

Methods 4 - 4

Chris Mathys



BSc Programme in Cognitive Science

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Categories, Curves & Splines

Linear models can do extra-linear things

Categories (dummy, indicator & index variables)

Polynomials and other simple curves

Splines and other additive structures



Drawing Inferences

How to use statistical models to get at scientific estimands?

Need to incorporate causal thinking into how we

- (1) draw the statistical models
- (2) process the results



Categories

How to cope with causes that are not continuous?

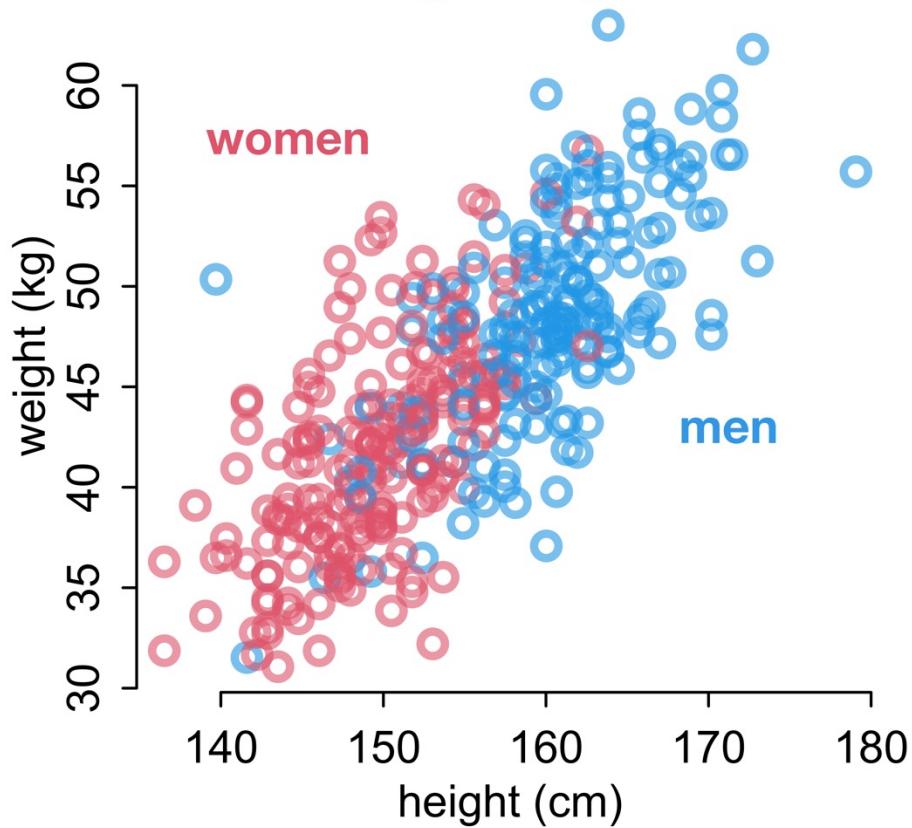
Categories: discrete, unordered types

Want to **stratify** by category:
Fit a separate line for each



```
> head(Howell1)
  height   weight age male
1 151.765 47.82561 63   1
2 139.700 36.48581 63   0
3 136.525 31.86484 65   0
4 156.845 53.04191 41   1
5 145.415 41.27687 51   0
6 163.830 62.99259 35   1
> █
```

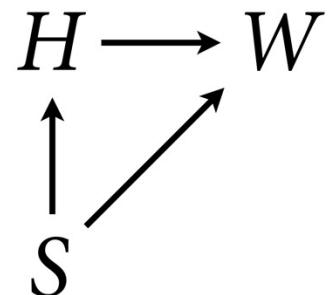
Adult height, weight, and sex



Think scientifically first

How are height, weight, and sex **causally** related?

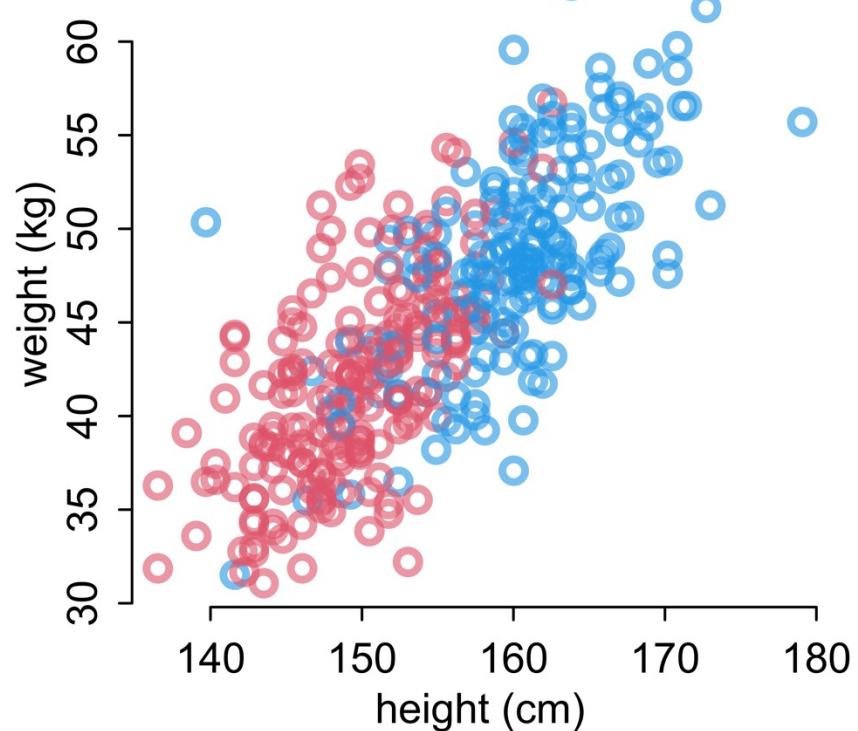
How are height, weight, and sex **statistically** related?



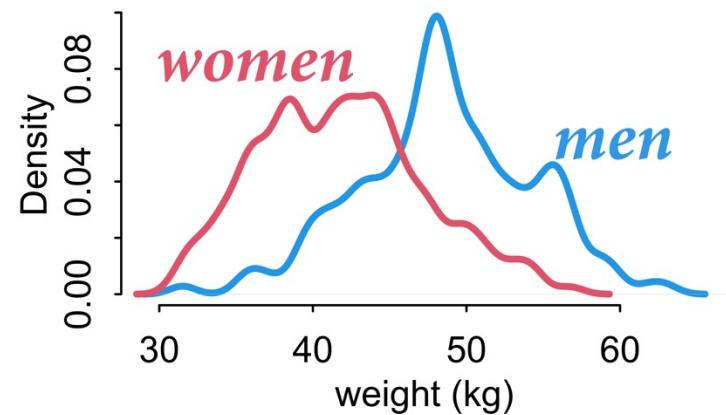
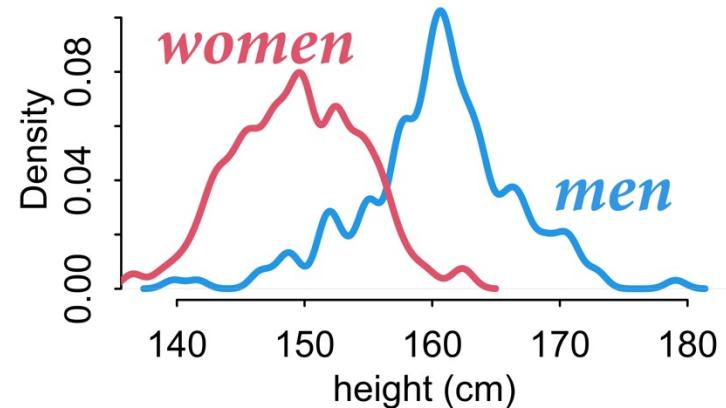
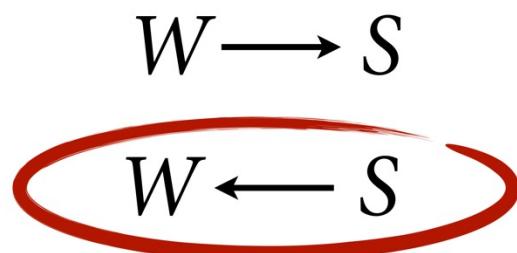
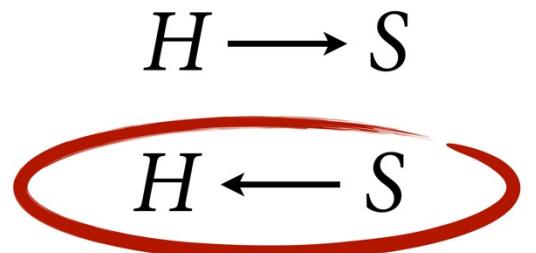
The causes aren't in the data

$$H \rightarrow W$$

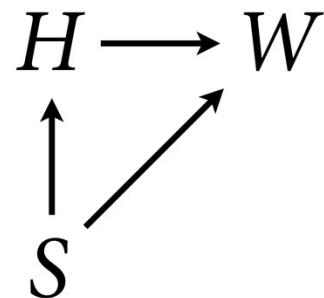
$$H \leftarrow W$$



The causes aren't in the data

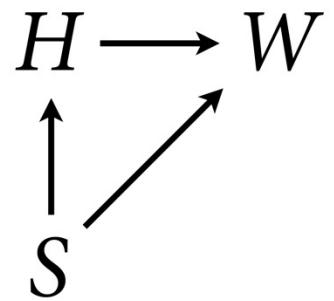


*height
influences
weight*



*weight is
influenced
by both
height & sex*

*sex influences both
height & weight*



$$H = f_H(S)$$

$$W = f_W(H, S)$$

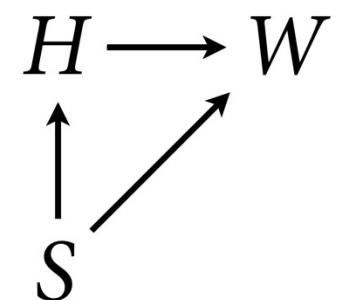
Think scientifically first

Different causal questions need
different statistical models

Q: Causal effect of H on W ?

Q: Causal effect of S on W ?

Q: Direct causal effect of S on W ?



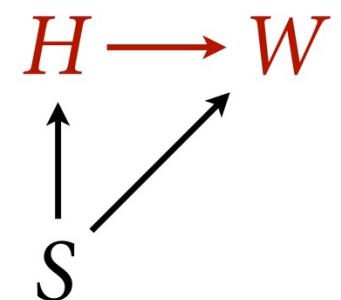
Think scientifically first

Different causal questions need
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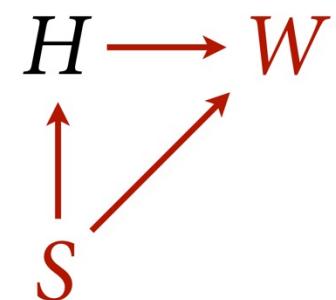
Think scientifically first

Different causal questions need
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Q: Causal effect of H on W ?

Q: Causal effect of S on W ?

Q: Direct causal effect of S on W ?



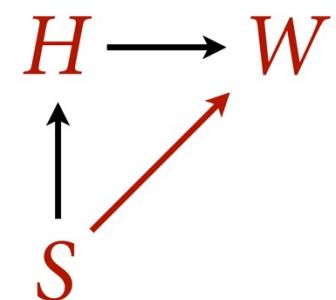
Think scientifically first

Different causal questions need
different statistical models

Q: Causal effect of H on W ?

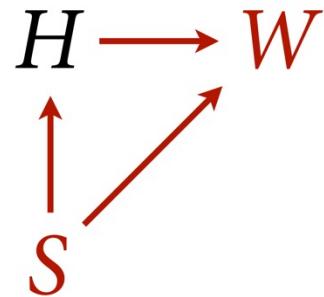
Q: Causal effect of S on W ?

Q: Direct causal effect of S on W ?

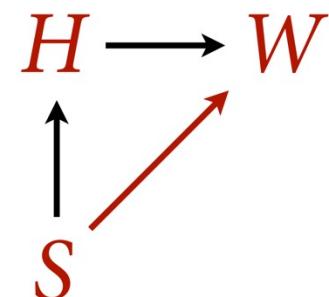


From estimand to estimate

Q: Causal effect of S on W ?



Q: Direct causal effect of S on W ?



*Need to model S as **categorical***

Drawing Categorical Owls

Several ways to code categorical variables

- (1) “dummy” and indicator (0/1) variables
- (2) index variables: 1,2,3,4,...

We will use index variables:

Extend to many categories with no change in code

Better for specifying priors

Extend effortlessly to multi-level models



Picasa

Drawing Categorical Owls

Example:

| | | | | |
|-------------|------|---------|--------|-------|
| Category | Cyan | Magenta | Yellow | Black |
| Index Value | 1 | 2 | 3 | 4 |



Influence of color coded by:

$$\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$$


Picasso

Drawing Categorical Owls

$$\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$$


$$y_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{\text{COLOR}[i]}$$

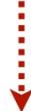


Picasso

Using Index Variables

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha$$



intercept

Using Index Variables

| | H | W | S |
|----|-----|----|---|
| 1 | 152 | 48 | 2 |
| 2 | 140 | 36 | 1 |
| 3 | 137 | 32 | 1 |
| 4 | 157 | 53 | 2 |
| 5 | 145 | 41 | 1 |
| 6 | 164 | 63 | 2 |
| 7 | 149 | 38 | 1 |
| 8 | 169 | 55 | 2 |
| 9 | 148 | 35 | 1 |
| 10 | 165 | 54 | 2 |
| 11 | 154 | 50 | 1 |
| 12 | 151 | 41 | 2 |

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]}$$



sex of i -th person $S = 1$ indicates female
 $S = 2$ indicates male

$$\alpha = [\alpha_1, \alpha_2]$$

Two intercepts, one for each value in S

Using Index Variables

| | H | W | S |
|----|-----|----|---|
| 1 | 152 | 48 | 2 |
| 2 | 140 | 36 | 1 |
| 3 | 137 | 32 | 1 |
| 4 | 157 | 53 | 2 |
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| 6 | 164 | 63 | 2 |
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| 8 | 169 | 55 | 2 |
| 9 | 148 | 35 | 1 |
| 10 | 165 | 54 | 2 |
| 11 | 154 | 50 | 1 |
| 12 | 151 | 41 | 2 |

$i = 1$

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]}$$

$\alpha = [\alpha_1, \alpha_2]$

A red arrow points from the text "S[1]=2" up towards the second element of the vector α , which is enclosed in a red box.

Using Index Variables

| | H | W | S |
|----|-----|----|---|
| 1 | 152 | 48 | 2 |
| 2 | 140 | 36 | 1 |
| 3 | 137 | 32 | 1 |
| 4 | 157 | 53 | 2 |
| 5 | 145 | 41 | 1 |
| 6 | 164 | 63 | 2 |
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| 8 | 169 | 55 | 2 |
| 9 | 148 | 35 | 1 |
| 10 | 165 | 54 | 2 |
| 11 | 154 | 50 | 1 |
| 12 | 151 | 41 | 2 |

$i = 2$

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha S[i]$$

$$S[2]=1$$

$$\alpha = [\alpha_1, \alpha_2]$$

Using Index Variables

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]}$$

Priors

$$\alpha = [\alpha_1, \alpha_2]$$

$$\alpha_j \sim \text{Normal}(60, 10)$$

Using Index Variables

```
data(Howell1)
d <- Howell1
d <- d[ d$age>=18 , ]
dat <- list(
  W = d$weight,
  S = d$male + 1 ) # S=1 female, S=2 male

m_SW <- quap(
  alist(
    W ~ dnorm(mu,sigma),
    mu <- a[S],
    a[S] ~ dnorm(60,10),
    sigma ~ dunif(0,10)
  ), data=dat )
```

quap() does the
indexing for you

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

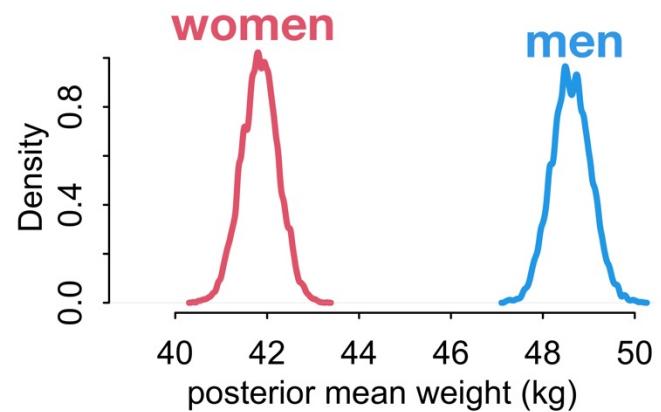
$$\mu_i = \alpha_{S[i]}$$

$$\alpha_j \sim \text{Normal}(60, 10)$$

$$\sigma \sim \text{Uniform}(0, 10)$$

Posterior means & predictions

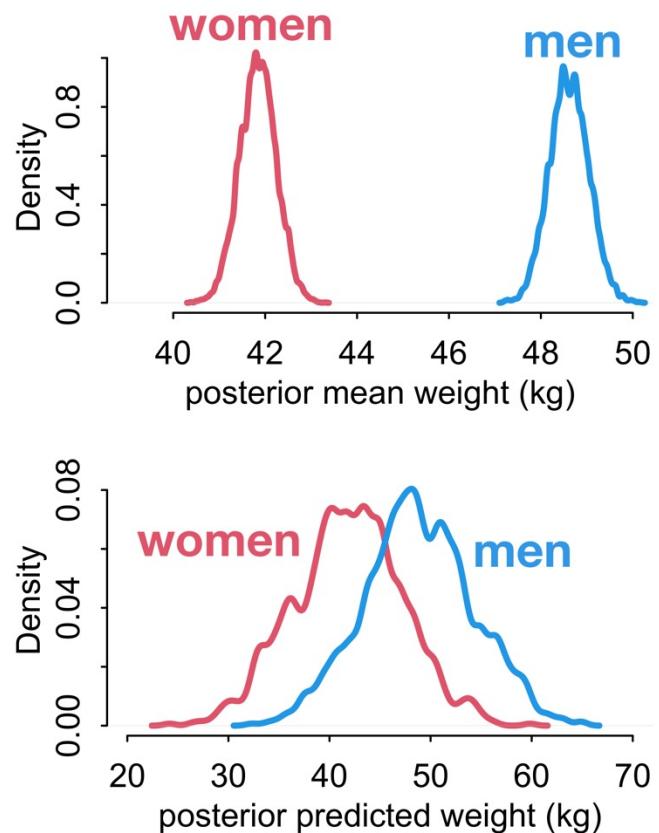
```
# posterior mean W
post <- extract.samples(m_SW)
dens( post$a[,1] , xlim=c(39,50) , lwd=3 ,
col=2 , xlab="posterior mean weight (kg)" )
dens( post$a[,2] , lwd=3 , col=4 , add=TRUE )
```



Posterior means & predictions

```
# posterior mean W
post <- extract.samples(m_SW)
dens( post$a[,1] , xlim=c(39,50) , lwd=3 ,
col=2 , xlab="posterior mean weight (kg)" )
dens( post$a[,2] , lwd=3 , col=4 , add=TRUE )

# posterior W distributions
W1 <- rnorm( 1000 , post$a[,1] , post$sigma )
W2 <- rnorm( 1000 , post$a[,2] , post$sigma )
dens( W1 , xlim=c(20,70) , ylim=c(0,0.085) ,
lwd=3 , col=2 )
dens( W2 , lwd=3 , col=4 , add=TRUE )
```

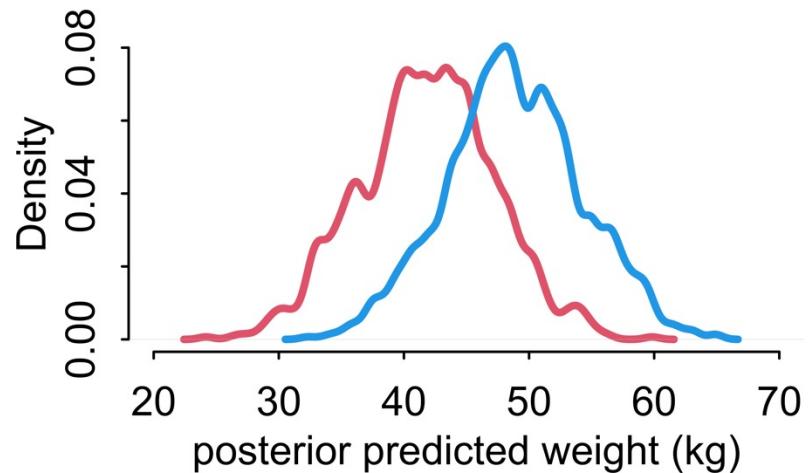


Always Be Contrasting

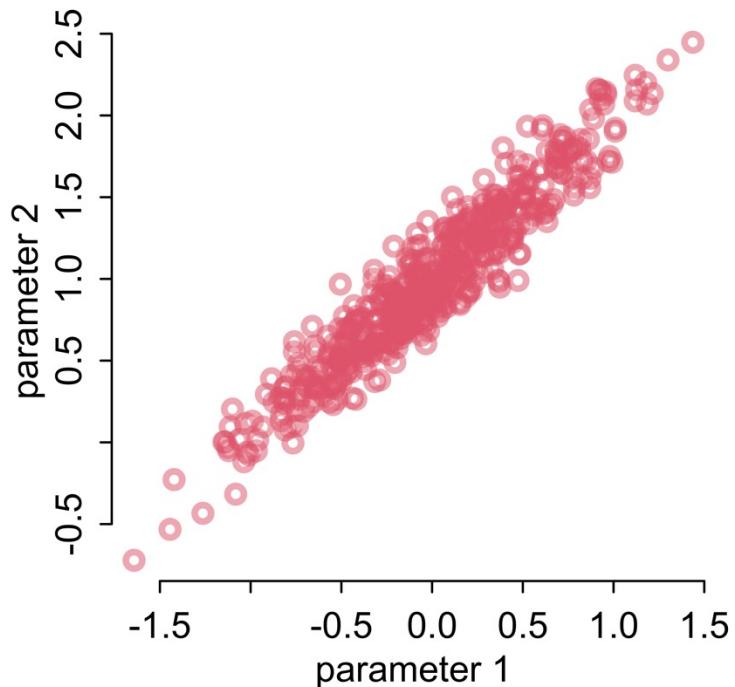
Need to compute **contrast**, the difference between the categories

It is **never** legitimate to compare **overlap** in parameters

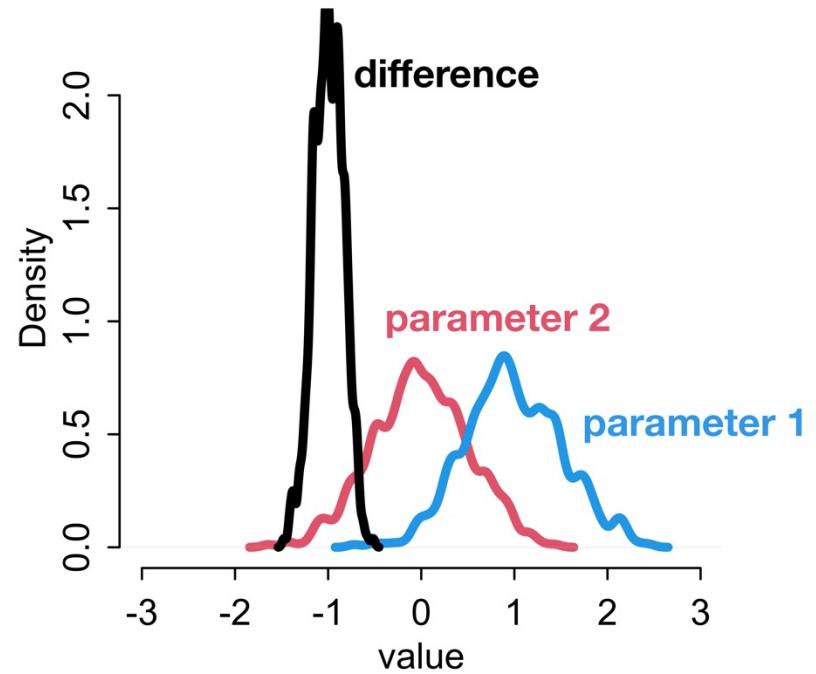
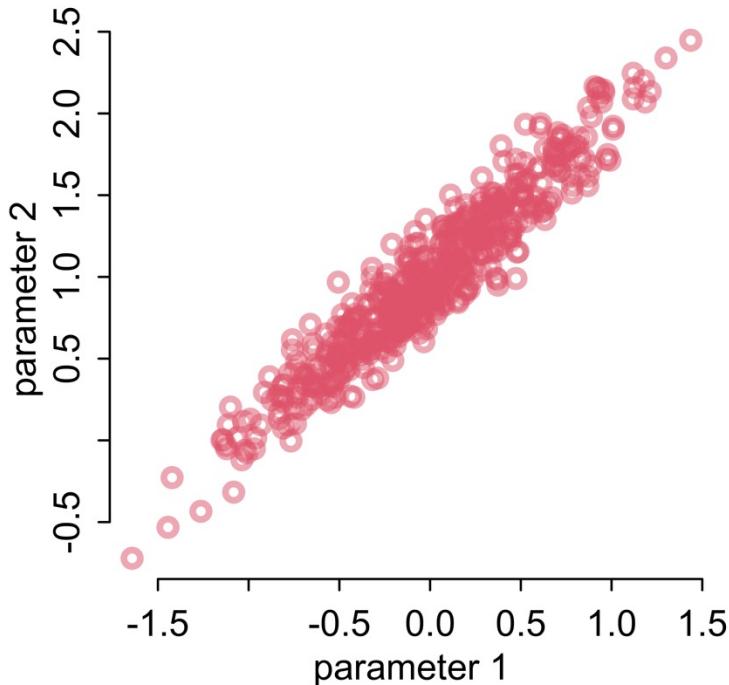
Must compute **contrast distribution**



It is **never** legitimate to compare **overlap** in parameters or lines



It is **never** legitimate to compare **overlap** in parameters or lines



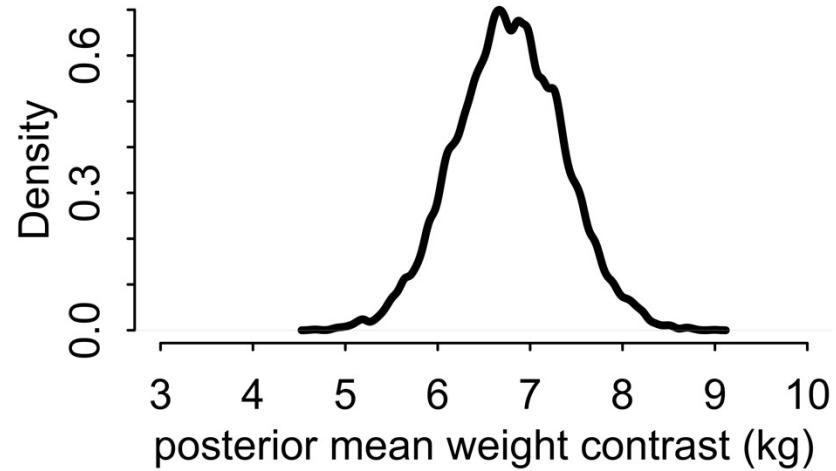
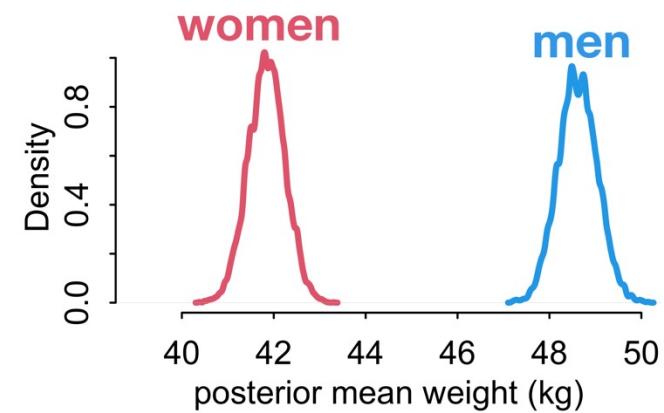
This means no comparing **confidence intervals** or **p-values** either!

Causal contrast

```
|> str(post)
List of 2
 $ sigma: num [1:10000] 5.38 5.54 5.45 5.72 5.28 ...
 $ a      : num [1:10000, 1:2] 42.8 41.5 41 41.9 42 ...
```

```
# causal contrast (in means)
mu_contrast <- post$a[,2] - post$a[,1]

dens( mu_contrast , xlim=c(3,10) , lwd=3 ,
col=1 , xlab="posterior mean weight contrast
(kg)" )
```

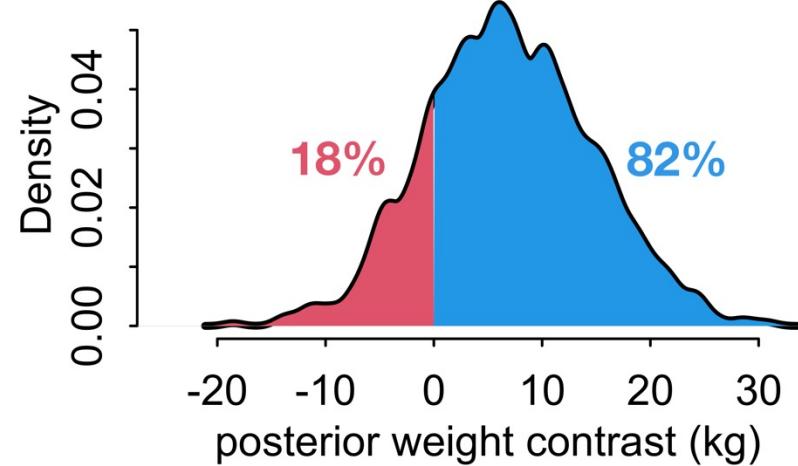
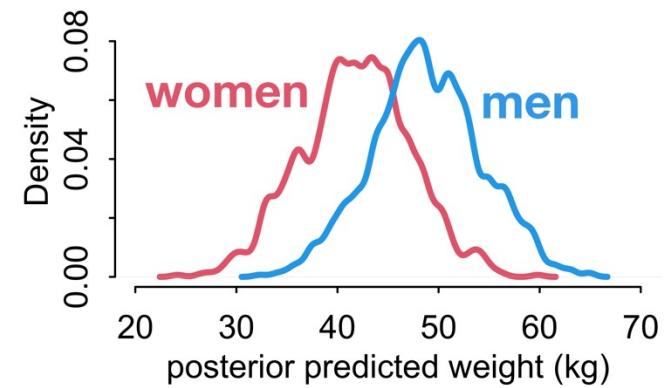


Weight contrast

```
# posterior W distributions
W1 <- rnorm( 1000 , post$a[,1] , post$sigma )
W2 <- rnorm( 1000 , post$a[,2] , post$sigma )

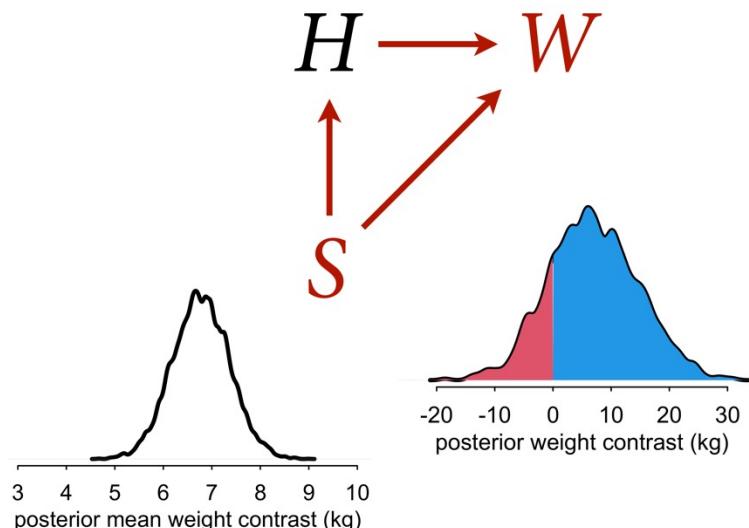
# contrast
W_contrast <- W2 - W1
dens( W_contrast , xlim=c(-25,35) , lwd=3 ,
col=1 , xlab="posterior weight contrast (kg)" )

# proportion above zero
sum( W_contrast > 0 ) / 1000
# proportion below zero
sum( W_contrast < 0 ) / 1000
```

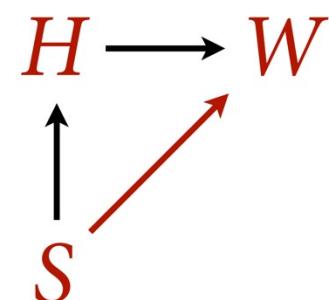


From estimand to estimate

Q: Causal effect of S on W ?



Q: Direct causal effect of S on W ?



Index Variables & Lines

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta(H_i - \bar{H})$$


intercept *slope*

Index Variables & Lines

| | H | W | S |
|----|-----|----|---|
| 1 | 152 | 48 | 2 |
| 2 | 140 | 36 | 1 |
| 3 | 137 | 32 | 1 |
| 4 | 157 | 53 | 2 |
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| 9 | 148 | 35 | 1 |
| 10 | 165 | 54 | 2 |
| 11 | 154 | 50 | 1 |
| 12 | 151 | 41 | 2 |

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]} + \beta_{S[i]}(H_i - \bar{H})$$



sex of i-th person *S = 1 indicates female*
S = 2 indicates male

$$\alpha = [\alpha_1, \alpha_2] \quad \beta = [\beta_1, \beta_2]$$

Two intercepts and two slopes, one for each value in S

Index Variables & Lines

| | H | W | S |
|----|-----|----|---|
| 1 | 152 | 48 | 2 |
| 2 | 140 | 36 | 1 |
| 3 | 137 | 32 | 1 |
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| 8 | 169 | 55 | 2 |
| 9 | 148 | 35 | 1 |
| 10 | 165 | 54 | 2 |
| 11 | 154 | 50 | 1 |
| 12 | 151 | 41 | 2 |

$i = 1$

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]} + \beta_{S[i]}(H_i - \bar{H})$$

$$\alpha = [\alpha_1, \alpha_2] \quad \beta = [\beta_1, \beta_2]$$

$S[1]=2$

```
graph TD; S[1] --> alpha2["alpha<br>[alpha1, alpha2]"]; S[1] --> beta2["beta<br>[beta1, beta2]"]
```

Index Variables & Lines

| | H | W | S |
|----|-----|----|---|
| 1 | 152 | 48 | 2 |
| 2 | 140 | 36 | 1 |
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| 12 | 151 | 41 | 2 |

$i = 2$

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]} + \beta_{S[i]}(H_i - \bar{H})$$

$$S[2]=1$$

$$\alpha = [\alpha_1, \alpha_2] \quad \beta = [\beta_1, \beta_2]$$

Index Variables & Lines

```
data(Howell1)
d <- Howell1
d <- d[ d$age>=18 , ]
dat <- list(
  W = d$weight,
  H = d$height,
  Hbar = mean(d$height),
  S = d$male + 1 ) # S=1 female, S=2 male

m_SHW <- quap(
  alist(
    W ~ dnorm(mu,sigma),
    mu <- a[S] + b[S]*(H-Hbar),
    a[S] ~ dnorm(60,10),
    b[S] ~ dlnorm(0,1),
    sigma ~ dunif(0,10)
  ), data=dat )
```

$$W_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha_{S[i]} + \beta_{S[i]}(H_i - \bar{H})$$

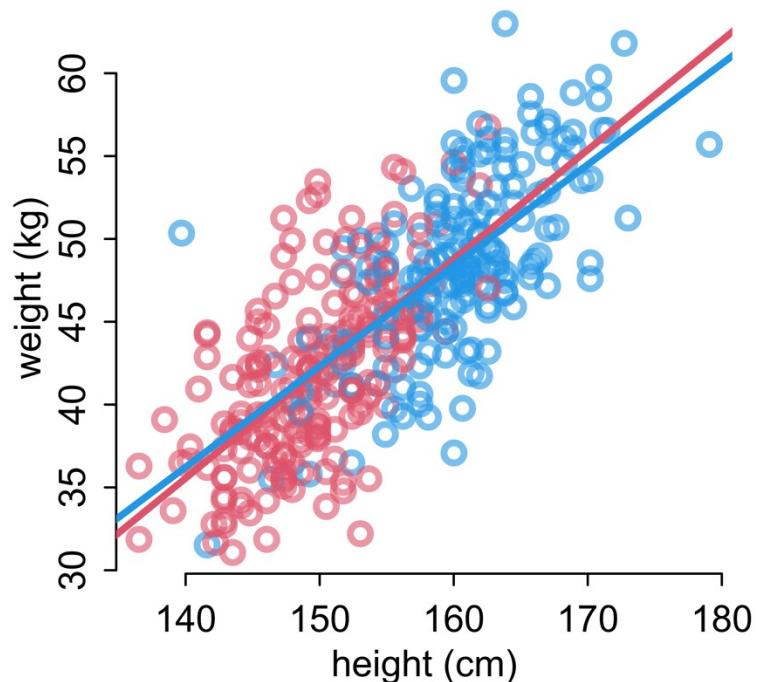
$$\alpha_j \sim \text{Normal}(60, 10)$$

$$\beta_j \sim \text{LogNormal}(0, 1)$$

$$\sigma \sim \text{Uniform}(0, 10)$$

Contrasts at each height

- (1) Compute posterior predictive for women
- (2) Compute posterior predictive for men
- (3) Subtract (2) from (1)
- (4) Plot distribution at each height (on right)



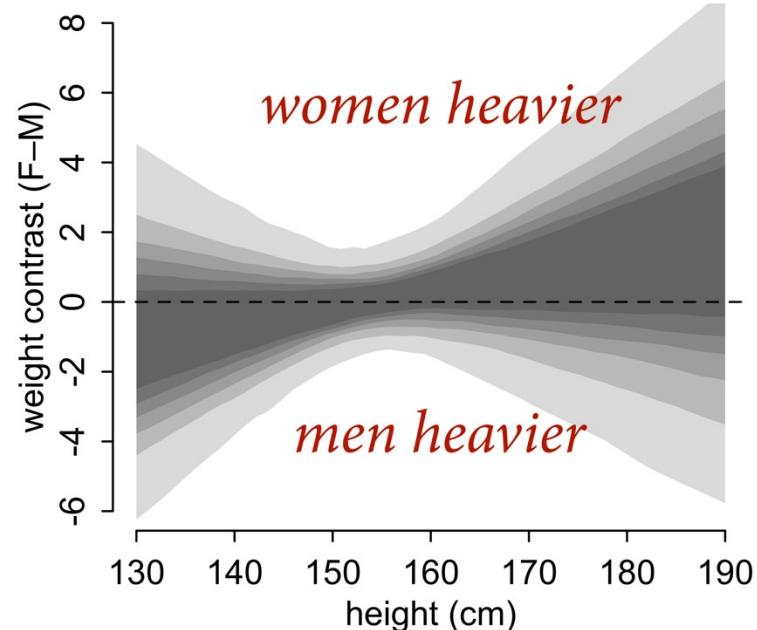
Contrasts at each height

```
xseq <- seq(from=130,to=190,len=50)

muF <-
link(m_adults2,data=list(S=rep(1,50),H=xseq,Hbar=mean(d$height)))
lines( xseq , apply(muF,2,mean) , lwd=3 , col=2 )

muM <-
link(m_adults2,data=list(S=rep(2,50),H=xseq,Hbar=mean(d$height)))
lines( xseq , apply(muM,2,mean) , lwd=3 , col=4 )

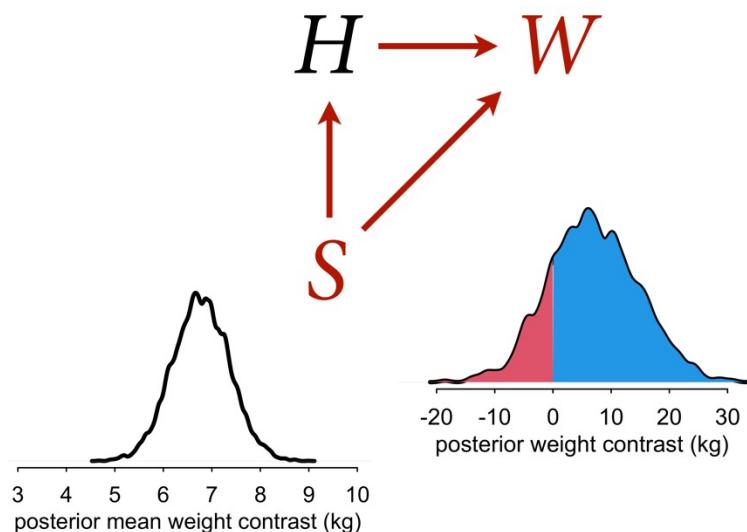
mu_contrast <- muF - muM
plot( NULL , xlim=range(xseq) , ylim=c(-6,8) , xlab="height (cm)"
, ylab="weight contrast (F-M)" )
for ( p in c(0.5,0.6,0.7,0.8,0.9,0.99) )
  shade( apply(mu_contrast,2,PI,prob=p) , xseq )
abline(h=0,lty=2)
```



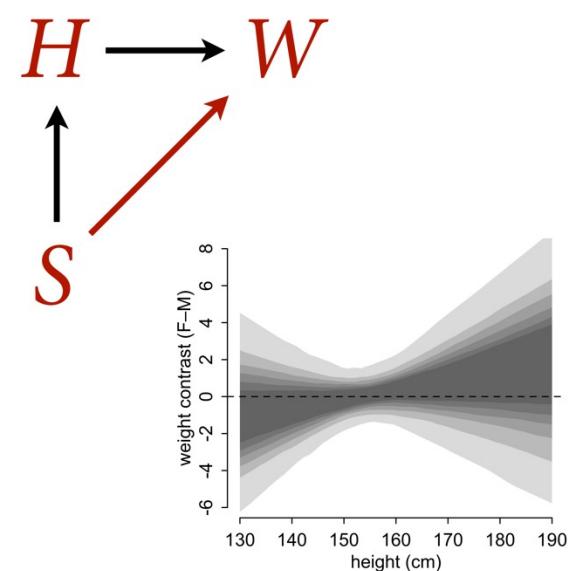
Nearly all of the causal effect of S acts through H

From estimand to estimate

Q: Causal effect of S on W ?



Q: Direct causal effect of S on W ?

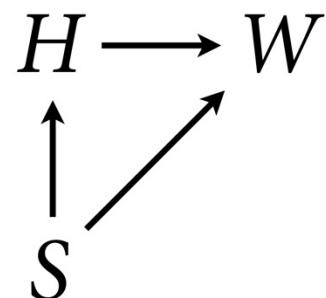


Full Luxury Bayes

We used two models for two estimands

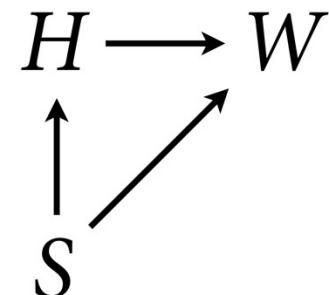
But alternative and equivalent approach is to use one model of entire causal system

Then use joint posterior to compute each estimand



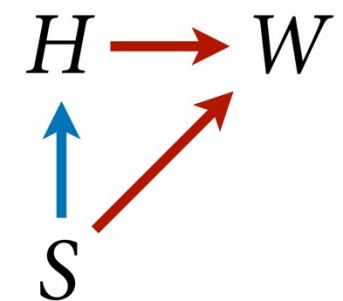
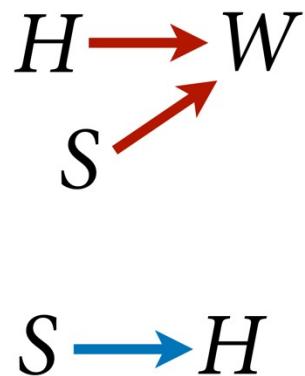
Full Luxury Bayes

```
m_SHW_full <- quap(  
  alist(  
  
    # weight  
    W ~ dnorm(mu,sigma),  
    mu <- a[S] + b[S] * (H-Hbar),  
    a[S] ~ dnorm(60,10),  
    b[S] ~ dlnorm(0,1),  
    sigma ~ dunif(0,10),  
  
    # height  
    H ~ dnorm(nu,tau),  
    nu <- h[S],  
    h[S] ~ dnorm(160,10),  
    tau ~ dunif(0,10)  
  
  ), data=dat )
```



Full Luxury Bayes

```
m_SHW_full <- quap(  
  alist(  
  
    # weight  
    W ~ dnorm(mu,sigma),  
    mu <- a[S] + b[S]*(H-Hbar),  
    a[S] ~ dnorm(60,10),  
    b[S] ~ dlnorm(0,1),  
    sigma ~ dunif(0,10),  
  
    # height  
    H ~ dnorm(nu,tau),  
    nu <- h[S],  
    h[S] ~ dnorm(160,10),  
    tau ~ dunif(0,10)  
  
  ), data=dat )
```



Full Luxury Bayes

```
m_SHW_full <- quap(  
  alist(  
  
    # weight  
    W ~ dnorm(mu,sigma),  
    mu <- a[S] + b[S]*(H-Hbar),  
    a[S] ~ dnorm(60,10),  
    b[S] ~ dlnorm(0,1),  
    sigma ~ dunif(0,10),  
  
    # height  
    H ~ dnorm(nu,tau),  
    nu <- h[S],  
    h[S] ~ dnorm(160,10),  
    tau ~ dunif(0,10)  
  
, data=dat )
```

```
> precis(m_SHW_full,depth=2)  
      mean   sd  5.5% 94.5%  
a[1]  45.17 0.44 44.47 45.87  
a[2]  45.09 0.46 44.37 45.82  
h[1]  149.53 0.40 148.89 150.18  
h[2]  160.36 0.43 159.67 161.04  
b[1]   0.66 0.06  0.56  0.75  
b[2]   0.61 0.05  0.52  0.70  
sigma  4.23 0.16  3.97  4.48  
tau    5.52 0.21  5.19  5.85  
> █
```

Causal effect is consequence
of intervention

Now simulate each
intervention

Simulating Interventions

```
post <- extract.samples(m_SHW_full)
Hbar <- dat$Hbar
n <- 1e4

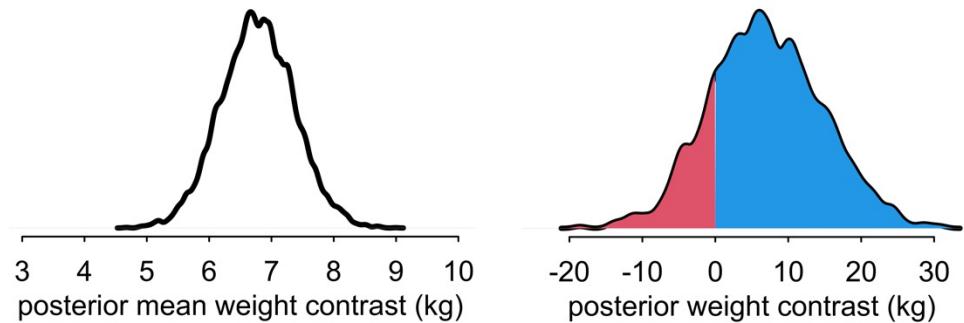
with( post , {
  # simulate W for S=1
  H_S1 <- rnorm(n, h[,1] , tau )
  W_S1 <- rnorm(n, a[,1] +
    b[,1]*(H_S1-Hbar) , sigma)

  # simulate W for S=2
  H_S2 <- rnorm(n, h[,2] , tau)
  W_S2 <- rnorm(n, a[,2] +
    b[,2]*(H_S2-Hbar) , sigma)

  # compute contrast
  W_do_S <-> W_S2 - W_S1
})
```

Total causal effect of S on W :
Consequence of changing S at birth

$$p(W|\text{do}(S))$$



Simulating Interventions

```
post <- extract.samples(m_SHW_full)
Hbar <- dat$Hbar
n <- 1e4

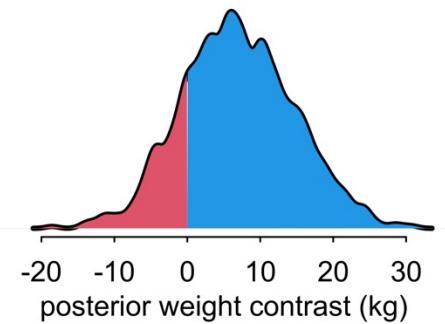
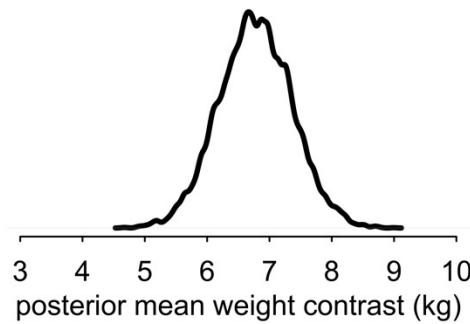
with( post , {
  # simulate W for S=1
  H_S1 <- rnorm(n, h[,1] , tau )
  W_S1 <- rnorm(n, a[,1] +
    b[,1]* (H_S1-Hbar) , sigma)

  # simulate W for S=2
  H_S2 <- rnorm(n, h[,2] , tau)
  W_S2 <- rnorm(n, a[,2] +
    b[,2]* (H_S2-Hbar) , sigma)

  # compute contrast
  W_do_S <-> W_S2 - W_S1
})
```

```
# automated way
HWsim <- sim(m_SHW_full,
  data=list(S=c(1,2)),
  vars=c("H","W"))
W_do_S_auto <- HWsim$W[,2] - HWsim$W[,1]
```

$$p(W|\text{do}(S))$$



Inference With Linear Models

With more than two variables, scientific (causal) model and statistical model not always same

- (1) State each estimand
- (2) Design **unique statistical model** for each
- (3) **Compute** each estimand

Or -----

- (1) State each estimand
- (2) Compute joint posterior for **causal system**
- (3) **Simulate** each estimand as an **intervention**

**ONE STAT MODEL
FOR EACH ESTIMAND**

**ONE SIMULATION
FOR EACH ESTIMAND**

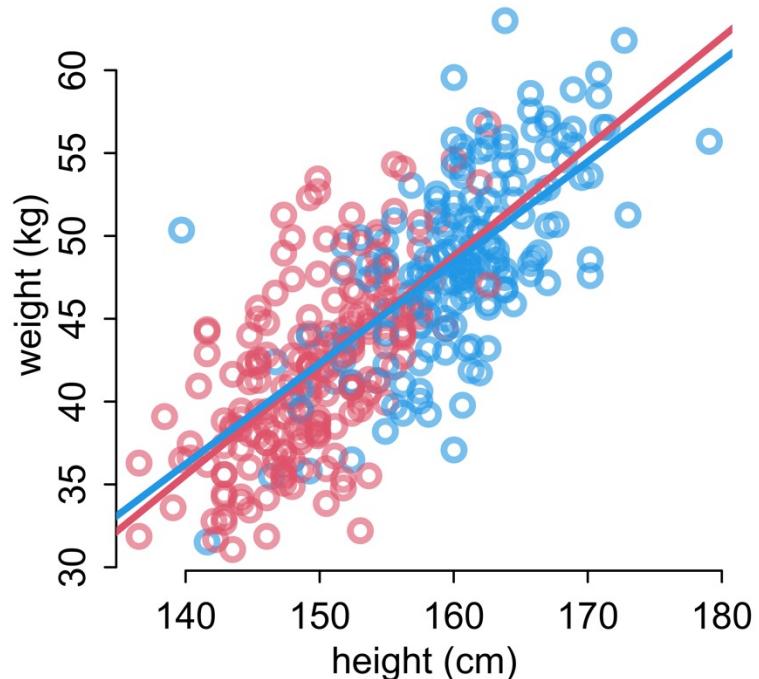
Categorical variables

Easy to use with index coding

Must later use samples to compute relevant contrasts

Always summarize (mean, interval) as the last step

Want **mean difference** and not **difference of means**



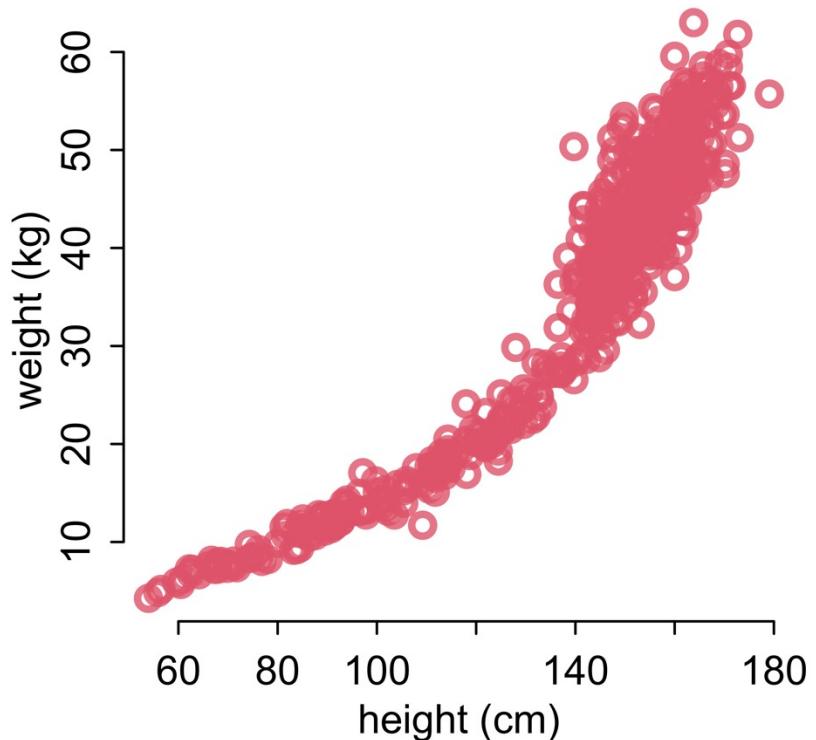
Curves from lines

$H \rightarrow W$ obviously not linear

Linear models can easily fit curves

But this is not **mechanistic**

```
library(rethinking)  
data(Howell1)
```



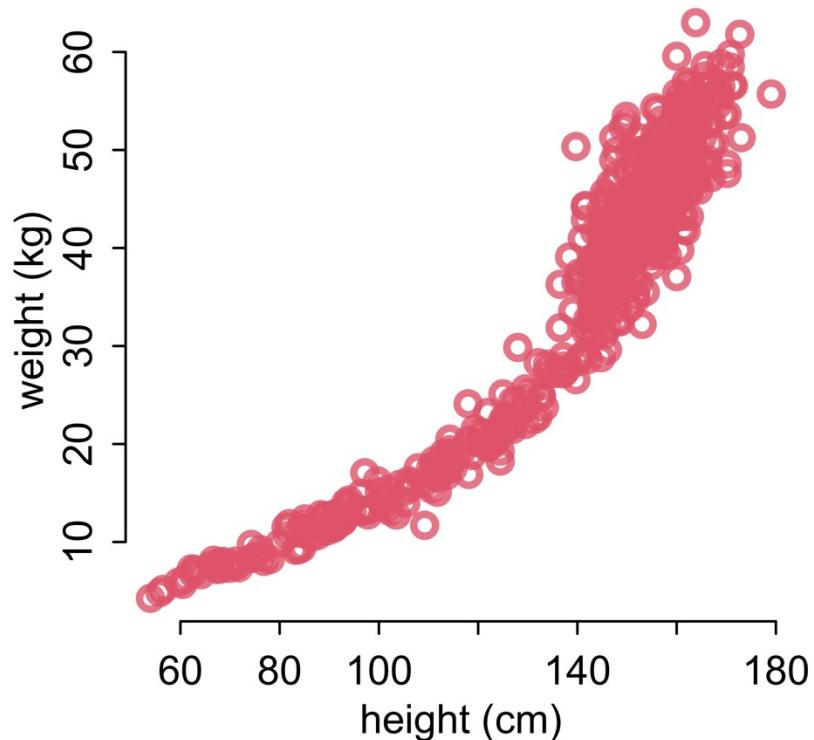
Curves from lines

Linear models can easily fit curves

Two popular strategies

- (1) polynomials — be wary
- (2) splines and generalized additive models — better

```
library(rethinking)  
data(Howell1)
```

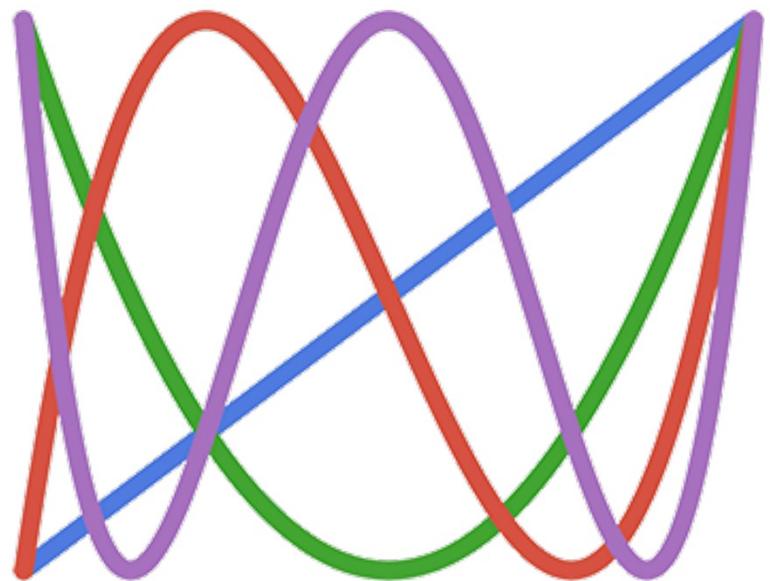


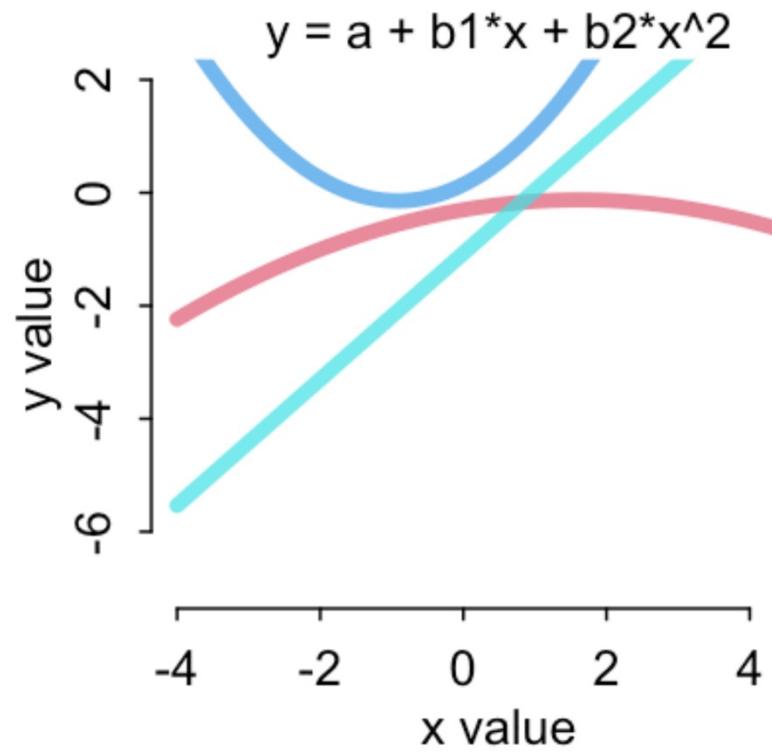
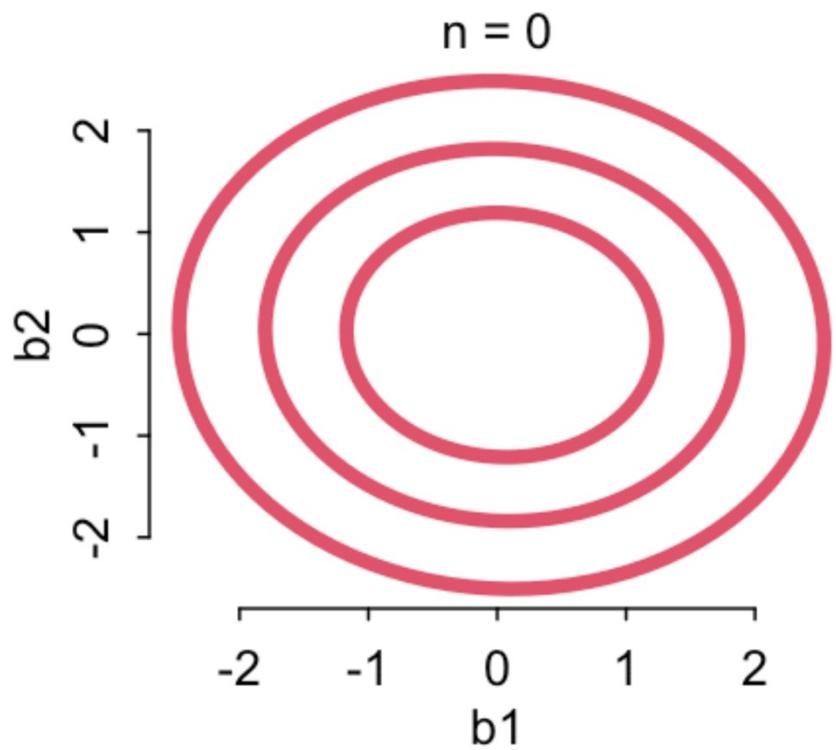
Polynomial linear models

Strategy: polynomial functions

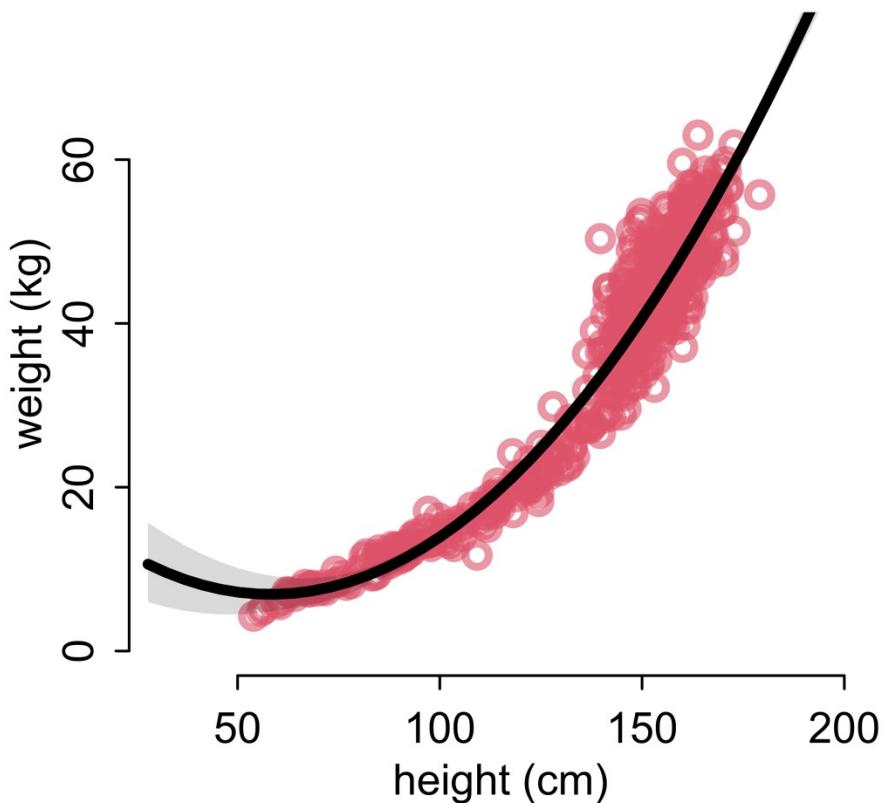
Problems: strange symmetries,
explosive uncertainty at edges

$$\mu_i = \alpha + \beta_1 x_i + \beta_2 x_i^2$$

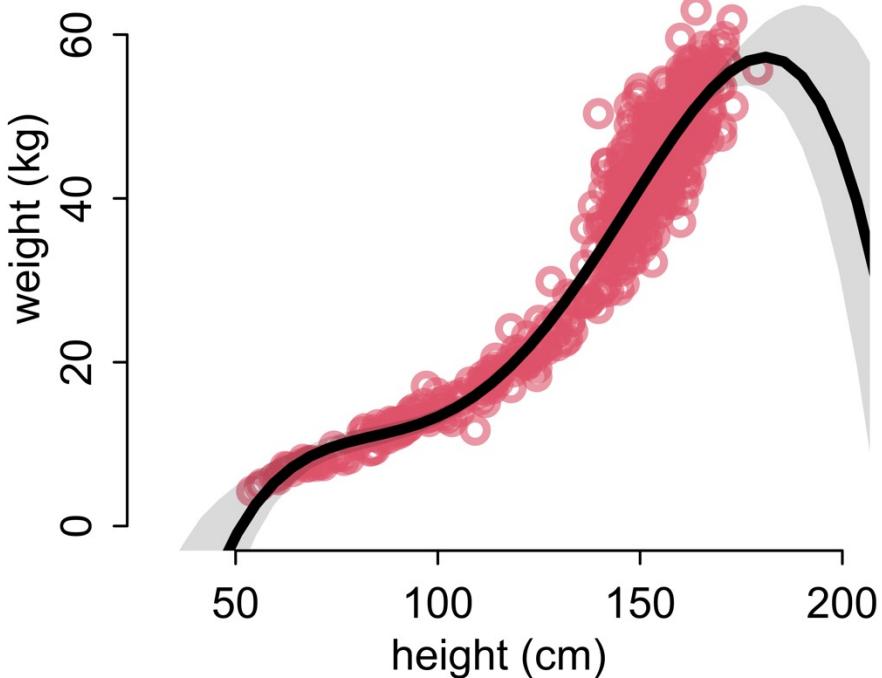




$$\mu_i = \alpha + \beta_1 H_i + \beta_2 H_i^2$$



$$\begin{aligned}\mu_i = & \alpha + \beta_1 H_i + \beta_2 H_i^2 \\ & + b_3 H_i^3 + b_4 H_i^4\end{aligned}$$



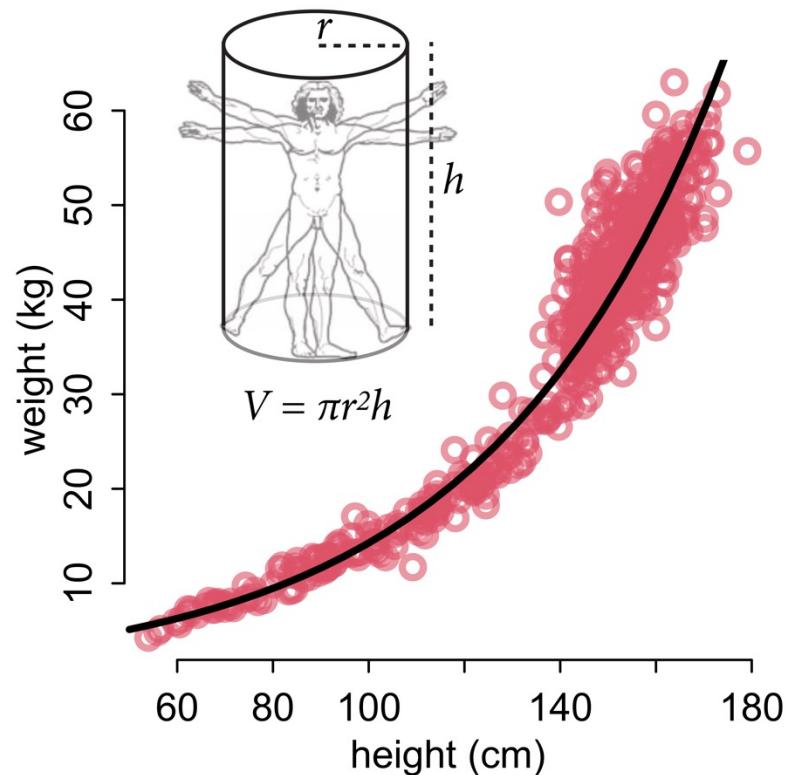
Thinking vs Fitting

Linear models can fit anything
(Geocentrism)

Better to think

$$\log W_i \sim \text{Normal}(\mu_i, \sigma)$$
$$\mu_i = \alpha + \beta(H_i - \bar{H})$$

Will revisit in lecture 19

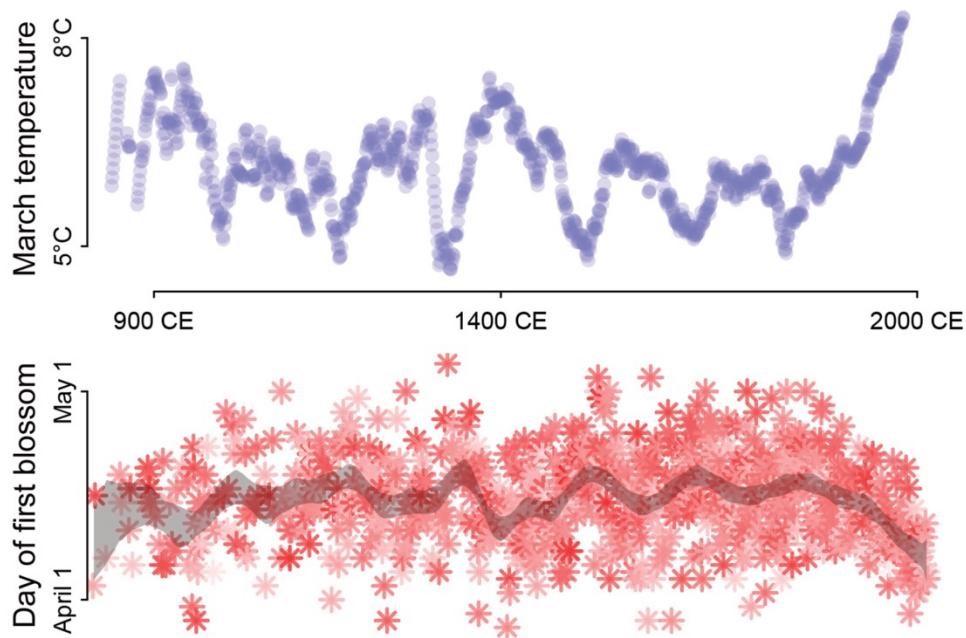


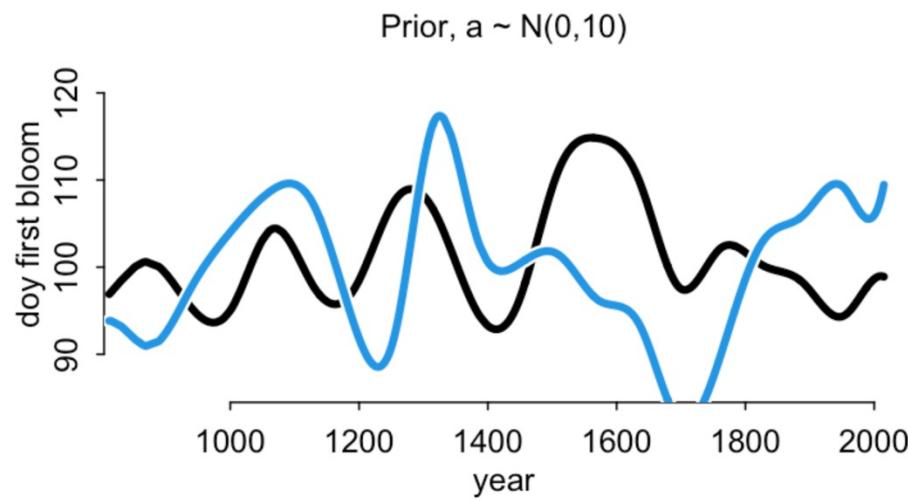
Splines

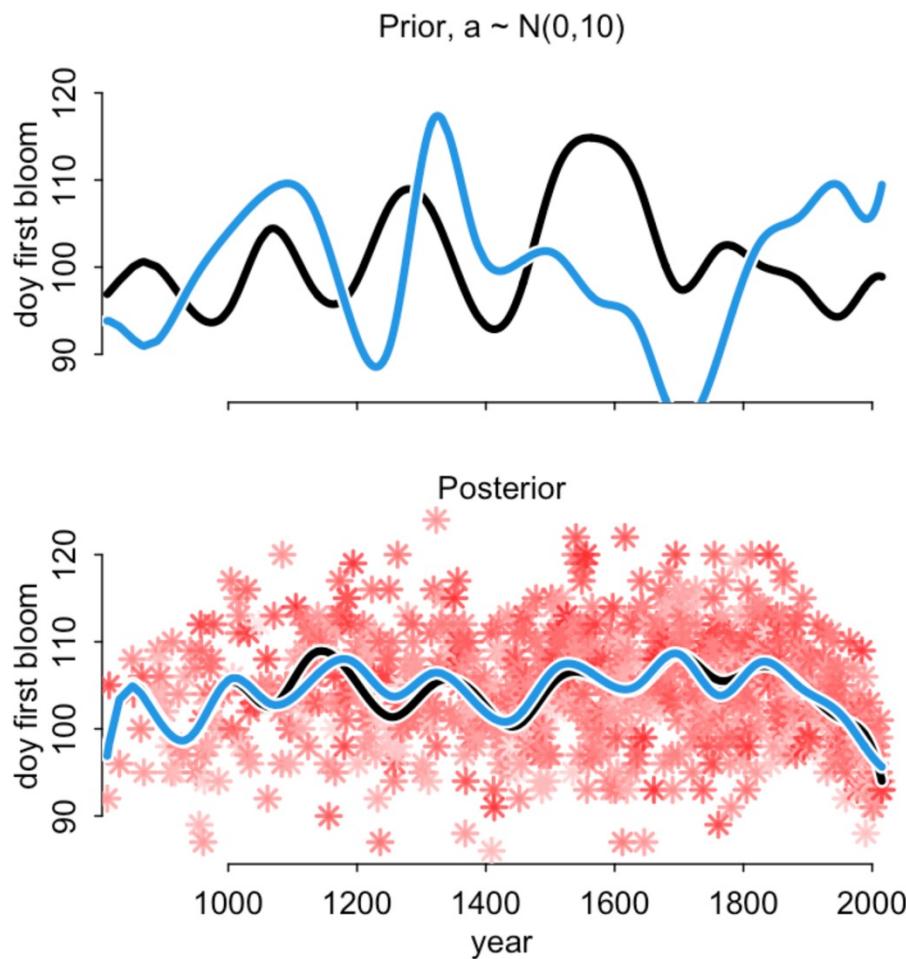
Basis-Splines: Wiggly function built from many local functions

Basis function: A local function

Local functions make splines better than polynomials, but equally geocentric







Going Local — B-Splines

B-Splines are just linear models, but with some weird synthetic variables:

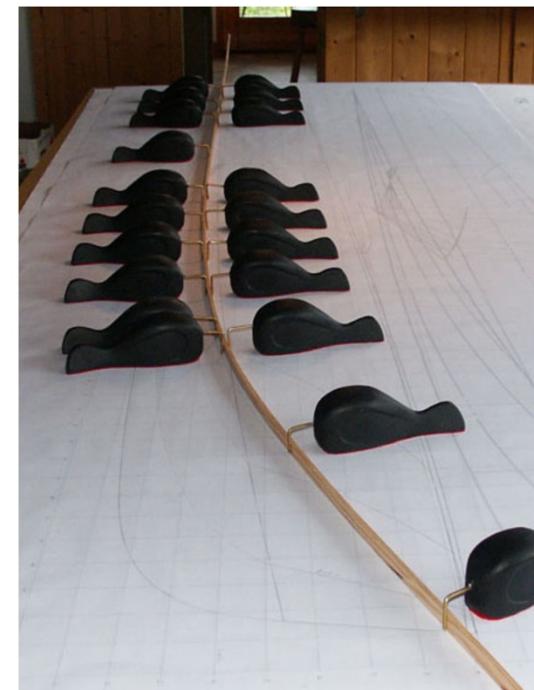
$$\mu_i = \alpha + w_1 B_{i,1} + w_2 B_{i,2} + w_3 B_{i,3} + \dots$$

Weights w are like slopes

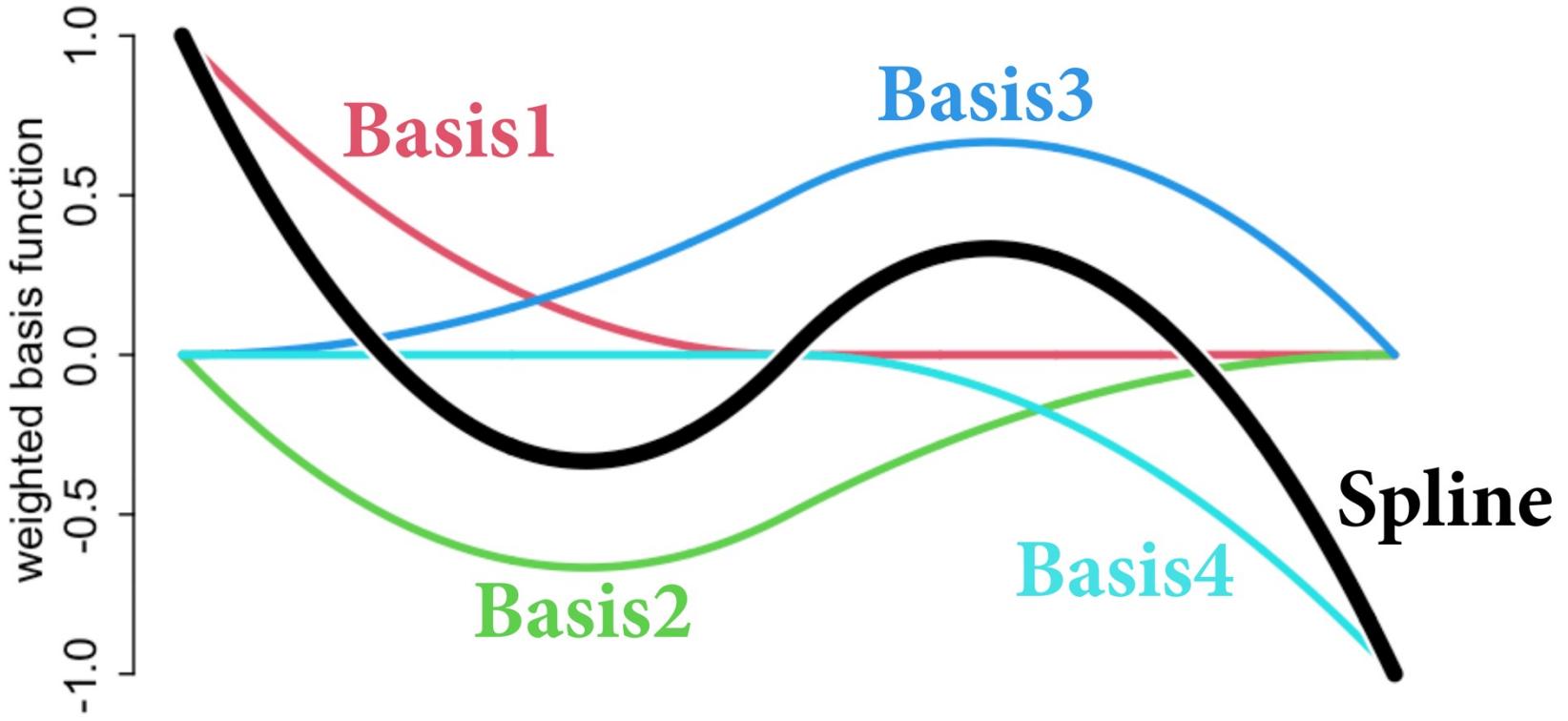
Basis functions B are synthetic variables

B values turn on weights in different regions

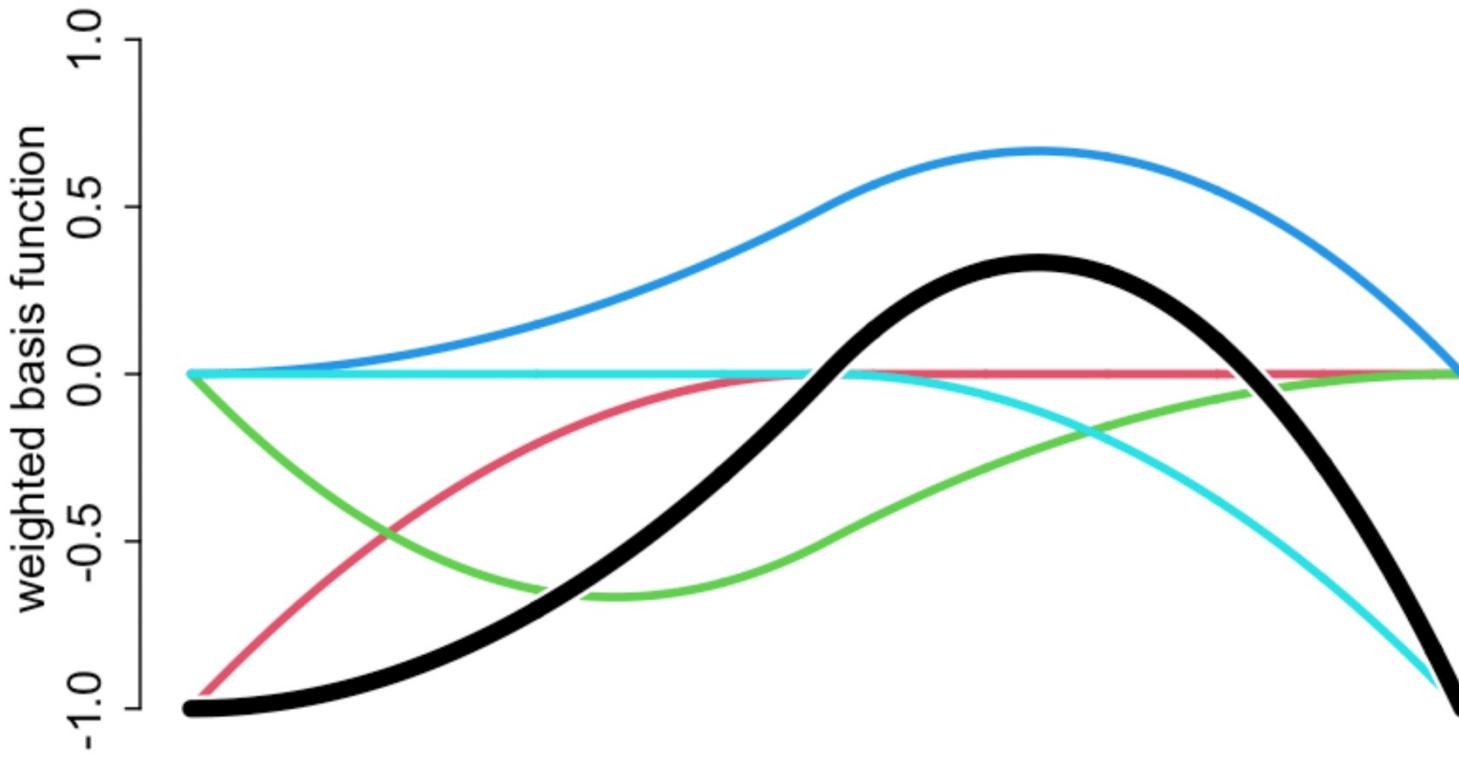
Detailed example starting on page 114



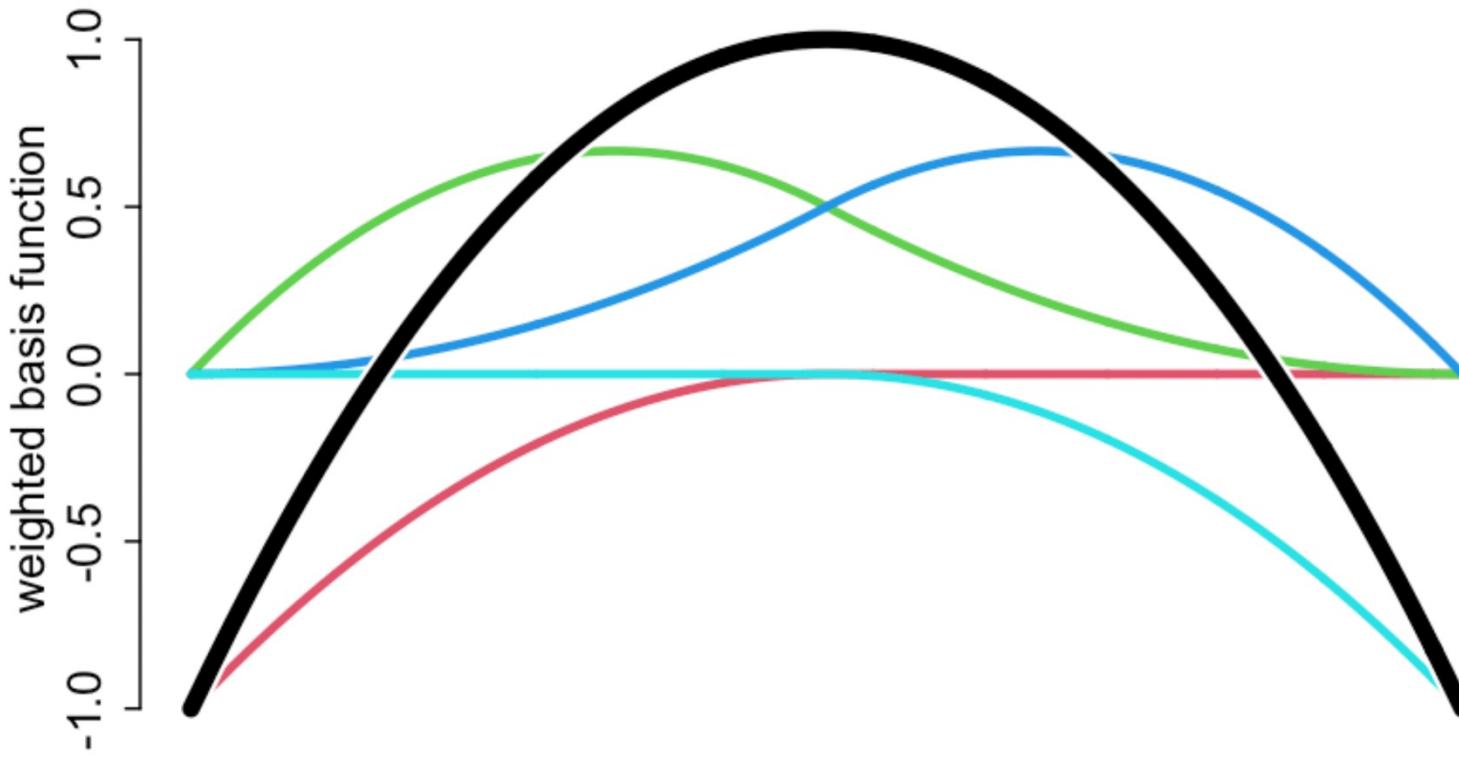
$$w = [1, -1, 1, -1]$$



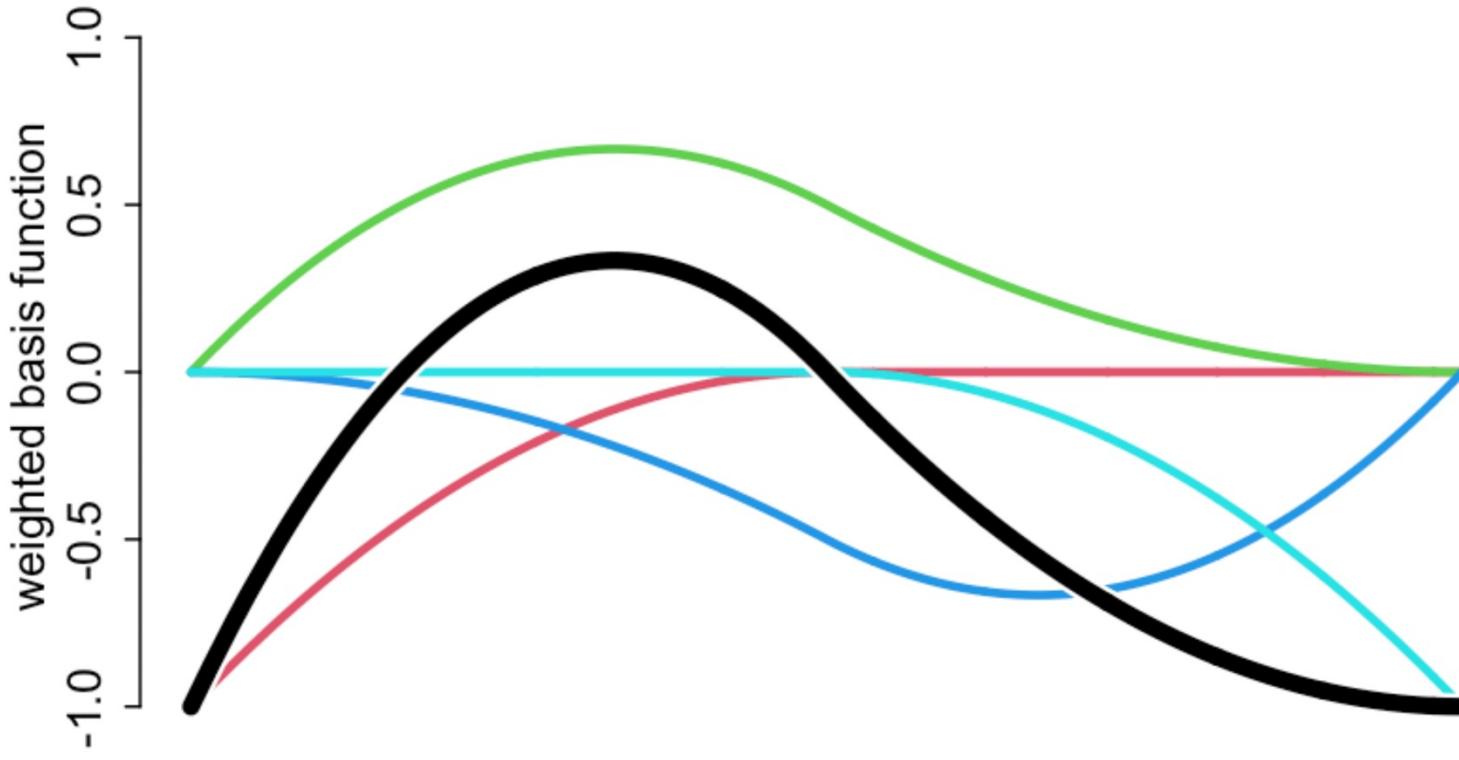
$$w = [-1, -1, 1, -1]$$



$$w = [-1, 1, 1, -1]$$



$$w = [-1, 1, -1, -1]$$

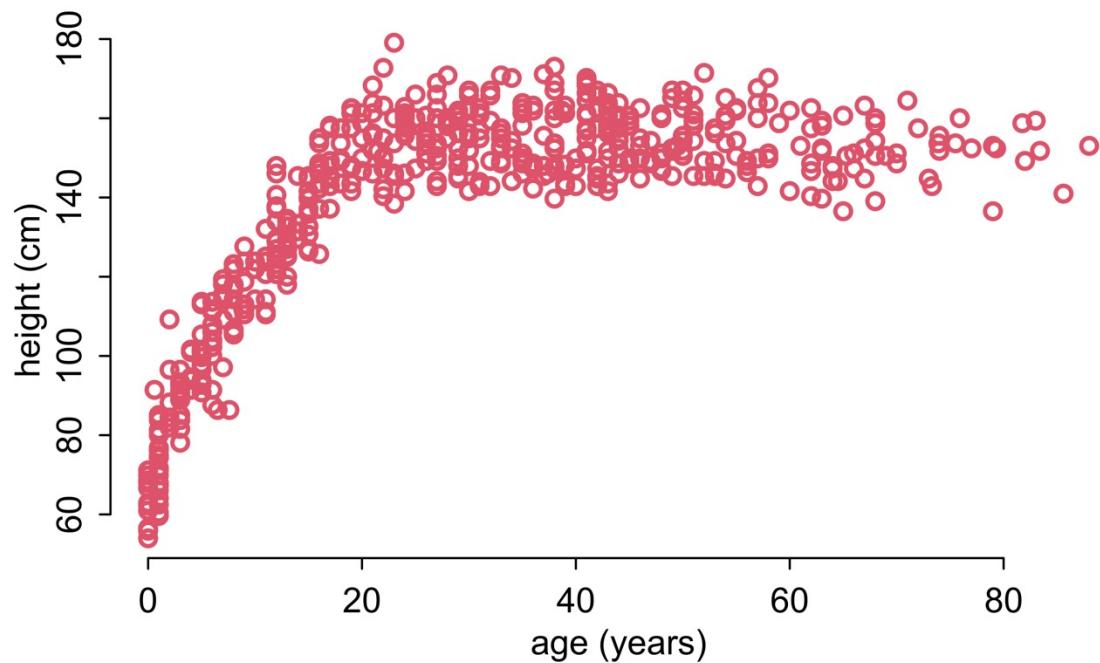


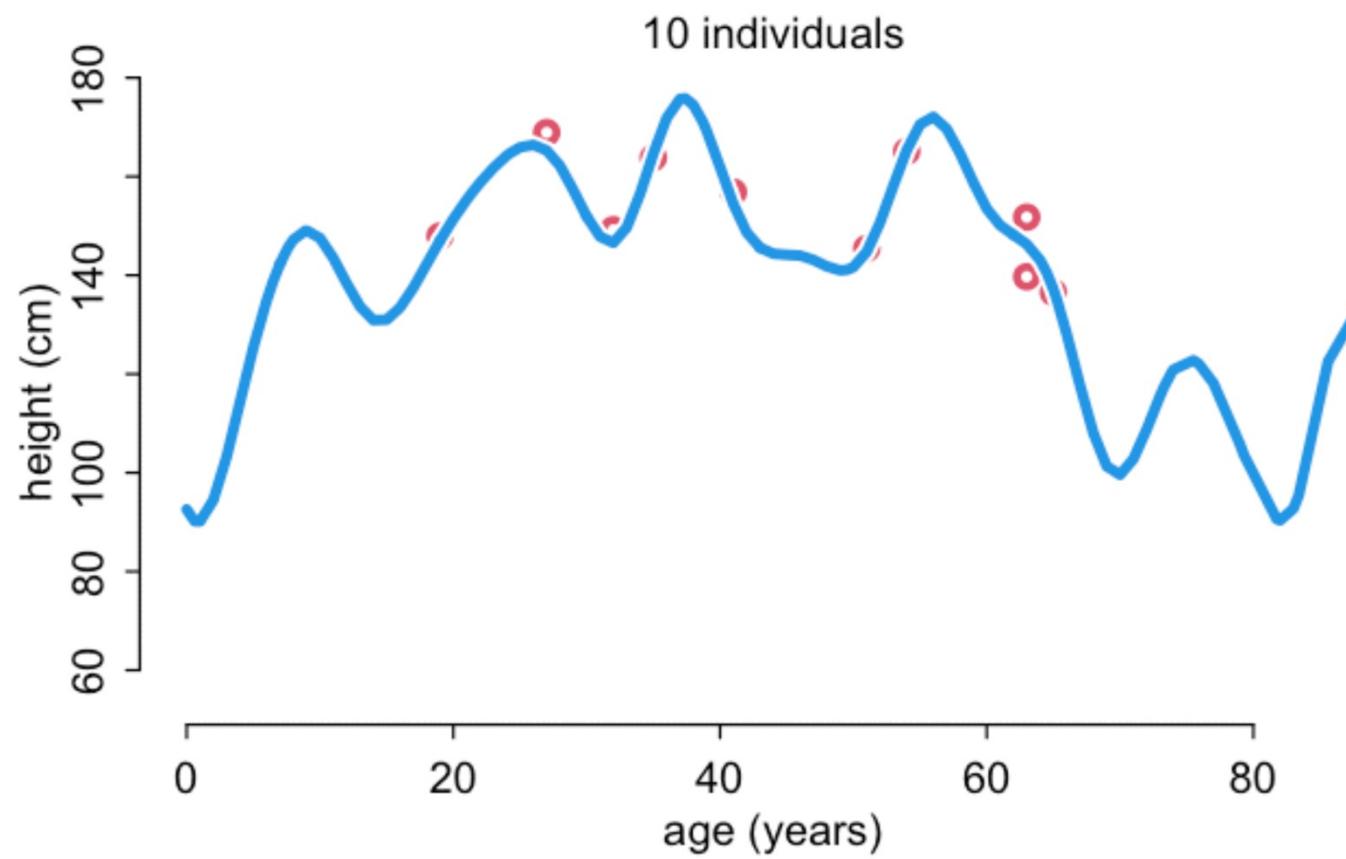
Height as a function of age

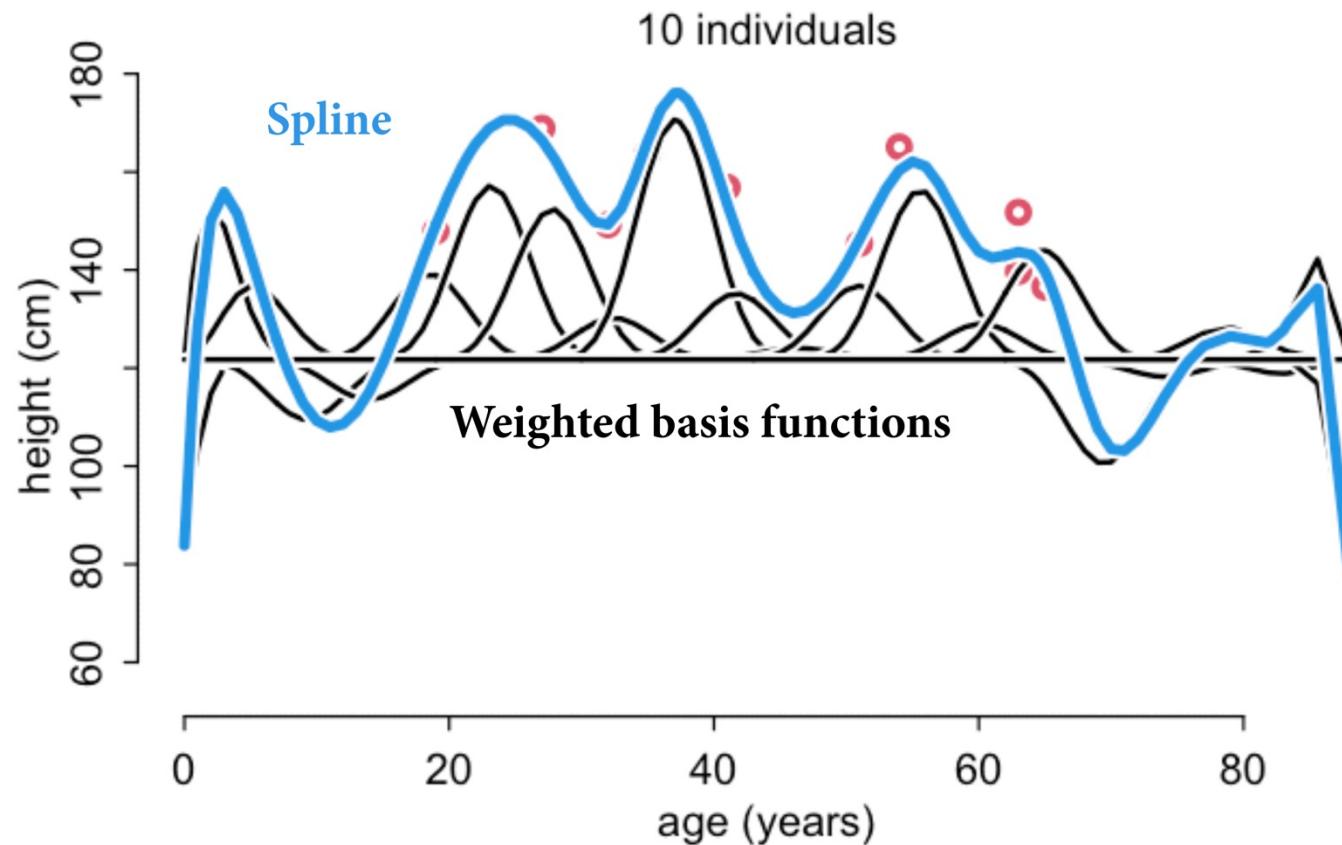
Obviously not linear

Fit spline as example

But biological model
would do a lot better







Curves and Splines

Can build very non-linear functions from linear pieces

Polynomials and splines are powerful geocentric devices

Adding scientific information always helps

e.g. Weight only increases with height

e.g. Height only increases with age, then levels off (or declines)

Ideally **statistical** model has some form as **scientific** model

