KTH Engineering Sciences

# SD1105 MATLAB

## Programming Task

# FoDyn04b

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**SD1105 MATLAB**

**Topic:** Vehicle Dynamics
**Problem:** FoDyn04b

**Title:** **RIDE VIBRATIONS IN VEHICLES II**

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## Background

Today's vehicle users have high demands for quality, economy, performance, comfort, and safety. Because these desirable attributes are often in mutual conflict, the final design of a vehicle represents a compromise. The task of the designer is to find the best possible technical solution accounting for all of these requirements. In the interest of cost effectiveness, computer simulation based on mathematical models is used ever more frequently to choose design parameters.

In this programming problem, you are to use two very simple models to analyze some of the fundamental oscillatory motion that occurs when an automobile drives over an uneven roadway surface.

## Model

The purpose of a model is to provide a simplified, yet relevant, description of the processes and parameters of interest. By that approach, relatively simple models used for parametric studies can replace expensive, time-consuming experimental studies of real systems. A well-formulated model makes it possible to discern how the essential parameters influence the process and how they can be selected to optimize the process in some sense. In the context of engineering and the natural sciences, models tend to be mathematical in character, and are therefore amenable to analysis by means of the powerful set of tools provided by mathematics and the various branches of applied mechanics

In applied mechanics, the simple spring-mass system, i.e., single degree-of-freedom oscillator, is an important building block that can be used to construct models of more complex engineering systems. The fundamental mathematical framework for the analysis of the spring-mass system primarily consists of Newton's laws of motion, as treated in the basic mechanics courses. The equations that describe the motions of the model can be analyzed by mathematical or numerical methods. A combination of two single degree-of-freedom systems taken together, thereby comprising a two degree-of-freedom system, can, for instance, be used to describe the suspension of an automobile, the sound insulation of a wall, and so forth (there is an endless supply of such examples; see figure 1).

In the area of vehicle dynamics, simple models from rigid body dynamics can be used to mathematically describe and simulate the on-road dynamics of the vehicle. The components of the vehicle, such as the wheel, suspension, and chassis, are described by a model in which rigid bodies are coupled by springs and dampers. The fundamentals of rigid body dynamics, and rigid body modeling, are taught in *Mechanics II.* A simple rigid body model that can be used to describe the behavior of an automobile traversing an uneven road surface is shown in figure 1. You can study modeling methods for vehicle dynamics in more depth in the various courses offered on roadway vehicle engineering, e.g., *Vehicle Dynamics*.

The model in figure 1 shows a car with a wheelbase dimension $L$ driven at a speed $V$ over a road surface with a sinusoidal profile $\zeta$ of wavelength $\lambda$. The mechanical model consists of several parts: *two point masses* $m_2$ and $m_3$, each of which describes a pair of wheels and their axle; *a rigid body*, of mass $m_1$ and mass moment of inertia $J$, which describes the chassis; and, finally, *a spring-damper system* which couples together the various components. In contrast to a point mass, a rigid body has finite dimensions, and can therefore

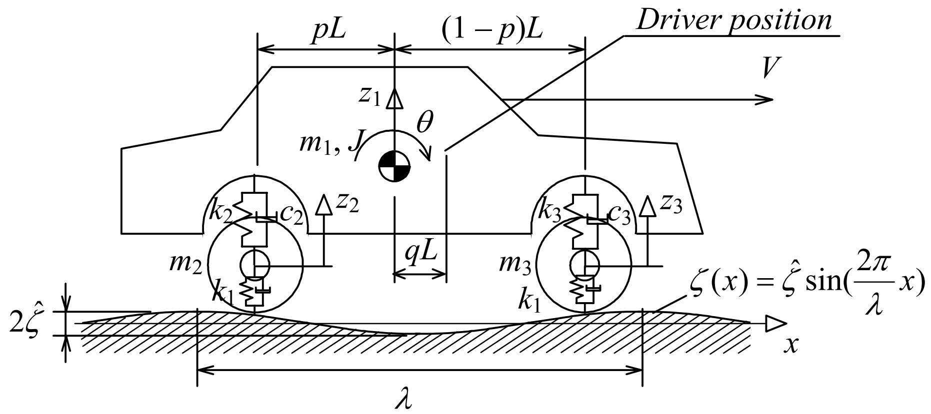


**Figure 1.** Simple rigid body dynamics model applicable to the study of an automobile's behavior as it traverses an uneven road surface.

be made to rotate about its center of mass when subjected to a moment. For that reason, rotations, as well as translations, are required to describe the motion of a rigid body.

Consider the model in figure 1. To indicate the position of the vehicle, i.e., its coordinates, we need to provide four numerical values: one for each of the point masses representing wheels, the displacements $z_2$ and $z_3$; and two for the chassis, the displacement $z_1$ and the rotation $\theta$. The three displacements specify the respective vertical motions of each of the components, and the rotation specifies the inclination of the chassis in the *xz*-plane. A model of that sort, using four values to indicate the positions of the components, is said to have four degrees of freedom. As such, the model in figure 1 is limited to describing the vehicle motions in the *xz*-planet. Motions outside of that plane, such as roll, are beyond the scope of that model.

With the mechanical model established, its characteristics must be specified in mathematical terms. In this case, they must be put in the form of equations describing how the motions of the components are coupled, and how they respond to external forces, in this case the fluctuating normal forces that arise in the tire-road contact patch. Applicable methods for setting up the equations of motion are introduced in the basic mechanics courses. Other examples are presented in the course *Sound and Vibrations.*

A model with four degrees of freedom is formulated, mathematically, by four equations. In this case, because the various displacements and the rotation are coupled, the mathematical model is a system of four coupled second order linear differential equations with constant coefficients,

$$\begin{aligned} m_1\frac{d^2z_1}{dt^2} + (c_2+c_3)\frac{dz_1}{dt} - c_2\frac{dz_2}{dt} - c_3\frac{dz_3}{dt} + (pc_2 - (1-p)c_3)L\frac{d\theta}{dt} + \\ + (k_2+k_3)z_1 - k_2z_2 - k_3z_3 + (pk_2 - (1-p)k_3)L\theta = 0 \end{aligned} \tag{1-a}$$

$$\begin{aligned} m_2\frac{d^2z_2}{dt^2} - c_2\frac{dz_1}{dt} + (c_1+c_2)\frac{dz_2}{dt} - pc_2L\frac{d\theta}{dt} - \\ - k_2z_1 + (k_1+k_2)z_2 - pk_2L\theta = (k_1\zeta_2 + c_1\frac{d\zeta_2}{dt}) \end{aligned} \tag{1-b}$$

$$m_3 \frac{d^2 z_3}{dt^2} - c_3 \frac{dz_1}{dt} + (c_1 + c_3)\frac{dz_3}{dt} + (1-p)c_3 L \frac{d\theta}{dt} - \\ - k_3 z_1 + (k_1 + k_3) z_3 + (1-p) k_3 L\theta = (k_1 \zeta_3 + c_1 \frac{d\zeta_3}{dt}) \tag{1-c}$$

$$J\frac{d^2\theta}{dt^2} + (pc_2 - (1-p)c_3)L\frac{dz_1}{dt} - pc_2 L\frac{dz_2}{dt} + (1-p)c_3 L\frac{dz_3}{dt} + (p^2 c_2 + (1-p)^2 c_3)L^2\frac{d\theta}{dt} + \\ + (pk_2 - (1-p)k_3)Lz_1 - pk_2 Lz_2 + (1-p)k_3 Lz_3 + (p^2 k_2 + (1-p)^2 k_3)L^2\theta = 0 \tag{1-d}$$

where $$\zeta_2 = \hat{\zeta}\sin(\frac{2\pi}{\lambda}x) = \{x = Vt\} = \hat{\zeta}\sin(\frac{2\pi}{\lambda}Vt), \tag{2-a}$$

$$\zeta_3 = \hat{\zeta}\sin(\frac{2\pi}{\lambda}(x+L)) = \{x = Vt\} = \hat{\zeta}\sin(\frac{2\pi}{\lambda}(Vt+L)), \tag{2-b}$$

are the local elevations of the roadway surface with respect to a datum plane (the plane representing average elevation), at the front and rear wheels, respectively. The parameter $\hat{\zeta}$ designates the magnitude of the roadway roughness, or its amplitude, as shown in figure 1. The contact force between the wheel and the roadway varies with a frequency $f$ determined by the number of times per second that a wheel passes a complete wavelength of roadway elevation variation; specifically, $f = V/\lambda$. Equation (2) can therefore be written

$$\zeta_2 = \hat{\zeta}\sin(2\pi f t) = \hat{\zeta}\sin(\omega t), \tag{3-a}$$

$$\zeta_3 = \hat{\zeta}\sin(2\pi f t + 2\pi\frac{L}{\lambda}) = \hat{\zeta}\sin(\omega t + 2\pi\frac{L}{\lambda}), \tag{3-b}$$

where the frequency parameter $\omega = 2\pi f$ is called the circular frequency.

The spring constants $k_1$, $k_2$ and $k_3$ are defined by the relation $F_{\text{spring}} = kz$ between the spring force and its deformation. The damping coefficients $c_1$, $c_2$ and $c_3$ are defined, in a similar fashion, by the relation $F_{\text{damp}} = c\, dz/dt$ between the damping force and the velocity of deformation of the damper.

Differential equations of that type can either be solved analytically, by methods to be covered in upcoming mathematics courses, or numerically integrated by methods that will be introduced in the course *Numerical Methods and Basic Programming*. The solution to equation set (1) has the four components $z_1$, $z_2$, $z_3$ and $\theta$, which together specify the positions of the components as functions of time. By changing the design parameters, such as the mass distribution, as well as stiffness and damping parameters, the design engineer has the capability to bring about vehicles with different dynamic characteristics.

Another more practical method with which to provide a mathematical description of a harmonic motion is to express it as

$$\zeta_2 = \hat{\zeta}e^{i\omega t} = \hat{\zeta}e^{i2\pi f t}, \tag{4-a}$$

$$\zeta_3 = \hat{\zeta}e^{i(\omega t + 2\pi L/\lambda)} = \hat{\zeta}e^{i2\pi f t}e^{i2\pi L/\lambda}, \tag{4-b}$$

where $i$ is the imaginary unit value $\sqrt{-1}$. Because the system of equations is linear in $z_1$, $z_2$, $z_3$ and $\theta$, and their respective derivatives, it can be shown that its solutions are

$$z_1(t) = \hat{z}_1 e^{i\omega t} = \hat{z}_1 e^{i2\pi ft}, \tag{5-a}$$

$$z_2(t) = \hat{z}_2 e^{i\omega t} = \hat{z}_2 e^{i2\pi ft}, \tag{5-b}$$

$$z_3(t) = \hat{z}_3 e^{i\omega t} = \hat{z}_3 e^{i2\pi ft}, \tag{5-c}$$

$$\theta(t) = \hat{\theta} e^{i\omega t} = \hat{\theta} e^{i2\pi ft}, \tag{5-d}$$

where $\hat{z}_1$, $\hat{z}_2$, $\hat{z}_3$ and $\hat{\theta}$ are unknown kinematic amplitudes. The unknown amplitudes can be determined by entering (4) and (5) into (1). Canceling the common factor $e^{i\omega t}$, we obtain

$$\begin{aligned} &-\omega^2 m_1 \hat{z}_1 + i\omega((c_2+c_3)\hat{z}_1 - c_2\hat{z}_2 - c_3\hat{z}_3 + (pc_2 - (1-p)c_3)L\hat{\theta}) + \\ &\qquad + (k_2+k_3)\hat{z}_1 - k_2\hat{z}_2 - k_3\hat{z}_3 + (pk_2 - (1-p)k_3)L\hat{\theta} = 0 \end{aligned} \tag{6-a}$$

$$\begin{aligned} &-\omega^2 m_2 \hat{z}_2 + i\omega(-c_2\hat{z}_1 + (c_1+c_2)\hat{z}_2 - pc_2L\hat{\theta}) - \\ &\qquad - k_2\hat{z}_1 + (k_1+k_2)\hat{z}_2 - pk_2L\hat{\theta} = (i\omega c_1 + k_1)\hat{\zeta} \end{aligned} \tag{6-b}$$

$$\begin{aligned} &-\omega^2 m_3 \hat{z}_3 + i\omega(-c_3\hat{z}_1 + (c_1+c_3)\hat{z}_3 + (1-p)c_3L\hat{\theta}) - \\ &\qquad - k_3\hat{z}_1 + (k_1+k_3)\hat{z}_3 + (1-p)k_3L\theta = (i\omega c_1 + k_1)\hat{\zeta} e^{i2\pi L/\lambda} \end{aligned} \tag{6-c}$$

$$\begin{aligned} &-\omega^2 J\hat{\theta} + i\omega((pc_2 - (1-p)c_3)L\hat{z}_1 - pd_2L\hat{z}_2 + (1-p)c_3L\hat{z}_3 + (p^2c_2 + (1-p)^2c_3)L^2\hat{\theta}) + \\ &\qquad + (pk_2 - (1-p)k_3)L\hat{z}_1 - pk_2L\hat{z}_2 + (1-p)k_3L\hat{z}_3 + (p^2k_2 + (1-p)^2k_3)L^2\hat{\theta} = 0 \end{aligned} \tag{6-d}$$

That is not a differential equation system, but an ordinary algebraic equation system with four unknowns: $\hat{z}_1$, $\hat{z}_2$, $\hat{z}_3$ and $\hat{\theta}$. Given input data in the form of masses, spring constants, damping coefficients, and a roadway roughness amplitude, the equation system can be solved for the four unknowns $\hat{z}_1$, $\hat{z}_2$, $\hat{z}_3$ and $\hat{\theta}$.

The solution of the system of equations is facilitated by incorporating a matrix-vector formalism. Equation (4) can be written in matrix-vector notation, as

$$(\mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M})\hat{\mathbf{z}} = \hat{\mathbf{f}} \tag{7}$$

where the stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} k_2+k_3 & -k_2 & -k_3 & (pk_2-(1-p)k_3)L \\ -k_2 & k_1+k_2 & 0 & -pk_2L \\ -k_3 & 0 & k_1+k_3 & (1-p)k_3L \\ (pk_2-(1-p)k_3)L & -pk_2L & (1-p)k_3L & p^2L^2k_2+(1-p)^2L^2k_3 \end{bmatrix}, \tag{7-a}$$

the mass matrix is

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & J \end{bmatrix}, \tag{7-b}$$

the damping matrix is

$$\mathbf{C} = \begin{bmatrix} c_2 + c_3 & -c_2 & -c_3 & (pc_2 - (1-p)c_3)L \\ -c_2 & c_1 + c_2 & 0 & -pc_2L \\ -c_3 & 0 & c_1 + c_3 & (1-p)c_3L \\ (pc_2 - (1-p)c_3)L & -pc_2L & (1-p)c_3L & p^2L^2c_2 + (1-p)^2L^2c_3 \end{bmatrix}, \tag{7-c}$$

the degree-of-freedom amplitude vector is

$$\hat{\mathbf{z}} = \begin{Bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \\ \hat{\theta} \end{Bmatrix}, \tag{7-d}$$

and the excitation force vector is

$$\hat{\mathbf{f}} = (k_1 + i\omega c_1)\hat{\zeta} \cdot \begin{Bmatrix} 0 \\ 1 \\ e^{i2\pi L/\lambda} \\ 0 \end{Bmatrix}. \tag{7-e}$$

With regard to vehicle behavior, an area of particular interest is the delimitation of situations in which resonant vibrations arise. Resonant vibrations are characterized by motions that become very large, even for relatively small excitation forces. Resonances are caused by excitation of the system at frequencies close to system eigenfrequencies. For that reason, it is important, in the design stage, to determine the eigenfrequencies of the vehicle. That is usually accomplished by studying the eigenfrequencies of the undamped system, i.e., the damping matrix **C** is set to zero.

As mentioned above, the eigenfrequencies are those particular frequencies at which the system is especially sensitive to disturbances in the form of exciting forces. Only a very small force is needed to bring about a large oscillatory motion. A means of determining the eigenfrequencies is therefore to simply set the force $\hat{\mathbf{f}}$ to 0 in equation (7). From that, we obtain a so-called eigenvalue problem, the solution of which gives the eigenfrequencies and eigenvectors. The eigenvalue problem is

$$(\mathbf{K} - \lambda_n \mathbf{M})\hat{\mathbf{x}}_n = \mathbf{0}, \tag{8}$$

where $\lambda_n$ is called the eigenvalue, and $\hat{\mathbf{x}}_n$ the eigenvector. The eigenvalue problem can be solved in MATLAB using the command **`eig`**. Because the system has four unknowns, the number of eigenvalues and eigenvectors is also four. The eigenfrequencies are found from the relation

$$f_n = \frac{\sqrt{\lambda_n}}{2\pi}, \quad n = 1...4. \tag{9}$$

To each eigenvalue, there is a corresponding eigenvector that describes how the system moves in the corresponding resonance (i.e., at that eigenfrequency). The input masses (chassis and wheel) move in different ways at different frequencies. The eigenvector describes how the masses move *relative* to one another. The motion of each degree-of-freedom is indicated by the corresponding element of the eigenvector. The ratio of any two elements gives a relation between the amplitudes of the degrees-of-freedom to which they correspond; if, for instance, the ratio is two, then the two amplitudes differ by a factor of two, regardless of their absolute values. If the elements have differing signs, that implies that the corresponding vibrations occur in antiphase, i.e., they move in opposite directions.

Once the eigenfrequencies $f_n$ are established, the critical speeds $V_{crit,n}$ , at which the car's vibrations become large when traversing the particular road surface in question, can also be determined from

$$V_{krit,n} = \lambda f_n . \tag{10}$$

If we ignore the inclination, the vertical motion of the center of mass of the car can be described by a so-called *quarter car model*. The quarter car model is simply obtained by distributing a fourth of the chassis mass over each of the four wheels, as shown in figure 2. Because the motion of the quarter car model can be described by indicating values of two coordinates, $z$ and $z_t$, it is therefore a two degree-of-freedom model.

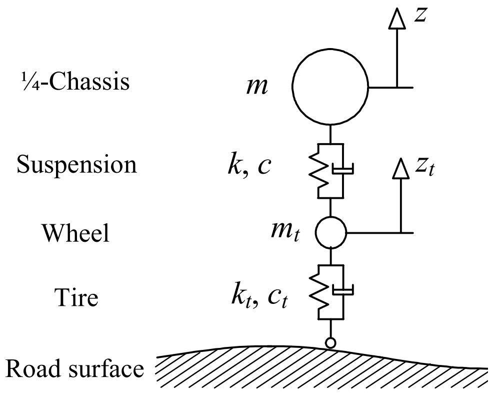


**Figure 2**. Quarter car model for the analysis of the vertical motions of a car.

To determine the motions of the quarter car model, the same methodology can be used as for the four degree-of-freedom model of figure 1. The amplitudes of the degrees-of-freedom can be found from matrix equation (7), in which have

a stiffness matrix
$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & k + k_t \end{bmatrix}, \tag{11-a}$$

a mass matrix
$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m_t \end{bmatrix}, \tag{11-b}$$

a damping matrix
$$\mathbf{C} = \begin{bmatrix} c & -c \\ -c & c + c_t \end{bmatrix}, \tag{11-c}$$

a displacement amplitude vector $\quad \hat{\mathbf{z}} = \begin{Bmatrix} \hat{z} \\ \hat{z}_t \end{Bmatrix}$ (11-d)

and a forcing vector $\quad \hat{\mathbf{f}} = \begin{Bmatrix} 0 \\ (k_t + i\omega c_t)\hat{\zeta} \end{Bmatrix}$. (11-e)

The displacement amplitudes can, for that case, be expressed as

$$\hat{z} = \frac{(k + i\omega c)(k_t + i\omega c_t)\hat{\zeta}}{(k + i\omega c - \omega^2 m)[(k + k_t) + i\omega(c + c_t) - \omega^2 m_t] - (k + i\omega c)^2}, \tag{12}$$

$$\hat{z}_t = \frac{(k + i\omega c - \omega^2 m)(k_t + i\omega c_t)\hat{\zeta}}{(k + i\omega c - \omega^2 m)[(k + k_t) + i\omega(c + c_t) - \omega^2 m_t] - (k + i\omega c)^2}. \tag{13}$$

**Problem Task**
You are tasked with investigating the oscillatory motions of an automobile using the models of figures 1 and 2. In particular, you are to study the vibrations at the driver position, when the vehicle traverses a road surface of sinusoidal profile, with wavelength $\lambda$, at speed $V$. The analysis is divided up into several steps.

***Parametric values***
*Four Degree-of-Freedom Model* (Figure 1):
$m_1$ = 600 kg, $m_2 = m_3$ = 45 kg, $J$ = 1020 kg·m$^2$, $L$ = 2.5 m, $p$ = 0.52, $q$ = 0.1,
$k_1$ = 500 kN/m, $k_2 = k_3$ = 20 kN/m, $c_1$ = 0 Ns/m, $c_2 = c_3$ = 1 kNs/m,

*Quarter Car Model* (Figure 2):
$m$ = 300 kg, $m_t$ = 45 kg, $k_t$ = 500 kN/m, $k$ = 20 kN/m, $c_t$ = 0 Ns/m, $c$ = 1 kNs/m,

*Roadway Surface* (Figure 1):
$\lambda$ = 16 m, $\hat{\zeta}$ = 4 cm.

***Sub-Task 1***
You are to investigate the vertical and rotational motion, as well as vibrations at the driver position, for a vehicle traveling at a constant speed $V$ over an uneven road surface. The analysis is to be carried out for a number of different speeds within the interval $V \in [10, 120]$ km/h. The results are then to be presented graphically with the absolute values of the displacement (translational or rotational) amplitude, plotted as a function of the car speed $V$. The speeds at which the amplitudes of motion at the driver position are relative maxima are to be indicated.

Input data for the four degree-of-freedom model is given in the form of an ASCII file, **bil0404.dat**, in which the various columns contain sequenced values of the velocity $V$ in km/h and the corresponding absolute values of the amplitudes of motion, i.e., for the car's chassis, $|\hat{z}_1|$ in meters and $|\hat{\theta}|$ in radians, and for the vertical motion at the driver position, $|\hat{z}_F|$, in meters. The car's amplitudes in the quarter car model are to be found using equation (12).

***Sub-Task 2***
By solving the eigenvalue problem for the four degree-of-freedom model, you are to determine the critical speeds $V_{crit}$ at which the vibrations of the car become large as the vehicle traverses the road surface in question. These speeds are to be indicated graphically in the figure that illustrates the amplitudes of the different degrees-of-freedom as functions of the speed $V$ of the car (the figure from sub-task 1). Secondly, the eigenvectors are to be reported in numerical form.

***Sub-Task 3***
You are to investigate how the ride vibrations of the car, when traveling at $V = 40$ km/h, are modified due to adjustment of the damping coefficient between the axles and the chassis in the four degree-of-freedom model. Present the results in the form of a curve showing the absolute values at the driver position as a function of the damping coefficient $c_2$, which varies on the interval [1, 10] kNs/m. Note that $c_3$ is varied in accordance with $c_2$. Determine the smallest possible value of the damping coefficient $c_2$ such that the absolute value of the vibration amplitude at the driver position not exceed 4 cm.

**Analysis Methodology**
To carry out the sub-tasks, the following approaches may be used:

***Sub-Task 1***
1. Read in data from the file **bil0404.dat**.
2. Plot the absolute value of the displacement amplitudes as functions of vehicle speed.
3. Determine, and graphically depict, the displacement amplitude with the help of the quarter car model.
4. Determine the maximum displacement amplitude at the driver position, and the car speed at which it occurs.

***Sub-Task 2***
1. Define the system matrices **K** and **M**.
2. Solve the eigenvalue problem, and determine the critical speeds.
3. Indicate the critical speeds in the diagram obtained in sub-task 1.
4. List the eigenvectors corresponding to each critical speed.

***Sub-Task 3***
1. Define the input parameters.
2. For each tested value of $c_2$ in the interval [1, 10] kNs/m :
    - define the damping matrix **C**,
    - calculate the amplitude vector $\hat{\mathbf{z}}$.
3. Determine the displacement amplitude at the driver position using the relation $\hat{z}_F = \hat{z}_1 - qL\hat{\theta}$, where $qL$ is the distance from the driver position to the center of mass (see figure 1).
4. Plot the absolute value of the driver position amplitude in a graph as a function of the damping $c_2$.
5. Determine the damping $c_2$ such that the amplitude at the driver position not exceed 4 cm.