National Institute for Astrophysics, Optics, & Electronics

Control Systems

"From Here to Infinity"

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The Kalman filter was originally designed to be applied in the aerospace area, mainly because this filter can be utilized to effectively estimate the states of a system which cannot be directly observed. This filter had a big and positive impact on the aerospace industry, which lead to the extrapolation of its use to other areas in the industry. Even though the use of this filter was widespread, certain problematics led to modifications that were paramount since in some industries there are assumptions that cant be satisfied to successfully apply the Kalman filter and for it to have a precise estimation. Normally there aren't exact mathematical models that describe industrial processes to optimize them, thus the H∞ Filter was born, designed with robustness in mind for these types of problems.

The biggest advantage that the H∞ filter has in comparison to the Kalman filter is that it doesn't make any assumptions about the noise or disturbances that the input of the system could get to have, in addition to minimizing the estimation error assuming the worst noise values that the system may have.

To describe the H∞ filter, we must have a basic notion of the Kalman filter, its disadvantages and the areas in the filter that can be improved so that we can take advantage of the H∞ filter. The Kalman filter estimates the states x of any linear dynamic system defined by the equations:

$$x_{k+1} = Ax_k + Bu_k + W_k$$

$$y_k = Cx_k + z_k$$

Where $A, B, and\ C$ are known matrices of the system, k is the time index defined in discrete values, and K is the Kalman gain matrix. We suppose that the state of x cannot be measured directly, only the value y can be measured directly. In this case we can use the Kalman filter to estimate the state, the estimated state will be referred as $\hat{x}$. The equations for the Kalman filter are such that

$$K_k = AP_kC^T(CP_kC^T + S_z)^{-1}$$

$$\hat{x}_{k+1} = (A\hat{x}_k + Bu_k) + K_k(y_{k+1} - C\hat{x}_k)$$

$$P_{k+1} = AP_kA^T + S_w - AP_kC^TS_z^{-1}CP_kA^T$$

Where S_w, S_z are the covariance matrices of $w, and\ z$, K is the Kalman gain matrix and P is the variance of the error estimation. For the Kalman filter to correctly work, the noise or disturbances of the system must be of average zero, and this must hold true for the whole process and in every iteration, otherwise it is not going to make appropriate and reliable estimations. We must also know the standard deviation of the disturbances in the system, if these are unknown, a suitable Kalman filter cannot be designed.

For the case in which the necessary assumptions to use the Kalman filter are not met, it is proposed to use the H∞ filter, since it doesn't make any assumption on the disturbances of the system and it minimizes the worst possible case of error estimation, in this project a problem as follows needs to be solved:

$$min_{\hat{x}}\ max_{w,v}\ J$$

Where J is the measure of how good the estimator is. We can visualize the disturbances $w, v,$ as a manifestation of Murphy's law, meaning that they will take the worst possible values. Given these values we require to find an estimation that minimizes the effects that the noise or disturbances can have on our estimation. We want to find an $\hat{x}$ that is as close as possible to the real value x, however, disturbances will have a detrimental effect on this, which is why the average of the norms of $w, and\ v,$ will be used, where each disturbance will have a damping factor to mitigate proportionally the impact these will have.

The equations for the H∞ filter are as follows:

$$L_k = (I - QP_k + C^T V^{-1} C P_k)^{-1}$$

$$K_k = AP_k L_k C^T V^{-1}$$

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + K_k(y_k - C\hat{x}_k)$$

$$P_{k+1} = AP_k L_k A^T + W$$

The damping factors $Q, W, and\ V$ are chosen by the designer of the filter depending on the results that are expected, taking into account previous knowledge and context of the problem. If we know that the noise w will be smaller than the noise v, the damping factor W should be smaller than V. This estimation problem is very difficult to solve mathematically, which is the reason that to solve the problem we propose:

$$J < \frac{1}{\gamma}$$

where γ is a constant chosen by the filter's designer. The only condition that must be met for the H∞ filter to find a solution is that the equations of the filter are only valid if γ is chosen such that all the eigenvalues of the matrix P have values below one.

Different strategies and/or heuristics exist to implement this filter in embedded systems in such a way that it is computationally more efficient and with results sufficiently precise, such as calculating L_k, P_k previously to obtain the Kalman gain matrix K_k in its stable state.

In this project we use the H∞ filter in a vehicle navigation problem in which the error measured of the velocity is $\frac{2ft}{s}$, the noise of acceleration due to irregularities on the road, air, etc. is of $0.2\frac{ft}{s^2}$, and the position is being measured 10 times per second. We have the model as follows:

$$x_{k+1} = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} 0.005 \\ 0.1 \end{pmatrix} u_k + w_k$$

$$y_k = \begin{pmatrix} 0 & 1 \end{pmatrix} x_k + z_k$$

The covariance matrices of the noise are:

$$S_z = 100$$

$$S_w = \begin{pmatrix} 10^{-6} & 2 \times 10^{-5} \\ 2 \times 10^{-5} & 4 \times 10^{-4} \end{pmatrix}$$

The simulation of the system is run using the H∞ filter, and Kalman filter as well to be used as reference, we obtain the following graphs.

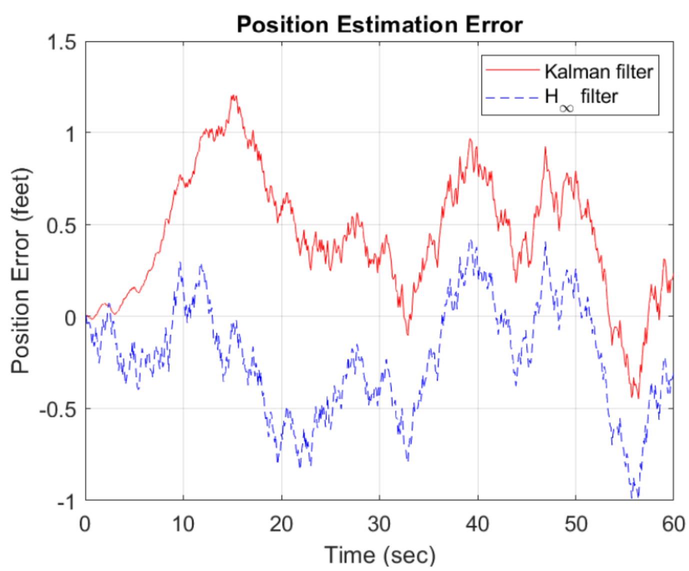

Figure 1. Position Estimation error

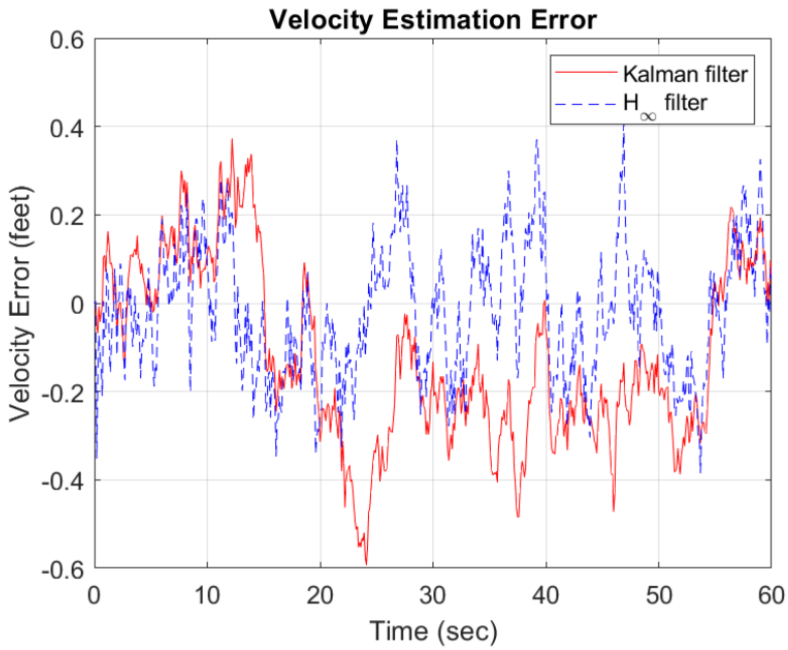

Figure 2. Velocity Estimation error

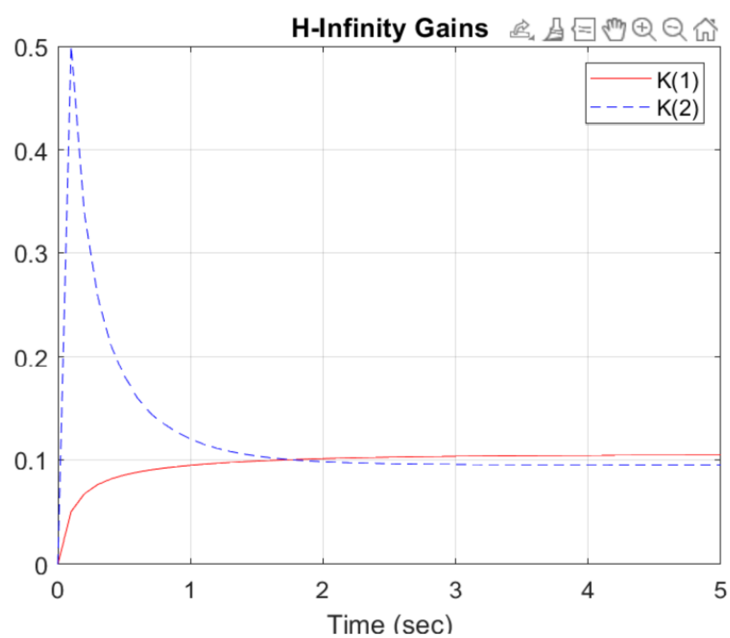

Figure 3. H∞ Filter gains

In figure 3 we can clearly observe the values of the gain matrix K reaches quickly a steady state at around 2 seconds. We can also see in both figure 1 and 2 that the error estimation of the H∞ filter has much better results in comparison to the traditional Kalman filter. We obtain results that are somewhat different to the original graphs from the article, however this is expected since in the simulation the noise is randomly generated using MATLAB's rand function. Given the random nature of this function, there are cases where the Kalman filter will have very similar performance compared to the H∞ filter, due to the noise average being around zero, which is one of the necessary conditions for the Kalman filter to correctly function. The H∞ filters robustness is clearly demonstrated.

Bibliography

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