Chapter 1

Bl-Homogeneous observers for MIMO linear time invariant systems

In this chapter we present the second part of main result in this work. We introduce the design of Bl-homogeneous observers for MIMO-LTI systems assuming strong observability. The idea is to transform the system in to a Special Coordinate Basis, (detailed in Chapter 2) obtaining a representation of the system without need to have a triangular structure. That is, despite the fact that such a triangular restructuring of the system in that special way is a great feature of linear systems and greatly simplifies the convergence analysis. Here, it is shown that the designed observers are able to deal with a more general type of interconnections between subsystems.

Here we use directly a discontinuous nonlinear observer instead of differentiators. This fact suppress the necessity of using a cascade scheme composed by a linear observer and a discontinuous differentiator.

As in SISO case, the nonlinear injection terms can be designed to accelerate the convergence as much as we want by selecting appropriate and sufficiently large gains. Even more, due to the assignability of bl-homogeneous degrees in the observer we can reach and assure exactly and finite-time (or moreover fixed-time) estimation of the states in presence of unknown inputs.

1.1 Unknown input observers for LTI-MIMO systems

Consider the MIMO-LTI system without feedthrough (for simplicity) given by

$$\Sigma : \begin{cases} \dot{x} & = Ax + D\omega \\ y & = Cx \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\omega \in \mathbb{R}^m$ the unknown input vector and $y \in \mathbb{R}^p$ is the output vector. Accordingly, the matrices A, D, C have appropriate dimensions. For simplicity in the development we do not consider a known input u, since it does not modify the (observability) properties and it is simple to include it in the observer design. The equations are understood in the Filippov sense [1] in order to provide for the possibility to use discontinuous signals in observers. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous.

The task is to build an unknown input observer (UIO) providing for finite-time (preferably fixed-time convergent and exact) estimation of the states in presence of the unknown inputs. In chapter 2 we have stated the necessary and sufficient conditions for the existence and of UIO. We assume to have strong observability only.

Special Coordinated Basis is a useful tool to represent a system in an appropriate form, see Section 2.XX. to more details.

1.1.1 Unknown Input Observer design

Given a strong observable system Σ in (1.1) under SCB transformation detailed in Section 2.1, if we take into account the Property 2.XX, then the transformed system $\Sigma_{SCB}(\Sigma_b, \Sigma_d)$ is described by the following set of equations.

Each subsystem $\Sigma_{b,\iota}$ with associated states $x_{b,\iota} \in \mathbb{R}^{n_{b,\iota}}, \iota = 1, ..., p_b$

$$\Sigma_{b,\iota} : \begin{cases} \dot{x}_{b,\iota,1} & = x_{b,\iota,2} + H_{bd,\iota,1} y_d, \quad y_{b,\iota} = x_{b,\iota,1}, \\ \dot{x}_{b,\iota,j} & = x_{b,\iota,j+1} + H_{bd,\iota,j} y_d, \\ & \vdots \quad j = 1, ..., n_{b,\iota} - 1 \\ \dot{x}_{b,\iota,n_{b,\iota}} & = A_{bb,\iota} x_b + H_{bd,\iota,n_{b,\iota}} y_d, \end{cases} \tag{1.2}$$

moreover $\sum_{\iota=1}^{p_b} n_{b,\iota} = n_b$.

And each subsystem $\Sigma_{d,i}$ with associated states $x_{d,i} \in \mathbb{R}^{n_{d,i}}, i = 1, ..., p_d$

$$\Sigma_{d,i} : \begin{cases} \dot{x}_{d,i,1} & = x_{d,i,2} + H_{dd,i,1} y_d, \quad y_{d,i} = x_{d,i,1} \\ \dot{x}_{d,i,j} & = x_{c,i,j+1} + H_{dd,i,j} y_d, \\ & \vdots \quad j = 1, ..., n_{d,i} - 1 \\ \dot{x}_{d,i,n_{d,i}} & = A_{db,i} x_b + A_{dd,i} x_d + w_{d,i}(t), \end{cases} \tag{1.3}$$

similarly $\sum_{i=1}^{p_d} n_{d,i} = n_d$.

Where $A_{bb,\iota}, H_{bd,\iota,j}, A_{dd,i}, A_{db,i}, H_{dd,i,j}$ are constant row vectors of appropriate dimensions. Recall that Σ_b corresponds to the observable dynamics of the system, and Σ_d with presence of unknown inputs corresponds to the strong observable dynamics. It is clear that Σ_b can be expressed in observer or observability canonical form, the first one allow us to apply directly and homogeneous differentiator as an observer[2], but it can not be applied in the observability form. On the other hand, the idea of this work is to show that observers with bl-homogeneous terms can deal with this kind of structures.

Assumption 1.1. *Unknown input $\omega(t)$ a is uniformly bounded function $\|\omega(t)\| \le \Delta$, i.e. each element of the vector $|w_{d,i}(t)| < \Delta_i \in \mathbb{R}_{\ge 0}$*

This allow us to relax the existence conditions of UIO in other to have an observer under strong observability only, see (**section 2.x). The relative degree of the outputs $y_{d,i}$ with respect to the unknown input is $n_{d,i}$.

The observer $\Omega(\Omega_b, \Omega_D)$ is given by

$$\Omega_{b,\iota} : \begin{cases} \dot{\hat{x}}_{b,\iota,1} & = -k_{b,\iota,1} L \tilde{\phi}_{b,\iota,1}(\hat{x}_{b,\iota,1} - y_{b,\iota}) + \hat{x}_{b,\iota,2} + H_{bd,\iota,1} y_d \\ \dot{\hat{x}}_{b,\iota,j} & = -k_{b,\iota,j} L^j \tilde{\phi}_{b,\iota,j}(\hat{x}_{b,\iota,1} - y_{b,\iota}) + \hat{x}_{b,\iota,j+1} + H_{bd,\iota,j} y_d \\ & \vdots \quad j = 1, ..., n_{b,\iota} - 1 \\ \dot{\hat{x}}_{b,\iota,n_{b,\iota}} & = -k_{b,\iota,n_{b,\iota}} L^{n_{b,\iota}} \tilde{\phi}_{b,\iota,n_{b,\iota}}(\hat{x}_{b,\iota,1} - y_{b,\iota}) + A_{bb,\iota} \hat{x}_b + H_{bd,\iota,n_{b,\iota}} y_d \end{cases} \tag{1.4}$$

$$\Omega_{d,i} : \begin{cases} \dot{\hat{x}}_{d,i,1} & = -k_{d,i,1} L \tilde{\phi}_{d,i,1}(\hat{x}_{d,i,1} - y_{d,i}) + \hat{x}_{d,i,2} + H_{dd,i,1} y_d, \\ \dot{\hat{x}}_{d,i,j} & = -k_{d,i,j} L^j \tilde{\phi}_{d,i,j}(\hat{x}_{d,i,1} - y_{d,i}) + \hat{x}_{d,i,j+1} + H_{dd,i,j} y_d, \\ & \vdots \quad j = 1, ..., n_{d,i} - 1 \\ \dot{\hat{x}}_{d,i,q_i} & = -k_{d,i,q_i} L^{n_{d,i}} \tilde{\phi}_{d,i,q_i}(\hat{x}_{d,i,1} - y_{d,i}) + A_{db,i} \hat{x}_b + A_{dd,i} \hat{x}_d \end{cases} \tag{1.5}$$

with positive external gains $k_{b,\iota,j} > 0, k_{d,i,j}$ and positive tuning gains $\alpha > 0, L > 0$, appropriately selected as it will be show latter.

The nonlinear output injection terms $\tilde{\phi}_{b,\iota,j}(\cdot)$ are obtained from the functions

$$\phi_{b,\iota,j}(s) = \kappa_{b,\iota j}\lceil s\rfloor^{\frac{r_{(b,\iota),0,j+1}}{r_{(b,\iota),0,1}}} + \theta_{b,\iota j}\lceil s\rfloor^{\frac{r_{(b,\iota),\infty,j+1}}{r_{(b,\iota),\infty,1}}} \tag{1.6}$$

by scaling the positive internal gains $\kappa_{b,\iota j} > 0, \theta_{b,\iota j} > 0$

$$\kappa_{b,\iota j} \to \left(\frac{L^{n_{b,\iota}}}{\alpha}\right)^{\frac{jd_0}{r_{(b,\iota),0,1}}} \kappa_{b,\iota j}, \qquad \theta_{b,\iota j} \to \left(\frac{L^{n_{b,\iota}}}{\alpha}\right)^{\frac{jd_\infty}{r_{(b,\iota),\infty,1}}} \theta_{b,\iota j} \tag{1.7}$$

with powers selected as $r_{(b,\iota),0,n_{b,\iota}} = r_{(b,\iota),\infty,n_{b,\iota}} = 1$, and

$$\begin{aligned} r_{(b,\iota),j,n_{b,\iota}} &= r_{(b,\iota),0,j+1} - d_0 = 1 - (n_{b,\iota} - j)d_0 \\ r_{(b,\iota),j,n_{b,\iota}} &= r_{(b,\iota),\infty,j+1} - d_\infty = 1 - (n_{b,\iota} - j)d_\infty \end{aligned} \tag{1.8}$$

which are completely defined by two parameters d_0, d_∞.

Similarly, for the observer Ω_d. The nonlinear output injection terms $\tilde{\phi}_{d,i,j}$ are obtained from the functions

$$\phi_{d,i,j}(s) = \kappa_{d,ij}\lceil s\rfloor^{\frac{r_{(d,i),0,j+1}}{r_{(d,i),0,1}}} + \theta_{d,ij}\lceil s\rfloor^{\frac{r_{(d,i),\infty,j+1}}{r_{(d,i),\infty,1}}} \tag{1.9}$$

by scaling the positive internal gains $\kappa_{d,ij} > 0, \theta_{d,ij} > 0$

$$\kappa_{d,ij} \to \left(\frac{L^{n_{d,i}}}{\alpha}\right)^{\frac{jd_0}{r_{(d,i),0,1}}} \kappa_{d,ij}, \qquad \theta_{d,ij} \to \left(\frac{L^{n_{d,i}}}{\alpha}\right)^{\frac{jd_\infty}{r_{(d,i),\infty,1}}} \theta_{d,ij} \tag{1.10}$$

with powers selected as $r_{(d,i),0,n_{d,i}} = r_{(d,i),\infty,n_{d,i}} = 1$, and

$$\begin{aligned} r_{(d,i),j,n_{d,i}} &= r_{(d,i),0,j+1} - d_0 = 1 - (n_{d,i} - j)d_0 \\ r_{(d,i),j,n_{d,i}} &= r_{(d,i),\infty,j+1} - d_\infty = 1 - (n_{d,i} - j)d_\infty \end{aligned} \tag{1.11}$$

which are completely defined by the same parameters d_0, d_∞. They have to satisfy

$$-1 \le d_0 \le d_\infty < \min_{\substack{\iota = 1,\ldots,p_b, \\ i = 1,\ldots,p_d}} \left\{\frac{1}{n_{b,\iota} - 1}, \frac{1}{n_{d,i} - 1}\right\} \tag{1.12}$$

We have to highlight the fact that nonlinear injection terms in (1.6) and (1.9) are very similar to them in the bl-homogeneous differentiator [3] but the terms given here are simpler. This simplifies the task of implementation.

1.1.2 Estimation in original coordinates

The estimated states obtained from the observer $\Omega(\Omega_b, \Omega_d)$ in (1.4), (1.5) correspond to the transformed system $\Sigma_{SCB}(\Sigma_b, \Sigma_d)$ in (1.2),(1.3) which has been represented in SCB coordinates through the state Γ_s, input Γ_i and output Γ_o transformations, moreover it was applied an extra transformation $T_{obv} = \operatorname{diag}\left\{\mathcal{O}_\iota^{-1}, \ldots, \mathcal{O}_i^{-1}\right\}, \iota = 1,\ldots,p_b, i = 1,\ldots,p_d$ that puts the subsystems in observability canonical form.

The estimated states in original coordinates for (1.1) are given by

$$x = \Gamma_s \Gamma_{obv} \hat{x} \tag{1.13}$$

1.1.3 Gain Selection

Each type of gains in the observer has a different role, and the idea in the gain tuning is very intuitive.

1. The internal gains $\kappa_{\psi,j} > 0, \theta_{\psi,j} > 0$ with $\psi = \{(b,\iota),(d,i)\}$ can be selected arbitrary. These positive real values correspond to the desired weighting of each term of low degree and high degree respectively in $\phi_{\psi,j}$.

2. The external gains $k_{\psi,j} > 0$ have the objective of stabilizing the observer in absence of interconnections and external perturbations, i.e. when $A_{dd} = A_{db} = A_{dd} = 0$ and $\omega(t) = 0$.

3. Parameter L is selected large enough to assure the convergence in presence of interconnections, but not of the bounded perturbations $\omega(t)$. Setting its value grater than minimal value to assure stability the convergence velocity will be increased.

4. The tuning parameter α is selected large enough to assure the convergence in presence of the unknown bounded inputs $\omega(t)$.

Theorem 1.1. *Let the strong observable MIMO-LTI system Σ (1.1) in original coordinates, there exist a set of transformations such that the transformed system Σ_{SCB} composed by a set of interconnected subsystems has an UIO given by (1.4) (1.5). Selecting d_0, d_∞ as in (1.12) and choosing arbitrary (internal gains) $\kappa_{\psi,j} > 0$ and $\theta_{\psi,j} > 0$, with $\psi = \{(b,\iota),(d,i)\}$. Suppose that either $\Delta = 0$ or $d_0 = -1$. Under this conditions, there exist appropriate gains $k_{\psi,j} > 0$ with $\psi = \{(b,\iota),(d,i)\}$, and parameters $L > 0, \alpha > 0$ sufficiently large such that the solutions of bl-homogeneous observer (1.4)(1.5) converge globally and asymptotically to the true states of Σ_{SCB}, i.e. $\hat{x}_j(t) \to x_j(t)$ as $t \to \infty$.*

In particular, it can converge globally and

- *exponentially if $d_0 = 0$,*
- *finite-time if $d_0 < 0$,*
- *fixed-time if $d_0 < 0$ and $d_\infty > 0$*

Proof. Theorem 1.1.
The proof, similar to the previous case will be carry out in a Lyapunov framework through a bl-homogeneous Lyapunov function (Bl-LF), composed by a sum of Bl-LFs of each subsystem.

Let the estimation error variables

$$\begin{aligned} e_{b,\iota,j} &= \hat{x}_{b,\iota,j} - x_{b,\iota,j}, \quad \iota = 1, ..., p_b, j = 1, ..., n_{b,\iota}, \\ e_{d,i,j} &= \hat{x}_{d,i,j} - x_{d,i,j}, \quad i = 1, ..., p_b, j = 1, ..., n_{d,i}, \end{aligned} \tag{1.14}$$

The dynamics error are described by

$$\Xi_{b,i} : \begin{cases} \dot{e}_{b,\iota,1} &= -k_{b,\iota,1} L \tilde{\phi}_{b,\iota,1}(e_{b,\iota,1}) + e_{b,\iota,2} \\ \dot{e}_{b,\iota,j} &= -k_{b,\iota,j} L^j \tilde{\phi}_{b,\iota,j}(e_{b,\iota,1}) + e_{b,\iota,j+1} \\ \quad \vdots & \quad j = 1, ..., n_{b,\iota} - 1 \\ \dot{e}_{b,\iota,n_{b,\iota}} &= -k_{b,\iota,n_{b,\iota}} L^{n_{b,\iota}} \tilde{\phi}_{b,\iota,n_{b,\iota}}(e_{b,\iota,1}) + A_{bb,\iota} e_b \end{cases} \tag{1.15}$$

$$\Xi_{d,i}: \begin{cases} \dot{e}_{d,i,1} & = -k_{d,i,1} L\tilde{\phi}_{d,i,1}(e_{d,i,1}) + e_{d,i,2} \\ \dot{e}_{d,i,j} & = -k_{d,i,j} L^j \tilde{\phi}_{d,i,j}(e_{d,i,1}) + e_{d,i,j+1} \\ \quad\vdots & j = 1, ..., n_{d,i} - 1 \\ \dot{e}_{d,i,q_i} & = -k_{d,i,q_i} L^{n_{d,i}} \tilde{\phi}_{d,i,q_i}(e_{d,i,1}) + A_{db,i} e_b + A_{dd,i} e_d - \omega_{d,i} \end{cases} \tag{1.16}$$

for $\iota = 1, 2, ..., p_b, i = 1, 2, ..., p_d$.

Applying the time scaling via the next transformation

$$\epsilon_{b,\iota,j} = \frac{L^{n_{b,\iota}-j+1}}{\alpha} e_{b,\iota,j}, \qquad \epsilon_{d,i,j} = \frac{L^{n_{d,i}-j+1}}{\alpha} e_{d,i,j} \tag{1.17}$$

we obtain

$$\Xi_{b,\iota}: \begin{cases} \dot{\epsilon}_{b,\iota,1} & = L\left[-k_{b,\iota,1}\phi_{b,\iota,1}(\epsilon_{b,\iota,1}) + \epsilon_{b,\iota,2}\right] \\ \dot{\epsilon}_{b,\iota,j} & = L\left[-k_{b,\iota,j}\phi_{b,\iota,j}(\epsilon_{b,\iota,1}) + \epsilon_{b,\iota,j+1}\right] \\ \quad\vdots & j = 1, ..., n_{b,\iota} - 1 \\ \dot{\epsilon}_{b,\iota,n_{b,\iota}} & = L\left[-k_{b,\iota,n_{b,\iota}}\phi_{b,\iota,n_{b,\iota}}(\epsilon_{b,\iota,1}) + \frac{1}{\alpha}\mu_\iota(\epsilon_b)\right] \end{cases} \tag{1.18}$$

$$\Xi_{d,i}: \begin{cases} \dot{\epsilon}_{d,i,1} & = L\left[-k_{d,i,1}\phi_{d,i,1}(\epsilon_{d,i,1}) + \epsilon_{d,i,2}\right] \\ \dot{\epsilon}_{d,i,j} & = L\left[-k_{d,i,j}\phi_{d,i,j}(\epsilon_{d,i,1}) + \epsilon_{d,i,j+1}\right] \\ \quad\vdots & j = 1, ..., n_{d,i} - 1 \\ \dot{\epsilon}_{d,i,q_i} & = L\left[-k_{d,i,q_i}\phi_{d,i,q_i}(\epsilon_{d,i,1}) + \frac{1}{\alpha}\Psi_i(\epsilon_b, \epsilon_d, \omega_{d,i})\right] \end{cases} \tag{1.19}$$

where

$$\begin{aligned} \mu_\iota(\epsilon_b) &= A_{bb,\iota} e_b = \sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} a_{(bb,\iota),j,k} e_{b,j,k} = \alpha \sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} \frac{a_{(bb,\iota),j,k}}{L^{n_{b,\iota}-k+1}} \epsilon_{b,j,k} \\ \Psi_i(\epsilon_b, \epsilon_d, \omega_{d,i}) &= A_{db,i} e_b + A_{dd,i} e_d - \omega_{d,i} \\ &= \sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} a_{(db,i),j,k} e_{b,j,k} + \sum_{j=1}^{p_d}\sum_{k=1}^{n_{d,j}} a_{(dd,i),j,k} e_{d,j,k} - \omega_{d,i} \\ &= \alpha \sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k}}{L^{n_{b,j}-k+1}} \epsilon_{b,j,k} + \alpha \sum_{j=1}^{p_d}\sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k}}{L^{n_{d,j}-k+1}} \epsilon_{d,j,k} \end{aligned} \tag{1.20}$$

the fact that $\tilde{\phi}_{\psi,j}(\frac{\alpha}{L^n}s) = \frac{\alpha}{L^n}\tilde{\phi}_{\psi,j}(s)$ has been used.

For the convergence proof, it is convenient to perform another state transformation

$$\begin{aligned} z_{b,\iota,j} &= \frac{\epsilon_{b,i,j}}{k_{b,\iota,j-1}}, \quad \iota = 1, ..., p_b, j = 1, ..., n_{b,\iota} \\ z_{d,i,j} &= \frac{\epsilon_{d,i,j}}{k_{d,i,j-1}}, \quad i = 1, ..., p_d, j = 1, ..., n_{d,i} \end{aligned} \tag{1.21}$$

Then (1.18) and (1.19) become

$$\Xi^*_{b,\iota}: \begin{cases} z'_{b,\iota,1} & = -\tilde{k}_{b,\iota,1}\left(\phi_{b,\iota,1}(z_{b,\iota,1}) + z_{b,\iota,2}\right) \\ z'_{b,\iota,j} & = -\tilde{k}_{b,\iota,j}\left(\phi_{b,\iota,j}(z_{b,\iota,1}) + z_{b,\iota,j+1}\right) \\ \quad\vdots & j = 1, ..., n_{b,\iota} - 1 \\ z'_{b,\iota,n_{b,\iota}} & = -\tilde{k}_{b,\iota,n_{b,\iota}}\phi_{b,\iota,n_{b,\iota}}(z_{b,\iota,1}) + \tilde{\mu}_\iota(z_b) \end{cases} \tag{1.22}$$

$$\Xi_{d,i}^{*}:\begin{cases} z_{d,i,1}' &= -\tilde{k}_{d,i,1}\left(\phi_{d,i,1}(z_{d,i,1}) + z_{d,i,2}\right) \\ z_{d,i,j}' &= -\tilde{k}_{d,i,j}\left(\phi_{d,i,j}(z_{d,i,1}) + z_{d,i,j+1}\right) \\ \quad\vdots & j = 1, ..., n_{d,i}-1 \\ z_{d,i,q_i}' &= -\tilde{k}_{d,i,q_i}\phi_{d,i,q_i}(z_{d,i,1}) + \tilde{\Psi}_i(z_b, z_d, \omega_{d,i}) \end{cases} \tag{1.23}$$

with $\tilde{k}_{b,\iota,j} = \frac{k_{b,\iota,j}}{k_{b,\iota,j-1}}$, $\tilde{k}_{d,i,j} = \frac{k_{d,i,j}}{k_{d,i,j-1}}$, $\quad k_{b,\iota,0} = k_{d,i,0} = 1$
where

$$\tilde{\mu}_\iota(z_b) = \frac{1}{k_{b,\iota,n_{b,\iota-1}}}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}}\frac{a_{(bb,\iota),j,k}k_{b,j,k-1}}{L^{n_{b,j}-k+1}}z_{b,j,k} \tag{1.24}$$

$$\begin{aligned}\tilde{\Psi}_i(z_b, z_d, \omega_{d,i}) = &\frac{1}{k_{d,i,n_{d,i-1}}}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}}\frac{a_{(db,i),j,k}k_{b,j,k-1}}{L^{n_{b,j}-k+1}}z_{b,j,k} \\ &+\frac{1}{k_{d,i,n_{d,i}-1}}\sum_{j=1}^{p_d}\sum_{k=1}^{n_{d,j}}\frac{a_{(dd,i),j,k}k_{d,j,k-1}}{L^{n_{d,j}-k+1}}z_{d,j,k} - \frac{1}{\alpha k_{d,i,n_{d,i}-1}}\omega_{d,i}\end{aligned} \tag{1.25}$$

Lyapunov analysis

Before presenting the Lyapunov function we have to recall that the output injection terms in (1.6)(1.9) are much simpler than those described in [3]. However, the stability proof in [3] for the differentiator before described in (∗∗section diff Moreno) is applicable to the case with the simpler injection terms (1.6)(1.9), since the same requirements and properties are fulfilled. These functions $\phi_{\psi,j}(s)$ can be written as a composition of functions $\varphi_{\psi,j}(s)$ with $\psi = \{(b,\iota),(d,i)\}$ where ψ again refers to the case of both subsystems. Such that

$$\phi_{\psi,j}(s) = \varphi_{\psi,j}\circ ...\circ\varphi_{\psi,2}\circ\varphi_{\psi,1}(s) \tag{1.26}$$

where

$$\begin{aligned}\varphi_{\psi,1}(s) &= \phi_{\psi,1}(s) \\ \varphi_{\psi,2}(s) &= \phi_{\psi,2}\circ\phi_{\psi,1}^{-1}(s) \\ &\vdots \quad j = 2, ..., n_\psi \\ \varphi_{\psi,j}(s) &= \phi_{\psi,j}\circ\phi_{\psi,j-1}^{-1}(s)\end{aligned} \tag{1.27}$$

It will be used a sum of (smooth) bl-homogeneous Lyapunov Functions (bl-LF) V, which were introduced in [3]. Selecting, for $n \geq 2$ two positive real numbers $p_0, p_\infty \in \mathbb{R}_+$ that correspond to the homogeneity degrees of the 0-limit and the ∞-limit approximations of V, such that

$$\begin{aligned} p_0 &\geq \max_{\substack{i=1,...p_b \\ j=1,...,n_{b,\iota}}}\left\{\frac{r_{(b,\iota),0,j}}{r_{(b,\iota),\infty,j}}\left(2r_{(b,\iota),\infty,j} + d_\infty\right)\right\} \\ p_0 &\geq \max_{\substack{i=1,...p_d \\ j=1,...,n_{d,i}}}\left\{\frac{r_{(d,i),0,j}}{r_{(d,i),\infty,j}}\left(2r_{(d,i),\infty,j} + d_\infty\right)\right\}\end{aligned} \tag{1.28}$$

$$
\begin{aligned}
p_\infty &\geq 2 \max_{\substack{i=1,...p_d \\ j=1,...,n_{d,i}}} \left\{ r_{(b,\iota),\infty,j} \right\} + d_\infty \\
p_\infty &\geq 2 \max_{\substack{i=1,...p_d \\ j=1,...,n_{d,i}}} \left\{ r_{(d,i),\infty,j} \right\} + d_\infty \\
\frac{p_0}{r_{(b,\iota),0,j}} &\leq \frac{p_\infty}{r_{(b,\iota),\infty,j}}, \quad \frac{p_0}{r_{(d,i),0,j}} \leq \frac{p_\infty}{r_{(d,i),\infty,j}}
\end{aligned}
\tag{1.29}
$$

For $\iota = 1,...,p_b$ and $j = 1,...,n_{b,\iota}$ choose arbitrary positive real numbers $\beta_{(b,\iota),0,j}, \beta_{(b,\iota),\infty,j} > 0$ such that the following functions are defined

$$
\begin{aligned}
Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) &= \sum_{k\in\{0,\infty\}} \beta_{(b,\iota),k,j} \left[\frac{r_{(b,\iota),k,j}}{p_k} |z_{b,\iota,j}|^{\frac{p_k}{r_{(b,\iota),k,j}}} - z_{b,\iota,j} \lceil \xi_{b,\iota,j} \rfloor^{\frac{p_k - r_{(b,\iota),k,j}}{r_{(b,\iota),k,j}}} \right. \\
&\quad \left. + \frac{p_k - r_{(b,\iota),k,j}}{p_k} |\xi_{b,\iota,j}|^{\frac{p_k}{r_{(b,\iota),k,j}}} \right], \\
\xi_{b,\iota,j} &= \varphi^{-1}_{b,\iota,j}(z_{b,\iota,j+1}) \quad j = 1,...,n_{b,\iota} - 1 \\
\xi_{b,\iota,n_{b,\iota}} &= z_{b,\iota,n_{b,\iota}+1} \equiv 0, \\
Z_{b,\iota,n_{b,\iota}}(z_{b,\iota,n_{b,\iota}}) &= \beta_{0,\iota,n_{b,\iota}} \frac{1}{p_0} |z_{b,\iota,n_{b,\iota}}|^{p_0} + \beta_{\infty,\iota,n_{b,\iota}} \frac{1}{p_\infty} |z_{b,\iota,n_{b,\iota}}|^{p_\infty}
\end{aligned}
\tag{1.30}
$$

and similarly, associated to Σ_d we construct for $i = 1,...,p_d$ and $j = 1,...,n_{d,i}$ choose arditrary positive real numders $\beta_{(d,i),0,j}, \beta_{(d,i),\infty,j} > 0$ such that the following functions are defined

$$
\begin{aligned}
Z_{d,i,j}(z_{d,i,j}, z_{d,i,j+1}) &= \sum_{k\in\{0,\infty\}} \beta_{(d,i),k,j} \left[\frac{r_{(d,i),k,j}}{p_k} |z_{d,i,j}|^{\frac{p_k}{r_{(d,i),k,j}}} - z_{d,i,j} \lceil \xi_{d,i,j} \rfloor^{\frac{p_k - r_{(d,i),k,j}}{r_{(d,i),k,j}}} \right. \\
&\quad \left. + \frac{p_k - r_{(d,i),k,j}}{p_k} |\xi_{d,i,j}|^{\frac{p_k}{r_{(d,i),k,j}}} \right], \\
\xi_{d,i,j} &= \varphi^{-1}_{d,i,j}(z_{d,i,j+1}) \quad j = 1,...,n_{d,i} - 1 \\
\xi_{d,i,n_{d,i}} &= z_{d,i,n_{d,i}+1} \equiv 0, \\
Z_{d,i,n_{d,i}}(z_{d,i,n_{d,i}}) &= \beta_{0,i,n_{d,i}} \frac{1}{p_0} |z_{d,i,n_{d,i}}|^{p_0} + \beta_{\infty,i,n_{d,i}} \frac{1}{p_\infty} |z_{d,i,n_{d,i}}|^{p_\infty}
\end{aligned}
\tag{1.31}
$$

it is easy to check the following

Lemma 1.1. *Similar to the SISO case we have*

[3] $Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) \geq 0$ for every $\iota = 1,...,p_b, j = 1,...,n_{b,\iota}$ and $Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) = 0$ if and only if $\varphi_{b,\iota,j}(z_{b,\iota,j}) = z_{b,\iota,j+1}$.

Similarly for $Z_{d,i,j}(z_{d,i,j}, z_{b,i,j+1})$, for every $i = 1,...,p_d, j = 1,...,n_{d,i}$

The Bl-LF for each subsystem of the error system $\Xi^*_{b,\iota}$ associated to observable one is defined as

$$V_{b,\iota}(z_{b,\iota}) = \sum_{j=1}^{n_{b,\iota}-1} Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) + Z_{b,\iota,n_{b,\iota}}(z_{b,\iota,n_{b,\iota}}) \tag{1.32}$$

So that, the Bl-LF for the error observable system Ξ^*_b is

$$V_b(z_b) = \sum_{\iota=1}^{p_b} V_{b,\iota}(z_{b,\iota}), \tag{1.33}$$

In a similar way, the Bl-LF for each subsystem of the error system associated to strongly observable one $\Xi^*_{d,i}$ is given by

$$V_{d,i}(z_{d,i}) = \sum_{j=1}^{n_{d,i}-1} Z_{d,i,j}(z_{d,i,j}, z_{d,i,j+1}) + Z_{d,i,n_{d,i}}(z_{d,i,n_{d,i}}) \tag{1.34}$$

and the Bl-LF for the error observable system Ξ^*_b is

$$V_d(z_d) = \sum_{i=1}^{p_d} V_{d,i}(z_{d,i}), \tag{1.35}$$

finally, the Bl-LF candidate for the whole system composed by the interconnection of the observable and strongly observable error systems Ξ^*_b in (1.22) and Ξ^*_d in (1.23) is given by the sum of them, i.e.

$$V(z) = V_b(z_b) + V_d(z_d) \tag{1.36}$$

For the partial derivatives we introduce the following variables associated with both type of systems, i.e. taking $\psi = \{(b,\iota),(d,i)\}$

$$\begin{aligned} \sigma_{\psi,j} &= \frac{\partial Z_{\psi,j}(z_{\psi,j}, z_{\psi,j+1})}{\partial z_{\psi,j}} = \sum_{k\in\{0,\infty\}} \beta_{k,i,j}\left(\lceil z_{\psi,j}\rfloor^{\frac{p_k - r_{\psi,k,j}}{r_{\psi,k,j}}} - \lceil \xi_{\psi,j}\rfloor^{\frac{p_k - r_{\psi,k,j}}{r_{\psi,k,j}}}\right) \\ s_{\psi,j} &= \frac{\partial Z_{\psi,j}(z_{\psi,j}, z_{\psi,j+1})}{\partial z_{\psi,j+1}} = \sum_{k\in\{0,\infty\}} -\beta_{k,i,j}\frac{p_k - r_{\psi,k,j}}{r_{\psi,k,j}}(z_{\psi,j} - \xi_{\psi,j})|\xi_{\psi,j}|^{\frac{p_k - 2r_{\psi,k,j}}{r_{\psi,k,j}}}\frac{\partial \xi_{\psi,j}}{z_{\psi,j+1}} \end{aligned} \tag{1.37}$$

Note that $Z_{\psi,j}, \sigma_{\psi,j}, s_{\psi,j}$ vanish when $\varphi_{\psi,j}(z_{\psi,j}) = z_{\psi,j+1}$.
Performing time derivative of $V(z)$ with respect to the new time variable τ

$$\begin{aligned} V'(z) = V'_b(z_b) + V'_d(z_d) &= \sum_{\iota=1}^{p_b} V'_{b,\iota}(z_{b,\iota}) + \sum_{i=1}^{p_d} V'_{d,i}(z_{d,i}) \\ &= -W_b(z_b) + \sum_{\iota=1}^{p_b}\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\tilde{\mu}_\iota(z_b) \\ &\quad - W_d(z_d) + \sum_{i=1}^{p_d}\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}}\tilde{\Psi}_i(z_b, z_d, \omega_{d,i}) \end{aligned} \tag{1.38}$$

where

$$W_b(z_b) = \sum_{\iota=1}^{p_b} W_{b,\iota}(z_{b,\iota}), \quad W_d(z_d) = \sum_{i=1}^{p_d} W_{d,i}(z_{d,i}) \tag{1.39}$$

$$
\begin{aligned}
W_{b,\iota}(z_{b,\iota}) = & \tilde{k}_{b,\iota,1}\sigma_{b,\iota,1}(\phi_{b,\iota,1}(z_{b,\iota,1}) - z_{b,\iota,2}) \\
& + \sum_{j=2}^{n_{b,\iota}-1} \tilde{k}_{b,\iota,j}\left[s_{b,\iota,j-1} + \sigma_{b,\iota,j}\right](\phi_{b,\iota,j}(z_1) - z_{b,\iota,j+1}) \\
& + \tilde{k}_{b,\iota,n_{b,\iota}}\left[s_{n_{b,\iota}-1} + \sigma_{n_{b,\iota}}\right]\phi_{n_{b,\iota}}(z_{b,\iota,n_{b,\iota}})
\end{aligned}
\tag{1.40}
$$

and

$$
\begin{aligned}
W_{d,i}(z_{d,i}) = & \tilde{k}_{d,i,1}\sigma_{d,i,1}(\phi_{d,i,1}(z_{d,i,1}) - z_{d,i,2}) \\
& + \sum_{j=2}^{n_{d,i}-1} \tilde{k}_{d,i,j}\left[s_{d,i,j-1} + \sigma_{d,i,j}\right](\phi_{d,i,j}(z_1) - z_{d,i,j+1}) \\
& + \tilde{k}_{d,i,n_{d,i}}\left[s_{n_{d,i}-1} + \sigma_{n_{d,i}}\right]\phi_{n_{d,i}}(z_{d,i,n_{d,i}})
\end{aligned}
\tag{1.41}
$$

Due to the definition of $s_{\psi,j}$ in (1.37), $s_{\psi,n} \equiv 0$ and functions $s_{\psi,j}, \sigma_{\psi,j} \in \mathcal{C}$ in $\mathbb{R}$ are r-bl-homogeneous of degrees $p_0 - r_{\psi,0,j}, p_0 - r_{\psi,0,j+1}$ for the 0-approximation and $p_\infty - r_{\psi,\infty,j}, p_\infty - r_{\psi,\infty,j+1}$ for the ∞-approximation, respectively. Additionally, we have $\sigma_{\psi,j} = 0$ on the same set as $s_{\psi,j} = 0$, i.e. they become both zero at the points where $Z_{\psi,j}$ achieves its minimum, $Z_{\psi,j} = 0$.

Each function $V_{b,\iota}(z_{b,\iota})$ and $V_{d,i}(z_{d,i})$ in (1.32),(1.34) are bl-homogeneous of degrees p_0 and p_∞ and $\mathcal{C}$ on $\mathbb{R}$. They are also non negative, since they are positive combinations of non negative terms $Z_{\psi,j}$ with $\psi = \{(b,\iota),(d,i)\}$ respectively. Moreover, $V_\psi(z_\psi)$ are positive definite since $V_\psi(z_\psi) = 0$ only if all $Z_{\psi,j} = 0$, what only happens at $z_b = 0$, $z_d = 0$ respectively. Then, $V_b(z_b)$, $V_d(z_d)$ and therefore $V(z)$ in (1.36) as a sum of them are positive definite. Due to bl-homogeneity they are also radially unbounded.

If we analyze the terms $W_{b,\iota}(z_{b,\iota}), W_{d,i}(z_{d,i})$, in (1.40),(1.41) they are both bl-homogeneous of degrees $p_0 + d_0$ for their 0-approximations and $p_\infty + d_\infty$ for their ∞-approximations. Therefore $W_b(z_b)$ and $W_d(z_d)$ in (1.39) as well as its sum $W_{bd}(z_b, z_d) = W_b(z_b) + W_d(z_d)$ are also bl-homogeneous of degrees $p_0 + d_0$ for their 0-approximations and $p_\infty + d_\infty$ for their ∞-approximations.

As a previous related result, it has been shown in [3] that there exists appropriate gains $\tilde{k}_{\psi,j}$ such that $W_\psi(z_\psi)$ in (1.40),(1.41) are rendered positive definite. Now, the idea in the following is to prove that there exist gains L, α sufficiently large such that the negative definiteness of the terms $-W_\psi(z_\psi)$ in (1.38) and therefore $V'(z)$ is hold.

We are now interested in finding an upper bound of $\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\tilde{\mu}_\iota(z_b)$. Assuming $L \geq 1$, and $\alpha \geq 1$. Taking into account (1.24), due to the power of L we can write

$$
\begin{aligned}
\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\tilde{\mu}_\iota(z_b) &= \frac{1}{k_{b,\iota,n_{b,\iota}-1}}\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}}\frac{a_{(bb,\iota),j,k}k_{b,j,k-1}}{L^{n_{b,j}-k+1}}z_{b,j,k} \\
&\leq \frac{1}{L}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}}\frac{a_{(bb,\iota),j,k}k_{b,j,k-1}}{k_{b,\iota,n_{b,\iota}-1}}\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}z_{b,j,k}
\end{aligned}
\tag{1.42}
$$

where $\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}$ is bl-homogeneous of degree $p_0 - r_{(b,\iota),0,n_{b,\iota}} = p_0 - 1$ for the 0-approximation and $p_\infty - r_{(b,\iota),\infty,n_{b,\iota}} = p_\infty - 1$ for the ∞-approximation, additionally, each term $z_{b,j,k}$ is bl-homogeneous of degree $r_{(b,j),0,k}$ and $r_{(b,j),\infty,k}$ for the 0 and ∞ approximation respectively.

Then, $\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}} z_{b,j,k}$ is bl-homogeneous of degree $p_0 - r_{(b,\iota),0,n_{b,\iota}} + r_{(b,j),0,k} = p_0 - (n_{b,j} - k)d_0$ for the 0-approximation and $p_\infty - r_{(b,\iota),\infty,n_{b,\iota}} + r_{(b,j),\infty,k} = p_\infty - (n_{b,j} - k)d_\infty$ for the $\infty-$approximation. We can conclude that

$$p_0 + d_0 \leq p_0 - (n_{b,j} - k)d_0, \qquad p_\infty - (n_{b,j} - k)d_\infty \leq p_\infty + d_\infty \tag{1.43}$$

And therefore, by the property of bl-homogeneous functions, there exist positive real numbers $\lambda_{b,\iota}$ such that

$$\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}} \tilde{\mu}_\iota(z_b) \leq \frac{\lambda_{b,\iota}}{L} W_b(z_b) \tag{1.44}$$

For the upper bound analysis of $\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} \tilde{\Psi}_i(z_b, z_d, \omega_{d,i})$, it is convenient to write $\tilde{\Psi}_i(z_b, z_d, \omega_{d,i})$ separately, i.e

$$\begin{aligned}
\tilde{\Psi}_i(z_b, z_d, \omega_{d,i}) &= \frac{1}{k_{d,i,n_{d,i-1}}} \sum_{j=1}^{p_b} \sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k} k_{b,j,k-1}}{L^{n_{b,j}-k+1}} z_{b,j,k} \\
&+ \frac{1}{k_{d,i,n_{d,i-1}}} \sum_{j=1}^{p_d} \sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k} k_{d,j,k-1}}{L^{n_{d,j}-k+1}} z_{d,j,k} - \frac{1}{\alpha k_{d,i,n_{d,i-1}}} \omega_{d,i} \\
&= \frac{1}{L} \tilde{\Psi}_{b,i} + \frac{1}{L} \tilde{\Psi}_{d,i} + \frac{1}{\alpha} \tilde{\Psi}_{\omega,i} \\
\tilde{\Psi}_{b,i}(z_b) &= \sum_{j=1}^{p_b} \sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k} k_{b,j,k-1}}{k_{d,i,n_{d,i-1}} L^{n_{b,j}-k}} z_{b,j,k} \\
\tilde{\Psi}_{d,i}(z_d) &= \sum_{j=1}^{p_d} \sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k} k_{d,j,k-1}}{k_{d,i,n_{d,i-1}} L^{n_{d,j}-k}} z_{d,j,k} \\
\tilde{\Psi}_{\omega,i}(\omega_d, i) &= -\frac{1}{k_{d,i,n_{d,i}-1}} \omega_{d,i}
\end{aligned} \tag{1.45}$$

where $\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}}$ is bl-homogeneous of degree $p_0 - r_{(d,i),0,n_{d,i}} = p_0 - 1$ for the 0-approximation and $p_\infty - r_{(d,i),\infty,n_{d,i}} = p_\infty - 1$ for the ∞-approximation, additionally, each term $z_{b,j,k}$ is bl-homogeneous of degree $r_{(b,j),0,k}$ and $r_{(b,j),\infty,k}$ for the 0 and ∞ approximation respectively, similarly each term $z_{d,j,k}$ is bl-homogeneous of degree $r_{(d,j),0,k}$ and $r_{(d,j),\infty,k}$ for the 0 and ∞ approximation respectively.

$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} z_{b,j,k}$ is bl-homogeneous of degree $p_0 - r_{(d,i),0,n_{d,i}} - r_{(b,j),0,k} = p_0 - (n_{b,j} - k)d_0$ for the 0-approximation and $p_\infty - r_{(d,i),\infty,n_{d,i}} - r_{(b,j),\infty,k} = p_\infty - (n_{b,j} - k)d_\infty$ for the ∞-approximation. In a similar way, $\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} z_{d,j,k}$ is bl-homogeneous of degree $p_0 - r_{(d,i),0,n_{d,i}} - r_{(d,j),0,k} = p_0 - (n_{d,j} - k)d_0$ for the 0-approximation and $p_\infty - r_{(d,i),\infty,n_{d,i}} - r_{(d,j),\infty,k} = p_\infty - (n_{d,j} - k)d_\infty$ for the ∞-approximation.

It is clear

$$\begin{aligned}
p_0 + d_0 \leq p_0 - (n_{b,j} - k)d_0 \qquad p_\infty - (n_{b,j} - k)d_\infty \leq p_\infty + d_\infty \\
p_0 + d_0 \leq p_0 - (n_{d,j} - k)d_0 \qquad p_\infty - (n_{d,j} - k)d_\infty \leq p_\infty + d_\infty
\end{aligned} \tag{1.46}$$

Therefore, by the property of bl-homogeneous functions, there exist $\lambda_{d,i}, \lambda_{bd,i}, \bar{\lambda}_{d,i} > 0$ such that

$$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}}\frac{1}{L}\tilde{\Psi}_{b,i}(z_b) \leq \frac{\lambda_{bd,i}}{L}W_{bd}(z_b,z_d) \tag{1.47}$$

$$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}}\frac{1}{L}\tilde{\Psi}_{d,i}(z_d) \leq \frac{\lambda_{d,i}}{L}W_d(z_d) \tag{1.48}$$

$$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} \leq \bar{\lambda}_{d,i}\left(W_d^{\frac{p_0-1}{p_0+d_0}}(z_d)+W_d^{\frac{p_\infty-1}{p_\infty+d_\infty}}(z_d)\right) \tag{1.49}$$

If we put everything together, $V'(z)$ in (1.38) can be bounded as

$$\begin{aligned} V'(z) \leq &- W_b(z_b) - W_d(z_d) + \frac{\lambda_b}{L}W_b(z_b) + \frac{\lambda_{bd}}{L}W_{bd}(z_b,z_d) + \frac{\lambda_d}{L}W_d(z_d) \\ &+ \frac{\bar{\lambda}_d}{\alpha}\left(W_d^{\frac{p_0-1}{p_0+d_0}}(z_d)+W_d^{\frac{p_\infty-1}{p_\infty+d_\infty}}(z_d)\right)\|\tilde{\Psi}_\omega(\omega_d)\|_\infty \\ = &-\left(1-\frac{\lambda_b+\lambda_{bd}}{L}\right)W_b(z_b) - \left(1-\frac{\lambda_d+\lambda_{bd}}{L}\right)W_d(z_d) \\ &+ \frac{\bar{\lambda}_d}{\alpha}\left(W_d^{\frac{p_0-1}{p_0+d_0}}(z_d)+W_d^{\frac{p_\infty-1}{p_\infty+d_\infty}}(z_d)\right)\|\tilde{\Psi}_\omega(\omega_d)\|_\infty \end{aligned} \tag{1.50}$$

where $\|\tilde{\Psi}_\omega(\omega_d)\|_\infty = \max\left\{\sum_{i=1}^{p_d}|\tilde{\Psi}_{\omega,i}(\omega_{d,i})|\right\}$.

If we apply Lyapunov stability arguments we conclude that it can be chosen L sufficiently large such that the first two terms become negative definite. In absence of $\tilde{\Psi}_\omega$, the origin $z=0$ is asymptotic stable. With $d_0<0$ it converges in finite time, moreover, with $d_\infty>0$ it converges in fixed-time.

In the case $\tilde{\Psi}_\omega \neq 0$ but ultimately bounded, by selecting $d_0=-1$ we can chose α sufficiently large, such that $V'(z)$ is negative definite. And then, finite-time or fixed-time stability is achieved. □

1.2 Example

Let a *strongly observable* system, given by a linearized model of the lateral motion of a light aircraft taken from [4][2] in original coordinates with unknown inputs

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu + D\omega \\ y &= Cx \end{cases} \tag{1.51}$$

where

$$A = \begin{bmatrix} -0.3 & 0 & -33 & 9.81 & 0 & -5.4 & 0 \\ -0.1 & -8.3 & 3.75 & 0 & 0 & 0 & -28.6 \\ 0.37 & 0 & -0.64 & 0 & 0 & -9.5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 10 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 20 \\ 0 \end{bmatrix}, \tag{1.52}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \omega(t) = 0.08 + 0.1sin(2t) + 0.02cos(13t), \quad |\omega(t)| \leq 0.2$$

The state vector $x = \begin{bmatrix} x_1 & \dots & x_7 \end{bmatrix}^T$ consists of sideslip velocity x_1, the roll rate x_2, the yaw rate x_3, the roll angle x_4, the yaw angle x_5, the rudder angle x_6 and the aileron angle x_6. The control input $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ is given by the rudder angle demand u_1 and the aileron angle demand u_2. The unknown input ω is a bounded actuator fault in the rudder is considered. The output $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ provide measurements of the roll rate x_2 and the yaw angle x_5.

It is important to note that the system (1.51) is unstable since the matrix A has to eigenvalues with non-negative real part in $s_1 = 0.1219$ and $s_2 = 0$, them all the states of the system can be unbounded. Therefore, the direct application of a homogeneous differentiator like the Levant´s one is impossible.

Applying the SCB change of coordinates, and expressing each subsystem in observability canonical form through $T = \text{diag}\left\{\mathcal{O}_1^{-1}, \mathcal{O}_2^{-1}\right\}$, the transformed system is given by

$$
\begin{gathered}
A = \begin{bmatrix} 0 & 1 & 0 & 0 & 2.006 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1.665 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2.6 & 0 & 0 \\ 0 & 0 & -7.009 & -6.401 & -3.645 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 488.789 & -3.81 & -14.933 & -183.158 & -87.597 & -17.838 \end{bmatrix}, \\
B = \begin{bmatrix} 0 & 0 \\ 0 & 77.44 \\ 0 & -456.244 \\ 0 & 2378 \\ 0 & 0 \\ 0 & -286 \\ -701.7 & 3803.8 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{gathered}
\tag{1.53}
$$

which can be written as

$$
\begin{gathered}
\Sigma_b : \begin{cases} \dot{x}_{b,1} & = x_{b,2} + [2.006]y_2, \qquad y_1 = x_{b,1} \\ \dot{x}_{b,2} & = x_{b,3} + [-1.665]y_2 + [0 \quad 77.44]u \\ \dot{x}_{b,3} & = x_{b,4} + [2.6]y_2 + [0 \quad -456.244]u \\ \dot{x}_{b,4} & = [0 \quad 0 \quad -7.009 \quad -6.401]x_1 + [-3.645]y_2 + [0 \quad 2378]u \end{cases} \\
\Sigma_d : \begin{cases} \dot{x}_{d,1} & = x_{d,2}, \qquad y_2 = x_{d,1} \\ \dot{x}_{d,2} & = x_{d,3} + [0 \quad -286]u \\ \dot{x}_{d,3} & = [0 \quad 488.789 \quad -3.81 \quad -14.933]x_b + \\ & \dots + [-183.158 \quad -87.597 \quad -17.838]x_d + [-701.7 \quad 3803.8]u + [1]\omega \end{cases}
\end{gathered}
\tag{1.54}
$$

where we have one observable subsystem $x_b \in \mathbb{R}^4$ and one strong observable subsystem $x_d \in \mathbb{R}^3$. In this example we omit the sub indices of the subsystem number, due to we have only $p_b = p_d = 1$. Note that the fact that the subsystems are in observability canonical form make impossible to apply the methodology and observer proposed in [2]. The terms in the last channel can be seen as interconnection ones, i.e. in some sense, the following methodology can be thought of as the observer design of linear interconnected systems.

The observer is given by

$$\Omega_b : \begin{cases} \dot{\hat{x}}_{b,1} &= -k_{b,1}L\tilde{\phi}_{b,1}(\hat{x}_{b,1} - y_1) + \hat{x}_{1,2} + [2.006]y_2 \\ \dot{\hat{x}}_{b,2} &= -k_{b,2}L^2\tilde{\phi}_{b,2}(\hat{x}_{b,1} - y_1) + \hat{x}_{1,3} + [-1.665]y_2 + [0 \quad 77.44]u \\ \dot{\hat{x}}_{b,3} &= -k_{b,3}L^3\tilde{\phi}_{b,3}(\hat{x}_{b,1} - y_1) + \hat{x}_{1,4} + [2.6]y_2 + [0 \quad -456.244]u \\ \dot{\hat{x}}_{b,4} &= -k_{b,4}L^4\tilde{\phi}_{b,4}(\hat{x}_{b,1} - y_1) + [0 \quad 488.789 \quad -3.81 \quad -14.933]\hat{x}_b + \\ & \quad ... + [-3.645]y_2 + [0 \quad 2378]u \end{cases} \tag{1.55}$$

$$\Omega_d : \begin{cases} \dot{\hat{x}}_{d,1} &= -k_{d,1}L\tilde{\phi}_{d,1}(\hat{x}_{d,1} - y_2) + \hat{x}_{1,2} \\ \dot{\hat{x}}_{d,2} &= -k_{d,2}L^2\tilde{\phi}_{d,2}(\hat{x}_{d,1} - y_2) + \hat{x}_{1,3} + [0 \quad -286]u \\ \dot{\hat{x}}_{d,3} &= -k_{d,3}L^3\tilde{\phi}_{d,3}(\hat{x}_{d,1} - y_2) + [0 \quad 488.789 \quad -3.810 \quad -14.933]\hat{x}_b + \\ & \quad ... + [-183.158 \quad -87.597 \quad -17.838]\hat{x}_d + [-701.7 \quad 3803.8]u \end{cases}$$

where the nonlinear output injection terms $\tilde{\phi}.(\cdot)$ are as follows

$$\tilde{\phi}_{b,j}(s) = \left(\frac{L^4}{\alpha}\right)^{\frac{jd_0}{1-3d_0}} \kappa_{b,\iota j}\lceil s \rfloor^{\frac{1-(3-j)d_0}{1-3d_0}} + \left(\frac{L^4}{\alpha}\right)^{\frac{jd_\infty}{1-3d_\infty}} \theta_{b,\iota j}\lceil s \rfloor^{\frac{1-(3-j)d_\infty}{1-3d_\infty}}, \quad j = 1, ..., 4 \tag{1.56}$$

and

$$\tilde{\phi}_{d,j}(s) = \left(\frac{L^3}{\alpha}\right)^{\frac{jd_0}{1-2d_0}} \kappa_{d,j}\lceil s \rfloor^{\frac{1-(2-j)d_0}{1-2d_0}} + \left(\frac{L^3}{\alpha}\right)^{\frac{jd_\infty}{1-2d_\infty}} \theta_{d,j}\lceil s \rfloor^{\frac{1-(2-j)d_\infty}{1-2d_\infty}}, \quad j = 1, ..., 3 \tag{1.57}$$

The assigned homogeneity degrees d_0, d_∞ in 0 and ∞ in these terms (1.56),(1.57) have to satisfy

$$-1 \leq d_0 \leq d_\infty < \min\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{3} \tag{1.58}$$

we present three cases:

1. Linear UIO. Homogeneity degrees $d_0 = d_\infty = 0$.
2. Continuous UIO. Homogeneity degrees $d_0 = -\frac{1}{9}, d_\infty = 0$
3. HOSM-UIO. Homogeneity degrees $d_0 = -1$, $d_\infty = \frac{1}{9}$. With this selection we get a discontinuous observer. Note that $d_0 < 0 > d_\infty$.

The initial conditions of the plant states are $x_0 = \begin{bmatrix}-0.5 & 0.1 & 0.02 & 0.2 & -0.1 & -0.3 & 0.2\end{bmatrix}$ and $\hat{x}_j = 0, j = 1, ..., 7$ for the observer. For all cases the values of gains $k_{b,j}, k_{d,j}$ are fixed as

$$\left\{k_{b,1} = 8.6k_{b,4}^{\frac{1}{4}} \quad k_{b,2} = 21k_{b,4}^{\frac{1}{2}} \quad k_{b,3} = 16.25k_{d,4}^{\frac{1}{3}} \quad k_{b,4} = 5\right\} \tag{1.59}$$

$$\left\{k_{d,1} = 3.34k_{d,3}^{\frac{1}{3}} \quad k_{d,2} = 5.3k_{d,3}^{\frac{2}{3}} \quad k_{d,3} = 10\right\} \tag{1.60}$$

In te case of Figures 1.1,1.2,1.3 a set of 5 seconds simulations are shown, where $L = 1, \alpha = 10$ and $\kappa_{b,j} = \kappa_{d,j} = 1$ have been set. We can see that in the cases 1 and 2 (Linear and continuous) the observer does not accurately estimate the states of the plant due to the non-compensation of unknown input, i.e. no matter how big the gains can be, with this homogeneity degrees selection, the observer will never be able to compensate the effect of the unknown input. This is cleaner in the last subfigure, where $\|e\|$ is illustrated.

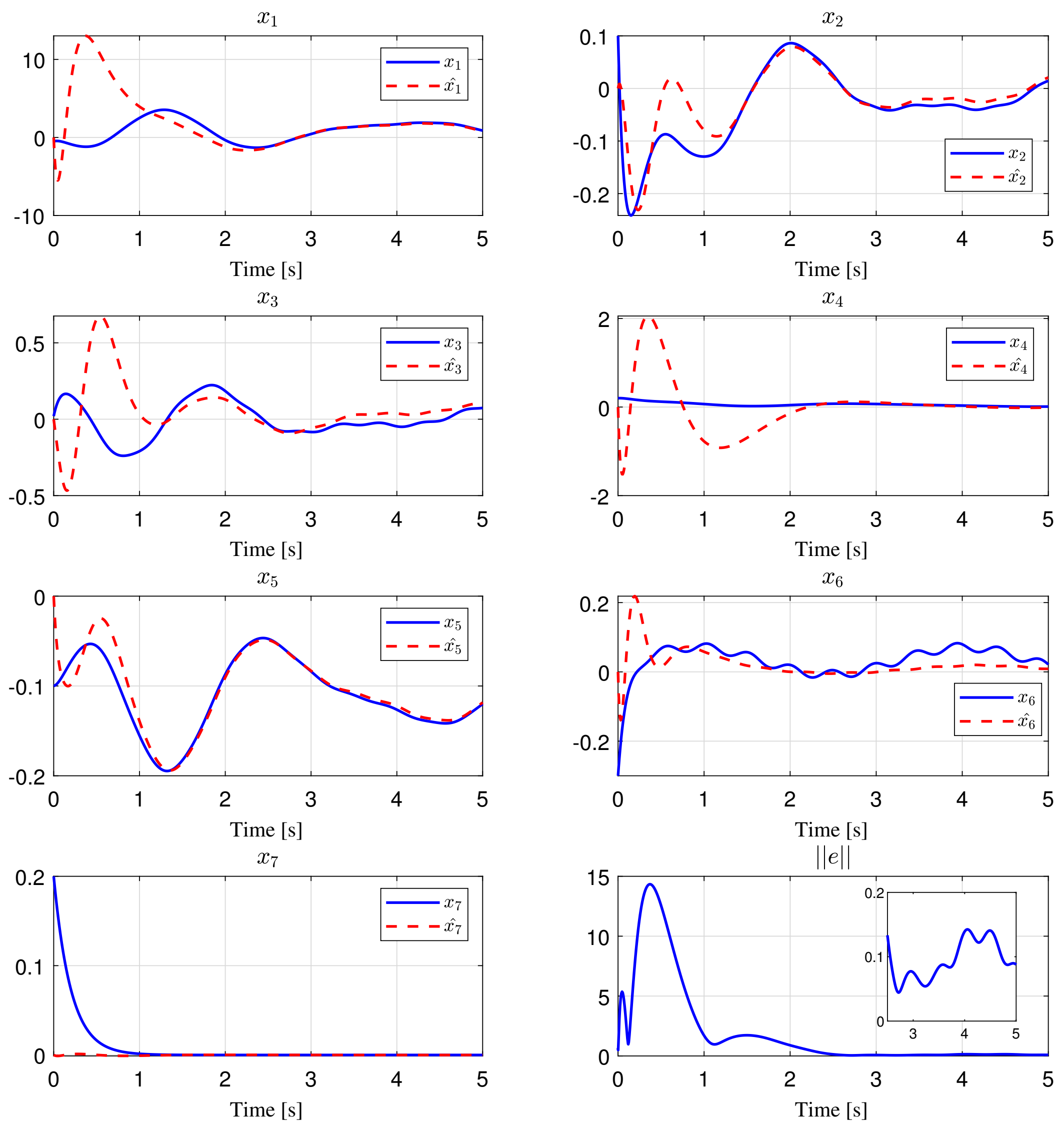

Figure 1.1: Estimation of plant states $x_1, ..., x_7$ and norm $\|e\|$ with $d_0 = d_\infty = 0$.

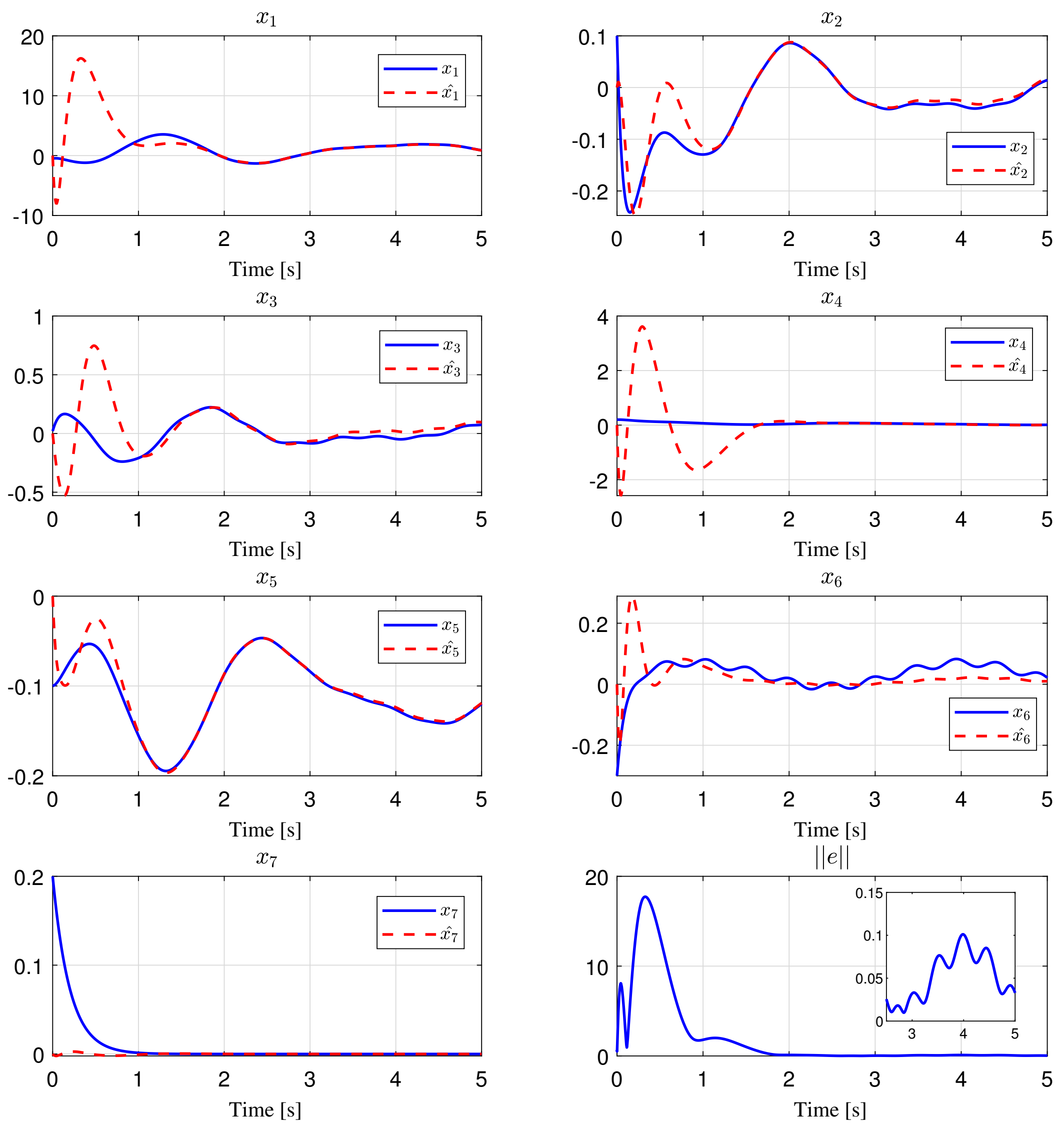

Figure 1.2: Estimation of plant states $x_1, ..., x_7$ and norm $\|e\|$ with $d_0 = 0, d_\infty = \frac{1}{9}$.

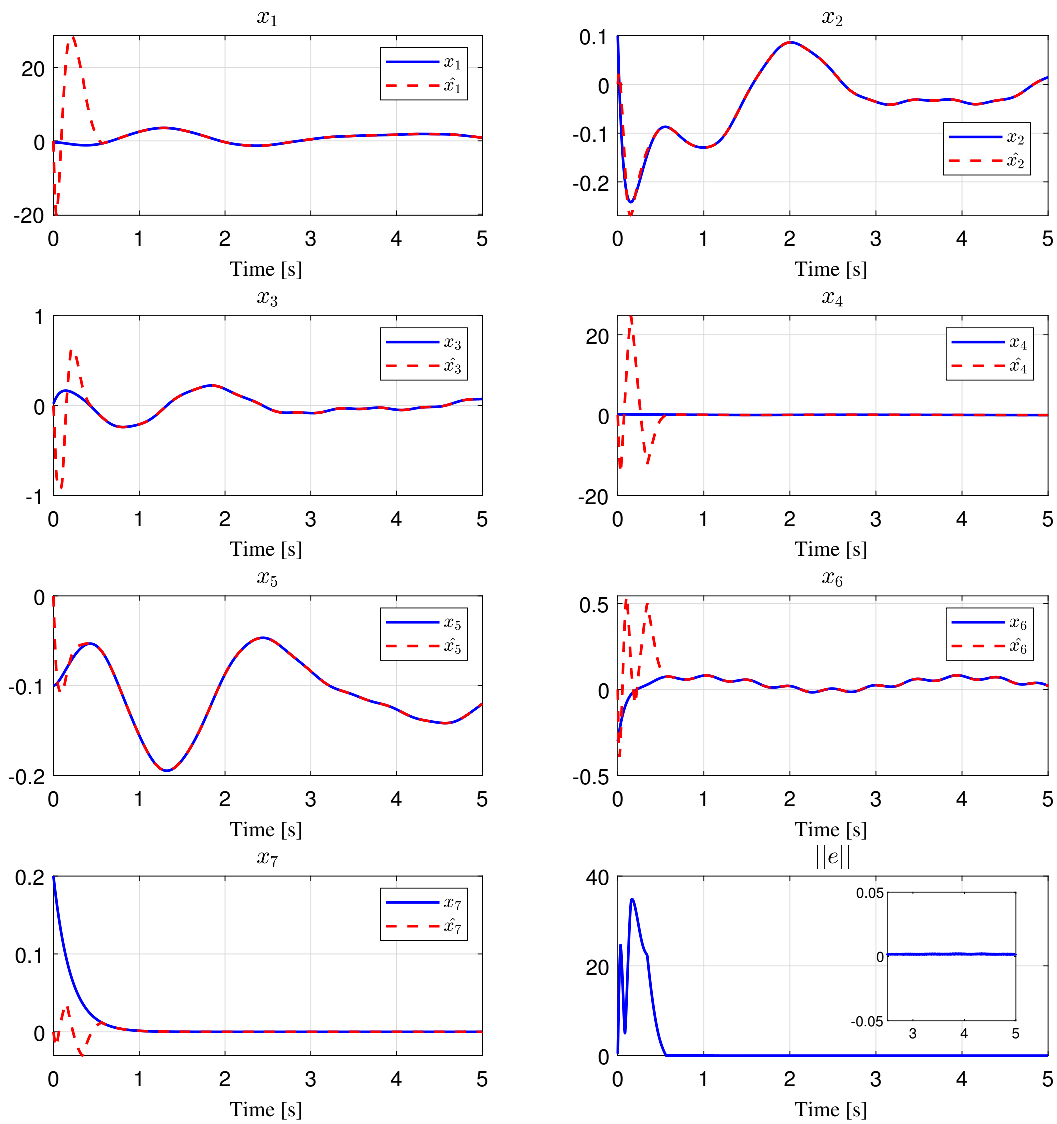

Figure 1.3: Estimation of plant states $x_1, ..., x_7$ and norm $||e||$ with $d_0 = 0, d_\infty = \frac{1}{9}$.

On the other hand, in the third case with $d_0 = -1$ (discontinuous observer), the induced HOSM allows the observer to compensate exactly the unknown inputs effect. It is shown in Figure 1.3, that after 0.5s an exact convergence of the error norm to zero even in presence of unknown input is achieved, therefore, the states are estimated exactly in finite time.

In all cases, with appropriately gains selection the error converges to a neighborhood of zero in about one second but it is not clear what happens when the initial estimate is not close to the true state. For the third case, i.e. with $d_0 = -1$, $d_\infty = \frac{1}{9}$ and setting now $L = 5, \alpha = 10$ we run a set of simulations, Figure 1.4 shows the norm of the error vector over a wide range orders of initial estimation error. Here, the error converge to zero in less than 4s for all cases.

In other words, we achieve more than finite-time, fixed-time stability of the estimation error, i.e. regardless of the initial magnitude of the error, the observer converges before certain time value $\bar{T}$, in this case we can see that the value of this upper cote of estimation time is $\bar{T} = 4s$, but this can be reduced arbitrary by increasing appropriately the value of parameter L as we said in the parameters tuning section.

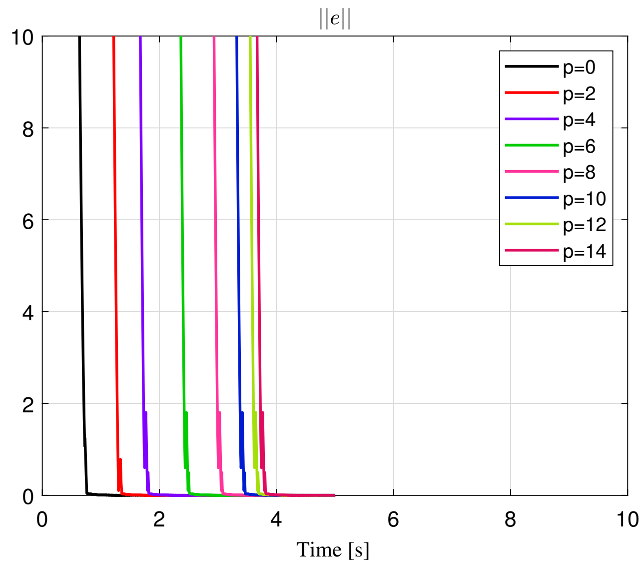

Figure 1.4: Norm of the estimation error $\|e\|$ with different orders at initial error $e_0 \times 10^p$ with $d_0 = -1, d_\infty = \frac{1}{9}$.

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