Chapter 1

Bl-Homogeneous observers for SISO Linear Time Invariant systems

In this chapter we present the first part of main result of this work. We introduce the design of Bl-homogeneous observers for SISO-LTI systems with bounded unknown inputs assuming strong observability. The idea is to transform the system in to a Special Coordinate Basis, (detailed in Chapter 2 for the MIMO general case) obtaining a representation of the system in which it is possible to design an UIO.

Here we use directly a discontinuous nonlinear observer instead of differentiators. This fact suppress the necessity of using a cascade scheme composed by a linear observer and a discontinuous differentiator.

The nonlinear injection terms can be designed to accelerate the convergence as much as we want by selecting appropriate and sufficiently large gains. Even more, due to the assignability of bl-homogeneous degrees in the observer we can reach and assure exactly and finite-time (or moreover fixed-time) estimation of the states in presence of unknown inputs.

1.1 Unknown input observers for LTI-SISO systems

Before attacking the MIMO case we will introduce the single-input single-output (SISO) case, which going to be useful in order to give the basic idea in solving the design problem.

Consider the SISO-LTI system without feedthrough (for simplicity) given by

$$\Sigma : \begin{cases} \dot{x} & = Ax + D\omega \\ y & = Cx \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\omega \in \mathbb{R}$ the unknown input and $y \in \mathbb{R}$ is the output. Accordingly, the matrices A, D, C have appropriate dimensions. For simplicity in the development we do not consider a known input u, since it does not modify the (observability) properties and it is simple to include it in the observer design. The equations are understood in the Filippov sense [1] in order to provide for the possibility to use discontinuous signals in observers. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous.

The task is to build an observer providing for finite-time (preferably fixed-time convergent and exact) estimation of the states in presence of the unknown input. In the previous chapters we have stated the general conditions for the existence and characterization of unknown input observers (UIO). Here we will recall this conditions in the particular case we are working on.

Accordingly to the definition of Rosenbrock's matrix in (**Rosenbrok) the system (1.1) is strongly observable if the triple (A, D, C) has no invariant zeros. Unfortunately, this definition does not give a specific form to the matrices. Special Coordinated Basis for SISO case (a particular case of MIMO-SCB presented in Chapter 2) clarifies this problem.

Theorem 1.1. *Consider the system* (1.1)*. There exist nonsingular state, input and output transformations* $\Gamma_s \in \mathbb{R}^{n\times n}, \Gamma_i \in \mathbb{R}, \Gamma_o \in \mathbb{R}$*, which decompose the state space of* Σ *into two subspaces,* x_a *and* x_d*. These two subspaces correspond to the finite zero and infinite zero structures of* Σ*, respectively. The new satate spaces, input and output spaces of the decomposed system are described by the following set of equations:*

$$x = \Gamma_s \bar{x}, \quad y = \Gamma_o \bar{y}, \quad u = \Gamma_i \bar{u}, \tag{1.2}$$

$$\bar{x} = \begin{bmatrix} x_a \\ x_d \end{bmatrix}, x_a \in \mathbb{R}^{n_a}, \quad x_d \in \mathbb{R}^{n_d}, \quad x_b = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_d} \end{bmatrix}, \tag{1.3}$$

and

$$\Sigma_{SCB} : \begin{cases} \dot{x}_a & = A_{aa}x_a + H_{ad}y \\ \dot{x}_{d,1} & = x_{d,2}, \quad y = x_1, \\ \dot{x}_{d,j} & = x_{d,j+1} \\ & \vdots \quad j = 1, ..., n_d - 1 \\ \dot{x}_{d,n_b} & = a_{d,1}x_{d,1} + a_{d,2}x_{d,2} + ... + a_{d,n_d}x_{d,n_d} + \omega \end{cases} \tag{1.4}$$

Similar to property (∗∗about Strong Obs) we have:

Property 1.1. *The system* Σ_{SCB} *in* (1.4) *is strong observable if and only if* x_a *is non-existent.*

This is equivalent to have relative degree n with respect to the unknown input $\omega(t)$. This latter is a sufficient condition of strong observability presented in [2].

If we assume strong observability, then we can apply an extra transformation $\Gamma_{obv} = \mathcal{O}^{-1}$ which puts the system in observability canonical form.

1.1.1 Unknown Input Observer design

Given a strong observable system Σ in (1.1) under the SCB transformation, then it is given by

$$\Sigma_s : \begin{cases} \dot{x}_1 & = x_2, \quad y = x_1, \\ \dot{x}_j & = x_{j+1} \\ & \vdots \quad j = 1, ..., n_d - 1 \\ \dot{x}_n & = A_{dd}x + \omega, \end{cases} \tag{1.5}$$

where $A_{dd} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ and we have by Property 1.1 that $n_d = n$ and it is supposed the following.

Assumption 1.1. *Unknown input* $\omega(t)$ *a is uniformly bounded function,* $|\omega(t)| \leq \Delta, \quad \Delta \in \mathbb{R}_{\geq 0}$

This allow us to relax the existence conditions of UIO's in other to have an observer under strong observability only, see Section 2.2. It has to be noted that the system (1.5) is in the observability canonical form, which requires the bl-homogeneity of the observer, while the observer canonical form can be implemented with a homogeneous differentiator (as already done in [3]). In this latter, the bl-homogeneous observer can also be implemented.

The observer is given by

$$\Omega : \begin{cases} \dot{\hat{x}}_1 & = -k_1 L \tilde{\phi}_1(\hat{x}_1 - y) + \hat{x}_2 \\ \dot{\hat{x}}_j & = -k_j L^j \tilde{\phi}_j(\hat{x}_1 - y) + \hat{x}_{j+1} \\ \vdots & j = 1, ..., n-1 \\ \dot{\hat{x}}_n & = -k_n L^n \tilde{\phi}_n(\hat{x}_1 - y) + A_{dd}\hat{x} \end{cases} \tag{1.6}$$

with positive external gains $k_j > 0$ and positive tuning gains $\alpha, L > 0$, appropriately selected as it will be show latter. The output injection terms $\tilde{\phi}_j(\cdot)$ are obtained from the functions

$$\phi_j(s) = \kappa_j \lceil s \rfloor^{\frac{r_{0,j+1}}{r_{0,1}}} + \theta_j \lceil s \rfloor^{\frac{r_{\infty,j+1}}{r_{\infty,1}}} \tag{1.7}$$

y scaling the positive internal gains $\kappa_j > 0, \theta_j > 0$

$$\kappa_j \to \left(\frac{L^n}{\alpha}\right)^{\frac{jd_0}{r_{0,1}}} \kappa_j, \qquad \theta_j \to \left(\frac{L^n}{\alpha}\right)^{\frac{jd_\infty}{r_{\infty,1}}} \theta_j \tag{1.8}$$

with powers selected as $r_{0,n} = r_{\infty,n} = 1$, and

$$\begin{aligned} r_{0,j} &= r_{0,j+1} - d_0 = 1 - (n-j)d_0 \\ r_{\infty,j} &= r_{\infty,j+1} - d_\infty = 1 - (n-j)d_\infty \end{aligned} \tag{1.9}$$

which are completely defined by two parameters d_0, d_∞. They have to satisfy $-1 \le d_0 \le d_\infty < \frac{1}{n-1}$.

We have to highlight the fact that injection terms in (1.6) and (1.7) are very similar to them in the bl-homogeneous differentiator (**bl-hom dif) but these latter are simpler. This simplifies the task of implementation. Then we can state the main result of this section.

1.1.2 Gain Selection

Each type of gain on the observer has a different role, and the idea in the gain tuning is very intuitive.

1. The internal gains $\kappa_j > 0, \theta_j > 0$ can be selected arbitrary. They can be selected as arbitrary positive values, and correspond to the desired weighting of each term of low degree and high degree respectively in ϕ_j.

2. The external gains $k_j > 0$ have the objective of stabilizing the observer in absence of interconnections and external perturbations, i.e. when $A_{dd} = 0$ and $\omega(t) = 0$.

3. Parameter L is selected large enough to assure the convergence in presence of interconnections, but not of the bounded perturbations $\omega(t)$. Setting its value grater than minimal value to assure stability the convergence velocity will be increased.

4. The tuning parameter α is selected large enough to assure the convergence in presence of the unknown bounded input $\omega(t)$.

1.1.3 Estimation in original coordinates

The estimated states obtained from the observer ω in (1.6) corresponds to the transformed system Σ_{SCB} in (1.5) represented in SCB coordinates through the state Γ_s, input Γ_i and output Γ_o transformations (1.2), moreover it was applied an extra transformation $T_{obv} = \mathcal{O}^{-1}$ which puts the system in observability canonical form. Therefore the estimated states in original coordinates for (1.1) are given by

$$x = \Gamma_s \Gamma_{obv} \hat{x} \tag{1.10}$$

Theorem 1.2. *Let the strong observable SISO-LTI system Σ (1.1) in original coordinates, there exist a set of transformations such that the transformed system Σ_s (1.5) has an UIO given by (1.6). Selecting $-1 \le d_0 \le d_\infty < \frac{1}{n-1}$ and chose arbitrary (internal gains) $\kappa_j > 0$ and $\theta_j > 0$, for $j = 1, ..., n$. Suppose that either $\Delta = 0$ or $d_0 = -1$. Under this conditions, there exist appropriate gains $k_j > 0$, for $j = 1, ..., n$, and parameters $L > 0, \alpha > 0$ sufficiently large such that the solutions of bl-homogeneous observer (1.5) converge globally and asymptotically to the true states of Σ_s, i.e. $\hat{x}_j(t) \to x_j(t)$ as $t \to \infty$. In particular, it can converge globally and*

- *exponentially if $d_0 = 0$,*
- *finite-time if $d_0 < 0$,*
- *fixed-time if $d_0 < 0$ and $d_\infty > 0$ subject to*

$$\begin{aligned}
&(a) \qquad -1 < d_0 < 0 < d_\infty < \frac{1}{n-1} \quad if \quad \omega(t) \equiv 0, \quad or \\
&(b) \qquad -1 = d_0 < 0 < d_\infty < \frac{1}{n-1} \quad if \quad \omega(t) \le \Delta.
\end{aligned}$$

Proof. Theorem 1.2.
The proof will be carry out in a Lyapunov framework through a bl-homogeneous Lyapunov function, this one can be used to realize an estimation fo the convergence time and calculation of gains k_j moreover in an optimal sense. Part of this work had been presented in [4]. This work does not address the problem.

Let the estimation error $e_j = \hat{x}_j - x_j$. The dynamics error are described by

$$\Xi : \begin{cases} \dot{e}_1 &= -k_1 L \tilde{\phi}_1(e_1) + \hat{e}_2 \\ \dot{e}_j &= -k_j L^j \tilde{\phi}_j(e_1) + \hat{e}_{j+1} \\ \ \vdots & j = 1, ..., n-1 \\ \dot{e}_n &= -k_n L^n \tilde{\phi}_n(e_1) + A_{dd}\hat{e} - \omega \end{cases} \tag{1.11}$$

where, by Assumption 1.1 $\omega(t) < \Delta$. Applying the time scaling via the next transformation

$$\epsilon_j = \frac{L^{n-j+1}}{\alpha} e_j, \quad j = 1, ..., n \tag{1.12}$$

we obtain

$$\Xi_{d,i} : \begin{cases} \dot{\epsilon}_1 &= L\left[-k_1 \phi_1(\epsilon_1) + \epsilon_2\right] \\ \dot{\epsilon}_j &= L\left[-k_j \phi_j(\epsilon_1) + \epsilon_{j+1}\right] \\ \ \vdots & j = 1, ..., n-1 \\ \dot{\epsilon}_n &= L\left[-k_n \phi_n(\epsilon_1) + \frac{1}{\alpha}\Psi_i(\epsilon, \omega)\right] \end{cases} \tag{1.13}$$

where

$$\Psi(\epsilon,\omega) = A_{dd}e - \omega = \sum_{j=1}^{n} a_j e_j - \omega = \alpha \sum_{j=1}^{n} \frac{a_j}{L^{n-j+1}} \epsilon_j - \omega \tag{1.14}$$

the fact that $\tilde{\phi}_j(\frac{\alpha}{L^n}s) = \frac{\alpha}{L^n}\tilde{\phi}_j(s)$ has been used.

For the convergence proof, it is convenient to perform another state transformation

$$z_j = \frac{\epsilon_j}{k_{j-1}}, \quad k_0 = 1, \quad j = 1, ..., n \tag{1.15}$$

Then (1.13) become

$$\Xi^* : \begin{cases} z_1' &= -\tilde{k}_1 \left(\phi_1(z_1) + z_2\right) \\ z_j' &= -\tilde{k}_j \left(\phi_j(z_1) + z_{j+1}\right) \\ \vdots & j = 1, ..., n-1 \\ z_n' &= -\tilde{k}_n \phi_n(z_1) + \tilde{\Psi}(z,\omega) \end{cases} \tag{1.16}$$

with $\tilde{k}_j = \frac{k_j}{k_{j-1}}, \quad k_0 = 1, \quad j = 1, ..., n$ and

$$\tilde{\Psi}(z,\omega) = \frac{1}{k_{n-1}} \sum_{j=1}^{n} \frac{a_j k_{j-1}}{L^{n-j+1}} z_j - \frac{1}{\alpha k_{n-1}} \omega \tag{1.17}$$

Lyapunov analysis

Before presenting the Lyapunov function we have to recall that the output injection terms in (1.7) are much simpler than those described in [4]. However, the stability proof in [4] for the differentiator before described in (**section diff Moreno) is applicable to the case with the simpler injection terms (1.7), since the same requirements and properties are fulfilled. The functions (1.7) can be written as a composition of functions $\varphi_j(s)$. Such that

$$\phi_j(s) = \varphi_j \circ ... \circ \varphi_2 \circ \varphi_1(s) \tag{1.18}$$

where

$$\begin{aligned} \varphi_1(s) &= \phi_1(s) \\ \varphi_2(s) &= \phi_2 \circ \phi^{-1}(s) \\ &\vdots \quad j = 2, ..., n \\ \varphi_j(s) &= \phi_j \circ \phi_{j-1}^{-1}(s), \quad j = 2, ..., n \end{aligned} \tag{1.19}$$

We will use a (smooth) bl-homogeneous Lyapunov Function (bl-LF) V, which was introduced in [4]. Selecting for $n \geq 2$ two positive real numbers $p_0, p_\infty \in \mathbb{R}_+$ that correspond to the homogeneity degrees of the 0-limit and the ∞-limit approximations of V, such that

$$\begin{aligned} p_0 &\geq \max_{j\in\{1,...,n\}} \{r_{0,j}\} + d_0 \\ p_\infty &\geq \max_{j\in\{1,...,n\}} \left\{ 2r_{\infty,j} + \frac{r_{\infty,j}}{r_{0,j}} d_0 \right\} \\ \frac{p_0}{r_{0,j}} &\leq \frac{p_\infty}{r_{\infty,j}} \end{aligned} \tag{1.20}$$

For $i = 1, ..., n$ choosing arbitrary positive real numbers $\beta_{0,i}, \beta_{\infty,i} > 0$ such that the following functions are defined

$$\begin{aligned} Z_j(z_j, z_{j+1}) &= \sum_{k\in\{0,\infty\}} \beta_{k,j} \left[\frac{r_{k,j}}{p_k} |z_j|^{\frac{p_k}{r_{k,j}}} - z_j \lceil \xi_j \rfloor^{\frac{p_k - r_{k,j}}{r_{k,j}}} + \frac{p_k - r_{k,j}}{p_k} |\xi_j|^{\frac{p_k}{r_{k,j}}} \right] \\ \xi_j &= \varphi_j^{-1}(z_{j+1}) \quad j = 1, ..., n-1 \\ \xi_j &= z_{n+1} = 0, \quad j = n \\ Z_n(z_n) &= \beta_{0,n} \frac{1}{p_0} |z_n|^{p_0} + \beta_{\infty,n} \frac{1}{p_\infty} |z_n|^{p_\infty} \end{aligned} \tag{1.21}$$

where we have

Lemma 1.1. *[4] $Z_j(z_j, z_{j+1}) \geq 0$ for every $j = 1, ..., n$ and $Z_j(z_j, z_{j+1}) = 0$ if and only if $\varphi_j(z_j) = z_{j+1}$.*

The Bl-homogeneous Lyapunov Function (Bl-LF) is defined as

$$V(z) = \sum_{j=1}^{n-1} Z_j(z_j, z_{j+1}) + Z_n(z_n) \tag{1.22}$$

For the partial derivatives we introduce the following variables

$$\begin{aligned} \sigma_j(z_j, z_{j+1}) &\triangleq \frac{\partial Z_j(z_j, z_{j+1})}{\partial z_j} = \sum_{k\in\{0,\infty\}} \beta_{k,j} \left(\lceil z_j \rfloor^{\frac{p_k - r_{k,j}}{r_{k,j}}} - \lceil \xi_j \rfloor^{\frac{p_k - r_{k,j}}{r_{k,j}}} \right) \\ s_j(z_j, z_{j+1}) &\triangleq \frac{\partial Z_j(z_j, z_{j+1})}{\partial z_{j+1}} = \sum_{k\in\{0,\infty\}} -\beta_{k,j} \frac{p_k - r_{k,j}}{r_{k,j}} (z_j - \xi_j) |\xi_j|^{\frac{p_k - 2r_{k,j}}{r_{k,j}}} \frac{\partial \xi_j}{z_{j+1}} \end{aligned} \tag{1.23}$$

where $\xi_i = \varphi_i^{-1}(z_{i+1})$. Note that $Z_{b,\iota,j}, \sigma_{b,\iota,j}, s_{b,\iota,j}$ vanish when $\varphi_{b,\iota,j}(z_{b,\iota,j}) = z_{b,\iota,j+1}$. Performing time derivative with respect the new time variable τ

$$V'(z) = -W(z) + \frac{\partial V(z)}{\partial z_n} \tilde{\Psi}(z, \omega) \tag{1.24}$$

where $\frac{\partial V(z)}{\partial z_n} = [s_{n-1} + \sigma_n]$ and

$$\begin{aligned} W(z) = & \tilde{k}_1 \sigma_1(\phi_1(z_1) - z_2) \\ & + \sum_{j=2}^{n-1} \tilde{k}_j [s_{j-1} + \sigma_j] (\phi_j(z_1) - z_{j+1}) \\ & + \tilde{k}_n [s_{n-1} + \sigma_n] \phi_n(z_1) \end{aligned} \tag{1.25}$$

Due to the definition of s_j in (1.23), $s_n \equiv 0$ and functions $s_j, \sigma_j \in \mathcal{C}$ in $\mathbb{R}$, are r-bl-homogeneous of degrees $p_0 - r_{0,j}, p_0 - r_{0,j+1}$ for the 0-approximation and $p_\infty - r_{\infty,j}, p_\infty - r_{\infty,j+1}$ for the ∞-approximation, respectively. Additionally, for $j = 1, ..., n$ we have $\sigma_j = 0$ on the same set as $s_j = 0$, i.e. they become both zero at the points where Z_j achieves its minimum, $Z_j = 0$.

V is bl-homogeneous of degrees p_0 and p_∞ and $\mathcal{C}$ on $\mathbb{R}$. It is also non negative, since it is a positive combination of non negative terms. Moreover, V is positive definite since $V(z) = 0$ only if all $Z_j = 0$, what only happens at $z = 0$. Due to bl-homogeneity it is also radially unbounded.

If we analyze (1.25), $W(z)$ is bl-homogeneous of degree $p_0 + d_0$ for the 0-approximation and $p_\infty + d_\infty$ for the ∞-approximation.

It has been shown in [4] that there exists appropriate gains $\tilde{k}_j$ such that $W(z)$ in (1.25) is rendered positive definite. The idea in the following is to prove that there exist gains L, α sufficiently large such that the negative definiteness of $-W(z)$ and therefore $V'(z)$ is hold.

From (1.24), we are now interested in finding an upper bound of $\tilde{\Psi}$. Assuming $L \geq 1$, and $\alpha \geq 1$. Due to the power of L we can write

$$\begin{aligned} \tilde{\Psi}(z,\omega) &= \sum_{j=1}^{n} \frac{a_j k_{j-1}}{k_{n-1} L^{n-j+1}} z_j - \frac{1}{\alpha k_{n-1}}\omega = \frac{1}{L}\tilde{\Psi}_s + \frac{1}{\alpha}\tilde{\Psi}_\omega \\ \tilde{\Psi}_s &= \sum_{j=1}^{n} \frac{a_j k_{j-1}}{k_{n-1} L^{n-j}} z_j, \quad \tilde{\Psi}_\omega = -\frac{1}{k_{n-1}}\omega \end{aligned} \tag{1.26}$$

The term $\frac{\partial V(z)}{\partial z_n}$ is bl-homogeneous of degree $p_0 - r_{0,n} = p_0 - 1$ for the 0-approximation and $p_\infty - r_{\infty,n} = p_\infty - 1$ for the ∞-approximation. Using the properties of bl-homogeneous functions, it is clear that each term $\frac{\partial V(z)}{\partial z_n} z_j$ is bl-homogeneous of degree $p_0 - r_{0,n} + r_{0,j} = p_0 - (n-j)d_0$ for the 0-approximation and $p_\infty - r_{\infty,n} + r_{\infty,j} = p_\infty - (n-j)d_\infty$ for the ∞-approximation. Finally, since $d_0 \leq 0$ and $d_\infty \geq 0$ we can conclude that

$$\begin{aligned} p_0 + d_0 &\leq p_0 - (n-j)d_0 \\ p_\infty + d_\infty &\geq p_\infty - (n-j)d_\infty \end{aligned} \tag{1.27}$$

and by the properties of bl-homogeneous functions, there exists a positive real number $\lambda_1 > 0$ which satisfy

$$\frac{\partial V(z)}{\partial z_n}\frac{1}{L}\tilde{\Psi}_s \leq \frac{\lambda_1}{L} W(z) \tag{1.28}$$

furthermore, there exists $\lambda_2 > 0$ such that

$$\frac{\partial V(z)}{\partial z_n} \leq -\lambda_2 \left[W(z)^{\frac{p_0 - 1}{p_0 + d_0}} + W(z)^{\frac{p_\infty - 1}{p_\infty + d_\infty}} \right] \tag{1.29}$$

If we put everything together, $V'(z)$ can be bounded as

$$\begin{aligned} V'(z) &\leq -W(z) + \frac{\lambda_1}{L}W(z) + \frac{\lambda_2}{\alpha}\left[W(z)^{\frac{p_0 - 1}{p_0 + d_0}} + W(z)^{\frac{p_\infty - 1}{p_\infty + d_\infty}} \right] \|\tilde{\Psi}_\omega\|_\infty \\ &= -\left(1 - \frac{\lambda_1}{L}\right) W(z) + \frac{\lambda_2}{\alpha}\left[W(z)^{\frac{p_0 - 1}{p_0 + d_0}} + W(z)^{\frac{p_\infty - 1}{p_\infty + d_\infty}} \right] \|\tilde{\Psi}_\omega\|_\infty \end{aligned} \tag{1.30}$$

If we apply Lyapunov arguments we can conclude that we can chose L sufficiently large such that the first term become negative definite. In absence of $\tilde{\Psi}_\omega$, $z = 0$ is asymptotic stable. With $d_0 < 0$ it converges in finite time, moreover, with $d_\infty > 0$ it converges in fixed-time.

In the case $\tilde{\Psi}_\omega \neq 0$ but ultimately bounded and selecting $d_0 = -1$ we can chose α sufficiently large, such that $V'(z)$ is negative definite. And then, finite-time or fixed-time stability is achieved. □

This idea will be applied in the MIMO case, but before that, it will be present some examples to illustrate the effectiveness of the observer.

1.2 Example

Let a *strongly observable* system taken from [2] and given by

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu + D\omega \\ y &= Cx \end{cases} \tag{1.31}$$

where $x \in \mathbb{R}^4, u \in \mathbb{R}, \omega \in \mathbb{R}, y \in \mathbb{R}$ are the states, known input, unknown input and output respectively. Note that the system is unstable since the matrix A has eigenvalues $\Lambda = \{-3, -2, -1, 1\}$ and one of them is positive, it means that one or more state trajectories can grow unboundedly.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix}, \quad B = D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ \omega(t) &= \cos(0.5t) + 0.5\sin(3t) + 0.5, \quad |\omega(t)| \leq 2 \end{aligned} \tag{1.32}$$

which can be written as

$$\Sigma : \begin{cases} \dot{x}_{b,1} &= x_{b,2}, \qquad y_1 = x_{b,1} \\ \dot{x}_{b,2} &= x_{b,3} \\ \dot{x}_{b,3} &= x_{b,4} \\ \dot{x}_{b,4} &= [6 \quad 5 \quad -5 \quad -5]x + \omega + u \end{cases} \tag{1.33}$$

The system is already in observability canonical form. Furthermore, the UIO can be designed as

$$\Omega : \begin{cases} \dot{\hat{x}}_1 &= -k_1 L \tilde{\phi}_1(\hat{x}_1 - y_1) + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -k_2 L^2 \tilde{\phi}_2(\hat{x}_1 - y_1) + \hat{x}_3 \\ \dot{\hat{x}}_3 &= -k_3 L^3 \tilde{\phi}_3(\hat{x}_1 - y_1) + \hat{x}_4 \\ \dot{\hat{x}}_4 &= -k_4 L^4 \tilde{\phi}_4(\hat{x}_1 - y_1) + [6 \quad 5 \quad -5 \quad -5]\hat{x} + u \end{cases} \tag{1.34}$$

where the nonlinear output injection terms $\tilde{\phi}_\cdot(\cdot)$ are as follows

$$\tilde{\phi}_j(s) = \left(\frac{L^4}{\alpha}\right)^{\frac{jd_0}{1-3d_0}} \kappa_j \lceil s \rfloor^{\frac{1-(3-j)d_0}{1-3d_0}} + \left(\frac{L^4}{\alpha}\right)^{\frac{jd_\infty}{1-3d_\infty}} \theta_j \lceil s \rfloor^{\frac{1-(3-j)d_\infty}{1-3d_\infty}}, \quad j = 1, ..., 4 \tag{1.35}$$

The assigned homogeneity degrees d_0, d_∞ in 0 and ∞ respectively in (1.35) have to satisfy

$$-1 \leq d_0 \leq d_\infty < \frac{1}{3} \tag{1.36}$$

we present three cases:

1. Linear UIO. Homogeneity degrees $d_0 = d_\infty = 0$.
2. Continuous UIO. Homogeneity degrees $d_0 = -\frac{1}{8}, d_\infty = 0$
3. HOSM-UIO. Homogeneity degrees $d_0 = -1$, $d_\infty = \frac{1}{8}$. With this selection we get a discontinuous observer. Note that $d_0 < 0 > d_\infty$.

The initial conditions of the plant states are $x_0 = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$ and $\hat{x}_j = 5, j = 1, ..., 4$ for the observer. For all cases the values of gains are fixed as

$$\left\{ k_1 = 8.6k_4^{\frac{1}{4}} \quad k_2 = 21k_4^{\frac{1}{2}} \quad k_3 = 16.25k_4^{\frac{1}{3}} \quad k_4 = 2 \right\} \tag{1.37}$$

internal gains $\kappa_j = \theta_j = 1, j = 1, ..., 4$ and parameters $L = 1, \alpha = 5$. We perform simulations along 5 seconds.

The Figures 1.2, 1.3 related to the linear and continuous cases respectively show that the observer can not exactly estimate the states of the plant, i.e. although the estimation error converges to a neighborhood of zero, it is not be able to converge exactly to zero, this is due to non-compensation of unknown input.

In the third case, with selection of homogeneity degree $d_0 = -1$ for the zero approximation (discontinuous observer) a High Order Sliding Mode (HOSM) is induced which allows the observer to compensate exactly the effect of unknown input. It is shown in Figure 1.4 that the observer achieve exact estimation of the estates, even in presence of the unknown input. This is illustrated in the last subfigures where the error norm converge to zero exactly in finite time.

In fact, in this case we get more than finite time stability, fixed time stability, i.e. there exist a $\bar{T}$ independent of e_0 such that for any initial error we have exact convergence in a time less than $\bar{T}$. In order to show this, With $L = 5$ now, the Figure 1.1 shows the norm of error vector $\|e\| = \sqrt{\sum_{j=1}^{4} e_j^2}$ for a wide range of magnitude orders in initial error $e_0 \times 10^p, p = 0, 2, ..., 14$. Despite of this, the convergence time does not increase beyond an upper bound in $\bar{T} = 5s$, more over, and as we said before this can be reduced arbitrary by increasing appropriately the value of parameter L.

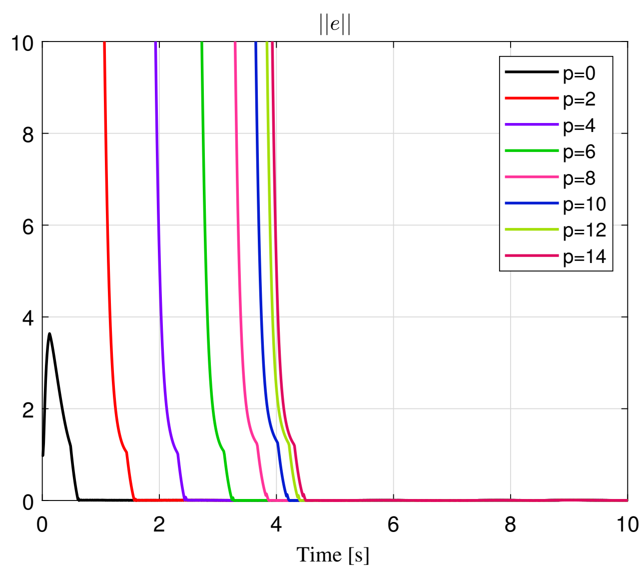

Figure 1.1: Norm of the estimation error $\|e\|$ with different orders at initial error. $e_0 \times 10^p$.

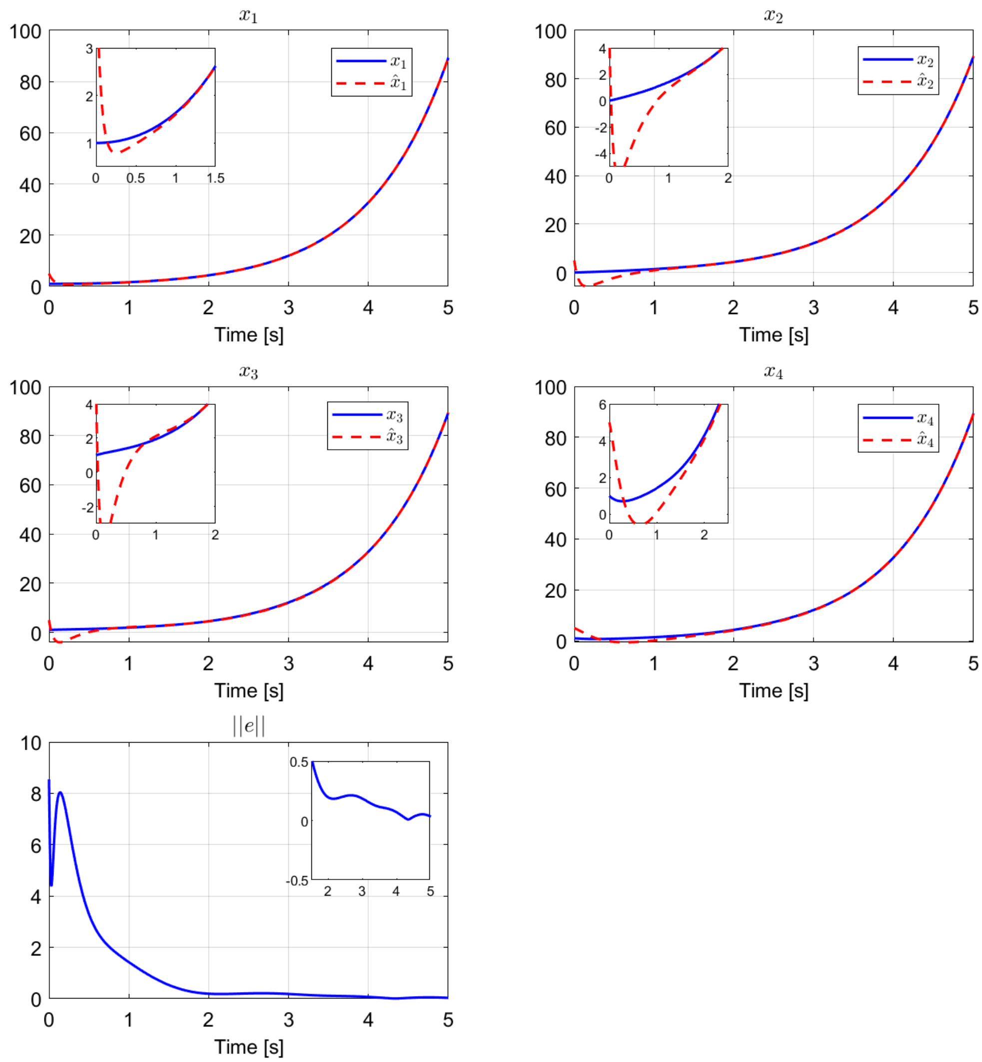

Figure 1.2: Estimation of plant states $x_1, ..., x_4$ and norm $\|e\|$.

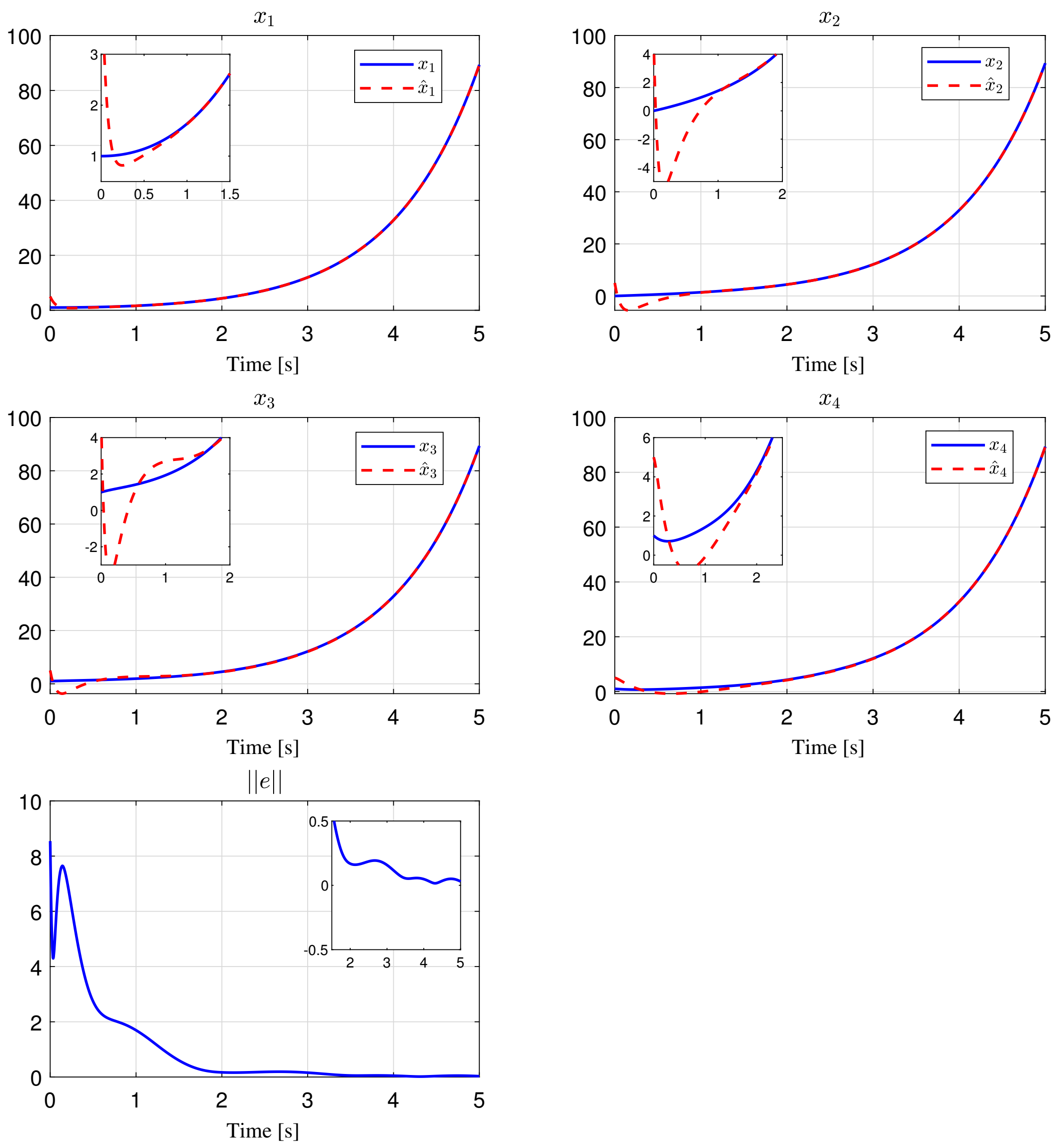

Figure 1.3: Estimation of plant states $x_1, ..., x_4$ and norm $||e||$.

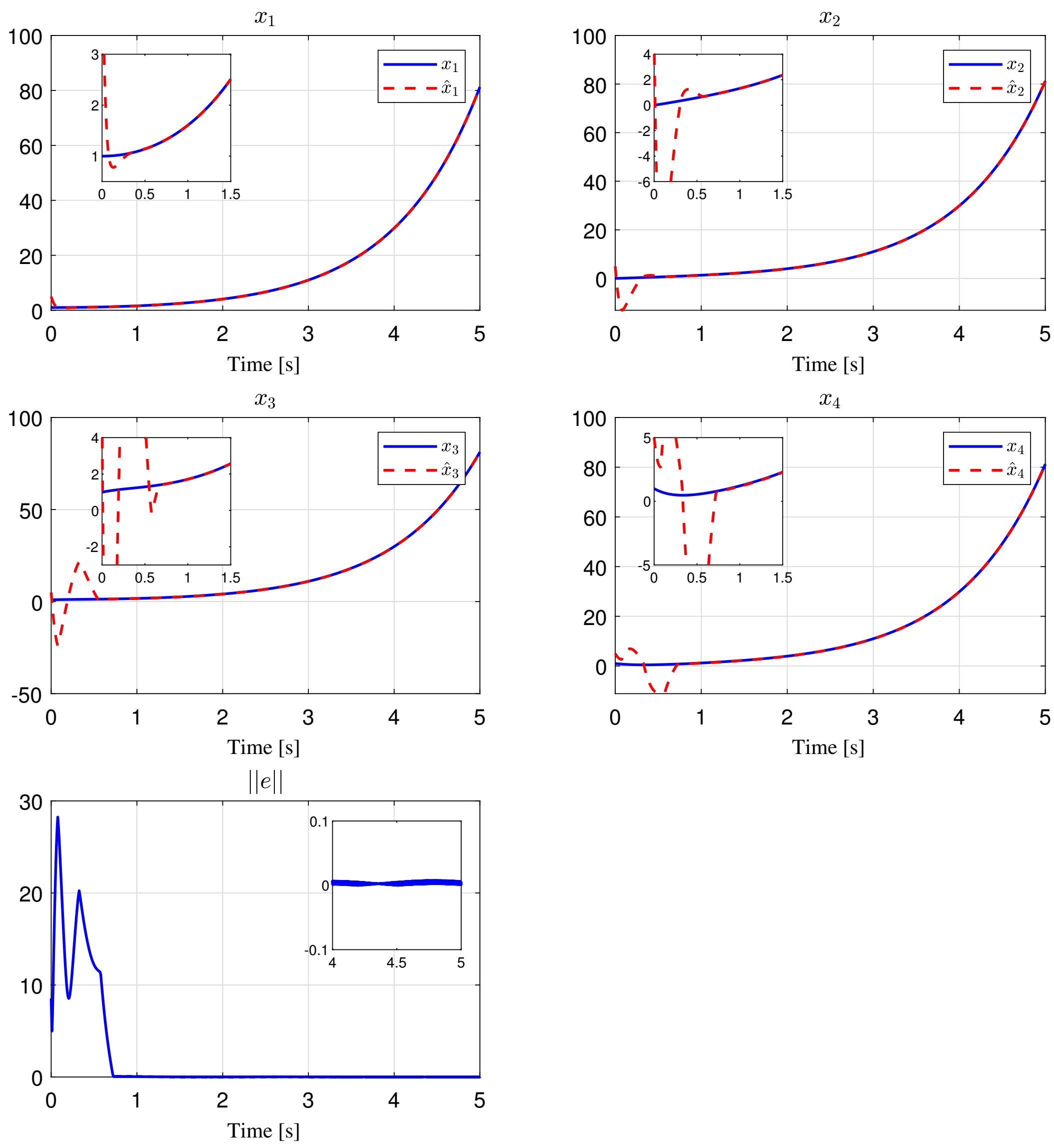

Figure 1.4: Estimation of plant states $x_1, ..., x_4$ and norm $\|e\|$.

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