Astrometry: Information Limits and Estimators

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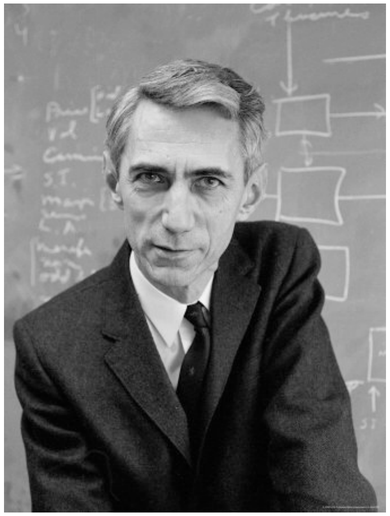
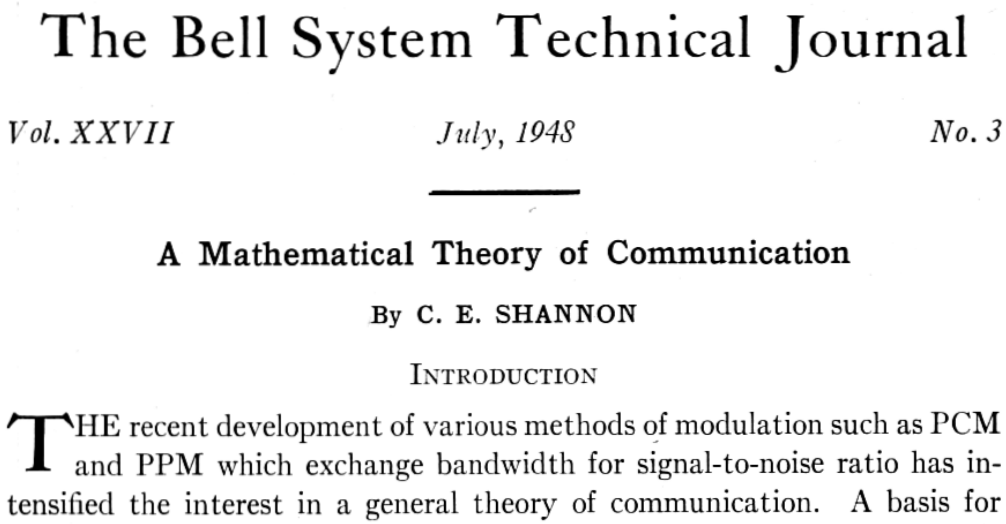
Information and Decision System Group

Information Theory





The Bell System Technical Journal

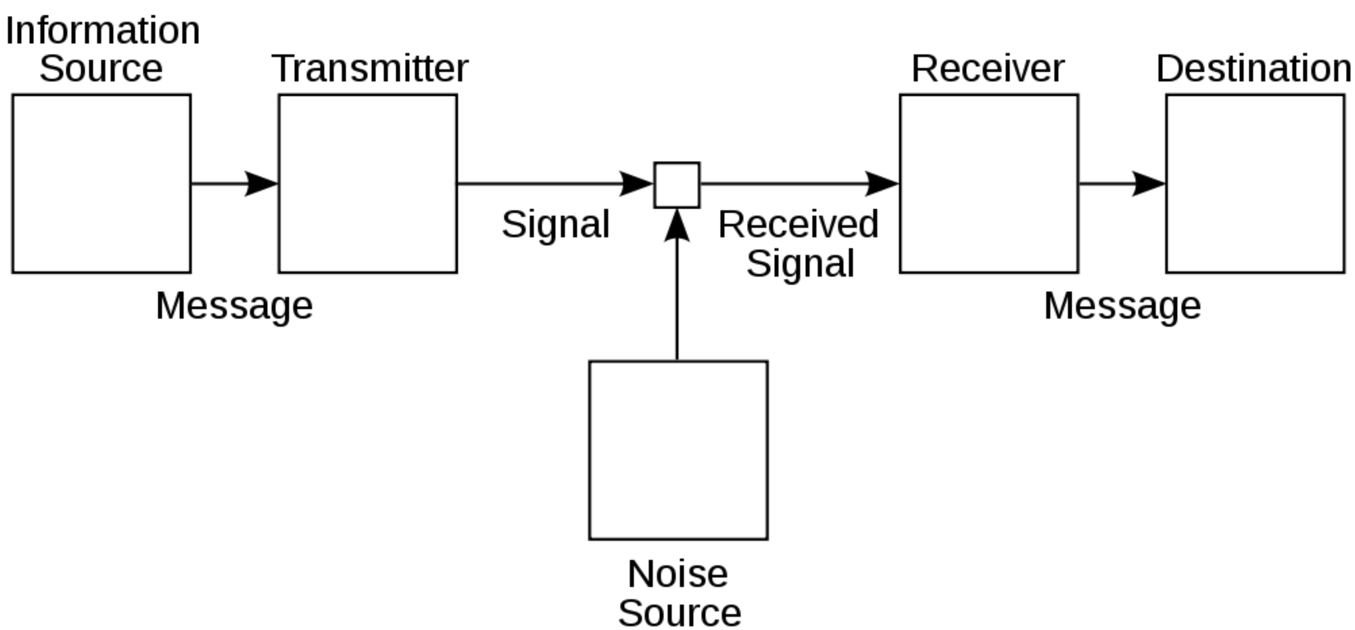
Vol. XXVII *July, 1948* *No. 3*

A Mathematical Theory of Communication

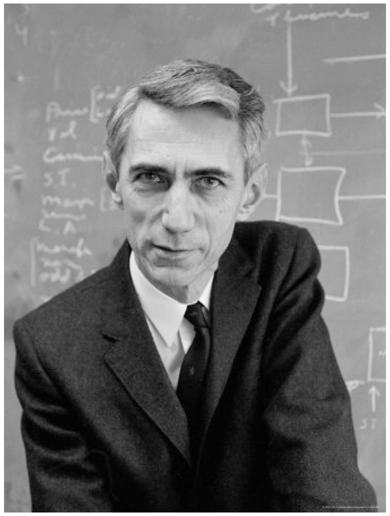
By C. E. SHANNON

Introduction

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for



Information Theory



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Channel Source Coding Theorem: [*Shannon, 1948*]

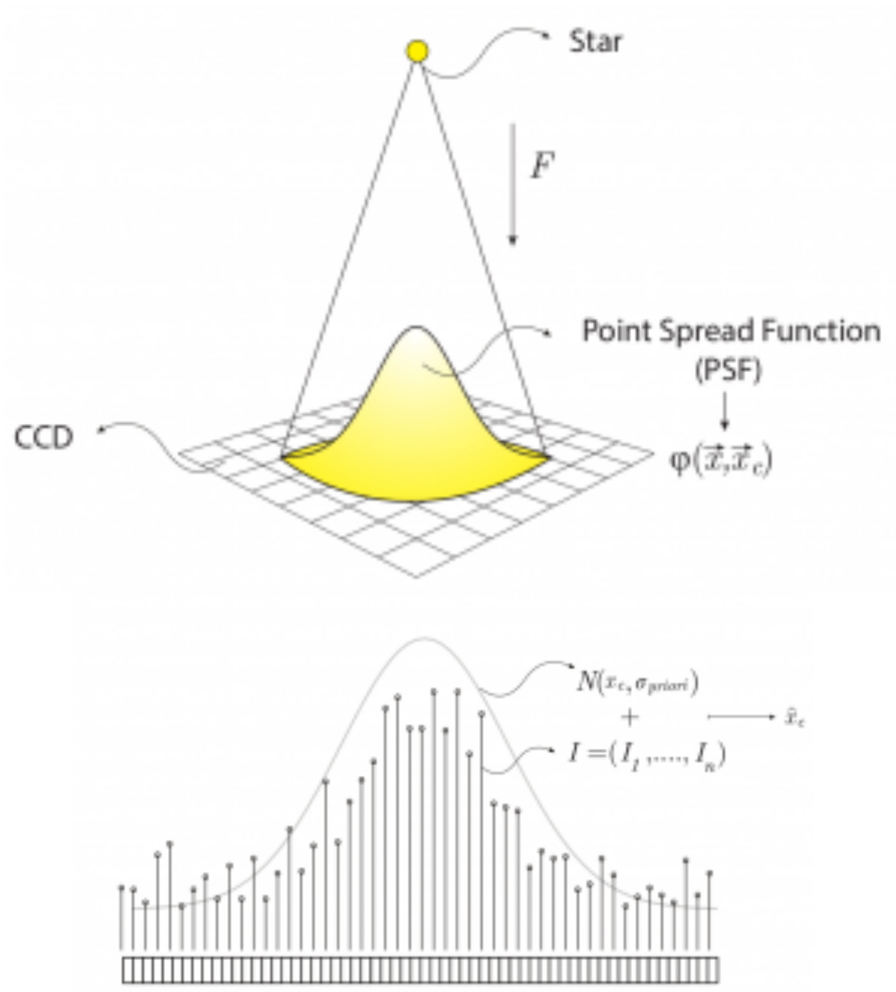
$$X(u) \longrightarrow \boxed{\text{channel}} \longrightarrow Y(u)$$

$$P_X \qquad P_{Y|X} \qquad P_Y$$

$$C = \sup_{P_X} I(X;Y)$$

Astrometry: An Information Flow Problem

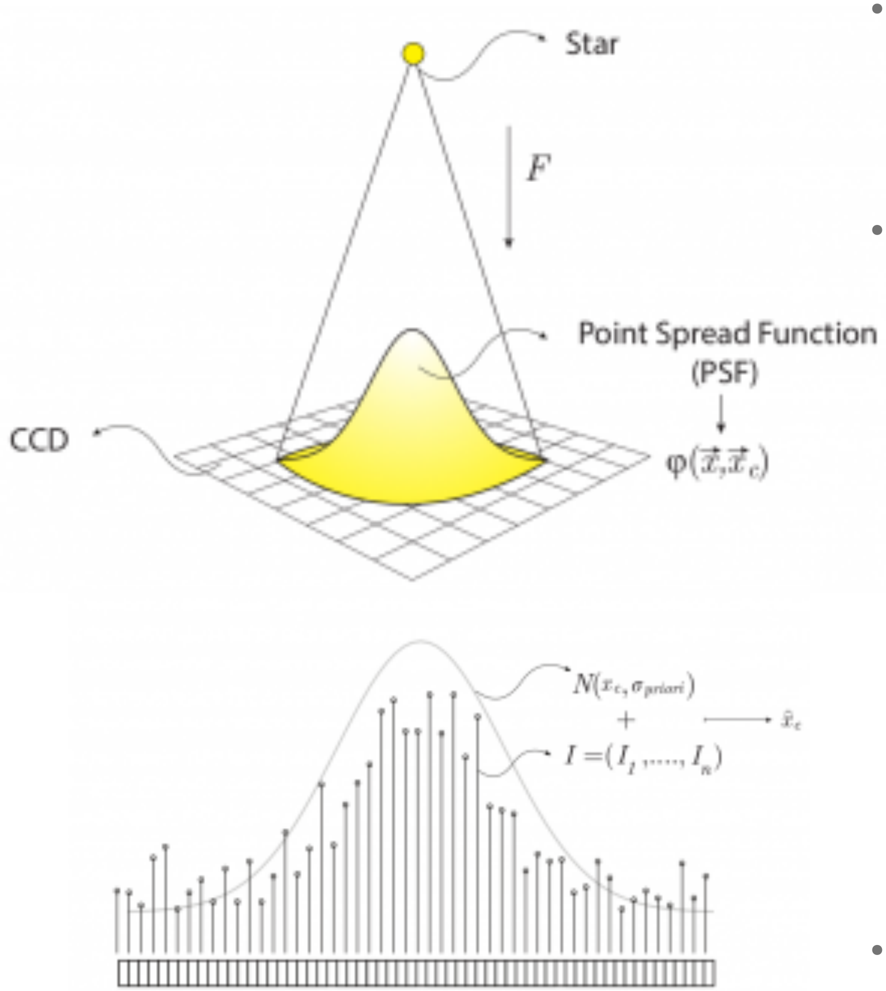
Source x_c



Data $I_1, .., I_n \sim p_{x_c}$

Astrometry: An Information Flow Problem

Source x_c



Data $I_1, .., I_n \sim p_{x_c}$

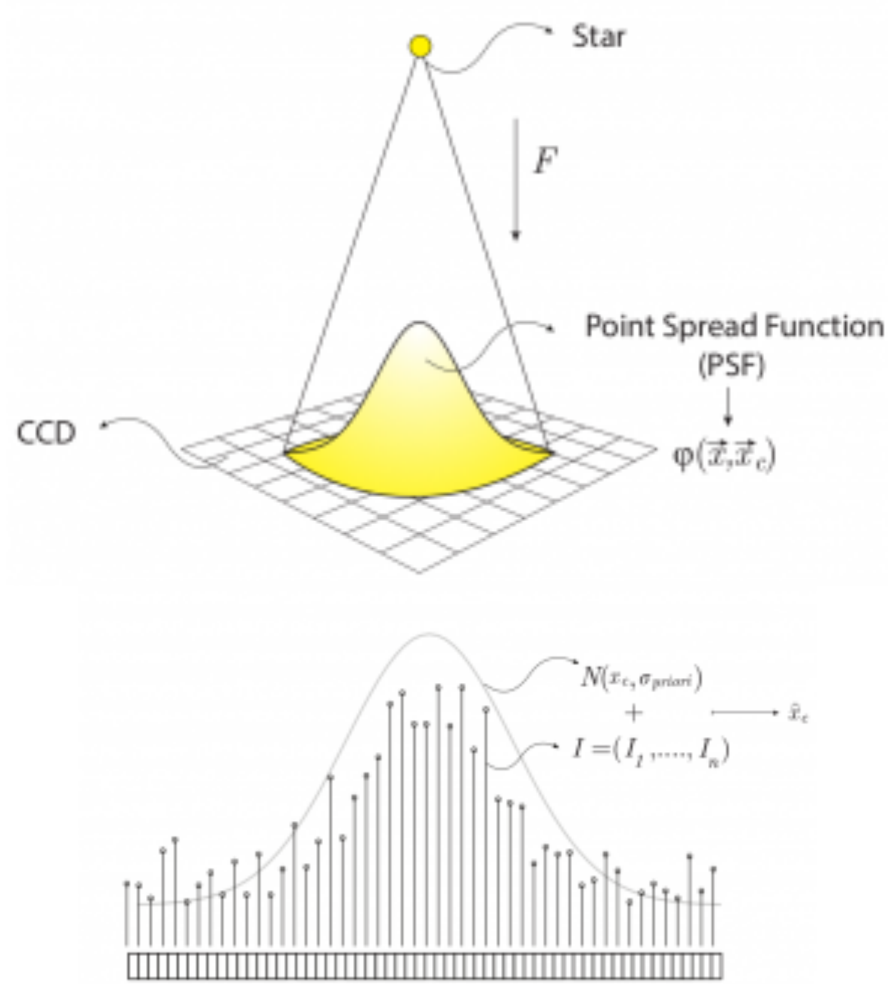
- What are the "fundamental bounds" for astrometry?
- How do the bounds depend on key aspects of the problem?
  - **quality of the site** (\sigma of the PSF)
  - **attributes of the objects** (F)
  - **quality of the instrument** (ROM, D, etc.)
  - **observation setting** (exposition time, etc)
- Are there data-processing schemes that "achieve" the limits?

Content

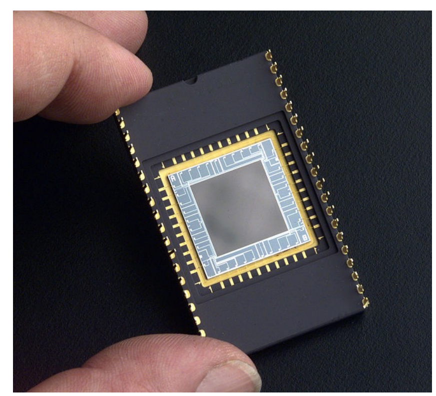
- Astrometry: A Statistical Decision Task
    - The Cramer Rao (CR) bound: Information limits
- Practical Estimators for Astrometry
    - LS and ML estimators
- The Role Priors: the Bayes Cramer Rao Bound

Astrometry: The Decision Setting

Source x_c



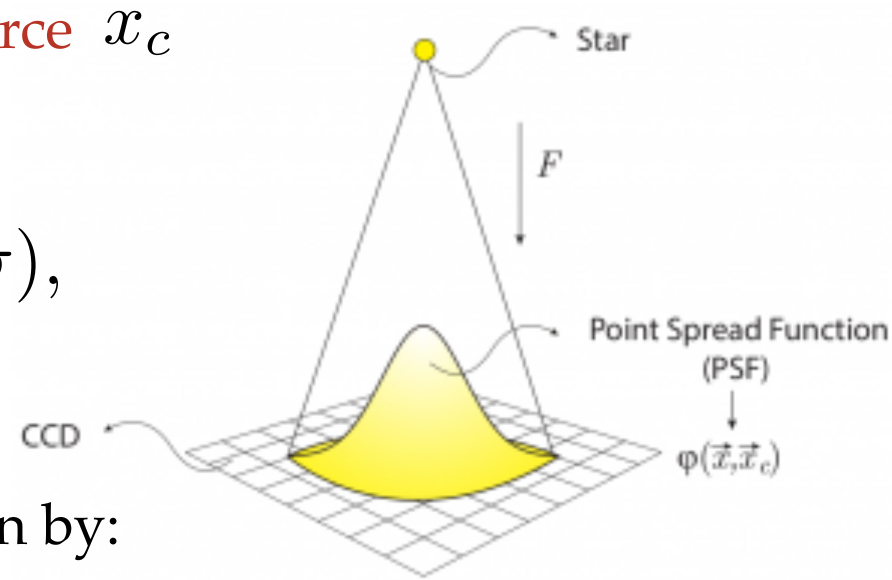
Data $I_1, .., I_n \sim p_{x_c}$



Astrometry: The decision setting

* Estimating the position of a Point Source x_c

$$\tilde{F}_{x_c,\tilde{F}}(x) = \tilde{F} \cdot \phi(x - x_c, \sigma),$$



the final expected pixel intensity is given by:

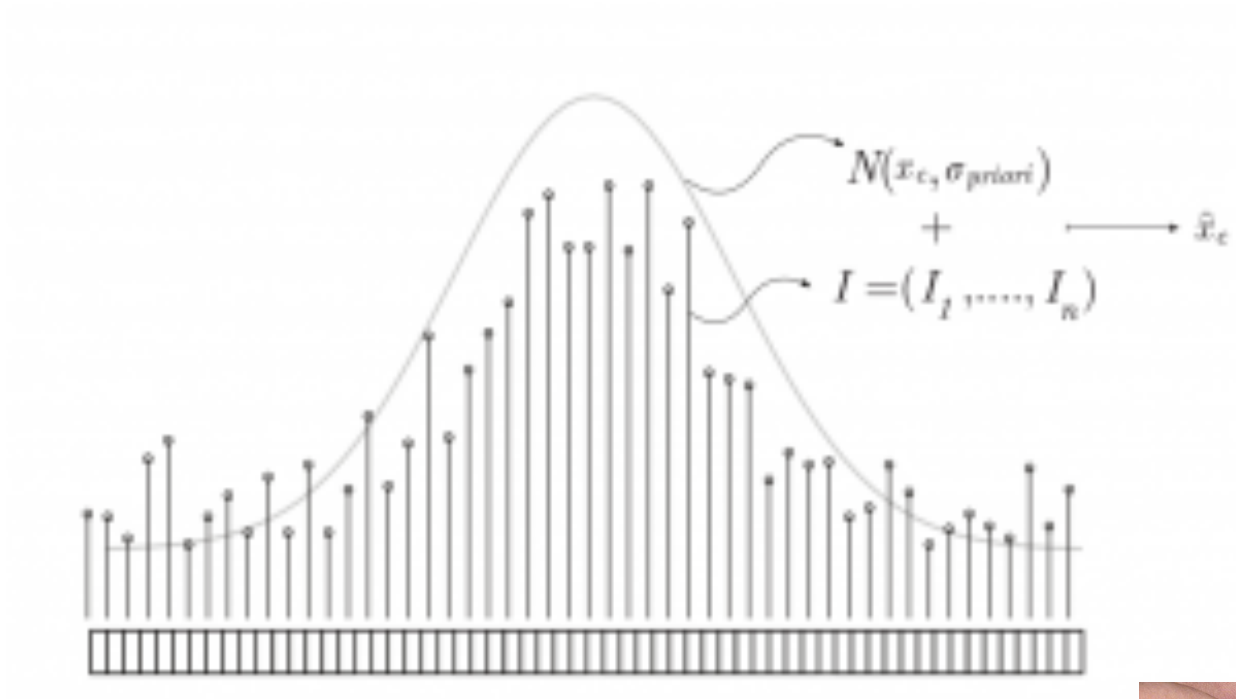
$$\lambda_k(x_c, \tilde{F}) \equiv \mathbb{E}\{I_k\} = \tilde{F} \cdot g_k(x_c) + \tilde{B}_k, \ \forall k \in \mathbb{Z}$$

where

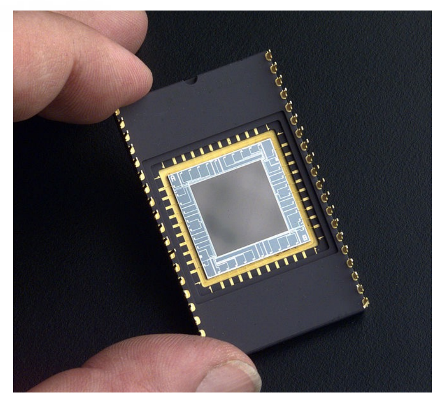
$$g_k(x_c) \equiv \underbrace{\int_{x_k - \Delta x/2}^{x_k + \Delta x/2}}_{quantization} \phi(x - x_c, \sigma)\, dx, \ \forall k \in \mathbb{Z},$$

Astrometry as an Inference Problem

Then we have $\left\{ \lambda_k(x_c, \tilde{F}), k = 1, .., n \right\}$



Randomness



Astrometry as an Inference Problem

Then we have $\left\{\lambda_k(x_c, \tilde{F}), k = 1, .., n\right\}$ where the observations

$$I^n = (I_1, .., I_n) \sim p_{x_c}$$

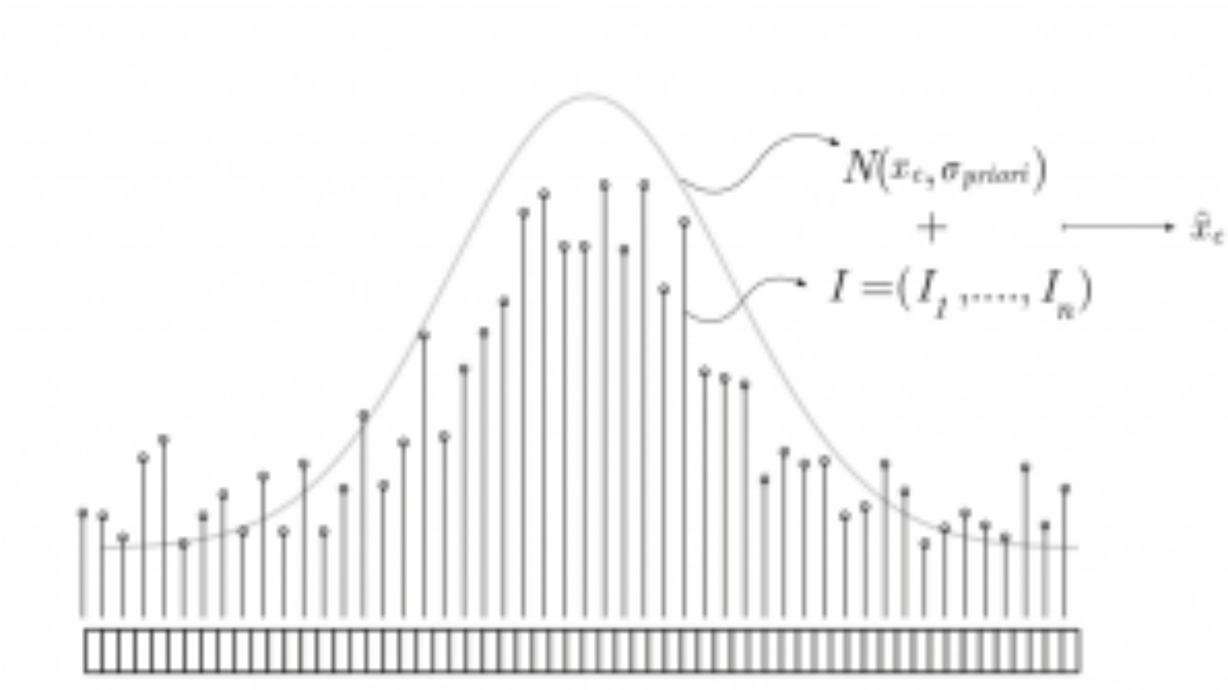
more precisely,

$$\underbrace{\mathbb{P}(I^n = i^n = (i_1, ..., i_n))}_{\equiv\, p_{x_c}(i^n)} = p_{\lambda_1(x_c,\tilde{F})}(i_1) \cdot p_{\lambda_2(x_c,\tilde{F})}(i_2) \cdots p_{\lambda_n(x_c,\tilde{F})}(i_n)$$

where $p_\lambda(x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$ (Poisson distribution).

Astrometry as an Inference Problem

Then we have $\left\{\lambda_k(x_c, \tilde{F}), k = 1, .., n\right\}$



Randomness

$$x_c \quad \longleftrightarrow \quad I^n \sim p_{x_c}$$

Astrometric Cramer Rao Bound

Theorem 1. (Rao (1945); Cramér (1946)) Let $\{I_k : k = 1, ..., n\}$ be a collection of independent observations that follow a parametric pmf p_{θ^m} defined on $\mathbb{N}$. The parameters to be estimated from $I^n = (I_1, ..., I_n)$ will be denoted in general by the vector $\theta^m = (\theta_1, \theta_2, ..., \theta_m) \in \Theta = \mathbb{R}^m$. Let

$$L(i^n;\theta^m) \equiv p_{\theta^m}(i_1)\cdot p_{\theta^m}(i_2)\cdots p_{\theta^m}(i_n)$$ The likelihood

be the likelihood of the observation $i^n \in \mathbb{N}^n$ given $\theta^m \in \Theta$. If the following condition is satisfied

$$\mathbb{E}_{I^n\sim p^n_{\theta^m}}\left\{\frac{\partial \ln L(I^n;\theta^m)}{\partial\theta_j}\right\} = 0, \quad \forall j \in \{1,\ldots,m\}, \tag{6}$$

then for any $\tau_n(\cdot) : \mathbb{N}^n \to \Theta$ unbiased estimator of θ^m (i.e., $\mathbb{E}_{I^n\sim p^n_{\theta^m}}\{\tau_n(I^n)\} = \theta^m$) it follows that

Minimum variance bound

$$Var(\tau_n(I^n)_j) \geq [\mathcal{I}_{\theta^m}(n)^{-1}]_{j,j}, \tag{7}$$

where $\mathcal{I}_{\theta^m}(n)$ is the *Fisher information* matrix given by

$$[\mathcal{I}_{\theta^m}(n)]_{j,l} = \mathbb{E}_{I^n\sim p^n_{\theta^m}}\left\{\frac{\partial \ln L(I^n;\theta^m)}{\partial\theta_j}\cdot\frac{\partial \ln L(I^n;\theta^m)}{\partial\theta_l}\right\}, \tag{8}$$

Information of the measurements

Astrometric Cramer Rao Bound

For the scalar case:

$$\min_{\tau_n(\cdot)\in\mathcal{T}^n} Var(\tau_n(I^n)) \geq \mathcal{I}_\theta(n)^{-1}$$

$$= \mathbb{E}_{I^n \sim p_\theta^n}\left\{\left[\left(\frac{d\ln L(I^n;\theta)}{d\theta}\right)^2\right]\right\}^{-1},$$

Exclusive function of the channel!

Astrometric Cramer Rao Bound

For the 1D Astrometric case we have that (Mendez et al 2013)

$$\mathcal{I}_{x_c}(n) = \sum_{k=1}^{n} \frac{\left(\tilde{F}\frac{dg_k(x_c)}{dx_c}\right)^2}{\tilde{F}g_k(x_c) + \tilde{B}_k},$$

and

$$\min_{\tau^n:\mathbb{N}^n \to \mathbb{R}} Var(\tau_n(I^n)) \geq \underbrace{\mathcal{I}_{x_c}(n)^{-1}}_{\equiv \sigma^2_{CR}}.$$

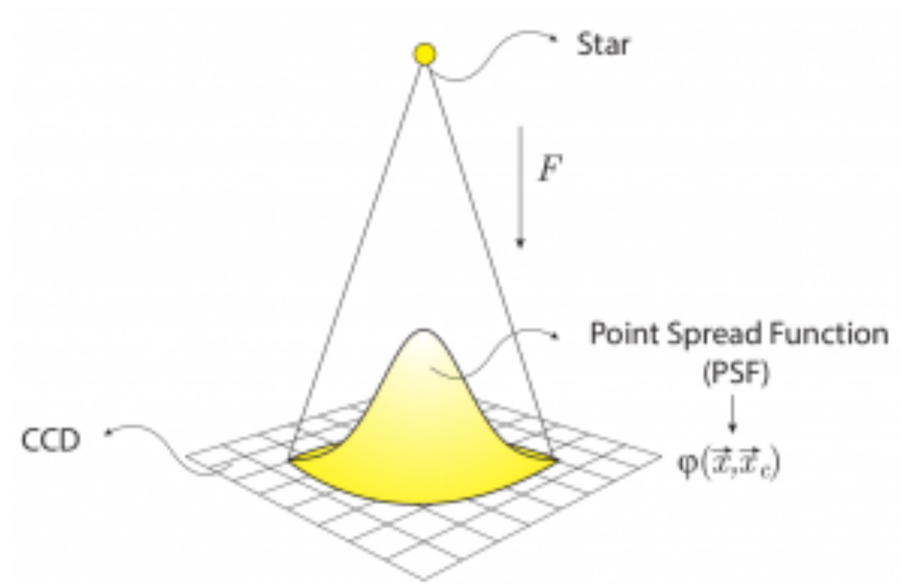
Numerical Analysis of the Bound

* Considering a Gaussian Profile for the PSF

$$\phi(x,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x)^2}{2\sigma^2}}$$

where $FWHM \equiv 2\sqrt{2\ln 2}\;\sigma$

Full-Width at Half-Maximum

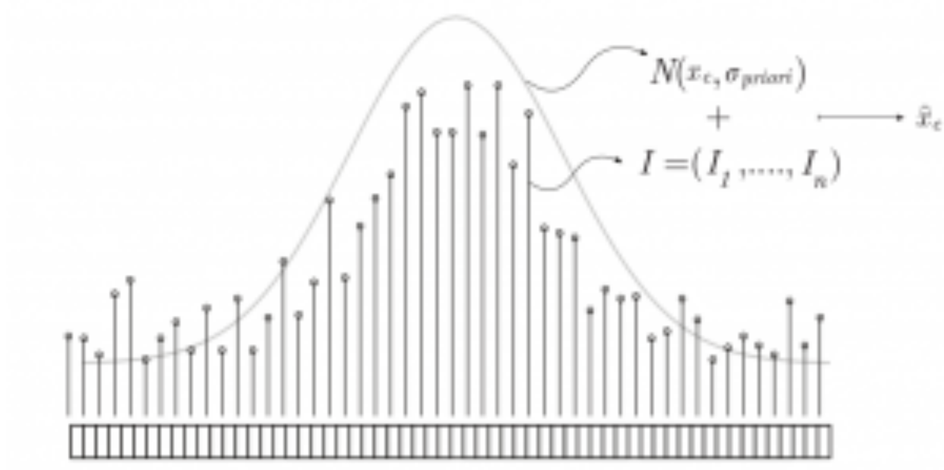


Numerical Analysis of the Bound

* Constant background model $\left\{\tilde{B}_i, i = 1, .., n\right\}$

$$B = f_s \Delta x + \frac{D + RON^2}{G} \; [ADU],$$

- f_s is the diffuse sky background (quality of the site)
- Δx pixel resolution (instrument)
- D, RON dark current and Read add Noise (instrument)



Numerical Analysis of the Bound

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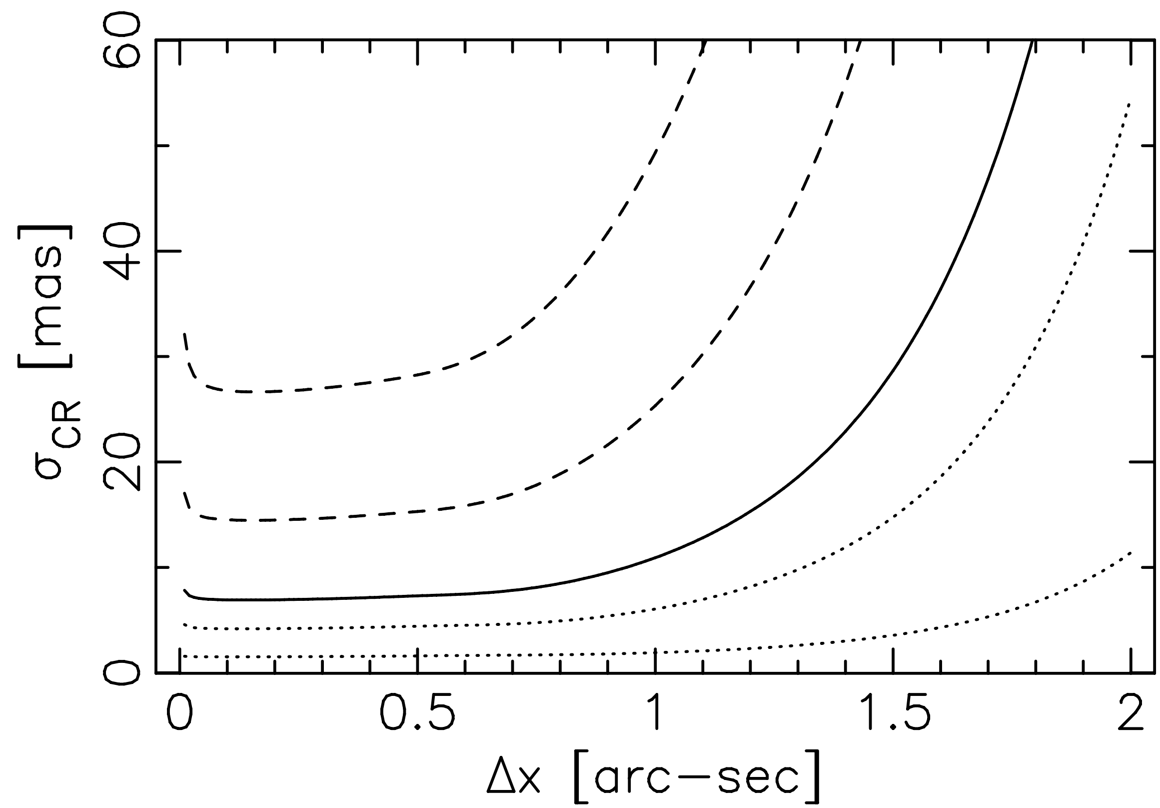
- D, RON dark current and Read add Noise (Instrument)

$$\underbrace{f_s \Delta x \gg \frac{D + RON^2}{G}}_{\text{ground based regime}}$$

Numerical Analysis of the Bound



Cramér-Rao bound σ_{CR} in milliarcseconds (mas), as a function of pixel size Δx in arcseconds. From top to bottom, we have F= 1000 ADU, S/N = 20, F=2000 ADU, S/N = 35 (both dashed lines); F=5000 ADU, S/N=74 (solid line).

$fs = 200 ADU/arcsecond$
$RON = 5e^-$
$FWHM = 1$
$G = 2e^-/ADU$

Asymptotic Regimes

On interesting case is the “high resolution regime“ $\Delta x / \sigma \ll 1$

$$\sigma_{x_c}^2 \approx \begin{cases} \frac{\sqrt{\pi}}{2\,(2\;\ln 2)^{3/2}} \cdot \frac{\tilde{B}}{\tilde{F}^2} \cdot \frac{FWHM^3}{\Delta x} & \text{if } \tilde{F} \ll \tilde{B} \\ \frac{1}{8\;\ln 2} \cdot \frac{1}{\tilde{F}} \cdot FWHM^2 & \text{if } \tilde{F} \gg \tilde{B}, \end{cases}$$

Mendez et al. 2013, PASP

… more of the 1D Astrometry analysis at

PUBLICATIONS OF THE ASTRONOMICAL SOCIETY OF THE PACIFIC, **125**:580–594, 2013 May

Analysis and Interpretation of the Cramér-Rao Lower-Bound in Astrometry: One-Dimensional Case

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AND

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ABSTRACT. In this article we explore the maximum precision attainable in the location of a point source imaged by a pixel array detector in the presence of a background, as a function of the detector properties. For this we use a well-known result from parametric estimation theory, the so-called Cramér-Rao lower bound. We develop the expressions in the one-dimensional case of a linear array detector in which the only unknown parameter is the source position. If the object is oversampled by the detector, analytical expressions can be obtained for the Cramér-Rao limit that can be readily used to estimate the limiting precision of an imaging system, and which are very useful for experimental (detector) design, observational planning, or performance estimation of data analysis software: In particular, we demonstrate that for background-dominated sources, the maximum astrometric precision goes as B/F^2, where B is the background in one pixel, and F is the total flux of the source, while when the background is negligible, this precision goes as F^{-1}. We also explore the dependency of the astrometric precision on: (1) the size of the source (as imaged by the detector), (2) the pixel detector size, and (3) the effect of source decentering. Putting

Mendez et al. 2014, PASP

… and extension to 2D Astrometry and joint Astrometry & Photometry

PUBLICATIONS OF THE ASTRONOMICAL SOCIETY OF THE PACIFIC, **126**:798–810, 2014 August

Analysis of the Cramér-Rao Bound in the Joint Estimation of Astrometry and Photometry

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Received 2014 April 07; accepted 2014 July 03; published 2014 August 19

ABSTRACT. In this paper, we use the Cramér-Rao lower uncertainty bound to estimate the maximum precision that could be achieved on the joint simultaneous (or two-dimensional) estimation of photometry and astrometry of a point source measured by a linear CCD detector array. We develop exact expressions for the Fisher matrix elements required to compute the Cramér-Rao bound in the case of a source with a Gaussian light profile. From these expressions, we predict the behavior of the Cramér-Rao astrometric and photometric precision as a function of the signal and the noise of the observations, and compare them to actual observations—finding a good correspondence between them. From the Cramér-Rao bound, we obtain the well-known fact that the uncertainty in flux on a Poisson-driven detector, such as a CCD, goes approximately as the square root of the flux. However, more generally, higher-order correction factors that depend on the ratio B/F or F/B (where B is the background flux per pixel, and F is the total flux of the source), as well as on the properties of the detector (pixel size) and the source (width of the light profile), are required for a proper calculation of the minimum expected uncertainty bound in flux. Overall, the Cramér-Rao bound predicts that the uncertainty in magnitude goes as $(\mathrm{S/N})^{-1}$ under a broad range of circumstan-

Achievability of the CR Bound

Achievability Analysis

- Is the CR bound attainable by any practical estimator?
- How far are practical estimators used in Astrometry (like LS, WLS, ML) from the CR bound?
- How previous questions depends on the observational regime (S/R, pixel resolution, site, PSF, etc.)?

Impossibility Result: Lobos et al. 2015, PASP

Proposition: For any fixed and unknown position x_c and unbiased estimator $\tau_n : \mathbb{N}^n \longrightarrow \Theta$

$$Var(\tau_n(I^n)) > \sigma^2_{CR}.$$

Least Square: Performance Approximation

The technical challenges is to find expression to approximate the performance of an "implicit estimator" solution of:

$$\tau_{LS}(I^n) = \arg\min_{\alpha \in \mathbb{R}} \underbrace{\sum_{i=1}^{n} (I_i - \lambda_i(\alpha))^2}_{\equiv J(I^n, \alpha)},$$

where $\lambda_i(\alpha) = \tilde{F} g_i(\alpha) + \tilde{B}_i.$

Main Result: Lobos et al. (2015) PASP

Theorem 1.— Let us consider a fixed and unknown parameter $x_c \in \mathbb{R}$, and that $I^n \sim f_{x_c}$. In addition, let us define the residual random variable $W(I^n, \alpha) \equiv \frac{J''(I^n,\alpha) - \mathbb{E}_{I^n \sim f_{x_c}}\{J''(I^n,\alpha)\}}{\mathbb{E}_{I^n \sim f_{x_c}}\{J''(I^n,\alpha)\}}$.[14] If there exists $\delta \in (0, 1)$ such that $\mathbb{P}(W(I^n, x_c) \in (-\delta, \delta)) = 1$, then:

$$|\mathbb{E}_{I^n \sim f_{x_c}}\{\tau_{\mathrm{LS}}(I^n)\} - x_c| \leq \epsilon(\delta), \tag{17}$$

$$\mathbb{E}_{I^n \sim f_{x_c}}\{(\tau_{\mathrm{LS}}(I^n) - x_c)^2\} \in \left(\frac{\sigma^2_{\mathrm{LS}}(n)}{(1+\delta)^2}, \frac{\sigma^2_{\mathrm{LS}}(n)}{(1-\delta)^2} \right), \tag{18}$$

where

$$\sigma^2_{\mathrm{LS}}(n) \equiv \frac{\mathbb{E}_{I^n \sim f_{x_c}}\{J'(I^n, x_c)^2\}}{(\mathbb{E}_{I^n \sim f_{x_c}}\{J''(I^n, x_c)\})^2} \tag{19}$$

and

$$\epsilon(\delta) \equiv \frac{\mathbb{E}_{I^n \sim f_{x_c}}\{|J'(I^n, x_c)|\}}{\mathbb{E}_{I^n \sim f_{x_c}}\{J''(I^n, x_c)\}} \cdot \frac{\delta}{1-\delta}. \tag{20}$$

Main Result

The "predicted nominal value" offers a closed-form

$$\underbrace{\sigma^2_{LS}(n) = \frac{\sum_{i=1}^{n}(\tilde{F}g_i(x_c) + \tilde{B}_i) \cdot (g_i'(x_c))^2}{\left(\tilde{F}\sum_{i=1}^{n}(g_i'(x_c))^2\right)^2}}_{\text{to compare with } \sigma^2_{CR}} .$$

Main Result

The "predicted nominal value" offers a closed-form

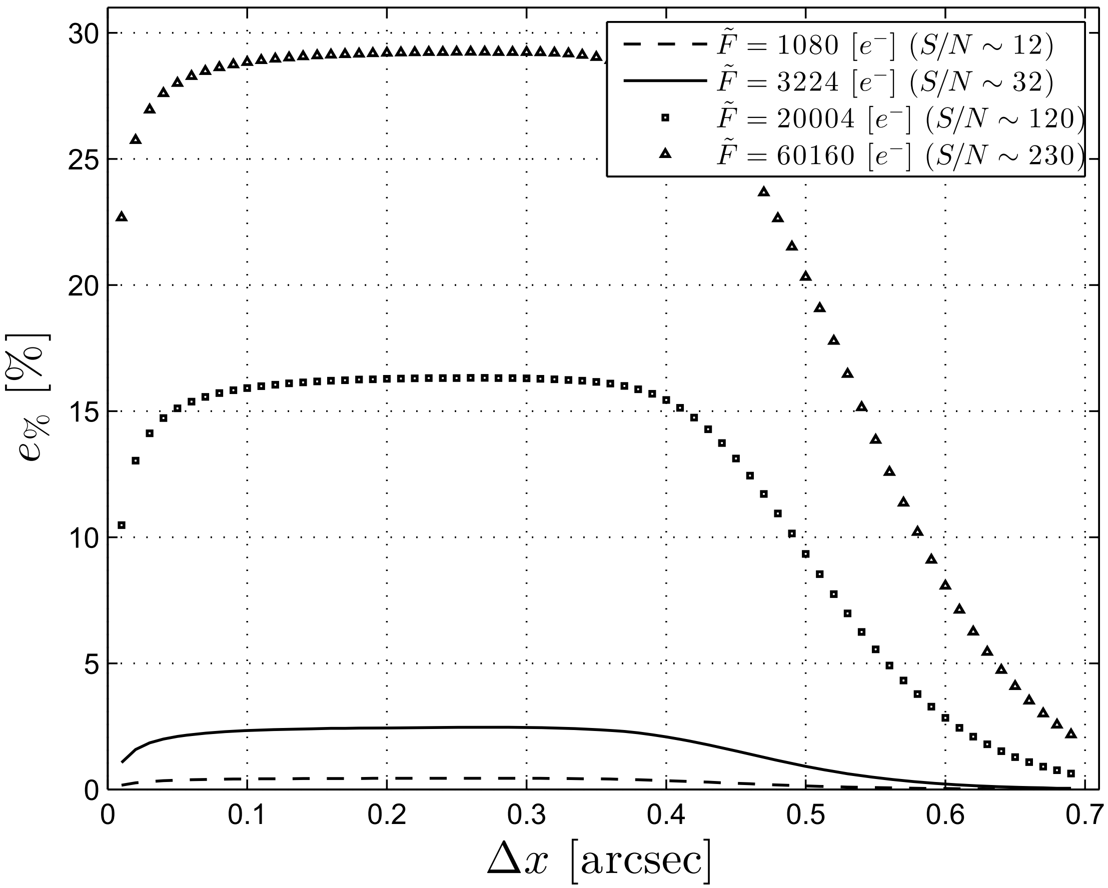
$$\underbrace{\sigma_{LS}^2(n) = \frac{\sum_{i=1}^{n}(\tilde{F} g_i(x_c) + \tilde{B}_i) \cdot (g_i'(x_c))^2}{\left(\tilde{F} \sum_{i=1}^{n} (g_i'(x_c))^2\right)^2}.}$$

Non-optimality for hight S/R:

Proposition 3.— Assuming the idealized high S/N regime, if we have a Gaussian-like PSF and $\Delta x/\sigma \ll 1$, then:

$$\frac{\sigma_{\mathrm{LS}}^2(n)}{\sigma_{\mathrm{CR}}^2} \approx \frac{8}{3\sqrt{3}} > 1. \tag{26}$$

Numerical Analysis: $e_{\%} = 100 \cdot \dfrac{\sigma^2_{LS} - \sigma^2_{CR}}{\sigma^2_{CR}}$



A significant performance gap between the LS technique and the CR bound is found for hight S/R regime.

More in Lobos et al., 2015, PASP.

Publications of the Astronomical Society of the Pacific, **127**:1166–1182, 2015 November

Performance Analysis of the Least-Squares Estimator in Astrometry

Rodrigo A. Lobos,[1] Jorge F. Silva,[1] Rene A. Mendez,[2] and Marcos Orchard[1]

Received 2015 May 08; accepted 2015 August 26; published 2015 October 22

ABSTRACT. We characterize the performance of the widely used least-squares estimator in astrometry in terms of a comparison with the Cramér–Rao lower variance bound. In this inference context the performance of the least-squares estimator does not offer a closed-form expression, but a new result is presented (Theorem 1) where both the bias and the mean-square-error of the least-squares estimator are bounded and approximated analytically, in the latter case in terms of a *nominal value* and an interval around it. From the predicted nominal value, we analyze how efficient the least-squares estimator is in comparison with the minimum variance Cramér–Rao bound. Based on our results, we show that, for the high signal-to-noise ratio regime, the performance of the least-squares estimator is significantly poorer than the Cramér–Rao bound, and we characterize this gap analytically. On the positive side, we show that for the challenging low signal-to-noise regime (attributed to either a weak astronomical signal or a noise-dominated condition) the least-squares estimator is near optimal, as its performance asymptotically approaches the Cramér–Rao bound. However, we also demonstrate that, in general, there is no unbiased estimator for the astrometric position that can precisely reach the Cramér–Rao bound. We validate our theoretical analysis through simulated digital-detector observations under typical observing conditions. We show that the *nominal value* for the mean-square-error of the least-squares estimator (obtained from our theorem) can be used as a benchmark indicator of the expected statistical performance of the least-squares method under a wide range of conditions. Our results are valid for an idealized linear (one-dimensional) array detector where intrapixel response changes are neglected, and where flat-fielding is achieved with very high accuracy.

The Maximum Likelihood Estimator

Given a set of observations I^n the maximum likelihood solves:

$$\tau_{ML}(I^n) = \arg\max_{\alpha \in \mathbb{R}} \ln p_\alpha(I^n)$$

that reduces to:

$$\tau_{ML}(I^n) = \arg\min_{\alpha \in \mathbb{R}} \sum_{i=1}^{n} -I_i \ln(\lambda_i(\alpha)) + \lambda_i(\alpha).$$

The Maximum Likelihood Estimator

Theorem 3 *Let us consider the ML estimator solution of (31), then we have that*

$$|\underbrace{\mathbb{E}_{I^n \sim f_{x_c}}\{\tau_{ML}(I^n)\} - x_c}_{bias}| \leq \epsilon_{ML}(n) \tag{32}$$

and

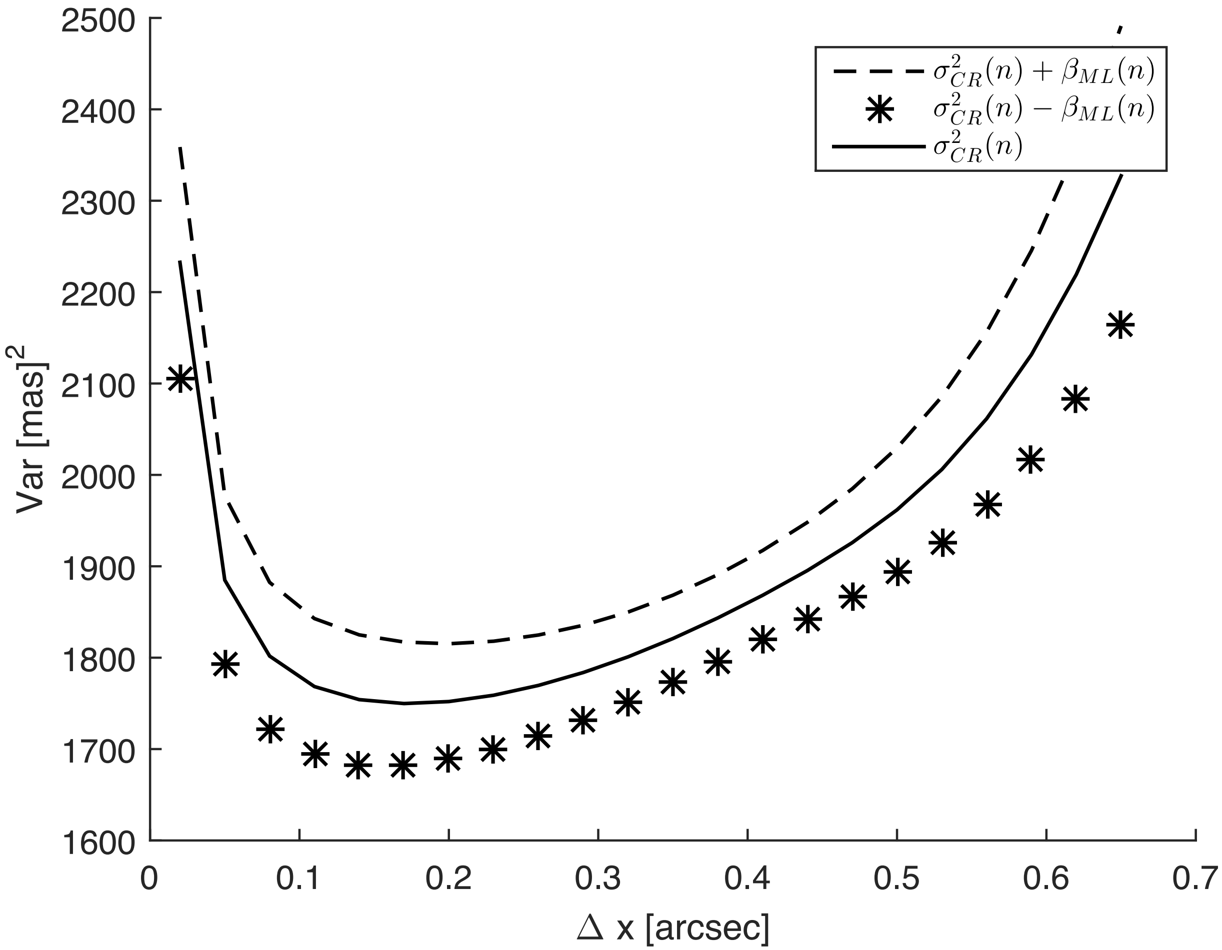
$$Var_{I^n \sim f_{x_c}}\{\tau_{ML}(I^n)\} \in \left(\sigma^2_{ML}(n) - \beta_{ML}(n), \sigma^2_{ML}(n) + \beta_{ML}(n)\right), \tag{33}$$

where

$$\sigma^2_{ML}(n) = \sigma^2_{CR}(n) = \left(\sum_{i=1}^{n} \frac{\left(\tilde{F}\frac{dg_i(x_c)}{dx_c}\right)^2}{\tilde{F}g_i(x_c) + \tilde{B}_i}\right)^{-1}, \tag{34}$$

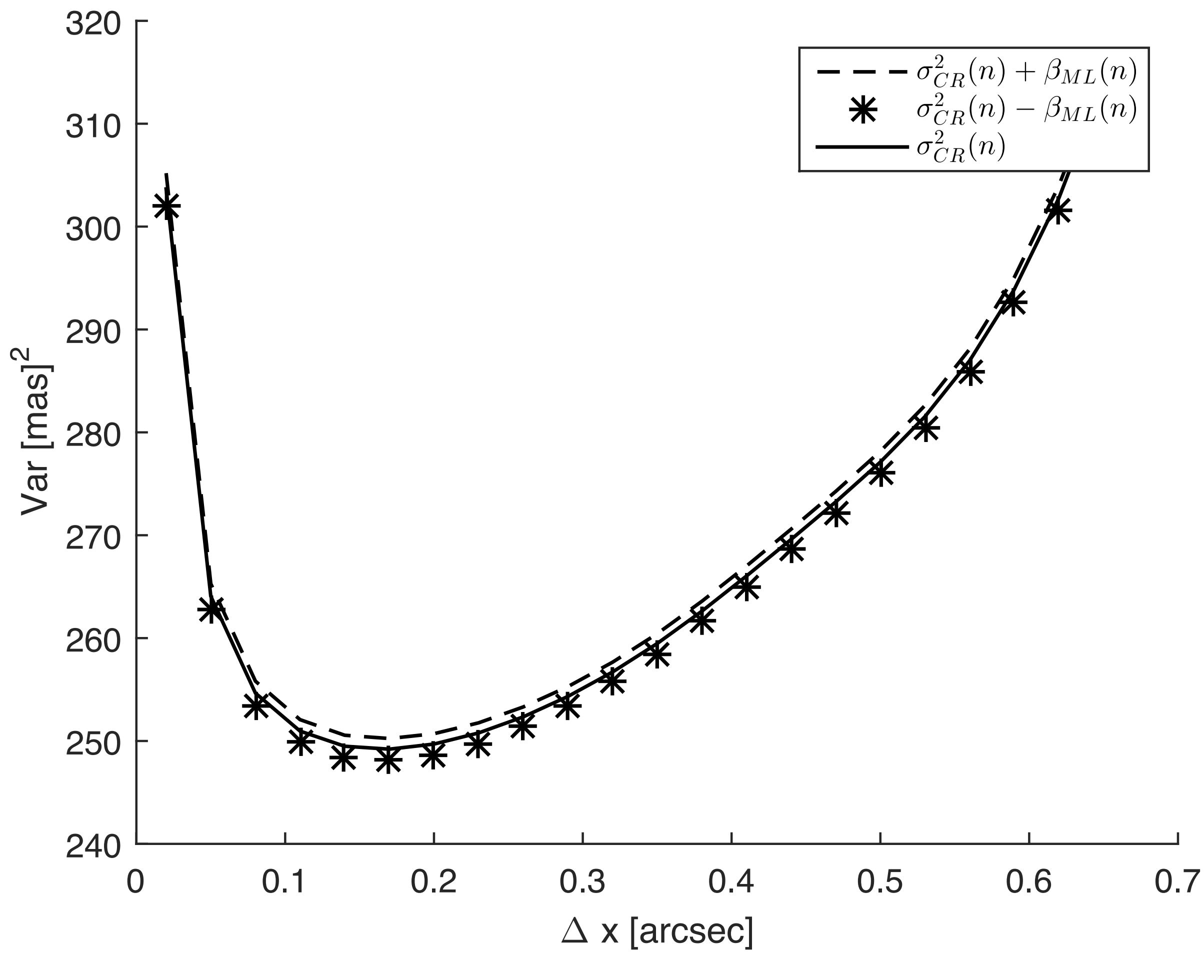
and $\beta_{ML}(n)$ and $\epsilon_{ML}(n)$ are well defined analytical expression of the problem (presented in Appendix C).

The Maximum Likelihood Estimator



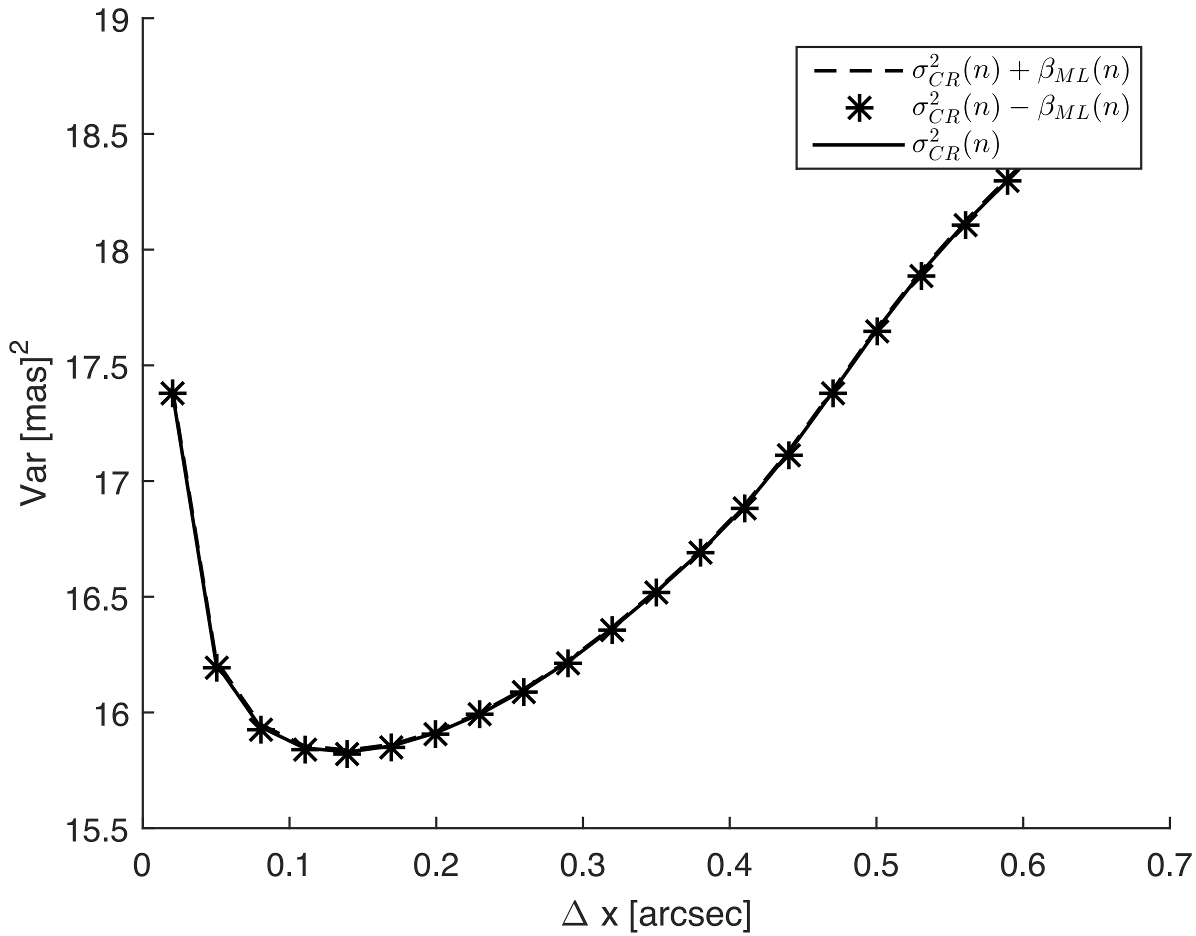
$\tilde{F} = 1080e^{-}$ small S/R

The Maximum Likelihood Estimator



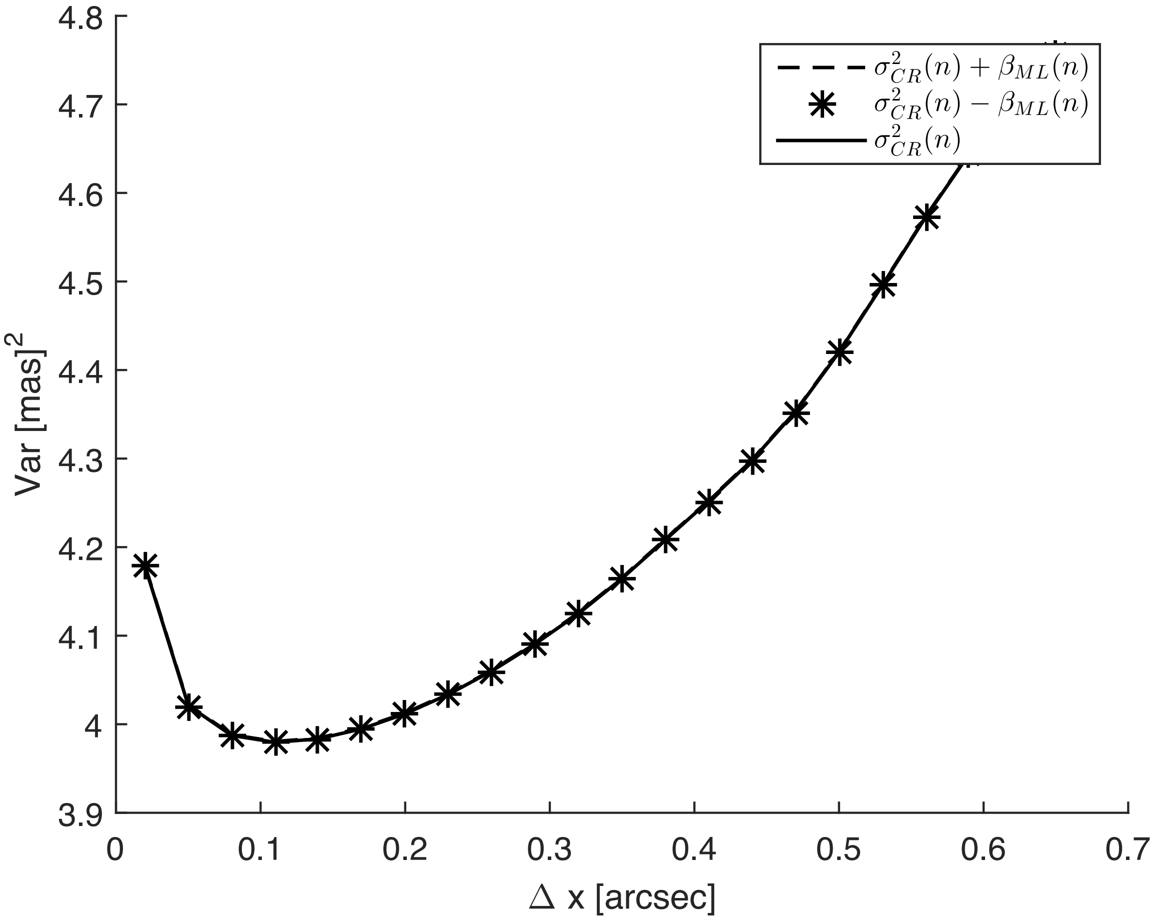
$\tilde{F} = 3224e^{-}$ medium S/R

The Maximum Likelihood Estimator



$\tilde{F} = 20004e^{-}$ high S/R

The Maximum Likelihood Estimator



$\tilde{F} = 60160e^{-}$ very high S/R

Integrating Prior Information

Prior Information in Astrometry

Recent work considers the use of "prior information" to improve the accuracy in Astrometry (Michalik et al. 2015 and Michalik &Lindegren 2016):

- Michalik &Lindegren 2016 proposes the use of priors from QSO proper motions to improve astrometry in the GAIA astrometric satellite.
- Michalik et al. 2015 uses prior information to improve astrometry in simulated GAIA observations for point source with poor observations (low S/R). Prior comes from reasonable assumptions about the distribution of proper motions and parallaxes.

Michalik, D. & Lindegren, L. 2016, A&A, 586, A26
Michalik, D., Lindegren, L., Hobbs, D., & Butkevich, A. G. 2015b, A&A, 583, A68

Bayes Setting

The position X_c is a r.v. with a "prior density function" $\psi(\cdot)$

$$\mathbb{P}(X_c \in B) = \int_B \psi(x)\, dx.$$

Bayes Setting

The position X_c is a r.v. with a "prior density function" $\psi(\cdot)$

$$\mathbb{P}(X_c \in B) = \int_B \psi(x)\, dx.$$

Then

$$\mathbb{P}(I^n = i^n | X_c = x_c) = p_{x_c}(i^n) = \underbrace{\Pi_{k=1}^n p_{\lambda_k(x_c)}(i_k)}_{\equiv L(i^n; x_c)}$$

and we have a joint r.v. (X_c, I^n)

$$\mathbb{P}((X_c, I^n) \in B \times A) = \int_B \sum_{i^n \in A} \underbrace{p_x(i^n) \cdot \psi(x)}_{\tilde{L}(x_c, i^n)}\, dx.$$

"joint likelihood"

The Bayes Cramer Rao Lower Bound

2. (Van Trees 2004, Sec. 2.4) For any possible decision rule $\tau^n : \mathbb{N}^n \to \mathbb{R}$, it is true that

$$\mathbb{E}_{(X_c, I^n)}\left\{(\tau^n(I^n) - X_c)^2\right\} \geq \left[\mathbb{E}_{(I^n, X_c)}\left\{\left(\frac{d \ln \tilde{L}(X_c, I^n)}{dx}\right)^2\right\}\right]^{-1}, \qquad (15)$$

"the Information Term"

where

$$\tilde{L}(x_c, i^n) \equiv p_{x_c}(i^n) \cdot \psi(x_c) = L(i^n; x_c) \cdot \psi(x_c) \qquad (16)$$

Van Trees, H. L. 2004, Detection, estimation, and modulation theory (John Wiley & Sons)

The Bayes Information

… after some algebra (Echeverria et al. 2016):

$$\underbrace{\mathbb{E}_{(X_c,I^n)}\left\{\left(\frac{d\ln\tilde{L}(X_c,I^n)}{dx}\right)^2\right\}}_{\text{Bayes Fisher Information}=BFI(F,\psi)} = \mathbb{E}_{X_c\sim\psi}\left\{\mathcal{I}_{X_c}(n)\right\} + \underbrace{\mathbb{E}_{X_c\sim\psi}\left\{\left(\frac{d\ln\psi(X_c)}{dx}\right)^2\right\}}_{\equiv\mathcal{I}(\psi)}.$$

The Bayes Information

… then:

$$\min_{\tau^n:\mathbb{N}^n\to\mathbb{R}} \mathbb{E}_{(X_c,I^n)}\left\{(\tau^n(I^n)-X_c)^2\right\} \geq \underbrace{\frac{1}{\mathbb{E}_{X_c\sim\psi}\left\{\mathcal{I}_{X_c}(n)\right\}+\mathcal{I}(\psi)}}_{\equiv\sigma^2_{BCR}},$$

- $\mathbb{E}_{X_c\sim\psi}\left\{\mathcal{I}_{X_c}(n)\right\}$ is the average Fisher Information
- $\mathcal{I}(\psi)$ information from the "prior"

Gain in Astrometric Precision from the Prior

Definition: The prior information $\mathcal{I}(\psi)$ is irrelevant if

$$\frac{\mathcal{I}(\psi)}{\mathbb{E}_{X_c \sim \psi}\left\{\mathcal{I}_{X_c}(n)\right\}} \approx 0,$$

otherwise is **relevant!**

Gain in Astrometric Precision from the Prior

More meaningful is the following "**operational**" indicator:

Definition: The gain in performance attributed to ψ is:

$$Gain(\psi) \equiv \min_{\tau^n_{un}\mathbb{N}^n \to \mathbb{R} \text{ and } \tau^n_{un} \text{ is unbiased}} \mathbb{E}\left\{\left(\tau^n_{unbias}(I^n) - x\right)^2\right\}$$

$$- \min_{\tau^n:\mathbb{N}^n \to \mathbb{R}} \mathbb{E}\left\{\left(\tau^n(I^n) - X_c\right)^2\right\},$$

Gain in Astrometric Precision from the Prior

……we can show that (Echeverria et al. 2016, Proposition 3)

$$Gain(\psi) \geq \underbrace{\mathbb{E}_{X_c \sim \psi}\left\{\mathcal{I}_{X_c}(n)^{-1}\right\}}_{\text{Mean CR Bound}} - \underbrace{\frac{1}{\mathbb{E}_{X_c \sim \psi}\left\{\mathcal{I}_{X_c}(n)\right\} + \mathcal{I}(\psi)}}_{\text{Bayes CR Bound}}$$

Gain in Astrometric Precision from the Prior

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where even for the "**worse prior**" (from Jensen´s Inequality):

$$Gain(\psi) \geq \mathbb{E}_{X_c \sim \psi}\left\{\mathcal{I}_{X_c}(n)^{-1}\right\} - \mathbb{E}_{X_c \sim \psi}\left\{\mathcal{I}_{X_c}(n)\right\}^{-1} \geq 0.$$

Gain in Astrometric Precision from the Prior

… further details in Echeverria et al., A&A, 2016.

Astronomy & Astrophysics manuscript no. Bayes_CR_astrometry_A&A
August 2, 2016

Analysis of the Bayesian Cramér-Rao lower bound in astrometry:

Studying the impact of prior information in the location of an object

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ABSTRACT

Context. The best precision that can be achieved to estimate the location of a stellar-like object is a topic of permanent interest in the astrometric community.

Aims. We analyze bounds for the best position estimation of a stellar-like object on a CCD detector array in a Bayesian setting where the position is unknown, but where we have access to a prior distribution. In contrast to a parametric setting where we estimate a parameter from observations, the Bayesian approach estimates a random object (i.e., the position is a random variable) from observations that are statistically dependent on the position.

Numerical Analysis

- We consider a Gaussian prior $\psi(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma_{priori}} e^{-\frac{(x-\mu)^2}{2\sigma_{priori}^2}}$

$$\mathcal{I}(\psi) = \mathbb{E}_{X_c \sim \psi} \left\{ \left(\frac{d \ln \psi(X_c)}{dx} \right)^2 \right\} = \frac{1}{\sigma_{priori}^2}.$$

-

Numerical Analysis

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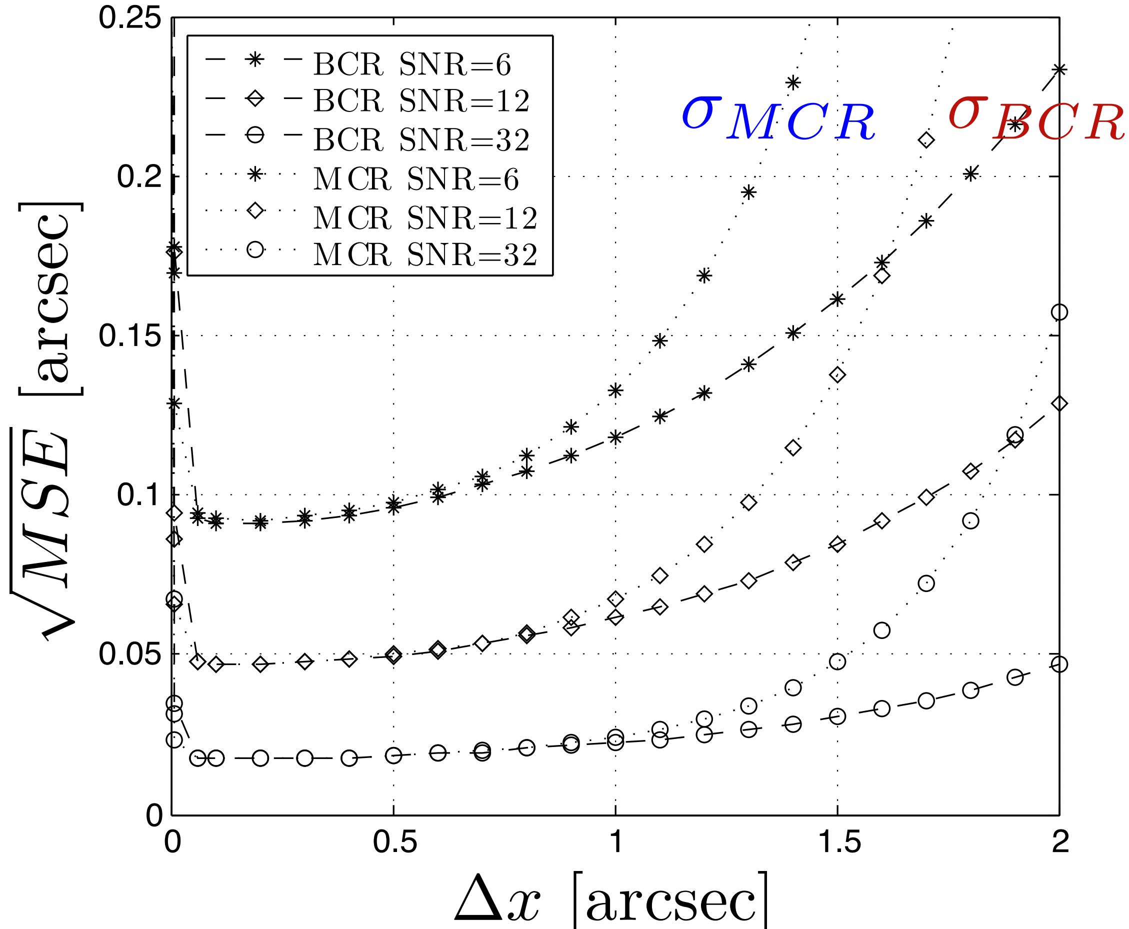
$$\mathcal{I}(\psi) = \mathbb{E}_{X_c \sim \psi}\left\{\left(\frac{d \ln \psi(X_c)}{dx}\right)^2\right\} = \frac{1}{\sigma_{priori}^2}.$$

- then we have an expression for the astrometric BCR bound

$$\sigma_{BCR}^2 = \left[\mathbb{E}_{(X_c, I^n)}\left\{\left(\frac{d \ln \tilde{L}(X_c, I^n)}{dx}\right)^2\right\}\right]^{-1}$$

$$= \left[\left(\frac{\tilde{F}^2}{2\pi\sigma^2 \tilde{B}} \cdot \mathbb{E}_{X_c \sim \mathcal{N}(\mu, \sigma_{priori})}\left\{\sum_{k=1}^{n} \frac{\left(e^{-\gamma(x_k^- - x_c)} - e^{-\gamma(x_k^+ - x_c)}\right)^2}{\left(1 + \frac{1}{\sqrt{2\pi}\,\sigma}\frac{\tilde{F}}{\tilde{B}} \cdot \int_{x_k^-}^{x_k^+} e^{-\gamma(x - x_c)}\, dx\right)}\right\} + \frac{1}{\sigma_{priori}^2}\right)\right]^{-1}.$$

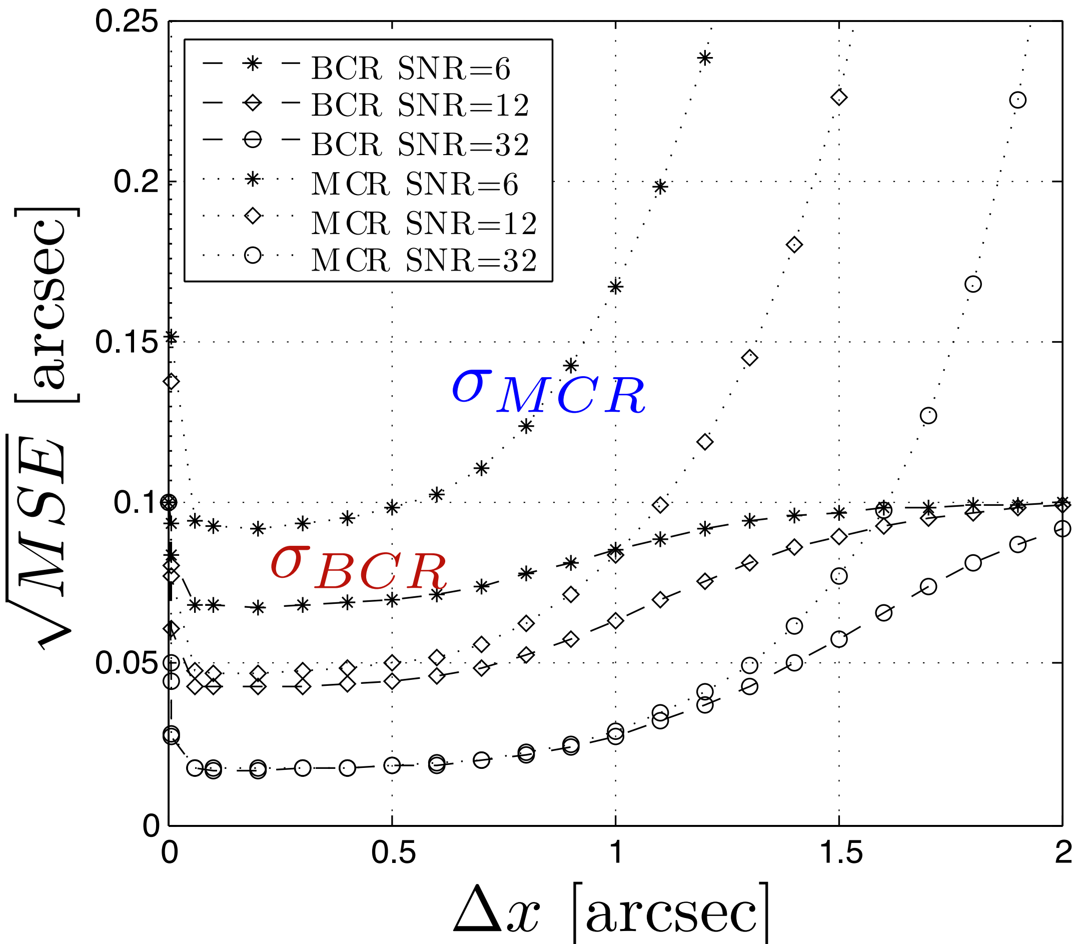
Numerical Analysis



$FWHM = 1$
$fs = 200ADU/arcsecond$
$RON = 5e^{-}\ \ D = 0$
$G = 2e^{-}/ADU$
$F \in \{268, 540, 1612\} \Rightarrow S/R \in \{6, 12, 32\}$

$\sigma_{priori} = 0.5arcsec.$

Numerical Analysis



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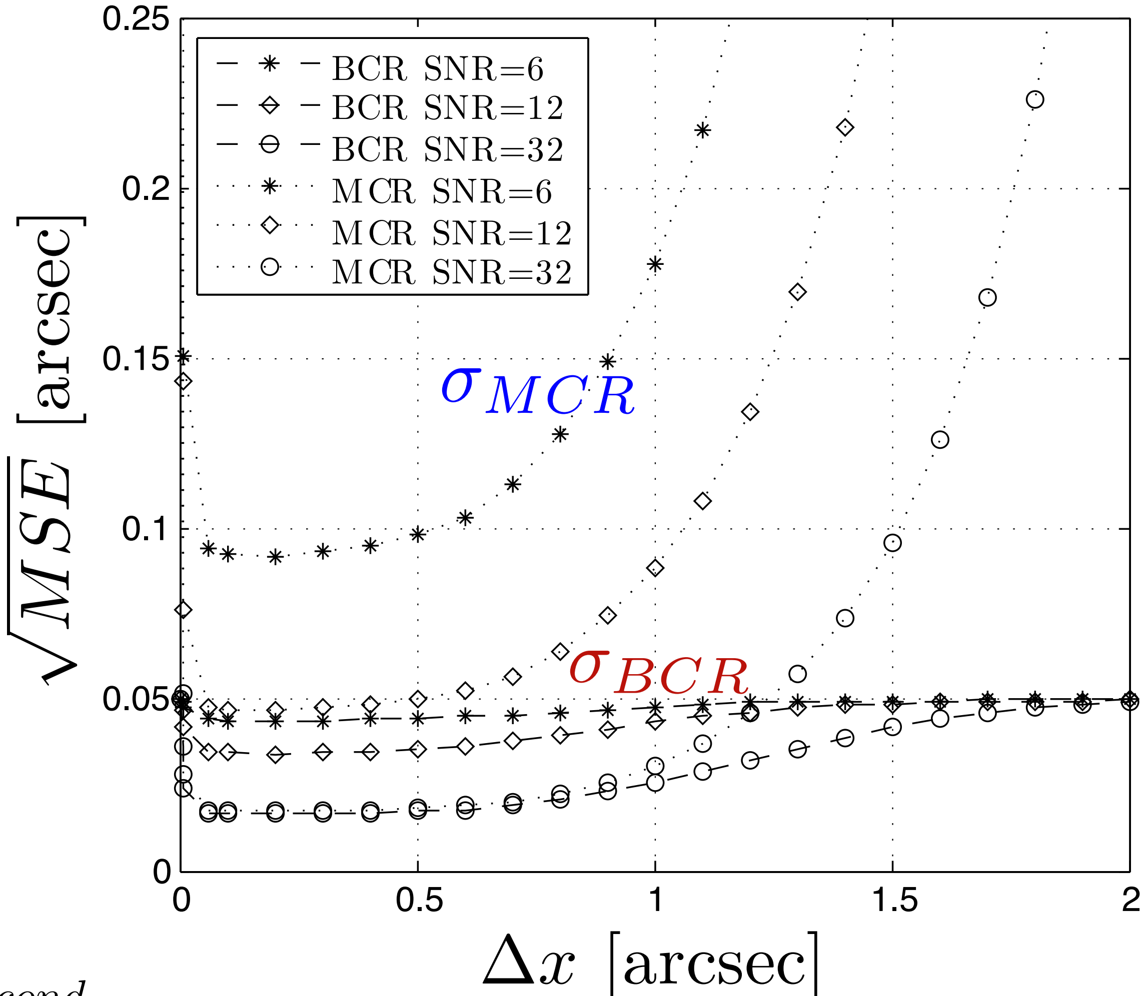
$RON = 5e^{-}\ D = 0$
$G = 2e^{-}/ADU$
$F \in \{268, 540, 1612\} \Rightarrow S/R \in \{6, 12, 32\}$

$\sigma_{priori} = 0.1arcsec.$

Numerical Analysis



$FWHM = 1$

$fs = 200 ADU/arcsecond$

$RON = 5e^- \quad D = 0$

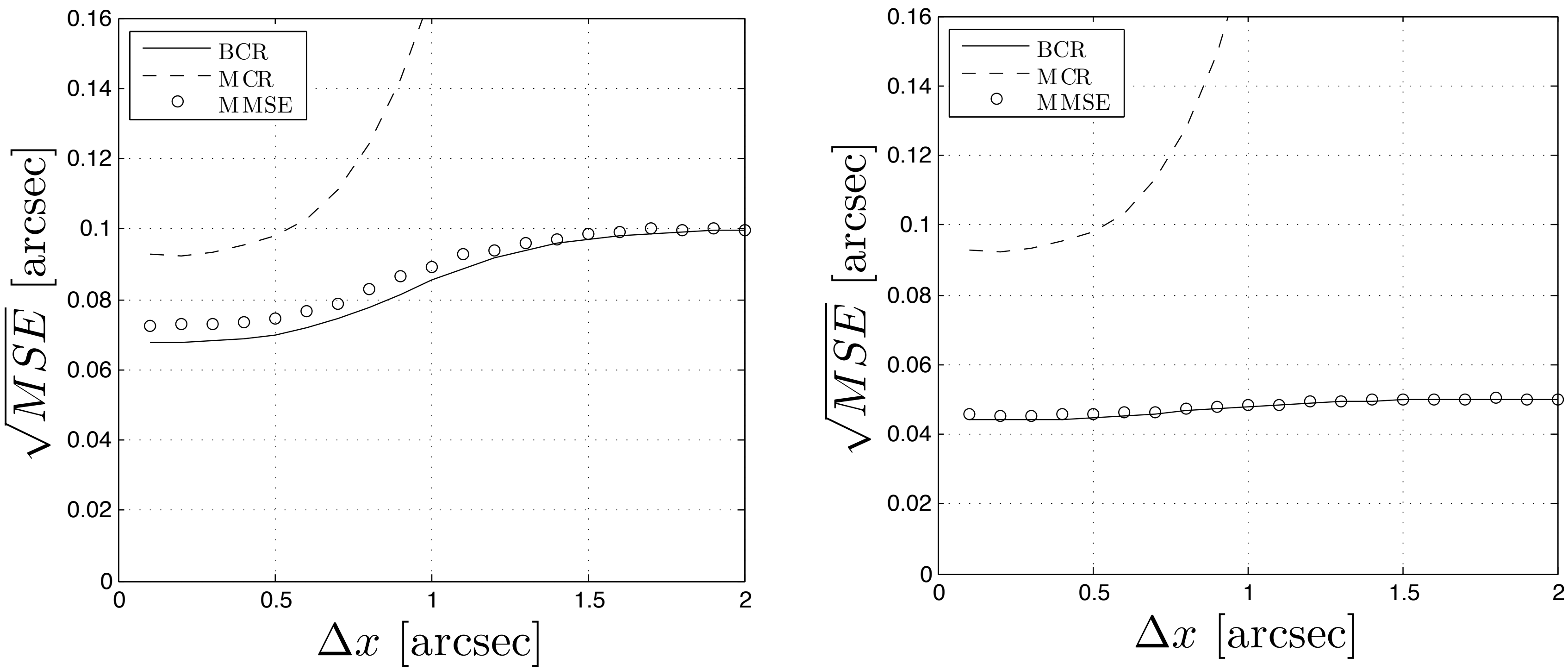
$G = 2e^-/ADU$

$F \in \{268, 540, 1612\} \Rightarrow S/R \in \{6, 12, 32\}$

$\sigma_{priori} = 0.05 arcsec.$

Numerical Analysis: Achievability with the MMSE

The conditional mean estimator (MMSE) achieves the σ^2_{BCR}



(a) σ_{priori}=0.1 arcsec

(b) σ_{priori}=0.05 arcsec

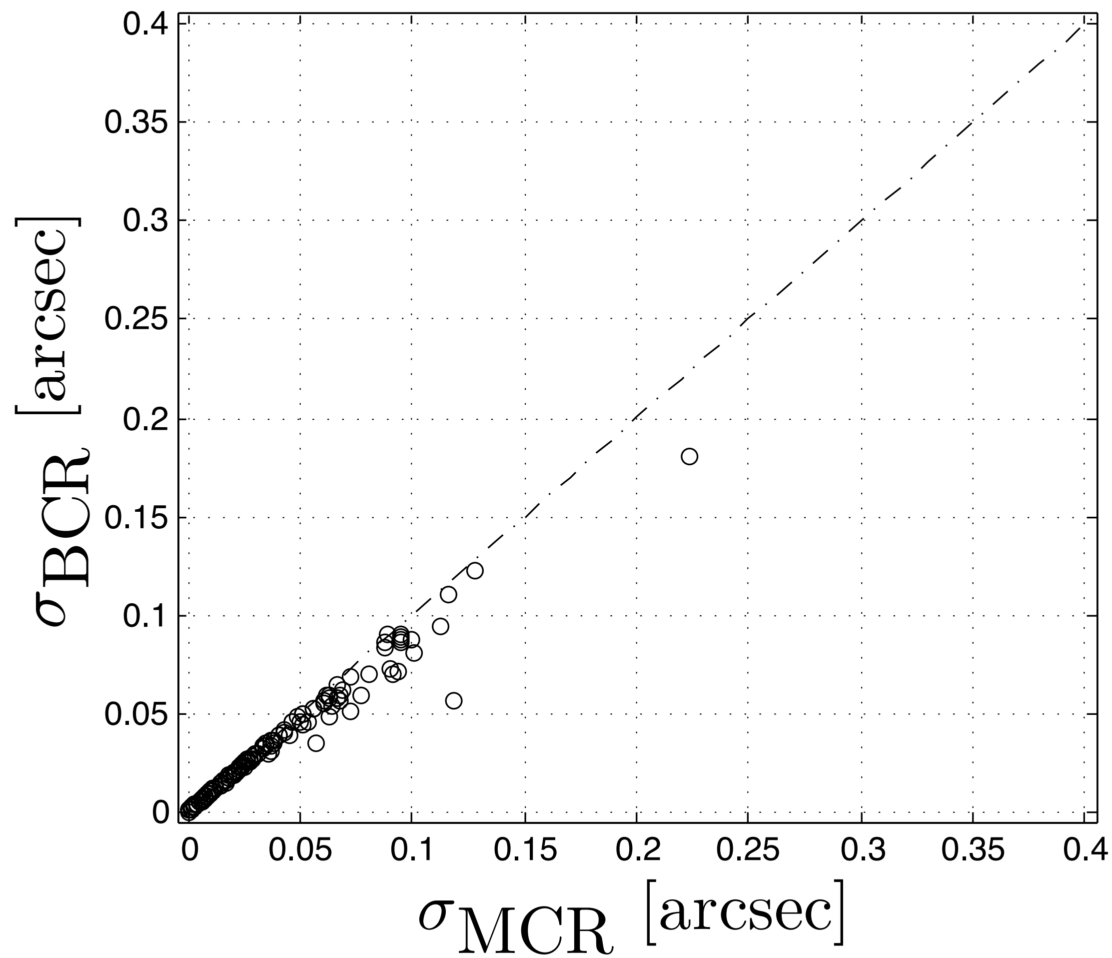
The Bayes CR bound can be attained in Astrometry!!

Bayes CR limits applied on Real Data

Gain in astrometric precision evaluated in a real data:

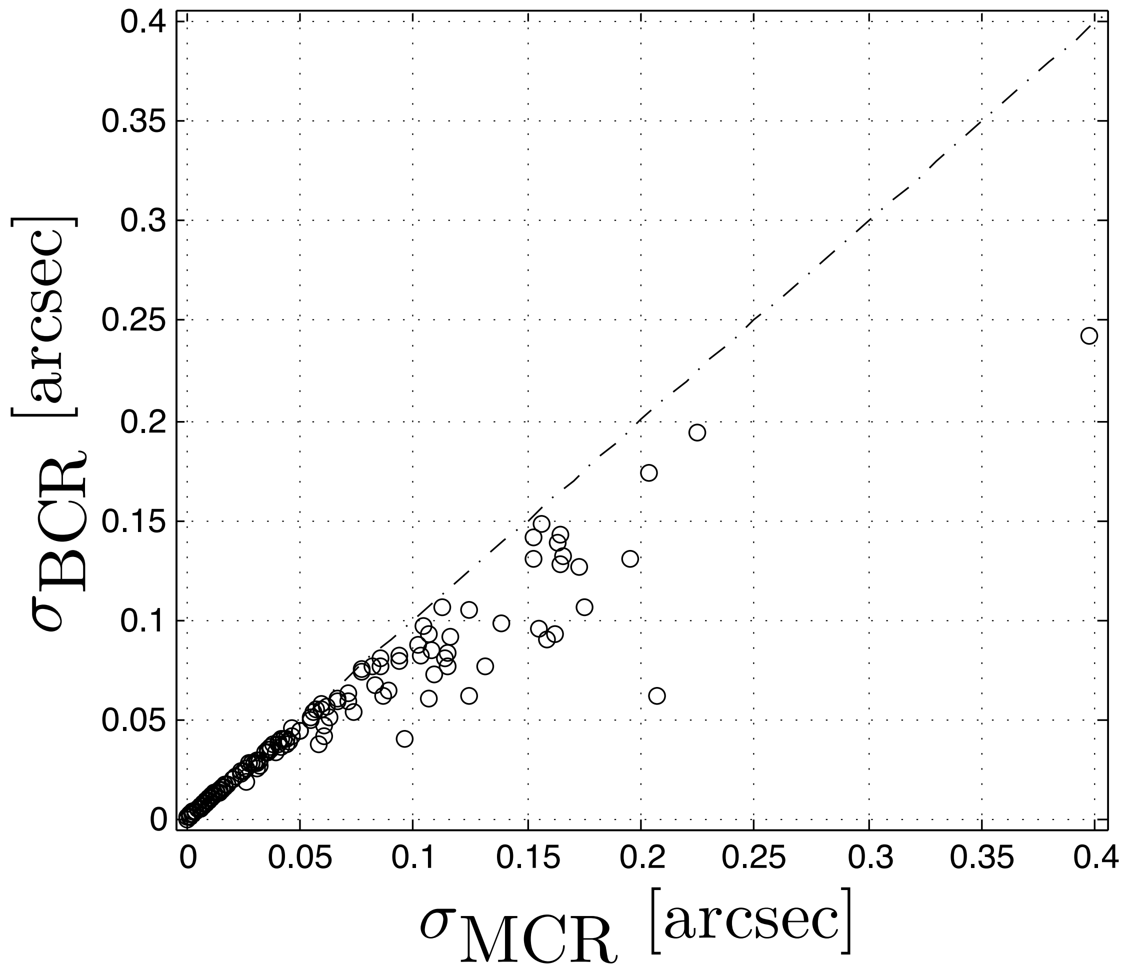
- (BCR vs. MCR) applied point-wise for all 226 objects in the USNO-B1 SPG stellar catalog (Echeverria et al. 2016, A&A)
  - the reported uncertainty of the catalog (in arcseg) is used as prior, under our Guassian assumption. (prior information)
  - New set of measurements assumed under our realistic observational ground-based conditions and realistic S/R derived from the catalog (data information)
    - specification of the EFOSC2 installed in NTT 3.58m telescope of ESO-La Silla observatory (RON of 9.2 e^-, D=7 e^-/pix/hr, G=1.33 e^-/ADU, pixel resolution of 0.24arcseg)

Bayes CR vs. Mean CR on a Real Scenario



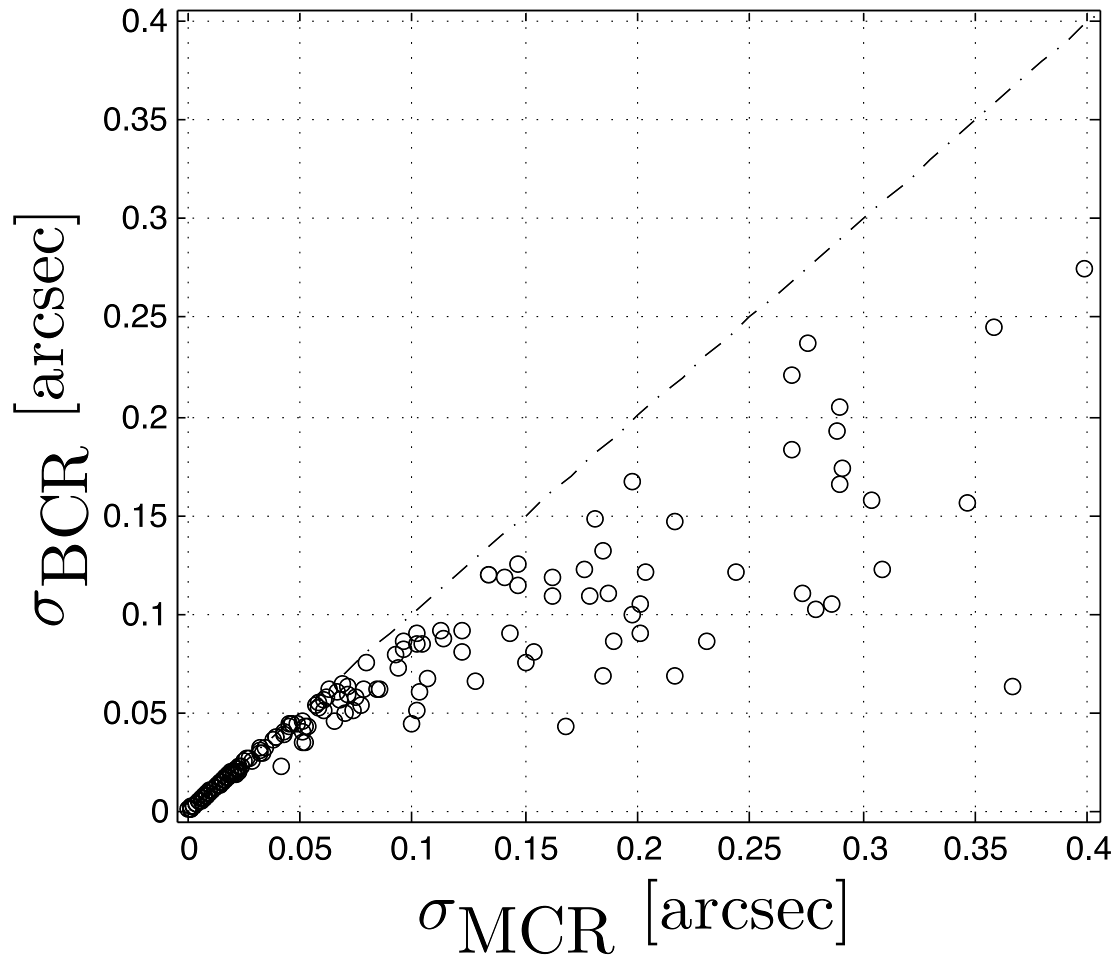
(b) FWHM=0.7 arcsec, aperture=1 m

Bayes CR vs. Mean CR on a Real Scenario



(d) FWHM=1.2 arcsec, aperture=1 m

Bayes CR vs. Mean CR on a Real Scenario



(e) FWHM=2 arcsec, aperture=1 m

Numerical Analysis

... further details in Echeverria et al., A&A, 2016.

Astronomy & Astrophysics manuscript no. Bayes_CR_astrometry_A&A
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Analysis of the Bayesian Cramér-Rao lower bound in astrometry:

Studying the impact of prior information in the location of an object

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Thanks