colorlinks, linkcolor=red!50!black, citecolor=blue!50!black, urlcolor=blue!80!black

TTK4130 – Modellering og Simulering

1 Modeling

2 Useful models

2.1 Mass spring damper

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1}$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \tag{2}$$

Note: A driving force $F(t)$ could be included on the right hand side.

2.2 Capacitor and inductor equations

$$i_C(t) = C\frac{dv_C}{dt}(t) \tag{3}$$

$$v_L(t) = L\frac{di_L}{dt}(t) \tag{4}$$

2.3 Pendulum equation

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{5}$$

For $\theta \ll 1$ we get the approximation

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{6}$$

with period

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} \tag{7}$$

3 Passivity

3.1 Definition

If the following inequality is satisfied for all u and T ≥ 0, then the system is passive.

$$\int_0^T y(t)u(t)dt \geq -E_0 \tag{8}$$

Note: If the roles of u and y are reversed, i.e. y is taken to be the input and u the output, then the equality still holds.

3.2 Interpretation

Interpretation based on energy conservation. The product uy is a power, thus we can think about the integral as the energy supplied by u or equivalently the energy absorbed by the system.

1. If $\int_0^T y(t)u(t)dt \geq 0$, energy is only absorbed. This inequality holds for a passive memory-less system (e.g. a circuit with only a resistor).
2. If $\int_0^T y(t)u(t)dt \geq -E_0$, the system can supply a limited amount of energy to the outside, due to initial conditions of energy storage elements such as capacitors and inductors.
3. If $\int_0^T y(t)u(t)dt \to -\infty$, the system is active

3.3 Passivity and transfer functions

A system is passive $\iff$ its transfer function is positive real.

3.4 Passivity and Lyapunov

Suppose there is a function $V(x) \geq 0$ and $g(x) \geq 0$ s.t.

$$\dot{V(x)} = u^T y - g(x) \tag{9}$$

for any u. The the system is passive

3.5 Preservation of passivity

If has two parallel systems S_1 and S_2 and u goes into both and $y = y_1 + y_2$ then if both systems are passive, their combination is passive.

4 Dynamics

5 Rigid body kinematics

5.1 Rotation Matrices

Let $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ and $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$ be the orthogonal bases of two coordinate frames. Then the *coordinate transform* from frame b to frame a is given by

$$\mathbf{R}_b^a = \begin{bmatrix} \mathbf{b}_1^a & \mathbf{b}_2^a & \mathbf{b}_3^a \end{bmatrix} \tag{10}$$

That is,

$$\mathbf{v}^a = \mathbf{b}_1^a \mathbf{v}^b \tag{11}$$

. The $\mathbf{R}_b^a$ can also be thought of simply as a rotation of a vector. Let's pretend forget that the b frame exists, and think only about the frame a. Example:

$$\mathbf{b}_1^a = \mathbf{R}_b^a \mathbf{a}_1^a \tag{12}$$

which shows that the first basis vector of the a frame is rotated to the first basis vector of the b frame, when everything is referred to frame a. In this sense, the matrix represents *a rotation from a to b*. Therefore the rotation matrix $\mathbf{R}_b^a$ is called both a *coordinate transformation from b to a* if we regard it from the first perspective and a *rotation from a to b* if we regard it from the second perspective.

5.2 Homogeneous transformation matrices

The homogeneous transformation matrix specifies the position and orientation of a coordinate frame w.r.t. a reference frame.

$$T_b^a = \begin{bmatrix} R_b^a & r_{ab}^a \\ 0 & 1 \end{bmatrix} \tag{13}$$

where r_{ab}^a is the origin of frame b in a coordinates. There are three use-cases for the matrix

1. Represent a configuration. This means that we can use T_b^a to represent coordinate system b with reference to coordinate system a or represent c from a $T_c^a = T_b^a T_c^b$
2. Change reference frame of a vector. See note 1 below.
3. Displace a vector or frame.

Note 1: We cannot write $p_a = T_b^a p_b$ as $T_b^a \in R^{4\times 4}$ and $p_b \in R^3$. We fix this with the homogeneous coordinate representation of the 3-vector.

$$\begin{bmatrix} p_a \\ 1 \end{bmatrix} = T_b^a \begin{bmatrix} p_b \\ 1 \end{bmatrix} \tag{14}$$

5.3 Euler angles

5.3.1 Axis-angle representation

Representing rotation with a rotation axis unit vector $\mathbf{v}$ and a rotation angle α. Formula to get from this representation to Rotation matrix is given by

$$R_b^a = \cos(\alpha)I + \sin(\alpha)(\mathbf{v}^a)^x + (1-\cos(\alpha))\mathbf{v}^a(\mathbf{v}^a)^T \tag{15}$$

The formula to build the skew-symmetric matrix above is:

$$\mathbf{u}^x := \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{16}$$

5.3.2 Quaternions

Unit quaternions are defined by η and ϵ, where

$$\eta = \cos(\frac{\alpha}{2}) \tag{17}$$

$$\theta = \mathbf{v}\sin(\frac{\alpha}{2}) \tag{18}$$

with α and ϵ from the angle-axis representation. To get from quaternions to rotation matrix, the following formula is used. It is based on the Axis-angle representation formula above.

$$\mathbf{R}_b^a = I + 2\eta\epsilon^\times + 2\epsilon^\times\epsilon^\times \tag{19}$$

5.4 Angular velocity

$$(\boldsymbol{\omega}_{ab}^a)^x = \dot{R_b^a} R_b^{aT} \tag{20}$$

$$\dot{R_b^a} = (\boldsymbol{\omega}_{ab}^a)^x R_b^a \tag{21}$$

$$\dot{R_b^a} = R_b^a(\boldsymbol{\omega}_{ab}^b)^x \tag{22}$$

If we have

$$R_b^a = R_1(\phi)R_2(\theta)R_3(\psi) \tag{23}$$

we derive ordinarily and get

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = M^{-1}\boldsymbol{\omega}_{ab}^a$$

Be aware of gimbal lock. With the representation above, the angle in the middle will be causing the gimbal lock for some angle. By doing the derivations and finding the M matrix we can find which angle and what value produces the gimbal lock based on when it becomes rank deficient.

5.4.1 Differentiation of coordinate vector

Starting point is

$$\mathbf{u}^a = R_b^a u^b \tag{24}$$

which we differentiate ordinarily

$$\dot{\mathbf{u}^a} = \dot{R_b^a}\mathbf{u}^b + R_b^a\dot{\mathbf{u}^b} \tag{25}$$

$$= R_b^a(\boldsymbol{\omega}_{ab}^b \times \mathbf{u}^b + \dot{\mathbf{u}^b}) \tag{26}$$

What we end up with is the formula

$$^a\frac{d\vec{u}}{dt} = {}^b\frac{d\vec{u}}{dt} + \vec{\omega}_{ab} \times \vec{u} \tag{27}$$

5.5 Solid Kinematic

Position of point p

$$\vec{r}_p = \vec{r}_0 + \vec{r} \tag{28}$$

Velocity of point p

$$\vec{v}_p = {}^a\frac{d\vec{r}_p}{dt} \tag{29}$$

$$= \vec{v}_0 + {}^b\frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r} \tag{30}$$

Acceleration of point p

$$\vec{a}_p = {}^b\frac{d^2\vec{r}}{dt^2} + 2\vec{\omega}_{ab} \times {}^b\frac{d\vec{r}}{dt} + \dot{\vec{\omega}}_{ab} \times \vec{r} + \vec{\omega}_{ab} \times (\vec{\omega}_{ab} \times \vec{r}) \tag{31}$$

Explanation of terms:

1. ${}^b\frac{d^2\vec{r}}{dt^2}$ is acceleration due to p accelerating in b
2. $2\vec{\omega}_{ab} \times {}^b\frac{d\vec{r}}{dt}$ Coriolis effect

3. $\dot{\vec{\omega}}_{ab} \times \vec{r}$ is acceleration to to b spinning faster and faster

4. $\vec{\omega}_{ab} \times (\vec{\omega}_{ab} \times \vec{r})$ centrifugal effects

Simplification: Whenever possible, we attach frame b to solid. This results in ${}^b\frac{d^2\vec{r}}{dt^2} = {}^b\frac{d\vec{r}}{dt} = 0$ which simplifies the formulas above.

5.6 The center of mass

The mass of a rigid body b is

$$m = \int_b dm = \int_b \rho(x,y,z)dV \tag{32}$$

The center of mass $\vec{r}_c$ is defined ass

$$\vec{r}_c = \frac{1}{m}\int_b \vec{r}_p dm \tag{33}$$

where $\vec{r}_p$ is the position of a mass element dm that is fixed in frame b. Note that

$$\int \vec{r} dm = \int \vec{r}_p dm - \int \vec{r}_c dm \tag{34}$$
$$= m\vec{r}_c - m\vec{r}_c \tag{35}$$
$$= 0 \tag{36}$$

5.7 Other useful formulas

Relation between linear and angular velocity

$$v = \omega r \tag{37}$$

6 Newton-Euler equations of motion

6.1 Equations of motion for a rigid body

$$\vec{F}_{bc} = m\vec{a}_c \tag{38}$$
$$\vec{T}_{bc} = \vec{M}_{b/c} * \dot{\vec{\omega}}_{ib} + \vec{\omega}_{ib} \times (\vec{M}_{b/c}\vec{\omega}_{ib}) \tag{39}$$

6.2 Kinetic energy

$$\mathcal{T} = \frac{1}{2}m(\mathbf{v}_c^b)^\top \mathbf{v}_c^b + \frac{1}{2}(\boldsymbol{\omega}_{ib}^b)^\top \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \tag{40}$$

Subscript c denotes center of mass and superscript b denotes a coordinate vector/matrix in frame b. ω_{ib}^b is the angular velocity of frame b relative to frame i. $\mathbf{M}_{b/c}^b$ is the inertia matrix of b about c given in b. I.e. the inertia matrix of the rigid body about the center of mass

6.3 Inertia matrix

$$\mathbf{M}_{b/c}^b = -\int_b (\mathbf{r}^b)^\times(\mathbf{r}^b)^\times dm = \int_b \left[(\mathbf{r}^b)^\top \mathbf{r}^b \mathbf{I} - \mathbf{r}^b(\mathbf{r}^b)^\top\right] dm \tag{41}$$

Note that $\mathbf{M}_{b/c}^b$ is positive definite since the kinteic energy $\mathcal{T} \geq 0$. Note also that the integral above is a triple integral of a $3 \times 3 - matrix$. About a specified axis the formula reduces to

$$I = \int_b (\mathbf{r}^b)^\top \mathbf{r}^b dm \tag{42}$$

6.4 Parallel axis theorem

The inertia matrix of b about a point o is given by

$$\mathbf{M}_{b/o}^b = \mathbf{M}_{b/c}^b - m(\mathbf{r}_g^b)^\times(\mathbf{r}_g^b)^\times = \mathbf{M}_{b/c}^b + m\left[(\mathbf{r}_g^b)^\top \mathbf{r}_g^b \mathbf{I} - \mathbf{r}_g^b(\mathbf{r}_g^b)^\top\right] \tag{43}$$

where $\mathbf{r}_g^b$ is the vector from the point o to the center of mass c. If o is the origin this corresponds to $\mathbf{r}_c^b$. In it's simplest form with two parallel axes, the formula reduces to

$$I = I_c + md^2 \tag{44}$$

where I_c is the moment of inertia about the axis through the center of mass and d is the distance between the axes.

Implicit Function Theorem, Jacobian $\frac{\partial \varphi(\bar{\mathrm{x}},\bar{\mathrm{y}})}{\partial \mathrm{x}}$ being full rank is required. ++

7 Lagrange Mechanics

Modeling mechanical systems using an "energy-based" approach

7.1 Kinetic energy

$$\mathcal{T} = \frac{1}{2}\dot{q}^T W(q)\dot{q} \tag{45}$$

where

$$W(q) = \sum_{i=1}^{N} m_i \frac{\partial p_i}{\partial q}^T \frac{\partial p_i}{\partial q} \tag{46}$$

7.2 Potential Energy

Comes in many forms, gravity is most commonly seen in this course

$$\mathcal{V} = mgz \tag{47}$$

where z is height in field. We have also seen potential energy in spring and combination.

7.3 Lagrange equation

The Lagrange function

$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - V(q) - z^T c(q) \tag{48}$$

is used in the Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}^T - \frac{\partial \mathcal{L}}{\partial q}^T = Q \tag{49}$$

where $c(q)$ are constraint equations, z are Lagrange multipliers (the forces that hold the constraints in place) and Q is generalized forces we get by using

$$\sum_{i=1}^{N} \frac{\partial p_i}{\partial q}^T F_i \tag{50}$$

Examples:

$q = x \in \mathbb{R}^1$, $q = \begin{bmatrix} x \\ \theta \end{bmatrix} \in \mathbb{R}^2$, $q = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$.

The purpose of the Lagrange equation is to derive model for system dynamics. Handy remarks:

- As long as the objects we model with regards to have mass, W(q) is always full rank.
- Model complexity due to W(q) not constant.
- Constrained Lagrange allows for use of Cartesian coordinates as general coordinates.
- W will always be constant when a collection of Cartesian coordinates are used as q

8 Balance Equations

9 Kinematics of Flow

9.1 The material derivative

Let $\mathbf{x}$ be the position of some fluid particle, with velocity $\dot{x} = \mathbf{v}$. Additionally, let $\phi(x,t)$ be some scalar field that varies in space and time i.e. temperature. The material derivative of $\phi(x,t)$ is defined as

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{v}^T \nabla \phi \tag{51}$$

9.2 Material control volume

A control volume that contains a specific set of particles. Moves together with the particles

$$\mathbf{v}_c = \mathbf{v} \tag{52}$$

where $\mathbf{v}_c$ is the velocity of the surface ∂V and $\mathbf{v}$ is the velocity of the particles.

9.3 Divergence theorem

For scalar field:

$$\int_{\partial V} \phi \mathbf{n} dA = \int_v \nabla \phi dV \tag{53}$$

where $\phi \in \mathbb{R}$ is field and $\mathbf{n} \in \mathbb{R}^3$ is unit normal of the surface at each of its points. For vector field

$$\int_{\partial V} \mathbf{u} \bullet \mathbf{n} dA = \int_V [1\;1\;1] \nabla u \tag{54}$$

9.4 Reynolds transport theorem

If V is an arbitrary control volume and phi (ϕ), a scalar field as before

$$\frac{d}{dt} \int_{v(t)} \phi dV = \int_{V(t)} dV + \int_{\partial V(t)} \phi \mathbf{v}_c \bullet \mathbf{n} dA \tag{55}$$

where $\mathbf{v}_c \in \mathbb{R}^3$ is velocity of the surface. By combining the Reynolds Transport Theorem with divergence theorem and material derivative definition we get

$$\begin{aligned} \frac{D}{Dt} \int_{V(t)} \phi dV &= \int_{V(t)} \frac{\partial \phi}{\partial t} + [1\;1\;1] \nabla (\phi \mathbf{v}) dV && (56) \\ &= \int_V (t) \frac{D\phi}{Dt} + \phi [1\;1\;1] \nabla \mathbf{v} && (57) \end{aligned}$$

10 Mass, momentum and energy balance

10.1 Mass balance

General formula

$$\frac{d}{dt} \int_{V(t)} \rho dV = - \int_{\partial V(t)} \rho \mathbf{v} \bullet \mathbf{n} dA \tag{58}$$

For a cylindrical tank i.e. this translates to

$$\frac{d}{dt}(\rho A h) = \dot{m} \tag{59}$$

$$\frac{d}{dt}(\rho A h) = \omega_0 - \omega_1 \tag{60}$$

$$\dot{h} = \frac{1}{\rho A}(\omega_1 - \omega_2) \tag{61}$$

10.2 Momentum balance

For material control volume $V(t)$

$$\rho(x,t) \frac{Dv(x,t)}{Dt} = p\mathbf{f} - \nabla p \tag{62}$$

where ρ is density of material control volume, v is particle velocity, $\mathbf{f}$ is vector representing a force per mass unit acting on ∂V and $v(x,t)$ is pressure at every point on the surface.

10.3 Energy balance

Energy in a volume element

$$dE = (u + \frac{1}{2} v^T v + \phi) \rho dV \tag{63}$$

11 Simulation

We are concerned with solving the IVP

$$\dot{y} = f(y,t), \quad y(t_0) = y_0 \tag{64}$$

The Jacobian of the system is defined as

$$J = \frac{\partial f}{\partial y}(y,t) \tag{65}$$

Note that the Jacobian is A for a linear, time-invariant system $\dot{x} = Ax + Bu$.

12 Explicit Runge-Kutta methods

An explicit Runge-Kutta method with stages for the system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) \tag{66}$$

is given by

$$\mathbf{k}_i = \mathbf{f}\left(\mathbf{y}_n + h\sum_{j=1}^{i-1} a_{ij}\mathbf{k}_j, t_n + c_i h\right), i = 1, \ldots, \sigma \tag{67}$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\sum_{j=1}^{\sigma} b_j \mathbf{k}_j \tag{68}$$

The explicit Runge-Kutta method can be written out as

$$\begin{aligned}
\mathbf{k}_1 =& \mathbf{f}\left(\mathbf{y}_n, t_n\right) \\
\mathbf{k}_2 =& \mathbf{f}\left(\mathbf{y}_n + h a_{21}\mathbf{k}_1, t_n + c_2 h\right) \\
\mathbf{k}_3 =& \mathbf{f}\left(\mathbf{y}_n + h\left(a_{31}\mathbf{k}_1 + a_{32}\mathbf{k}_2\right), t_n + c_3 h\right) \\
&\vdots \\
\mathbf{k}_\sigma =& \mathbf{f}\left(\mathbf{y}_n + h\left(a_{\sigma 1}\mathbf{k}_1 + \ldots + a_{\sigma,\sigma-1}\mathbf{k}_{\sigma-1}\right), t_n + c_\sigma h\right) \\
\mathbf{y}_{n+1} =& \mathbf{y}_n + h\left(b_1\mathbf{k}_1 + \ldots + b_\sigma \mathbf{k}_\sigma\right)
\end{aligned} \tag{69}$$

Each explicit Runge-Kutta method is described by its parameters, a_{ij} , b_i and c_i, which can be arranged in a *Butcher array* of the form

$$\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_\sigma & a_{\sigma 1} & a_{\sigma 2} & \ldots & a_{\sigma,\sigma-1} & \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
\end{array}$$

If all non-zero entires of the matrix A are below the diagonal, then the method is explicit. Otherwise, the method is implicit.

Example: The 4th-order Runge-Kutta method

$$\begin{aligned}
k_1 &= f\left(t_n, y_n\right) \\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\
k_4 &= f\left(t_n + h, y_n + hk_3\right) \\
y_{n+1} &= y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned} \tag{70}$$

is equivalent to

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array} \tag{71}$$

Example: Task 3 (15th May, Exam 2019)

$$\begin{array}{c|cc}
\gamma & \gamma & 0 \\
1-\gamma & 1-2\gamma & \gamma \\
\hline
 & 1/2 & 1/2
\end{array} \tag{72}$$

is equivalent to

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}\left(\mathbf{y}_n + \gamma h\mathbf{k}_1, t_n + \gamma h\right), \\
\mathbf{k}_2 &= \mathbf{f}\left(\mathbf{y}_n + (1-2\gamma)h\mathbf{k}_1 + \gamma h\mathbf{k}_2, t_n + (1-\gamma)h\right), \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2}h\mathbf{k}_1 + \frac{1}{2}h\mathbf{k}_2
\end{aligned} \tag{73}$$

Example: Task 1 (June, Exam 2016)
Showing an implicit Runge-Kutta with three stages. (Implicit as non-zero elements on the diagonal of the A-matrix implies that the method is implicit

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}\left(\mathbf{y}_n, t_n\right) \\
\mathbf{k}_2 &= \mathbf{f}\left(\mathbf{y}_n + \frac{h}{4}\mathbf{k}_1 + \frac{h}{4}\mathbf{k}_2, t_n + \frac{h}{2}\right) \\
\mathbf{k}_3 &= \mathbf{f}\left(\mathbf{y}_n + h\mathbf{k}_2, t_n + h\right) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{6}\left(\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3\right)
\end{aligned} \tag{74}$$

is equivalent to

$$\begin{array}{c|ccc}
0 & & & \\
1/2 & 1/4 & 1/4 & \\
1 & 0 & 1 & \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array} \tag{75}$$

13 Stability functions

The stability of a numerical method is ensured if $|R(h\lambda_i)| \leq 1$ for all eigenvalues λ_i.
Higher order stability function : $R(h\lambda) = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2$

13.1 ERK methods

$$R_E(h\lambda) = \det\left[I - h\lambda(A - \mathbf{1}b^\top)\right], \text{ where } \mathbf{1} = (1, ..., 1)^\top \tag{76}$$

Note that $R_E(h\lambda)$ will be a polynomial in $h\lambda$ of order less than or equal to σ (the number of stages).

13.2 IRK methods

$$R(h\lambda) = \left[1 + h\lambda b^\top (I - h\lambda A)^{-1}\mathbf{1}\right] \tag{77}$$

$$R(h\lambda) = \frac{\det\left[I - h\lambda(A - \mathbf{1}b^\top)\right]}{\det(I - h\lambda A)} \tag{78}$$

14 Stability of RK methods

14.1 Aliasing

The *Nyquist frequency* is half of the sampling rate

$$\omega_{\text{Nyquist}} = \frac{1}{2}\cdot\frac{2\pi}{h}, \quad \text{where } h \text{ is the step size.} \tag{79}$$

Two systems oscillating at a low frequency $\omega < \omega_{\text{Nyquist}}$ and a high frequency $\omega + 2k\frac{\pi}{h} > \omega_{\text{Nyquist}}$ (k integer) will intercept at all sampling points, and therefore a solver will not be able to distinguish them. More specifically, the solver will believe that the system with higher frequency is the system with lower frequency, when fitting the curve.

14.2 A- and L-stability

Definition: A method is A-stable if
$|R(h\lambda)| \leq 1 \;\forall\;$ Re $\lambda \leq 0$.
This definitions means that an A-stable method is stable for all stable test systems $\dot{y} = \lambda y$. Note also that no ERK method can be A-stable, since $|R_E(h\lambda)| \to \infty$ as $|\lambda| \to \infty$.
Definition: A method is L-stable if it is A-stable and $|R(j\omega h)| \to 0$ when $\omega \to \infty$ $\forall$ systems $\dot{y} = j\omega y$.
A-stable methods can suffer from aliasing for systems with fast dynamics (faster than Nyquist frequency), whereas an L-stable method will simply damp out these fast dynamics. This means that the L-stable method might give a better qualitative representation of what the actual solution looks like.

14.3 Stiffly accurate methods and algebraic stability

Definition: A method is stiffly accurate if

$$\det(A) \neq 0 \text{ and } b = A^\top[0, 0, ..., 1]^\top \tag{80}$$

Note: A-stable and stiffly accurate $\implies$ L-stable.
Definition: A method is algebraically stable if

$$M = \text{diag}(b)A + (\text{diag}(b)A)^\top + bb^\top \tag{81}$$

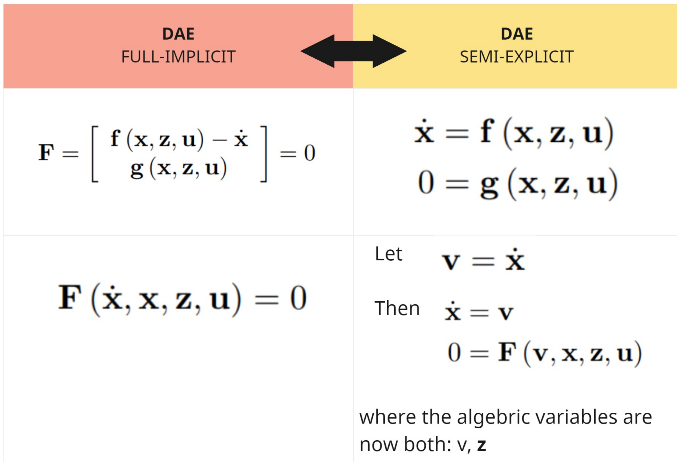
is positive semi-definite. Note: Algebraically stable $\implies$ A-stable.

15 DAEs

Used a lot in the modelling of complex and large-scale systems. DAEs are a set of equations that do not directly define the entire state. Has many forms, but usually: if the implicit ODE $F(\dot{x}, x, u, t) = 0$ and it's $det(\frac{\partial F}{\partial \dot{x}}) = 0$ then its a DAE. Method for finding index (one way to go about it):

- Differentiate algebraic equation(s) $g(x, z, u)$ until you can solve for the algebraic variable(s).
- The DAE system is now index 1. If you differentiated p times in the previous step, the index is $p + 1$.

Differential index, number of times we need to differentiate to transform DAE to ODE.
Index reduction, is the differentiate the DAE to be one differentiation from ODE.



Example: Finding differential index
Given eq. 82

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{f}(\mathrm{x}, \mathrm{z}, \mathrm{u}) \\ 0 &= g(\mathrm{x}, \mathrm{z}, \mathrm{u}) \end{aligned} \tag{82}$$

if a single time differentiation on the algebraic equation g, yields an ODE - then this means eq. 82 is an Index-1 DAE.

Example: Fully-Implicit to Semi-Explicit
Given eq. 83

$$\begin{aligned} m\ddot{x} &= -2x\mu \\ m\ddot{y} &= -mg - 2y\mu \\ l^2 &= x^2 + y^2 \end{aligned} \tag{83}$$

let $q = (x, y)^T, v = (x', y')^T$, then

$$\begin{aligned} q' &= v \\ v' &= \frac{1}{m}\left(-2\mu q + \begin{pmatrix} 0 \\ -mg \end{pmatrix}\right) \\ 0 &= x^2 + y^2 - l^2 = q^T q - l^2 \end{aligned} \tag{84}$$

15.0.1 Simulation

To simulate them, we need consistency conditions for it to be well-defined. We need to reduce it twice (differentiate the constrains twice, e.g. DAE Idx 3 => DAE Idx 1). If we reduce the constraints, we get "hidden constraints". $\dot{c}(q(0)) = 0$ (might have linear drift), $\ddot{c}(q(0)) = 0$.

We might also get constraint drift. Can be reduced using "Baumgarte" stabilization.

$$\ddot{c} + 2\alpha\dot{c} + \alpha^2 c > 0 \tag{85}$$

(we substitute the $\ddot{c}(q(0))$ constraint)

16 Advanced topics

16.1 Automatic adjustment of step size

The step size h can be selected so that the desired accuracy is obtained. Variable-step methods are useful for stiff systems (large spread in eigenvalues of Jacobian) and systems with strong nonlinearities (eigenvalues of Jacobian of linearization change a lot for each time step).

Idea: Estimate local error and adjust h such that the local error is less than the specified tolerance.

Implementation:

1. Compute the next iteration with two different methods: y_{n+1} with a method of order p and $\hat{y}_{n+1}$ with a method of order $\hat{p} = p + 1$.

2. The local exact solution is then

$$y_L(t_n; t_{n+1}) = y_{n+1} + e_{n+1} = \hat{y}_n + \hat{e}_{n+1} \tag{86}$$

   with $e_{n+1} = O(h^{p+1})$ and $\hat{e}_{n+1} = O(h^{p+2})$.

3. Since $\hat{e}_{n+1} \ll e_{n+1}$, we get the following

$$y_{n+1} - \hat{y}_n = e_{n+1} - \hat{e}_{n+1} \approx e_{n+1} \tag{87}$$

   h can then be chosen such that the local error e_{n+1} is as small as desired.

Since $\hat{y}_{n+1}$ is computed with a higher-order method than y_{n+1}, it would make sense to use that for the next iteration instead, this is called local extrapolation. Whichever solution is chosen as $\hat{y}_{n+1}$ is called the *embedded solution*.

16.2 Event detection

Let the event be given by

$$g(y, t) = 0 \tag{88}$$

e.g. a bouncing ball hitting the floor (crossing the x-axis). By checking for sign changes in g for each iteration, the time $t_n + \alpha$ of the event can be found by solving

$$g[y_n(\alpha), t_n + \alpha h] = 0 \tag{89}$$

for $\alpha \in [0, 1]$, where $y_n(\alpha)$ is the *dense output* found with interpolation (see page 565).

16.3 Multistep methods

A one-step method only uses the previous value y_n to compute y_{n+1}. A multistep method, on the other hand, uses y_{n-1}, y_{n-2}, etc. as well. The scheme looks like this:

$$\begin{aligned} y_{n+1} &= \alpha_1 y_n + \alpha_2 y_{n-1} + \dots \\ &\quad + h(\beta_0 f(y_{n+1}, t_{n+1}) + \beta_1 f(y_n, t_n) + \beta_2 f(y_{n-1}, t_{n-1}) + \dots) \end{aligned} \tag{90}$$

The parameters/weights are derived by curve fitting polynomials to the previous time steps. The known stability concepts from one-step methods apply to multistep methods as well.

17 Other topics

18 Modeling friction

Most models are ad-hoc and empirical.

18.1 Coloumb friction model

$$F_f = -\mu F_N sgn(v) \tag{91}$$

Problem if we miss the exact time when v=0

18.2 Viscous friction

Has several models here, I will not go in detail.

18.3 Static friction

Karnopp + Coloumb model

$$F_f = \min(-u, F_N) \quad if\ v = 0 \tag{92}$$
$$F_f = -\mu F_N sgn(v) \quad if\ v \neq 0 \tag{93}$$

19 Hybrid systems

Combination of continuous states and discrete states.

19.1 Guard conditions

$$G(q_i, q_j) : Q \times Q \to X_G \subseteq X \tag{94}$$

defines the set of X for which a jump from a discrete state q_i to q_j is triggered.

19.2 Reset map

$$\mathcal{R}(\bullet, \bullet, \bullet) : Q \times Q \times X \to X_R \subseteq X \tag{95}$$

defines how the continuous states are affected via a jump from a discrete state q_i to q_j. Example:

$$\mathcal{R}(q_1, q_2, x) = \frac{2}{3}x \tag{96}$$

typical example from car where a gear shift from gear q_1 to q_2 on the state $x = rpm$ reduces the rpm.

20 Event-Based Integrators

The method for properly simulating hybrid systems. An event is hitting a guard condition so that we get a jump in discrete state. Event conditions $e(x) = 0$,$e(x)$ are positive on one side of guard and negative on the other. Integrator simply monitors $e(x)$ and if it changes sign the integrator reduces the step s.t. we hit the $e(x) = 0$

21 Tables

22 Inverse of 3x3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix} \tag{97}$$

23 Trig functions

$$\cot(\theta) = \frac{1}{\tan(\theta)} \tag{98}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \tag{99}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \tag{100}$$

$$\sin(\theta) = \frac{\text{motstående}}{\text{hypotenus}} \tag{101}$$

$$\cos(\theta) = \frac{\text{hosliggende}}{\text{hypotenus}} \tag{102}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \tag{103}$$

24 Trig identities

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{104}$$

$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \tag{105}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{106}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{107}$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \tag{108}$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin \tag{109}$$

$$\tag{110}$$

25 Geometric series

$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right) \tag{111}$$

26 Partial integration

$$\int u\,dv = uv - \int v\,du \tag{112}$$