE) Estimator:** Regress $\overline{\mathbf{Y}}$ on $\overline{\mathbf{X}}$.
- **Does it work?** Check POLS assumptions on transformed data
- **Exogeneity:** Requires $\mathrm{E}\left[\overline{\mathbf{x}}_i'(c_i + \overline{u}_i)\right] = \mathbf{0}_{K\times 1}$.

BE Estimator

BE Assumptions

1. $\mathrm{E}(\overline{\mathbf{x}}_i \overline{u}_i) = \mathbf{0}$ for all t, s, and $\mathrm{E}(\overline{\mathbf{x}}_i c_i) = \mathbf{0}$,
2. $\overline{\mathbf{X}}'\overline{\mathbf{X}}$ must have full rank.

Translating to model primitives

- **Exogeneity** sufficient conditions are
  - Strict exogeneity: $\mathrm{E}(u_{it}|\mathbf{x}_i) = 0$ for all t
  - Uncorrelated individual effects: $\mathrm{E}(\mathbf{x}_{it} c_i) = \mathbf{0}$ for all t
    - $\Rightarrow$ i.e. we are in the *random effects* model.
- **Rank condition:** unlike FE or FD, we can now allow regressors with zero time-variation.
  - **Bonus question:** can you think of a variable allowed by BE but not FE rank condition?

Random Effects Model

Model

$$y_{it} = \mathbf{x}_{it}\beta + c_i + u_{it}, \quad c_i \sim \text{IID}(0, \sigma_c^2), u_{it} \sim \text{IID}(0, \sigma_u^2).$$

Assumptions

- **RE 1(a):** $\text{E}(u_{it}|\mathbf{X}_i) = 0$
- **RE 1(b):** $\text{E}(c_i|\mathbf{x}_{it}) = \mathbf{0}$
- **RE 2:** $\text{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i)$ has full rank, where $\mathbf{\Omega} \equiv \text{E}(\mathbf{v}_i\mathbf{v}_i')$
- **RE 3(a):** $\text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{X}_i, c_i) = \sigma_u^2\mathbf{I}_T$
- **RE 3(b):** $\text{E}(c_i^2|\mathbf{x}_i) = \sigma_c^2$

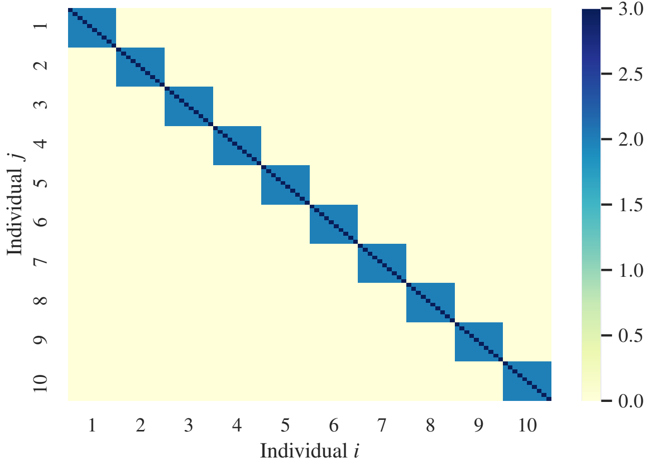
Random Effects Assumptions

- **POLS consistent:** with both $\mathrm{E}(\mathbf{x}_{it}u_{it}) = \mathbf{0}$ and $\mathrm{E}(\mathbf{x}_{it}c_i) = \mathbf{0}$, we have consistency of pooled OLS.
  - RE furthermore requires *strict* (and not just contemporaneous) exogeneity.
- **Claim:** POLS is not efficient when $\sigma_c > 0$.
- **Proof:** POLS is efficient under IID errors. In particular, $\mathrm{Cov}(v_{it}, v_{js}|\mathbf{X}) = 0$ for all i, j, t, s.
- Consider the composite error term, $v_{it} \equiv c_i + u_{it}$
- The covariance is

$$\mathrm{Cov}(v_{it}, v_{js}|\mathbf{X}) = \begin{cases} 0 & \text{if } i \neq j \text{ by independence over } i, \\ \sigma_c^2 & \text{if } i = j \text{ and } t \neq s, \\ \sigma_c^2 + \sigma_u^2 & \text{if } i = j \text{ and } t = s. \end{cases}$$

  - I.e. $\mathrm{V}(\mathbf{v}|\mathbf{X})$ is *block diagonal* with N consecutive $T \times T$ blocks.
  - **QED**.

Visualizing the composite error covariance: Cov$(c_i + u_{it}, c_j + u_{js})$



Random Effects Assumptions

- **Rank condition:** assumes that $\mathbf{X}'\mathbf{X}$ is invertible.
  - Recall: FE/FD assumes that the *transformed* matrices are invertible.
  - **Implication:** time-invariant variables are *allowed*.
    - **Implicit assumption**: they are independent of c_i.
- **Efficiency:** our IID assumptions imply **homoskedasticity** for both u and c:
  - $\mathrm{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i) = \sigma_u^2 \mathbf{I}_{T\times T}$: i.e. homoskedasticity ans serial uncorrelatedness of u_{it}
    - where $\mathbf{I}_{T\times T}$ is the identity matrix.
  - $\mathrm{E}(c_i^2|\mathbf{x}_i) = \sigma_c^2$
- **Then** the variance of $v_{it} \equiv c_i + u_{it}$ becomes

$$\boldsymbol{\Omega} \equiv \mathrm{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i) = \begin{pmatrix} \sigma_c^2+\sigma_u^2 & \cdots & \sigma_c^2 \\ \vdots & \ddots & \vdots \\ \sigma_c^2 & \cdots & \sigma_c^2+\sigma_u^2 \end{pmatrix} = \sigma_c^2 \mathbf{1}_T \mathbf{1}_T' + \sigma_u^2 \mathbf{I}_{T\times T}$$

Estimation

- **As shown:** error terms are correlated across rows when $\sigma_c > 0$, so POLS is inefficient.
- **However:** we derived the error variance, so we have a recipe for a weighted least squares estimator (which recovers efficiency).
- **Efficient estimation** is performed with weighted least squares (GLS),

$$\hat{\beta}_{GLS} = \left(\sum_{i=1}^{N} \mathbf{X}_i'\Omega^{-1}\mathbf{X}_i\right)^{-1}\left(\sum_{i=1}^{N} \mathbf{X}_i'\Omega^{-1}\mathbf{y}_i\right).$$

Random Effects Estimator

Random Effects (RE) Estimator

1. **Estimate σ_u, σ_c:** Compute the FE and BE estimators, and obtain the residuals

$$\hat{\sigma}_u^2 = \frac{1}{NT - N - K} \sum_i \sum_t \left(\hat{\ddot{u}}_{it}^{FE}\right)^2 \quad \hat{\ddot{u}}_{it}^{FE} \equiv (y_{it} - \overline{y}_i) - (\mathbf{x}_{it} - \overline{\mathbf{x}}_i)\hat{\beta}_{FE}$$

$$\hat{\sigma}_c^2 = \frac{1}{N - K - 1} \sum_i (\hat{u}_{it}^{BE})^2 - T^{-1}\hat{\sigma}_u^2, \quad \hat{u}_{it}^{BE} \equiv \overline{y}_i - \overline{\mathbf{x}}_i'\hat{\beta}_{BE}.$$

(note that in the BE, $\overline{\mathbf{x}}_i$ should contain an intercept (thus $K+1$ columns), while $\mathbf{x}_{it}$ in the FE should not)

2. **Estimate $\hat{\beta}_{RE}$** using (F)GLS

$$\hat{\beta}_{RE} = \left(\sum_i \mathbf{X}_i'\hat{\mathbf{\Omega}}^{-1}\mathbf{X}_i\right)^{-1} \sum_i \mathbf{X}_i'\hat{\mathbf{\Omega}}^{-1}\mathbf{y}_i,$$

where $\hat{\mathbf{\Omega}} = \hat{\sigma}_c^2 \mathbf{1}_T \mathbf{1}_T' + \hat{\sigma}_u^2 \mathbf{I}_{T \times T}$.

RE Estimator

Random Effects (RE) Estimator

$$\hat{\beta}_{RE} = \left(\sum_i \mathbf{x}_i \hat{\mathbf{\Omega}}^{-1} \mathbf{x}_i'\right)^{-1} \sum_i \mathbf{x}_i \hat{\mathbf{\Omega}}^{-1} \mathbf{y}_i,$$

$$\text{where } \hat{\mathbf{\Omega}} = \hat{\sigma}_c^2 \mathbf{1}_T \mathbf{1}_T' + \hat{\sigma}_u^2 \mathbf{I}_{T\times T},$$

- **Extremes** wrt. σ_c vs. σ_u:
  - $\sigma_c = 0$ (no random effects): $\hat{\beta}_{RE} = \hat{\beta}_{POLS}$ since $\hat{\Omega}$ becomes diagonal.
  - $\sigma_u = 0$ (only random effects): $\hat{\beta}_{RE} \to \hat{\beta}_{FE}$... easier to see later.
- **Turns out:** $\hat{\beta}_{RE}$ can also be obtained by running POLS on a transformed dataset.

Quasi-demeaning

- **Define:**

$$\lambda \equiv 1 - \sqrt{\frac{\sigma_u^2}{T\sigma_c^2 + \sigma_u^2}} \quad \text{(note that } \lambda \in [0;1].\text{)}$$

- **It can be shown** that

$$\boldsymbol{\Omega}^{-\frac{1}{2}} = \frac{1}{\sigma_u}\left(\mathbf{I}_T - \lambda \mathbf{P}_T\right), \quad \mathbf{P}_T \equiv \frac{1}{T}\mathbf{j}_T\mathbf{j}_T'.$$

  - Define $\mathbf{C}_T \equiv (\mathbf{I}_T - \lambda \mathbf{P}_T)$ $(T \times T)$
- **RE Estimator** arises from estimation on the transformed system

$$\mathbf{C}_T\mathbf{y}_i = \mathbf{C}_T\mathbf{X}_i\beta + \mathbf{C}_T\mathbf{v}_i.$$

- **Crucially:** The new errors, $\check{\mathbf{v}}_i \equiv \mathbf{C}_T\mathbf{v}_i$, satisfy homoskedasticity:

$$\mathrm{E}(\check{\mathbf{v}}_i\check{\mathbf{v}}_i') = \mathbf{C}_T\boldsymbol{\Omega}\mathbf{C}_T = \sigma_u^2\mathbf{I}_T.$$

Estimation of σ_c, σ_u by POLS

- **Challenge:** Obtaining estimates of σ_c, σ_u.
- **Simpler problem:** estimating σ_c and σ_v
  - $\Rightarrow$ then $\sigma_u^2 = \sigma_v^2 - \sigma_c^2$ under RE.3
- **Estimating σ_v^2:** Use POLS

$$\hat{\sigma}_v^2 = \frac{1}{NT - K} \sum_{i=1}^{N} \sum_{t=1}^{T} (\hat{v}_{it}^{POLS})^2,$$

where $\hat{v}_{it}^{POLS} \equiv y_{it} - \mathbf{x}_{it}\hat{\beta}_{POLS}$, $\hat{\beta}_{POLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$.
  - Consistent? Yes, under RE.1
- **Estimating σ_c^2:** Still using POLS residuals
  - Note, $\boldsymbol{\Omega}$ has elements σ_c^2 everywhere in the lower triangle, which has $T(T-1)\frac{1}{2}$ elements

$$\hat{\sigma}_c^2 = \frac{1}{NT(T-1)\frac{1}{2} - K} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{t-1} \hat{v}_{it}^{POLS} \hat{v}_{is}^{POLS}.$$

Alternative: Using FE and BE

- **Alternative idea** for estimating (σ_u, σ_c).
  - **Idea:** Note that both FE and BE are consistent under RE1.
- **FE residuals:** information about u_{it},

$$\hat{\sigma}_u^2 = \frac{1}{NT - N - K} \sum_i \sum_t \left(\hat{\ddot{u}}_{it}^{FE}\right)^2, \quad \hat{\ddot{u}}_{it}^{FE} \equiv (y_{it} - \overline{y}_i) - (\mathbf{x}_{it} - \overline{\mathbf{x}}_i)\hat{\beta}_{FE}$$

- **BE residuals:** information about c_i, ...
  - ... but there is still $T^{-1} \sum_{t=1}^{T} u_{it}$ in the BE residual.

$$\hat{\sigma}_c^2 = \frac{1}{N - K - 1} \sum_i (\hat{u}_{it}^{BE})^2 - T^{-1}\hat{\sigma}_u^2, \quad \hat{u}_{it}^{BE} \equiv \overline{y}_i - \overline{\mathbf{x}}_i \hat{\beta}_{BE}.$$

RE Estimator by Quasi-demeaning

Random Effects (RE) Estimator

1. Estimate the FE and BE estimators, and compute

$$\begin{aligned}\hat{\sigma}_u^2 &= \frac{1}{NT-N-K}\sum_i \sum_t \left(\hat{\ddot{u}}_{it}^{FE}\right)^2, \quad \hat{\ddot{u}}_{it}^{FE} \equiv (y_{it} - \bar{y}_i) - (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)\hat{\beta}_{FE} \\ \hat{\sigma}_c^2 &= \frac{1}{N-K-1}\sum_i (\hat{u}_{it}^{BE})^2 - T^{-1}\hat{\sigma}_u^2, \quad \hat{u}_{it}^{BE} \equiv \bar{y}_i - \bar{\mathbf{x}}_i\hat{\beta}_{BE}.\end{aligned}$$

2. Transform data by quasi-demeaning: $\check{y}_{it} \equiv y_{it} - \hat{\lambda}\bar{y}_i$ and $\check{\mathbf{x}}_{it} \equiv x_{it} - \hat{\lambda}\bar{\mathbf{x}}_i$, where

$$\hat{\lambda} = 1 - \sqrt{\frac{\hat{\sigma}_u^2}{T\hat{\sigma}_c^2 + \hat{\sigma}_u^2}}.$$

   (Note that $\hat{\lambda} \in [0;1]$.)

3. Estimate by POLS on transformed dataset

RE Interpretation: Revisited

- **RE:** POLS on quasi-demeaned data, $\check{y}_{it} \equiv y_{it} - \hat{\lambda}\overline{y}_i$ and $\check{\mathbf{x}}_{it} \equiv x_{it} - \hat{\lambda}\overline{\mathbf{x}}_i$, where

$$\hat{\lambda} = 1 - \sqrt{\frac{\hat{\sigma}_u^2}{T\hat{\sigma}_c^2 + \hat{\sigma}_u^2}} = 1 - \sqrt{\frac{1}{T\frac{\hat{\sigma}_c^2}{\hat{\sigma}_u^2} + 1}}$$

  Focus on $\frac{\sigma_c^2}{\sigma_{u^2}}$
- **Intuitively:** The RE estimator uses both within and between variation.
- **POLS:** $\frac{\sigma_c^2}{\sigma_u^2} \to 0 \Rightarrow \lambda \to 0$: no individual effects so demeaning is not necessary.
  - RE and POLS estimates will be close.
- **FE:** $\frac{\sigma_c^2}{\sigma_{u^2}} \to \infty$ or when $T \to \infty$.
  - $\frac{\sigma_c^2}{\sigma_u^2} \to \infty$: almost all variation occurs between individuals.
  - $T \to \infty$: enough variation within individuals to discard the between variation.

Variance Estimator

Variance Estimator

Assuming that u_{it} and c_i are IID,

$$\widehat{\text{Avar}}(\hat{\beta}_{RE}) = \hat{\sigma}_u^2 \left(\sum_i \sum_t \check{\mathbf{x}}_{it}' \check{\mathbf{x}}_{it}\right)^{-1}.$$

Robust Variance Estimator

$$\widehat{\text{Avar}}(\hat{\beta}_{RE}) = \left(\sum_i \sum_t \check{\mathbf{x}}_{it}' \check{\mathbf{x}}_{it}\right)^{-1} \sum_i \sum_{t=1}^{T} \sum_{s=1}^{T} \check{\mathbf{x}}_{it}' \hat{\check{u}}_{it} \hat{\check{u}}_{is} \check{\mathbf{x}}_{is} \left(\sum_i \sum_t \check{\mathbf{x}}_{it}' \check{\mathbf{x}}_{it}\right)^{-1}$$

where $\hat{\check{u}}_{it} \equiv \hat{u}_{it} - \hat{\lambda} \hat{u}_{it}$, and $\hat{u}_{it}$ is the RE residual. This allows arbitrary autocorrelation and heteroskedasticity.

- Note: somewhat awkward to use an RE estimator if one fears that RE3 is not satisfied...

Blitz Discussion

Random Effects Model

Consider the model

$$y_{it} = \mathbf{x}_{it}\beta + c_i + u_{it},$$

$$\text{and assume } c_i \sim \mathrm{IID}(0, \sigma_c^2), \quad u_{it} \sim \mathrm{IID}(0, \sigma_u^2),$$

(meaning that $\mathbf{x}_{it}$ is independent of both c_i and u_{it}).

Discuss

Assume that $T = 1$ and that $\mathrm{Rank}[\mathrm{E}(\mathbf{x}_{it}'\mathbf{x}_{it})] = K$.

1. Why is $\hat{\beta}_{POLS}$ consistent? Because x_it is uncorrelated with c_i and u_it
2. Why can we *not* identify both σ_c and σ_u? We have no way to separate the individual heterogeneity and "idiosyncratic" errors(?) In fact, because we are in a single time period, the idiosyncratic errors are in fact no longer idiosyncratic! The 'noise' do not change over time - we can only observe one time period - so c_i + u_it
3. Why would panel data ($T > 1$) help?
   This would allow to us identify the variance of the idiosyncratic noise, allowing for noise to vary over time and unit of interests

FE vs. RE

Hausman Test

- **Question:** is FE or RE model appropriate?
  - $\Rightarrow$ question posed by the **Hausman Test**.
- **The null hypothesis:** the RE model is appropriate, i.e.

$$\mathcal{H}_0 : \text{RE.1-3 and FE.1-3 hold.}$$

- **Idea:** Under $\mathcal{H}_0$:
  - *both* $\hat{\boldsymbol{\beta}}_{RE}$ and $\hat{\boldsymbol{\beta}}_{FE}$ are consistent,
  - both become normal with rate $\sqrt{N}$.
- **Result:** because the difference of two normals is normal:

$$\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{RE} - \hat{\boldsymbol{\beta}}_{FE}\right) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{V}_H).$$

- **Test statistic:** Idea: "$z_k \sim \mathcal{N}(0, \sigma^2) \Rightarrow \sum_{k=1}^{K} \frac{z_k^2}{\sigma^2} \sim \chi^2(K)$"

$$H = \left(\hat{\boldsymbol{\beta}}_{RE} - \hat{\boldsymbol{\beta}}_{FE}\right)' \mathbf{V}_H^{-1} \left(\hat{\boldsymbol{\beta}}_{RE} - \hat{\boldsymbol{\beta}}_{FE}\right) \sim \chi^2\left(\dim(\boldsymbol{\beta})\right).$$

Hausman Test: Variance

- **Missing piece:** How do we compute Avar$(\hat{\beta}_{RE} - \hat{\beta}_{FE})$?
- **In general**, $\mathrm{V}(X - Y) = \mathrm{V}(X) + \mathrm{V}(Y) - 2\mathrm{Cov}(X, Y)$
- **Full efficiency case:** it can be shown that when $\hat{\beta}_{RE}$ is fully efficient, Cov $\left(\hat{\beta}_{RE}, \hat{\beta}_{FE}\right) = \mathrm{V}(\hat{\beta}_{RE})$, so

$$\mathrm{Avar}(\hat{\beta}_{RE} - \hat{\beta}_{FE}) = \mathrm{Avar}\left(\hat{\beta}_{FE}\right) - \mathrm{Avar}\left(\hat{\beta}_{RE}\right).$$

  - $\mathbf{V}_H$ is always positive definite – in the scalar case, $\mathrm{V}\left(\hat{\beta}_{FE}\right) > \mathrm{V}\left(\hat{\beta}_{RE}\right)$ both under the null and the alternative.
  - However, if $\hat{\sigma}_u$ estimates come from different models, it can cause trouble.
- **When** $\hat{\beta}_{RE}$ is **not fully efficient**, "bootstrapping" can be used. *BUT!* Then the variance formula changes...

Hausman Test: Efficient case

Hausman Test

The test statistic is

$$H = \left(\hat{\beta}_{RE} - \hat{\beta}_{FE}\right)' \left[\mathrm{V}\left(\hat{\beta}_{FE}\right) - \mathrm{V}\left(\hat{\beta}_{RE}\right)\right]^{-1} \left(\hat{\beta}_{RE} - \hat{\beta}_{FE}\right).$$

Under the null hypothesis that assumptions RE.1-3 and FE.1-3 are true, we have

$$H \overset{a}{\sim} \chi^2(K),$$

where $K = \dim(\beta_{FE})$ (number of time-invariant regressors)

The Bootstrap

The Bootstrap: Motivation

- **Hausman's insight:** $\mathrm{V}(\hat{\beta}_{FE} - \hat{\beta}_{RE})$ takes a particularly simple form under the null.
- **Objection:** What if the null doesn't hold, e.g. due to heteroskedasticity?
  - What if $\mathrm{V}(\hat{\beta}_{FE} - \hat{\beta}_{RE}) \neq \mathrm{V}(\hat{\beta}_{FE}) - \mathrm{V}(\hat{\beta}_{RE})$?
- **Analytic expressions** may be beyond our capability.
- **More generally,** The Bootstrap procedure provides help in seemingly helpless situations.

The Bootstrap Procedure

- **Write** our estimator as $\hat{\beta} = f(\{y_i, \mathbf{x}_i\}_{i=1}^N)$
  - It is a function of a dataset

Bootstrapping

For replications $r = 1, ..., R$ do:

- Take N draws from $\{1, ..., N\}$ with replacement, yielding $\{y_i^{(r)}, \mathbf{x}_i^{(r)}\}_{i=1}^N$.
  - Note: some individuals, i, may be drawn multiple times, and some not at all.
- Compute $\hat{\beta}^{(r)} = f(\{y_i^{(r)}, \mathbf{x}_i^{(r)}\}_{i=1}^N)$

The bootstrapped standard error is then

$$\hat{\mathrm{V}}^{Boot}(\hat{\beta}) = \frac{1}{R}\sum_{r=1}^{R}\left(\hat{\beta}^{(r)} - \overline{\hat{\beta}}\right)^2, \quad \overline{\hat{\beta}} = \frac{1}{R}\sum_{r=1}^{R}\hat{\beta}^{(r)}.$$

In Python

```python
import numpy as np
N,K = X.shape
for r in range(R):
        #select rows randomly with replacement
        ii = np.random.choice(range(N), N, replace=True)
        X_r = X[ii, :]
        y_r = y[ii] # assuming y.ndim == 1
        betas[r, :] = estimate(X_r, y_r)

# compute empirical covariances and put into a matrix
C = np.cov(betas, rowvar=False) # variables are in columns
se = np.sqrt(np.diag(C)) # alternative: np.std(betas,0)
```

Bootstrapping the Hausman Test

- **In the Hausman tes**t, we require $\text{Var}(\hat{\beta}_{RE} - \hat{\beta}_{FE})$...
- ... **so** in each replication, **instead** of computing a single estimate,
  - we we **estimate both** $\hat{\beta}_{RE}$ and $\hat{\beta}_{FE}$ and compute the difference
- **From scalar to vector:** Simply compute the empirical Covariance matrix: `np.cov()`
  - Note that cov by default expects rows to be variables.