HW5f

Math 172: Galois Theory

Jun 14 at 1:00

Prof. Thonkers

Stephen Xu

Problem 1

24.3.7

Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbb{C}$.

a) Compute $f = \min_{\mathbb{Q}}(\alpha)$.

b) Find $E \subseteq \mathbb{C}$ such that E is the splitting field for f over $\mathbb{Q}$. Compute $|E : \mathbb{Q}|$.

c) Show that $\operatorname{Gal}(E/\mathbb{Q})$ contains an element of order 4.

a) Consider $\alpha^2 - \sqrt{2} = 2$. Rearranging, we obtain $\alpha^2 = 2 + \sqrt{2}$. We thus have $\alpha^4 = \left(2+\sqrt{2}\right)^2 \Rightarrow \alpha^4 - 4\sqrt{2} - 6 = 0$. We can rewrite $4\sqrt{2} + 6$ in terms of α^2, obtaining $4\sqrt{2} + 6 = 4\alpha^2 - 2$. Therefore $\alpha^4 - 4\alpha^2 + 2$. As such we obtain a potential minimal polynomial $x^4 - 4x^2 + 2$. Using Eisenstein's, we see it's irreducible and monic with α as a root. Therefore $f = x^4 - 4x^2 + 2$.

b) We can find the roots as $\pm\sqrt{2+\sqrt{2}}, \pm\sqrt{2-\sqrt{2}}$. We note that because they share the same minimal polymial, $|E : \mathbb{Q}| = \deg\left(\min_{\mathbb{Q}}(\alpha)\right) = 4$, and we have that $E = \mathbb{Q}\left(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}\right)$.

c) Consider the automorphism $\sigma \in \operatorname{Gal}(E/\mathbb{Q})$ such that $\varphi(\alpha) = \beta$, where $\beta = \sqrt{2-\sqrt{2}}$. Consider

$$\varphi\left(\sqrt{2}\right) = \varphi\left(\sqrt{\alpha\beta}\right) = \varphi(\alpha^2 - 2) = \varphi(\alpha)^2 - \varphi(2) = \beta^2 - 2 = -\sqrt{2}$$

Using this, we see that we can use σ recursively to obtain $\beta, \alpha, -\alpha, -\beta$. We thus show that σ is an element of order 4 in $\operatorname{Gal}(E/\mathbb{Q})$. ❤️

Problem 2

New Problem
Content 1 from part 1
Content 2 from part 2

a) Answering question posed from part 1.

b) Answering question posed from part 2.